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A Characterization of the state space of quantum mechanics.

by

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§1. Introduction

In mathematical foundation of quantum mechanics, one tries to formulate the basic axioms in a form which allows some direct physical interpretation and then mathematically derive the Hilbert space structure for the description of quantum mechanics. We present here one such axiomatization in terms of the geometry of the set of all states.

The notion of a mixture of physical states naturally arises from the probabilistic (but not necessarily quantum mechanical) description of the results of physical measurements and an application of Tychonof Theorem shows the existence of a theory with a compact convex set of states among physically equivalent theories in the sense of Haag and Kastler (but not necessarily in the C*-algebra framework). Thus we deal with a compact convex set K of states (in a locally convex Hausdorff topological real vector space).

In order to avoid topological complication, we limit our attention to a finite dimensional K. We shall imbed K in a hyper-plane $H = \{\varphi \in V;\ e(\varphi)=1\}$ in a real vector space V such that $e \neq 0$ is in the dual of V and K affinely span H. The cone

$$V_+ = \bigcup_{\lambda \geqq 0} \lambda K$$

is then a convex cone, making V a base-ordered space.

The convex cone V_+ which we shall be discussing has been characterized by Vinberg [1] as a homogeneous selfdual cone in a finite dimensional Euclidean space and corresponds to the dual cone of the positive part $\mathfrak{A}_+ = \{a^2; a \in \mathfrak{A}\}$ of a formally real, finite-dimensional real Jordan algebra $\mathfrak{A}$, completely classified by Jordan, von Neumann and Wigner [2]. The characterization of the natural cone for von Neumann algebras by Connes [3] has led Bellissard and Iochum to the characterization of the corresponding cone for an infinite-dimensional Jordan algebra in terms of the facial homogeneity and self-duality [4]. On the other hand Alfsen and Shultz [5] have given a characterization of the state space of an infinite-dimensional Jordan algebra, utilizing the notion of P-projections.

Our characterization is closely related to that of Alfsen and Shultz. We replace P-projections by a similar but somewhat weaker notion, first proposed by Wittstock, which has more physical appeal. Thus we obtain a characterization, which is a mathematical improvement (in the finite-dimensional case) over that of Alfsen and Shultz (the same results from individually weaker assumptions) and at the same time an axiomatization with a physical appeal.

§2. Basic Notions.

A quantum mechanical measurement (of the first kind in the sense of Pauli) of an observable A will produce eigenstates of A with a certain probability distribution when it is measured on any given state φ. If A is an observable with only two values 0 and 1 (so-called question with 0 and 1 corresponding to answers no and yes), the measurement can be viewed as a (quantum mechanical) selection procedure of states corresponding to the "yes" answer, a state $\hat{p}\varphi$ coming out with the probability $\hat{p}(\varphi)$ when the state φ is sent in. The follwoing is then a natural axiomatization of this physical concept.

Definition 2.1. A filtering $\hat{p}$ is a mapping from K into a pair of a point $\hat{p}\varphi$ in K and a number $\hat{p}(\varphi) \in [0,1]$ except $\hat{p}\varphi$ will be a new point 0 (for a notational purpose) when $\hat{p}(\varphi)=0$, such that the following conditions are satisfied:

(1) If $\hat{p}\varphi \neq 0$, $\hat{p}(\hat{p}\varphi) = \hat{p}\varphi$ and $\hat{p}(\hat{p}\varphi) = 1$. (The state $\hat{p}\varphi \in K$ is an eigenstate corresponding to the yes answer.)

(2) If $\varphi = \lambda\varphi_1+(1-\lambda)\varphi_2$ with $0 \leqq \lambda \leqq 1$, then

$$\hat{p}(\varphi) = \lambda\hat{p}(\varphi_1)+(1-\lambda)\hat{p}(\varphi_2),$$

$$\hat{p}\varphi = \alpha\hat{p}\varphi_1+(1-\alpha)\hat{p}\varphi_2$$

where

$$\alpha = \hat{p}(\varphi)^{-1}\lambda\hat{p}(\varphi_1).$$

(The mixture φ behaves as the state φ_1 with the probability λ and φ_2 with the probability $1-\lambda$.)

(3) If $\hat{p}(\varphi) = 1$, then $\hat{p}\varphi = \varphi$. (The state is an eigenstate corresponding to the yes answer in the sense that if it surely yields the answer yes, then the state is not altered by the measuring process.)

(4) There exists another map $\hat{p}'$ with all the above properties of $\hat{p}$, satisfying, in addition, the following relation for all states φ:

$$\hat{p}(\varphi)+\hat{p}'(\varphi) = 1.$$

($\hat{p}'$ corresponds to the selection of states with the answer no and is called a complement of $\hat{p}$.)

A filtering is in one-to-one correspondence with a filtering projection p defined as follows:

Definition 2.2. A filtering projection p is a linear mapping of V into V with the following properties:

(a) $p^2=p$. (Projection.)

(b) $pV_+ \subset V_+$. (Positive.)

(c) $e(p\varphi)=e(\varphi)$ and $\varphi \in V_+$ imply $\varphi=\varphi$. (Neutral.)

(d) There exists another positive, neutral projection p' satisfying

$$e(p\varphi)+e(p'\varphi) = e(\varphi)$$

for all $\varphi \in V_+$.

The correspondence between a filtering $\hat{p}$ and a filtering projection p is given by $\hat{p}(\varphi)=e(p\varphi)$ and $\hat{p}\varphi=e(p\varphi)^{-1}p\varphi$ (if $e(p\varphi)\neq 0$) for $\varphi \in K$ on one hand and $p(\lambda\varphi)=\lambda\hat{p}(\varphi)p\varphi$

for $\varphi \in K$ and $\lambda \geqq 0$ (and hence $\lambda\varphi \in V_+$) on the other.

A face f of a convex set C (K or V_+ for example) is a convex subset of C such that $\varphi = \lambda\varphi_1 + (1-\lambda)\varphi_2$ with $\varphi \in f$, $\varphi_1 \in K$, $\varphi_2 \in K$ and $0 < \lambda < 1$ implies $\varphi_1 \in f$ and $\varphi_2 \in f$. (Physically it is a set of states stable under mixing and purification.) The positive image pV_+ $(=(pV) \cap V_+)$ is always a face of V_+. A non-empty face $\hat{f}$ of V_+ is in one-to-one correspondence with a face f of K through the following relation:

of a filtering projection p

$$f = \hat{f} \cap K,$$

$$\hat{f} = \bigcup_{\lambda \geqq 0} \lambda f \qquad (\hat{f} = 0 \text{ if } f \text{ is empty.})$$

A face consisting of a single point is called an extremal point. (A pure state in physics.) A one-dimensional face of a convex cone is called an extremal rays. If φ and ψ are both pure states (i.e. extremal points of K), then the following quantity (between 0 and 1) is called the transition probability from ψ to φ:

$$t(\psi, \varphi) = \hat{p}_\varphi(\psi)$$

where $\hat{p}_\varphi$ is a filtering such that $\hat{p}\psi$ is φ for any ψ (unless $\hat{p}(\psi) = 0$).

§3. Postulates.

The first postulate requires the existence of sufficiently many filterings satisfying a certain consistency condition:

Postulate $\mathfrak{O}$ For each face f of K, there exists a filtering $\hat{p}_f$ satisfying the following conditions:

(i) $\hat{p}_f K \equiv \{\hat{p}_f \varphi\ ;\ \varphi \in K,\ \hat{p}_f(\varphi) \neq 0\} = f.$

(ii) If $f_1 \leqq f_2$, $\hat{p}_{f_1}\hat{p}_{f_2}\varphi = \hat{p}_{f_1}\varphi$ and $\hat{p}_{f_1}(\hat{p}_{f_2}\varphi)\hat{p}_{f_2}(\varphi) = \hat{p}_{f_1}(\varphi)$ for any $\varphi \in K$.

(iii) For each face f, there exists another face f' of K such that

$$\hat{p}_f(\varphi) + \hat{p}_{f'}(\varphi) = 1 \quad (\text{any } \varphi \in K).$$

The filtering projection corresponding to the filtering $\hat{p}_f$ will be denoted by p_f. The condition (ii) can be expressed as $p_{f_1} p_{f_2} = p_{f_1}$ if $f_1 \leqq f_2$, i.e. the selection of states in f_2 does not affect the selection of states in f_1 if f_1 is a subset of f_2. The condition (iii) is slightly stronger than mere existence of a complement, namely the assignment of the filtering $\hat{p}_f$ for each face f is such that there exists an f' for each f with $\hat{p}_f$ and $\hat{p}_{f'}$ complementary.

The second postulate is the symmetry of the transition probability.

Postulate $\mathfrak{N}$ $t(\psi,\varphi) = t(\varphi,\psi)$ for any pure states φ and ψ.

The above two postulates together are much stronger than the individual postulate as can be seen from examples. (In the case of postulate $\mathcal{H}$ alone, we only assume the existence of a filtering $\hat{p}_\varphi$ for each pure state φ, satisfying $\hat{p}_\varphi K = \varphi$.)

It is possible that Postulates $\mathcal{O}$ and $\mathcal{H}$ are already sufficient to reach our goal. Since we do not have either proof or a counter-example at present, we introduce the following additional postulate.

Postulate $\mathcal{P}$ If φ is a pure state, $\hat{p}_f\varphi$ is pure for any face f unless $\hat{p}_f(\varphi) = 0$.

§4. Results.

The main result is that Postulates $\mathcal{O}$, $\mathcal{H}$ and $\mathcal{P}$ for a finite dimensional compact convex cone V_+ is equivalent to V_+ being isomorphic (to) the state space (i.e. the dual cone of the positive part) of a formally real, finite-dimensional real Jordan algebra.

Our analysis is actually concentrated on the consequence of Postulates $\mathcal{O}$ and $\mathcal{H}$. By a general argument, any finite-dimensional compact convex cone can be written as a direct sum of its minimal split faces, where a face f of K is called a split face if there exists another face f' of K such that any point $\psi \in K$ can be written uniquely as $\psi = \lambda\varphi+(1-\lambda)\varphi'$ with $\varphi \in f$, $\varphi' \in f'$ and $0 \leq \lambda \leq 1$ (φ unique for $\lambda \neq 0$ and φ' unique for $\lambda \neq 1$). Therefore it is sufficient to analyze an irreducible K (i.e. K

without any non-trivial split face).

Two pure states φ and ψ are called lattice orthogonal if $\varphi' \geqq \psi$ (φ' in the sense of Postulate $\mathcal{O}$ (iii)) and metrically orthogonal if $t(\varphi,\psi) = 0$. These two notions are shown to be identical under Postulates $\mathcal{O}$ and $\mathcal{H}$. The rank of a face f (and of K itself in particular) is the maximal number of mutually orthogonal pure states in f.

There are two important technical steps in our analysis of Postulates $\mathcal{O}$ and $\mathcal{H}$. First, the lattice of all faces of K (in terms of inclusion ordering) is shown to be an orthocomplemented, orthomodular lattice under postulate $\mathcal{O}$ alone. Second, the transition probability induces an inner product in V, which is shown to be non-degenerate and positive definite, under Postulates $\mathcal{O}$ and $\mathcal{H}$.

The case of K with rank 2 can be handled by Postulates $\mathcal{O}$ and $\mathcal{H}$ (without $\mathcal{P}$). An irreducible K with rank 2 is always a ball (of radius $2^{-1/2}$) and corresponds to a Jordan algebra called a spin factor. (The algebra of $x_0+\sum x_j\sigma_j$ with real x's and totally anticommuting σ_j satisfying $\sigma_j^2 = 1$.)

The case of K with rank 3 is crucial. This is the only case where we actually use Postulate $\mathcal{P}$. There are only 4 different irreducible K with rank 3. They corresponds to the Jordan algebra of hermitian 3×3 matrices over real, complex or quaternion, or the exceptional Jordan algebra (hermitian 3×3 matrices over octanion).

For a general K, Postulates $\mathcal{O}$ and $\mathcal{H}$ (without $\mathcal{P}$

are shown to be equivalent to the following 4 properties of V_+ relative to some positive definite inner product denoted by $<\varphi,\psi>$. (This coincides with the inner product given by the transition probability.)

(α) Every non-empty face F of V_+ is self-polar in its linear span Lin F:

$$F = \{\varphi \in \text{Lin } F;\ <\varphi,\psi> \geqq 0 \ \text{ for all } \ \psi \in F\}.$$

(β) Every non-empty face F of V_+ satisfies $(F^0)^0=F$ where

$$F^0 = \{\varphi \in V_+;\ <\varphi,\psi>= 0 \ \text{ for all } \ \psi \in F\}.$$

(γ) There exists a vector e in V_+ whose orthogonal projection on any extremal ray of V_+ has a unit length.

(δ) The vector e above is in the convex span of F and F^0 for every 0 face F of V_+.

The proofs of all assertions are refered to [6].

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