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Are All Smooth Solutions of Euclidean $SU(2)$-Yang-Mills Equations Selfdual or Antiselfdual?

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I. The conjecture formulated in the titel is commonly believed to be true among physicists. The main support for it comes from an analogy relating SU(2)-Yang-Mills (Y.M.)-fields in four dimensions to the nonlinear O(3)-σ- model in two dimensions. The list of similarities between the two theories comprises on the classical level the topological structure of the S_2 and S_4 compactified versions resp. of the two models, the analytic structure of the (anti-)self-dual solutions of the corresponding field equations and on the quantum level features for which the key words are asymptotic freedom and infrared behaviour.

It is well known |1|, that all classical solutions of the nonlinear O(3)-σ- model on S_2 are in fact selfdual or antiselfdual.

The proof for an analogous theorem for SU(2) - Y.M.$_4$ is an open problem. For the time being only local theorems on the uniqueness of (anti-)self-dual solutions are known. One of these theorems will be discussed in following (cf. |2|). For another approach leading to equivalent results see ref. |3|. After the R.C.P. no 25 has taken place, I learnt of an interesting paper by Bourguignon, Lawson and Simons |4|. These authors derive a global C^0 - bound for the integrand of the Y.M. action (i.e. a pointwise estimate) by which non-(anti-) self-dual solutions are excluded.

Our results, presented in this note, are based on an estimate involving a higher Sobolev norm. We have not been able to sharpen this estimate so that it is uniformly applicable to all Pontryagin classes. In this sense our results are weaker than those of Bourguignon et al..
However for each single Pontryagin class one can give also within our approach a uniform estimate (which becomes weaker for increasing Pontryagin number) for a Sobolev norm neighbourhood of the manifold of (anti-) self-dual connections where non-(anti-) self-dual solutions do not appear. Since a C^0-estimate on one hand and a Sobolev norm estimate on the other hand yield rather different sets of functions, the results of Bourguignon et.al. and ours may be considered as complementing each other.

II. The main idea of our approach is to consider a Dirac eigenvalue equation instead of attacking the Y.M. equations directly. The Dirac eigenvalue equation transformed from S_4 to R_4 is:

$$\mathcal{D}\psi_n \equiv \frac{1+\|x\|^2}{2}\,\gamma\cdot D\,\psi_n = \lambda_n\,\psi_n,$$

where $\gamma D = \gamma_\mu(\partial_\mu + A_\mu \otimes)$,

and $\|x\|^2 = \sum_{i=1}^{4} x_i^2$. $\{\gamma_\mu\}$ denotes a representation of the Euclidean Clifford algebra: $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ The spinors ψ_n are chosen to carry the adjoint representation of SU(2)." $\otimes$ " denotes the ordinary 3-vector product. A_μ stands for a SU(2) connection over R_4, which has a smooth counterpart on S_4.

Let $n_+(n_-)$ be the number of zero modes of $\not{D}$ with positive (negative) chirality. As a consequence of the Atiyah-Singer index theorem |5| one obtains the relation:

$$|n_+ - n_-| = 4|n|, \tag{1}$$

where n is the Pontryagin number of the connection A_μ. Suppose now that A_μ is a selfdual connection. From the squared Dirac operator (we omit the factors $(1 + \|x\|^2/2)$)

$$(\gamma \cdot D)^2 = D^2 + \frac{1}{4}[\gamma_\mu, \gamma_\nu] F_{\mu\nu}, \tag{2}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \otimes A_\nu$,

one can read off that all zero modes of $\not{D}$ have positive chirality. Indeed, the part of $[\gamma_\mu, \gamma_\nu]$ carrying negative chirality is antiselfdual and therefore vanishes if contracted with the (by assumption) self-dual field strength tensor $F_{\mu\nu}$. The remaining term on the r. h. s. of (2) with negative chirality is a positive definite operator, which has no zero eigenvalues. One has therefore in the special case that A_μ is self-dual the more precise result substituting equ. (1):

$$n_+ = 4n \;, \quad n_- = 0 \,. \tag{3}$$

The crucial point is now that all connections solving the Y.M. equations and rendering the zero-modes characteristics as specified in (3) for the associated Dirac operator are necessarily self-dual. The reason for that is the following: from any solution of the Y.M. equation one can construct zero modes of the associated Dirac operator by setting

$$\psi = F_{\mu\nu} [\gamma_\mu, \gamma_\nu] \chi \, , \qquad (4)$$

where χ is a constant spinor carrying no SU(2) indices. ψ is a zero mode of the Dirac operator in virtue of the Y.M. equ. of motion satisfied by A_μ i.e.: $D_\mu(A)F_{\mu\nu} = 0$. The construction (4) renders zero modes of either chirality if $F_{\mu\nu}$ is neither self-dual nor anti-self-dual and therefore rel. (3) is not satisfied in this case.

To obtain a local uniqueness theorem for (anti-) self-dual solutions of the Y.M. equations one has to find a criterion for deformations of (anti-)self-dual connections which guarantees that the zero mode characteristics (3) is not changed under the deformations. For this purpose we define suitable Sobolev spaces H_0 and H_1 *) such that D maps H_0 continously into H_1 and the inverse D^{-1} maps the complement of the (finite dimensional) zero mode space of D continously into H_0. D and D^{-1} acting on these spaces are Fredholm operators for which an easily accesible perturbation theory is available. In particular the following lemma, well known from the functional analysis of Fredholm operators, is useful:

Lemma: Let $\|D\|$ and $\|D^{-1}\|$ be the operator norms of D: $H_0 \to H_1$. D^{-1}: range $D(c\, H_1) \to H_0/$ (ker D). If D' is a deformation of D such that $\|D - D'\| \leq \frac{1}{\|D^{-1}\|}$ then D' has not more zero modes than D.

The Dirac operator associated to a (anti-)self-dual connection, say A, has the minimal number of zero modes compatible with the Atiyah-Singer index theorem (cf. equ. (3)). This is also true according to the above lemma for any connection A' satisfying $\|D(A)-D(A')\| \leq \frac{1}{\|D(A)^{-1}\|}$ and therefore A' is only a solution of the

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*) For H_0 we take a L_2-space. The constituting norm of H_1 contains the modulus of first derivatives.

Y.M. equations if it is (anti-)self-dual.
This is the local uniqueness theorem for (anti-) selfdual connections announced in the introduction.

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