R. Temam

Qualitative Properties of Navier-Stokes Equations

Les rencontres physiciens-mathématiciens de Strasbourg - RCP25, 1978, tome 26
« Conférences de : A. Andreotti, A. Koranyi, J.L. Lebowitz, L. Michel et R. Teman », ,
exp. n° 5, p. 49-58

<http://www.numdam.org/item?id=RCP25_1978__26__49_0>

Communication at the International Symposium on Partial Differential Equations and Continuum Mechanic, Rio de Janeiro, August 1977,
Proceedings edited by L.A. MEDEIROS, NOrth Holland Publishers

QUALITATIVE PROPERTIES OF NAVIER-STOKES EQUATIONS

R. TEMAM

Département de Mathématiques
Université de Paris Sud
91405 - Orsay, France

The purpose of this lecture is to present some new properties of the set of steady-state solutions to the Navier-Stokes equations of a viscous incompressible fluid.

It is known that for small Reynolds numbers, if a steady excitation is applied to the fluid then there is a unique stable steady state which actually appears for t large $(t \to \infty)$. If the Reynold's number increases, then it is conjectured and experimentally well-known (cf. B.T. Benjamin [1,2], D. Joseph [7]) that new steady states appear some being stable and some being unstable. As far as we know, very little has been proved concerning the set of all stationary solutions of the equations. In joint works with C. Foias (cf [4][5][6]) the author has attempted to find some qualitative informations on this set, and we are going to summarize the main results of [5][6].

Section 1 contains the description of Navier-Stokes equations and their functional setting. Section 2 contains the description of the results. The plan is the following:

1. Steady-state Navier-Stokes Equations.
2. Properties of $S(f,\varphi,\nu)$
   2.1 General properties

1. Steady-State Navier-Stokes equations.

Soient Ω be the domain filled by the fluid, $\Omega \subset \mathbb{R}^{\ell}$, $\ell = 2$ or 3. We assume that Ω is bounded with a smooth boundary Γ.

Let $u(x) = (u_1(x),\ldots,u_k(x))$, and $p(x)$ be the velocity of the particle of fluid at point x and the pressure at x $(x \in \Omega)$; then u and p satisfy the equations

$$(1.1) \qquad -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f \quad \text{in} \quad \Omega,$$

$$(1.2) \qquad \nabla u = 0 \quad \text{in} \quad \Omega,$$

$$(1.3) \qquad u = \varphi \quad \text{on} \quad \Gamma,$$

where f represents volumic forces, φ is the given velocity of Γ which is assumed to be materialized and solid, $\nu = Re^{-1}$ is the inverse of a Reynolds number. For f, φ, ν (and Ω) given, the problem is to find u and p satisfying (1.1)-(1.3).

In the functional setting of the equation, it is usual to introduce the space $\mathbb{L}^2(\Omega) = L^2(\Omega)^{\ell}$ and to consider the orthogonal decomposition of $\mathbb{L}^2(\Omega)$ (cf O.A. Ladyzhenskaya [8] or R. Temam [14]) ;

$$\mathbb{L}^2(\Omega) = H \oplus G,$$

$$G = \{\nabla p \mid p \in L^2(\Omega),\ \frac{\partial p}{\partial x_i} \in L^2(\Omega), \quad 1 \le i \le \ell\}$$

$$H = \{u \in \mathbb{L}^2(\Omega) \mid \nabla u = 0, \quad u\cdot\nu\big|_{\Gamma} = 0\},$$

with ν the unit outward normal vector on Γ. We denote by

P the projection in $\mathbb{L}^2(\Omega)$ onto H.

If u and p satisfy (1.1)-(1.3) and are sufficiently regular, then u clearly satisfies

$$P(-\nu\Delta u + (u\cdot\nabla)u) = f \quad \text{in} \quad \Omega, \tag{1.4}$$

assuming that $Pf = f$, which is always possible and amounts to modifying p. Conversely it is classical (cf [8]) that if u satisfies (1.4) together with (1.2) and (1.3), then there exists a scalar function p such that u, p satisfy (1.1)-(1.3). Therefore the equations (1.2)-(1.4) for u are equivalent to the original problem.

Now we write $u = \bar{u} + \Phi$ where Φ is some extension of the function φ inside Ω. It is convenient to define $\Phi = \Lambda\varphi$ as the solution of the nonhomogeneous Stokes problem

$$\begin{cases} -\Delta\Phi + \nabla\pi = 0 \quad \text{in} \quad \Omega, \\ \nabla\Phi = 0 \quad \text{in} \quad \Omega, \\ \Phi = \varphi \quad \text{on} \quad \Gamma. \end{cases} \tag{1.5}$$

In this case $\bar{u}$ satisfies

$$P[-\nu\Delta\bar{u} + ((\bar{u}+\Phi)\cdot\nabla)(\bar{u}+\Phi)] = f \quad \text{in} \quad \Omega, \tag{1.6}$$

$$\nabla\bar{u} = 0 \quad \text{in} \quad \Omega, \tag{1.7}$$

$$\bar{u} = 0 \quad \text{on} \quad \Gamma. \tag{1.8}$$

We introduce the linear unbounded operator A in H, whose domain is

$$D(A) = \mathbb{H}^2(\Omega) \cap \mathbb{H}^1_0(\Omega) \cap H,$$

and

$$Av = -P\Delta v, \qquad \forall\, v \in D(A);$$

here $H^m(\Omega)$ is the Sobolev space of order m, $\mathbb{H}^m(\Omega) = [H^m(\Omega)]^{\ell}$

and $H_0^1(\Omega)$ is the space of $u \in H^1(\Omega)$ vanishing on Γ (for the theory of Sobolev spaces cf J.L. Lions - E. Magenes [11]). It is known (cf [14]) that A is a self-adjoint strictly positive and invertible operator in H, and that $A^{-1} \in \mathcal{L}(H)$ is compact (Ω bounded).

We also introduce the operators B such that

$$B(u,v) = P[(u\cdot\nabla)v],$$
$$B(u) = B(u,u).$$

For $\ell = 2$ or 3, we infer from the Sobolev imbedding theorems that $B(\cdot,\cdot)$ maps $\mathbb{H}^2(\Omega) \times \mathbb{H}^2(\Omega)$ into H, and in particular $D(A) \times D(A)$ into H.

The equations (1.6)-(1.8) are now equivalent to the problem

$$(1.9) \qquad \begin{cases} \text{To find } \bar{u} \text{ in } D(A) \text{ such that} \\ \nu A\bar{u} + B(\bar{u}+\Lambda\varphi) = f. \end{cases}$$

This is the functional form of the steady-state Navier-Stokes equations which we had in view and on which is based the following study. We denote by $S(f,\varphi,\nu)$ the set of $\bar{u} \in D(A)$ satisfying (1.9).

2. Properties of $S(f,\varphi,\nu)$.

We assume that f is given in H, φ is given in $\mathbb{H}^{3/2}(\Gamma) = [H^{3/2}(\Gamma)]^\ell$. By the Stokes formula, and because of (1.2), φ must verify

$$\int_\Omega \nabla u \, dx = \int_\Gamma \varphi\cdot\nu \, d\Gamma = 0.$$

We will impose a slightly stronger condition on φ:

(2.1) $$\int_{\Gamma_i} \varphi\cdot\nu \, d\Gamma = 0, \qquad 1 \le i \le N,$$

where $\Gamma_1,\ldots,\Gamma_N$ are the connected components of Γ ($\Gamma = \Gamma_1$ and $N=1$ if Γ is connected). We denote by $\dot{\mathbb{H}}^{3/2}(\Gamma)$ the set of φ in $\mathbb{H}^{3/2}(\Gamma)$ satisfying (2.1).

2.1 General properties.

For every f given in H and φ given in $\dot{\mathbb{H}}^{3/2}(\Gamma)$, the set $S(f,\varphi,\nu)$ is nonempty. This is an existence theorem for the steady state Navier-Stokes equations. This existence result appears in O.A. Ladyzhenskaya [8], J. Leray [9], J.L. Lions [10] with stronger assumptions on f and/or φ; for the weaker assumption $f,\varphi \in H \times \dot{\mathbb{H}}^{3/2}(\Gamma)$, cf. C.Foias-R.T.[6].

The set $S(f,\varphi,\nu)$ is reduceed to a single point if ν is sufficiently large, more precisely if

(2.2) $$\nu > \sigma_0\,(|f|_H, \|\varphi\|_{\mathbb{H}^{3/2}(\Gamma)}),$$

where the function $\sigma_0 : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is increasing with respect to each of its two arguments.

Now let w_j, $j \ge 1$, be the orthomormal basis in H consisting of the eigenvectors of A^{-1} (A^{-1} is compact self-adjoint in H). Let P_m denote the orthogonal projector in H onto the space spanned by $w_1,\ldots,w_m$. We have the following:

(2.3) For m sufficiently large, $m \ge m_*(f,\varphi,\nu)$, P_m is a one to one mapping on $S(f,\varphi,\nu)$ and $P_m\, S(f,\varphi,\nu)$ is a real compact C-analytic set.

This means that $P_m\, S(f,\varphi,\nu)$ (and in some sense $S(f,\varphi,\nu)$ itself) is a finite union of points, regular analytic curves, regular analytic manifolds of higher dimensions (cf Bruhat-Whitney [3]). As a corollary we get:

(2.4) Either $S(f,\varphi,\nu)$ is the union of a finite number of points or $S(f,\varphi,\nu)$ contains at least an analytic curve.

2.2 Generic Properties.

We are now going to describe generic properties of $S(f,\varphi,\nu)$. A result which is typical of the results established in [4][5] is the following

Theorem 1 - For every $\nu > 0$ and $\varphi \in \dot{H}^{3/2}(\Gamma)$ fixed, there exists an open dense set $\mathcal{O}_1 \subset H$, and for every $f \in \mathcal{O}_1$, $S(f,\varphi,\nu)$ is finite, card $S(f,\varphi,\nu)$ is odd, and card $S(f,\varphi,\nu)$ is constant on every connected component of $\mathcal{O}$.

The principle of the proof is as follows: we consider the non linear mapping N_1 from $D(A)$ into H:

$$\bar{u} \mapsto N_1(\bar{u}) = \nu A\bar{u} + B(\bar{u}).$$

When f is a regular value of N_1, the Fréchet derivative N_1' of N_1 is regular at every preimage point (i.e. every $\bar{u}$ such that $N_1(\bar{u}) = f$). Whence the $\bar{u}$'s in $N_1^{-1}(f)$ are isolated and there is a finite number of such $\bar{u}$'s, since $N_1^{-1}(f)$ is compact. The fact that the set $\mathcal{O}_1$ of regular values of N_1 is dense is the less trivial result and follows from the infinite dimensional version of Sard's theorem due to Smale [13]. The other properties are consequences of the implicit function theorem and some specific properties of N_1. Finally the oddness of card $S(f,\varphi,\nu)$ follows from a topological degree argument.

A similar result when f, φ, ν, are simultaneously allowed to vary is this one:

Theorem 2 - There exists a dense open set $\mathcal{O}_2$ in $H\times\dot{\mathbb{H}}^{3/2}(\Gamma)\times\mathbb{R}_+$, and for every $f,\varphi,\nu\in\mathcal{O}_2$, card $S(f,\varphi,\nu)$ is finite and odd. Furthermore card $S(f,\varphi,\nu)$ is constant on every connected component of $\mathcal{O}_2$.

Same proof as Theorem 1.

We may now think of a result symmetrical to Theorem 1 in the sense of a generic result with respect to φ when f and ν are fixed. We have:

Theorem 3 - For every $\nu > 0$ fixed, for every fixed f in $C^\alpha(\bar{\Omega})^\ell \cap H$ $(\alpha > 0)$, there exists a dense open set $\mathcal{O}_3 \subset \subset \mathring{\mathbb{C}}^{2+\alpha}(\Gamma)$ (1), such that card $S(f,\varphi,\nu)$ is finite and odd, $\forall\,\varphi\in\mathcal{O}_3$, and card $S(f,\varphi,\nu)$ is constant on every connected component of $\mathcal{O}_3$.

The proof of this result due to J.C. Saut and the author (cf [12]) involves different technics. In particular Sard-Smale's theorem is replaced by a transversality theorem due to Abraham and Quinns. As a tool for this proof we also need the following uniqueness theorem for a Cauchy problem associated to Stokes equations:

For b given in $H\cap H^{1,\infty}(\Omega)^\ell$, if v and q satisfy

(1) $\dot{\mathbb{C}}^s(\Gamma)$ is the set of functions φ in $C^2(\Gamma)^\ell$ which satisfy (2.1).

$$(2.5)\qquad \begin{cases} -\Delta v + (b\cdot\nabla)v + (v\cdot\nabla)b + \nabla q = 0 & \text{in } \Omega \\ \nabla v = 0 \quad \text{in } \Omega \\ v = 0 \quad \text{and} \quad \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \Gamma \end{cases}$$

then $v = 0$ and q is a constant.

Remark 1 - The fact that $S(f,\varphi,\nu)$ is generally finite was conjectured by B.T. Benjamin.

2.3 Generic Bifurcation.

We now describe a result of generic bifurcation for the equation (1.9). Similar results are proved in [6] for the classical Taylor and Bénard problems.

Theorem 4 - We assume that $\varphi \in \dot{H}^{3/2}(\Gamma)$ is fixed. There exists $\mathcal{O}_4(\varphi)$ a dense G_δ - subset of H and for every $f \in \mathcal{O}_4(\varphi)$, the manifold

$$S = \bigcup_{\nu>0} S(f,\varphi,\nu)$$

has the following form:

(i) It is constituted of isolated points and isolated analytic manifolds which lye above isolated values of ν. The number of such values of ν is finite on every semi axis $\nu \geqslant \nu_0 > 0$.

(ii) It is constituted of one (or more) analytic manifold(s) of dimension 1, whose projection on the ν axis is the whole interval $]0,+\infty[$. The set of singular points of this manifold is finite in every region $\nu \geq \nu_0 > 0$.

As a Corollary we get

(2.6) Generically, the set of (primary and secundary) bifurcating values of ν for (1.9) is countable and can only accumulate at $\nu = 0$.

Remark 2 - As far as we know, this is the first information available concerning all the primary and secundary bifurcating points of a nonlinear equation.

Remark 3 - The methods used for the proof of the above results are quite general and probably apply to the equations of nonlinear elasticity.

References

[1] T.B. Benjamin, Applications of Leray-Schauder degree theory to problems of hydrodynamic stability, Math. Proc. Camb. Phil. Soc. 79, 1976, p.373-392.

[2] T.B. Benjamin, Bifurcation Phenomena in Steady flows of a viscous fluid, Part 1 Theory, Part 2 Experiments, Reports nº 83, 84, University of Essex, Fluid Mechanics Research Institute, May 1977.

[3] F. Bruhat - H. Whitney, Quelques propriétés fondamentales des ensembles analytiques Réels, Comm. Math. Helv. 33, 1959, p. 132-160.

[4] C. Foias - R. Temam, On the stationary statistical solutions of the Navier-Stokes equations and Turbulence, Publication Mathématique d'Orsay, nº 120-75-28, Université de Paris - Sud, Orsay, 1975.

[5] C. Foias - R. Temam, Structure of the set of stationnary solutions of the Navier-Stokes equations, Comm. Pure Appl. Math., XXX, 1977, p. 149-164.

[6] C. Foias - R. Temam, Remarques sur les equations de Navier-Stokes stationnaires et les phénomènes successifs de bifuraction, Annali di Sc. Norm. Sup. di Pisa, vol. dédié à J. Leray, à paraître.

[7] D. Joseph, Stability of fluid motions, vol.I and II, Springer-Verlag New York - Heildeberg, 1976.

[8] O.A. Ladyzhenskaya, The mathematical theory of viscous incompressible flow, Gordon and Breach, New-York, 1969.

[9] J. Leray, Etude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique, J. Math. Pures Appl. XIII, 193 , p. 1-82.

[10] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linaires, Gauthier-Villars-Dunod, Paris 1969.

[11] J.L. Lions - E. Magenes, Non-homogeneous boundary value problems and applications, Springer-Verlag, Heidelberg/New-York, 1973.

[12] J.C. Saut - R. Temam, Propriétés de l'ensemble des solutions stationnaires ou périodiques des équations de Navier-Stokes: généricité par rapport aux données aux limites, Comptes Rendus Ac. Sc. Paris, 1977.

[13] S. Smale, An infinite dimensional version of Sard's Theorem, Amer. J. Math., 87, 1965, p. 861-866.

[14] R. Temam, Navier-Stokes equations, Theory and Numerical Analysis, North-Holland-Elsevier, Amsterdam, New-York, 1977.