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Highest Weights of Semisimple Lie Algebras

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HIGHEST WEIGHTS OF SEMISIMPLE LIE ALGEBRAS

by

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ABSTRACT

The nine well known semisimple Lie algebras are partitioned in two classes: $W_{\ell pc=1}$ (all roots have the same length) and $W_{\ell zc\neq 1}$ (the roots have two different lengths of ratio equal to $\sqrt{c}$).

For each of these two classes a general expression is given for few elements of interest as the highest weight vector (h.w.v.) L and its power $\delta(L)$, the eigen values of the second order Casimir operator, the width of a weight diagram, the dimensions and the matrix elements of irreducible representations of semi simple Lie algebras.

In appendix are given two examples of application of this paper.

Highest weights of semisimple Lie algebras.

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Introduction. This paper is concerned with semisimple Lie algebras defined over an algebraically closed field of characteristic zero only (in brief s.*L*.a.), i.e. with the type of algebras widely used by physicists. Calculations of highest weight vectors in particular cases [4,11-13] have of course been done already. However here the use of a general procedure yields general formulas which give a very simple proof that no other s.*L*.a. than the well known ones do exist.

To make the paper relatively self contained and to define notations we first recall the usual definitions of roots of an algebra, the Dynkin diagram and the highest weight vector (in brief h.w.v.) of a given representation of that algebra [1-14]..

In the second part the calculation of the h.w.v. is performed firstly when all the roots have the same length and secondly when the roots have two different lengths of ratio equal to $\sqrt{c}$; these two cases correspond respectively to the two classes of s.*L*.a. $W_{\ell pc=1}$ and $W_{\ell zc\neq 1}$(c = 2 or 3) .

The third part is devoted to the interpretation of the results obtained in the second part; in a first step[20] it is very simply shown that no other semisimple Lie algebras (defined over an algebraically closed field of characteristic zero) than the ones already known do exist: the four series

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A_ℓ, B_ℓ, C_ℓ, D_ℓ and the five "exceptional" Lie algebras $\{E_6, E_7, E_8, F_4, G_2\}$ that we reclassify according to our scheme as

$$W_{\ell pc=1} = \{A_\ell,\ D_\ell,\ E_\ell \quad \text{with} \quad \ell = 6,7,8 \text{ only}\}$$

$$W_{\ell zc\neq 1} = \{B_\ell,\ C_\ell,\ F_4\ ,\ G_2\}$$

In a second step we calculate and tabulate the power $\delta(L)$ of the highest weight vector L and link it to $R = \frac{1}{2}\sum_{\rho>0} \rho$; hence the eigen values of the Casimir operator and the width of a weight diagram can be deduced.

In a third step the results so obtained are used to build up the matrices of representations for the two classes of algebras (dimensions and matrix elements).

In appendix two examples are briefly studied as applications of this paper.

II. Roots, Dynkin diagram and highest weight.

The following fundamental facts are well known :

§1. If $\Sigma = \{\alpha_1,\ldots,\alpha_i,\ldots,\alpha_j,\ldots,\alpha_\ell\}$ is an irreducible fundamental system of simple roots we have

i) $\alpha_1,\ldots,\alpha_\ell$ are linearly independant ;

ii) $$\frac{2\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_i,\alpha_i\rangle} = -m\ , \qquad \frac{2\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle} = -c \qquad (m,c \in \mathbb{Z} > 0)\ ; \tag{1}$$

iii) Σ is not decomposable into two mutually orthogonal subsets.

Consequently

$$\frac{[2\langle\alpha_i,\alpha_j\rangle]^2}{\langle\alpha_i,\alpha_i\rangle\langle\alpha_j,\alpha_j\rangle} = 4\cos^2\theta = mc \leq 4 \tag{2}$$

and for $m = 1$ one only gets :

$c = 0, (\theta = 90^\circ)$;	$c = 1, (\theta = 120^\circ)$;	$c = 2, (\theta = 135^\circ)$;	$c = 3, (\theta = 150^\circ)$;
0 line (i.e. no connection)	1 line	2 lines	3 lines

$$c = 4, \begin{cases} \theta = 0 & \alpha_j = \alpha_i \\ \theta = \pi & \alpha_j = -\alpha_i\ . \end{cases} \tag{3}$$

Also

$$\frac{\dfrac{2<\alpha_i,\alpha_j>}{<\alpha_j,\alpha_j>}}{\dfrac{2<\alpha_i,\alpha_j>}{<\alpha_i,\alpha_i>}} = \frac{<\alpha_i,\alpha_i>}{<\alpha_j,\alpha_j>} = c \qquad (4)$$

i.e. the roots have only two possible lengths.

Hence

$$<\alpha_j,\alpha_j> = \lambda, \quad <\alpha_i,\alpha_i> = c\lambda, \quad <\alpha_i,\alpha_j> = \begin{cases} -\frac{c\lambda}{2} & \text{if } \alpha_i,\alpha_j \text{ are connected roots ;} \\ 0 & \text{if } \alpha_i,\alpha_j \text{ are not connected roots.} \end{cases} \qquad (5\text{-a})$$

Normalizing α_j such $\lambda = \frac{2}{c}$ yields the following relations :

$$<\alpha_j,\alpha_j> = \lambda = \frac{2}{c}, \; <\alpha_i,\alpha_i> = 2, \; <\alpha_i,\alpha_j> = \begin{cases} -1 & \text{if } \alpha_i,\alpha_j \text{ are connected roots} \\ 0 & \text{if } \alpha_i,\alpha_j \text{ are not connected roots} \end{cases} \qquad (5\text{-b})$$

§2. To every given irreducible representation (denoted I.R.) corresponds a unique vector L (in the idempotent $\mathcal{D}$) called the highest weight vector (denoted h.w.v.) of the given I.R. From this h.w.v. L all the properties of the I.R. can be deduced ;

for instance the H. Weyl formula giving the dimension N is well known :

$$N = \prod_{\mu\in\Sigma_+} \frac{(L+R,\mu)}{(R,\mu)} = \prod_{\mu\in\Sigma_+} \left[\frac{(L,\mu)}{(R,\mu)} +1 \right] \qquad (6)$$

Σ_+ being the subset of positive root and

$$R = \frac{1}{2} \sum_{\mu\in\Sigma_+} \mu \; . \qquad (7)$$

From the h.w.v. L , a set of N ordinary weight vectors $\{\lambda_1,\ldots,\lambda_r,\ldots,\lambda_N\}$ can be deduced (all distincts if there is no degeneracy) and used in turn to calculate matrices of the I.R. diagonal ones

$$(F_\mu)^r_r = (\mu,\lambda_r) \qquad \mu \in \Sigma_+ \tag{8}$$

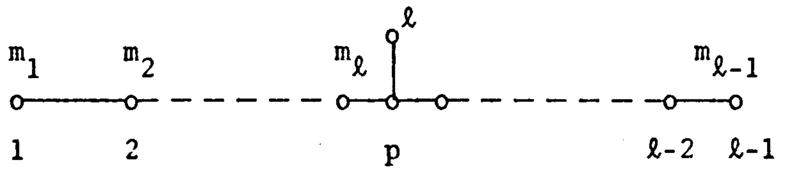
and non diagonal ones

$$(E_\alpha)^t_r = \pm\sqrt{(F_\alpha)^r_r + [(E_\alpha)^r_s]^2} \quad \text{where} \quad (E_\alpha)^r_s \neq 0 \quad \text{if} \quad \lambda_s = \lambda_r + \alpha \tag{9}$$

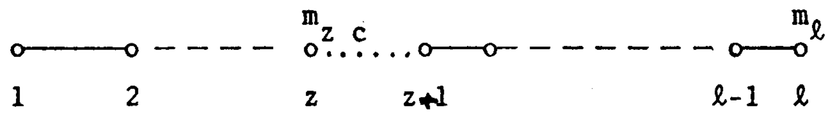
using
$$(E_{-\alpha})^r_s = -(E_\alpha)^s_r \ . \tag{10}$$

II. Calculation of the highest weight vector.

Having emphisized the importance of the h.w.v., it seems natural to calculate its expression for each of the two type of I.R. given by the following Dynkin diagrams :



(Type I)



(Type II)

where

$$m_i = L\alpha_i = \frac{2\langle L,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle} \qquad (m_i \in \mathbb{Z} > 0 \ ; \ i = 1,2,\ldots,\ell-1,\ell) \tag{11}$$

Writing $L = \sum_{k=1}^{\ell} a_k\alpha_k$ and using (5) we get the a_k's as solution of a system of ℓ linear equations :

$$m_i = \frac{2}{\langle\alpha_i,\alpha_i\rangle}[a_{i-1}\langle\alpha_{i-1},\alpha_i\rangle + a_i\langle\alpha_i,\alpha_i\rangle + a_{i+1}\langle\alpha_{i+1},\alpha_i\rangle + a_\ell\langle\alpha_\ell,\alpha_i\rangle\,\delta_{ip}] \tag{12}$$

the last term occuring only for diagrams of type (I) when i = p.

The system (12) has been solved for each of the two types of diagrams (I) and (II). The corresponding results are given in tables I and II for diagrams (I) and (II) respectively. If one writes $a_k = \frac{1}{\Delta} \sum_{i=1}^{\ell} \xi_k^i m_i$, then one gets two different expressions of Δ according to the type of diagram, say Δ_p for (I) and Δ_z for (II). These expressions will be analyzed in §3 to give the reason for the limitation of the number of simple Lie algebras. As a consequence of Chevalley's theorem[7,14] the classification of Dynkin diagram is equivalent to that of simple algebraic groups over algebraically closed fields of zero characteristic.

Table I : ξ_k^i for Type I (algebras $W_{\ell pc=1}$)

$$\Delta_p \equiv \Delta = p^2 + (2 - p)\ell \qquad \delta = \ell - p - 2 \qquad \Delta + p\delta = 2(\ell - p)$$

ξ_k^i	$1 \leq i \leq p-1$	$p \leq i \leq \ell-1$	ℓ
$1 \leq k \leq p-1$	$(\Delta+i\delta)k$ $(\Delta+k\delta)i$	$2k(\ell-i)$	$k(\ell-p)$
$p \leq k \leq \ell-1$	$(\ell-k)2i$	$(\ell-i)[p^2+(2-p)k]$ $[p^2+(2-p)i](\ell-k)$	$(\ell-k)p$
ℓ	$(\ell-p)i$	$p(\ell-i)$	ℓ

Table II : ξ^i_k for Type II (algebras $W_{\ell zc}$)

$\Delta_z(j) = \ell + 1 - j + (1 - c)(\ell - z)(z - j) \qquad j = i,k,z$ or 0.

ξ^i_k	$1 \leq i \leq z-1$	$z \leq i \leq \ell$
$1 \leq k \leq z-1$	$k\Delta_z(i)$ $i\Delta_z(k)$	$k(\ell+1-i)$
z	$i(\ell+1-z)$	$z(\ell+1-i)$
$z+1 \leq k \leq \ell$	$ic(\ell+1-k)$	$(\ell+1-i)[k+(1-c)z(k-z-1)]$ $(\ell+1-k)[i+(1-c)z(i-z-1)]$

III. Analysis of results and applications.

1. The two sets of algebras $W_{\ell pc=1}$ and $W_{\ell zc\neq 1}$.

As the h.w.v. has been written $L = \sum_{k=1}^{\ell} a_k \alpha_k$ with $a_k = \frac{1}{\Delta} \sum_{i=1}^{\ell} \xi_k^i m_i$

we must have $\Delta \neq 0$ and $\Delta > 0$.

In the case of diagrams of Type I i.e. of $W_{\ell pc=1}$ we have

$$\Delta \equiv \Delta_p = p^2 + (2 - p)\ell = \ell + 1 + (p - 1)(p - \ell + 1) > 0$$

$p = \ell - 1$	(or 1)	$\Delta = \ell + 1$	> 0	for all ℓ	A_ℓ
$p = 2$	(or $\ell - 2$)	$\Delta = p^2 = 4$	> 0	for all ℓ	D_ℓ
$p = 3$	(or $\ell - 3$)	$\Delta = 9 - \ell$	> 0	for $\ell = 6,7,8$	E_6, E_7, E_8
p big		$\Delta \sim p(p - \ell)$	> 0	for $p > \ell$	nonsense.

In the case of diagrams of Type II i.e. of $W_{\ell zc}$ we have

$$\Delta \equiv \Delta_z = \ell + 1 + (1 - c)(\ell - z)z$$

$c = 1$ we come back to the previous case where all the roots have the same length with a linear diagram ($\ell = p - 1$) i.e. to A_ℓ

$c = 2$ $\Delta = \ell + 1 - (\ell - z)z = 2 + (z - 1)(z - \ell + 1) > 0$

$z = \ell - 1$	$\Delta = 2$	> 0	for all ℓ	B_ℓ
$z = 1$	$\Delta = 2$	> 0	for all ℓ	C_ℓ
$z = 2$	$\Delta = 5 - \ell$	> 0	for $\ell = 4$	F_4

$c = 3$ $\Delta = \ell + 1 - 2z(\ell - z)$

	$z = 1$	$\Delta = 3 - \ell$	> 0	for $\ell = 2$	G_2
$c > 1$	z big	$\Delta \sim z(z - \ell)$	> 0	for $z > \ell$	nonsense.

When it is written for instance $9 - \ell > 0$, of course one can take $\ell = 5$ (or 4) which gives D_5 (or A_4) already seen ; similarly for $5 - \ell > 0$ $\ell = 3$ gives B_3 already seen.

As all other diagrams lead to a null h.w.v.; one is left with the only 9 s.L.a. already known and widely used by physicists; these 9 s.L.a. can be classified in two sets:

$$W_{\ell pc=1} = \{ A_\ell, D_\ell, E_\ell \text{ with } \ell = 6,7,8 \}$$
$$W_{\ell zc\neq 1} = \{ B_\ell, C_\ell, F_4, G_2 \}.$$

IV. 2. Power of weight vector. (Freudenthal[10] and Jacobson[11] use equivalently the word 'level').By definition the power $\delta(\lambda_r)$ of weight vector $\lambda_r = \sum_{k=1}^{\ell} \lambda_r^k \alpha_k$ is

$$\delta(\lambda_r) = \sum_{k=1}^{\ell} \lambda_r^k \ . \tag{14}$$

IV. 2.1. Power of the h.w.v..

The power $\delta(L)$ of the h.w.v. $L = \sum_{k=1}^{\ell} a_k \alpha_k$ is

$$\delta(L) = \sum_{k=1}^{\ell} a_k = \frac{1}{\Delta} \sum_{k=1}^{\ell} \sum_{i=1}^{\ell} \xi_k^i \, m_i = \frac{1}{\Delta} \sum_{i=1}^{\ell} \Big(\sum_{k=1}^{\ell} \xi_k^i \Big) m_i \tag{15}$$

Let us write $\sum_{k=1}^{\ell} \xi_k^i = \Lambda^i$ so that in general

$$\delta(L) = \frac{1}{\Delta} \sum_{i=1}^{\ell} \Lambda^i m_i \ . \tag{16}$$

For $W_{\ell p c=1}$ the calculation of $\delta(L)$ implies three steps (and of course $\Delta = \Delta_p$)

$1 \leq i \leq p-1$

$$\Lambda^i = \frac{i}{2} \left[2(\ell+1)(\ell-p) + (p-i)\Delta_p \right] . \tag{17}$$

$p \leq i \leq \ell-1$

$$\Lambda^i = \frac{\ell-i}{2} \left[2(\ell+1)p + (i-p)\Delta \right] . \tag{18}$$

$i = \ell$

$$\Lambda^\ell = \frac{\ell}{2} \left[2(\ell+1) - \Delta \right] . \tag{19}$$

Hence we get for the power of the h.w.v. of $W_{\ell p c=1}$ algebras:

$$\delta(L_p) = \frac{1}{\Delta_p} \left\{ \sum_{i=1}^{p-1} \frac{i}{2} \left[2(\ell+1)(\ell-p) + (p-i)\Delta_p \right] m_i + \sum_{i=p}^{\ell-1} \frac{\ell-i}{2} \left[2(\ell+1)p + (i-p)\Delta_p \right] m_i + \frac{\ell}{2} \left[2(\ell+1) - \Delta_p \right] m_\ell \right\} . \tag{20,a}$$

Specializing p to $\ell-1, 2, 3$ we get Λ^i (hence $\delta(L_p)$) for A_ℓ , D_ℓ , E_ℓ ($\ell = 6,7,8$) respectively; the results are given in Table III.

It is remarkable that due to the symmetry in i and k of Table I the h.w.v. R of the I.R. given by the Dynkin diagram for which $m_i = 1$ for all $i = 1,2,\ldots,\ell$ will have the same coefficients as $\delta(L_p)$ i.e.:

$$\mathbf{R} = \frac{1}{\Delta_p} \left\{ \sum_{k=1}^{p-1} \frac{k}{2} \left[2(\ell+1)(\ell-p) + (p-k)\Delta_p \right] \alpha_k + \sum_{k=p}^{\ell-1} \frac{\ell-k}{2} \left[2(\ell+1)p + (k-p)\Delta_p \right] \alpha_k + \frac{\ell}{2} \left[2(\ell+1) - \Delta_p \right] \alpha_\ell \right\} . \tag{20,b}$$

It will be seen later (§ IV.2.2. Theorem I) that R is also the half sum of the positive roots.

Table III : $\boxed{\delta(L_p) = \frac{1}{\Delta_p}\sum_{i=1}^{\ell} \Lambda_p^i m_i \quad \text{for} \quad W_{\ell p c = 1}}$

$$\Delta_p = p^2 + (2-p)\ell = \ell+1 + (1-p)(\ell-1-p)$$

p	$W_{\ell p 1}$	Δ_p	$1 \leq i \leq p-1$ $\frac{\Lambda_p^i}{\Delta_p} = \frac{i}{2\Delta_p}\left\{2(\ell+1)(\ell-p) + (p-i)\Delta_p\right\}$	$p \leq i \leq \ell-1$ $\frac{\Lambda_p^i}{\Delta_p} = \frac{\ell-i}{2\Delta_p}\left\{2(\ell+1)p + (i-p)\Delta_p\right\}$	$i = \ell$ $\frac{\Lambda_p^\ell}{\Delta_p} = \frac{\ell}{2\Delta_p}\left[2(\ell+1) - \Delta_p\right]$
$\ell-1$	A_ℓ	$\ell+1$	$\frac{i(\ell+1-i)}{2}$	$\frac{(\ell-i)(\ell+1-i)}{2}$	$\frac{\ell}{2}$
2	D_ℓ	4	$\frac{i(\ell^2-\ell+2-2i)}{4}$	$\frac{(\ell-i)(\ell+i-1)}{2}$	$\frac{\ell(\ell-1)}{4}$
3	E_ℓ	$9-\ell$ $(\ell=6,7,8)$	$\ell=6: \frac{i}{2}(17-i)$ $\frac{i}{2}\left\{\frac{2\ell^2 - 7\ell + 21 - i(9-\ell)}{9-\ell}\right\} \Rightarrow \ell=7; \frac{i}{2}(35-i)$ $\ell=8: \frac{i}{2}(93-i)$	$\ell=6: \frac{6-i}{2}(11+i)$ $\frac{\ell-i}{2}\left\{\frac{9\ell-21+i(9-\ell)}{9-\ell}\right\} \Rightarrow \ell=7: \frac{7-i}{2}(21+i)$ $\ell=8: \frac{8-i}{2}(51+i)$	$\ell=6: 11$ $\frac{\ell}{2}\left(\frac{3\ell-7}{9-\ell}\right) \Rightarrow \ell=7: \frac{49}{2}$ $\ell=8: 68$

For $W_{\ell zc}$ the calculation of $\delta(L)$ implies only two steps (and of course $\Delta = \Delta_z$)

$1 \leq i \leq z-1$

$$\Lambda^i = \frac{i}{2}\left\{c(\ell+1)(\ell-z) + (z+1-i)\Delta_z\right\}. \qquad (21)$$

$z \leq i \leq \ell$

$$\Lambda^i = \frac{\ell+1-i}{2}\left\{(\ell+1)(z+1) + (i-z-1)\Delta_z\right\}. \qquad (22)$$

Hence we get for the power of the h.w.v. of $W_{\ell zc}$ algebras:

$$\delta(L_z) = \frac{1}{\Delta_z}\left\{\sum_{i=1}^{z-1} \frac{i}{2}\left[c(\ell+1)(\ell-z) + (z+1-i)\Delta_z\right]m_i + \sum_{i=z}^{\ell}\left(\frac{\ell+1-i}{2}\right)\left[(\ell+1)(z+1) + (i-z-1)\Delta_z\right]m_i\right\}. \qquad (23,a)$$

Due to the properties of Table II the h.w.v. R of the I.R. given by the Dynkin diagram for which $m_i = 1$ for all $i = 1,2,\dots,\ell$ will be:

$$R = \frac{1}{\Delta_z}\left\{\sum_{k=1}^{z-1} \frac{k}{2}\left[c(\ell+1)(\ell-z) + (z+1-k)\Delta_z\right]\alpha_{\ell+1-k} + \sum_{i=z}^{\ell}\left(\frac{\ell+1-k}{2}\right)\left[(\ell+1)(z+1) + (k-z-1)\Delta_z\right]\alpha_{\ell+1-k}\right\}. \qquad (23,b)$$

The connection of R and $\delta(L_z)$ is so established; that R is the half sum of the positive roots will be seen in theorem I as before.

The formulas obtained for A_ℓ from $W_{\ell zc=1}$ as well as from $W_{\ell pc=1}$ are evidently the same for $p = z = \ell-1$.

Now for $c = 2$ we get the Λ^i's for B_ℓ when $z = \ell-1$, for C_ℓ when $z = 1$, for F_4 when $z = 2$;

for $c = 3$ we get the Λ^i's for G_2 when $z = 1$ and $\ell = 2$.

The results are given in Table IV.

It is worth while writing the formulas for $\ell = 2$ and $z = 1$ considering the frequent use of algebras of order 2. In that case, we get $\Delta_z = 4-c$, and Table II gives $\xi_1^1 = 2$, $\xi_1^2 = 1$; $\xi_2^1 = c$, $\xi_2^2 = 2$;

hence for the h.w.v.:

$$L_{2,c} = \frac{1}{4-c}\left\{(2m_1+m_2)\alpha_1 + (cm_1+2m_2)\alpha_2\right\} \qquad (24)$$

and its power

$$\delta(L_{2,c}) = \frac{1}{4-c}\left\{(2+c)m_1 + 3m_2\right\} \qquad (25)$$

which checks with Table IV.

These formulas can be used for A_2 ($c = 1$, $\delta(L_{2,1}) = m_1 + m_2$),

for B_2 or for C_2 ($c = 2$, $\delta(L_{2,2}) = 2m_1 + \frac{3}{2}m_2$),

and for G_2 ($c = 3$, $\delta(L_{2,3}) = 5m_1 + 3m_2$).

Table IV: $\boxed{\delta(L_z) = \frac{1}{\Delta_z} \sum_{i=1}^{\ell} \Lambda_z^i m_i \qquad \text{for } W_{\ell z c}}$

$$\Delta_z = \ell+1 + (1-c)(\ell-z)z$$

c	z	$W_{\ell z c}$	Δ_z	$1 \leq i \leq z-1$ $\frac{\Lambda_z^i}{\Delta_z} = \frac{i}{2\Delta_z}\left\{(z+1-i)\Delta_z + c(\ell+1)(\ell-z)\right\}$	$z \leq i \leq \ell$ $\frac{\Lambda_z^i}{\Delta_z} = \frac{\ell+1-i}{2\Delta_z}\left\{(i-z-1)\Delta_z + (\ell+1)(z+1)\right\}$
1	$\ell-1$	A_ℓ	$\ell+1$	$\frac{i}{2}(\ell+1-i)$	$\left(\frac{\ell+1-i}{2}\right)i$
2	$\ell-1$	B_ℓ	2	$\frac{i}{2}(2\ell+1-i)$	$\left(\frac{\ell+1-i}{2}\right)\left\{i + \frac{\ell(\ell-1)}{2}\right\}$
	1	C_ℓ	2	$\frac{i}{2}(\ell^2+1-i)$	$\left(\frac{\ell+1-i}{2}\right)(\ell-1+i)$
	2	$F_{\ell=4}$	$5-\ell=1$	$\frac{i}{2(5-\ell)}\left\{(3-i)(5-\ell) + 2(\ell+1)(\ell-2)\right\}$ only valid for $\ell=4$ and $i=1$ i.e. $\frac{\Lambda_z^1}{\Delta_z}(F_4) = 11$	$\frac{\ell+1-i}{2(5-\ell)}\left\{(i-3)(5-\ell) + (\ell+1)3\right\}$ which gives for F_4 $\frac{\Lambda_z^i}{\Delta_z}(F_4) = \frac{5-i}{2}(i+12) \quad i=2,3,4$
3	1	$G_{\ell=2}$	$3-\ell=1$	$\frac{i}{2(3-\ell)}\left\{(2-i)(3-\ell) + 3(\ell+1)(\ell-1)\right\}$ gives for $\ell=2$ and $i=1$ $\frac{\Lambda_z^1}{\Delta_z}(G_2) = 5$	$\frac{\ell+1-i}{2(3-\ell)}\left\{(i-2)(3-\ell) + (\ell+1)2\right\}$ which gives for G_2 $\frac{\Lambda_z^i}{\Delta_z}(G_2) = \frac{3-i}{2}(i+4) \quad i=1,2$

11.2. The formulas obtained for A_ℓ from $W_{\ell p c=1}$ as well as from $W_{\ell z c=1}$ are evidently the same for $p = z = \ell-1$.

The most important fact which comes out from Tables III & IV is that $\delta(L)$ is either integer or half integer so that $2\delta(L)+l = T$ is always an integer either odd or even respectively. As we shall see below T is the number of layers of the weight system constituted by all the weight vectors; the dimension N of the representation is equal to the cardinal of the set of weight vectors denoted by $\{w.v.\} = \{\lambda_1 = L, \lambda_2, \ldots\ldots, \lambda_N\}$.

Ordinary weight vectors are obtained by subtracting simple roots one by one from the h.w.v. **L** subject to rule (I):

If α_k is a simple root and $\lambda_s \in \{w.v.\}$ then $\lambda_r = \lambda_s - \alpha_k \in \{w.v.\}$ if and only if the integer $Q(\lambda_s, \alpha_k)$ determined by the two conditions

$$\begin{aligned} &\lambda_s + Q(\lambda_s, \alpha_k)\alpha_k \in \{w.v.\} \\ &\lambda_s + (Q+1)\alpha_k \notin \{w.v.\} \end{aligned} \qquad (26, a)$$

is such that

$$\frac{2(\lambda_s, \alpha_k)}{(\alpha_k, \alpha_k)} + Q(\lambda_s, \alpha_k) > 0 \qquad (26, b)$$

One can define the vector $S_r = \sum_{j=1}^{l} i_r^j \alpha_j$ $\qquad (27)$

where i_r^j are l positive or nul integers ($j = 1, 2, \ldots, l$) such that if

$$\lambda_r = \lambda_1 - S_r \in \{w.v.\} \qquad (28)$$

then $\qquad \delta(\lambda_r) = \delta(\lambda_1) - \delta(S_r) \qquad (29)$

i.e. the power of λ_r differs from the power of $\lambda_1 = L$ by the integer

$$\delta(S_r) = \sum_{j=1}^{l} i_r^j = i_r^1 + \cdots + i_r^l \rightrightarrows r - 1 \qquad (30)$$

which is the number of simple roots subtracted from λ_1 to give λ_r.

In others words for any $\lambda_r \in \{w.v.\}$, $\delta(L)$ and $\delta(\lambda_r)$ are either both integers or both half integers so that for a given representation all the powers of the weight system are of the same nature (corresponding to Wigner's integer or half integer representations).

Now equation (30) might have many independent solutions, say q_r solutions satisfying conditions (26 a,b); in that case the q_r ordinary weight vectors (in brief o.w.v.) $\lambda_r^{(1)},\ldots\ldots,\lambda_r^{(q_r)}$ form the r-th layer of o.w.v. all with the same power $\delta(\lambda_r)=\delta(\lambda_1)-(r-1)$.

The r-th layer is said to be power degenerate of order q_r.

In particular for the first layer corresponding to the uniqe h.w.v. $\lambda_1 = L$ one has $r=1$,$S_1=0$, $\delta S_1=0$, $q_1=1$ and the first layer is never degenerate.

If $\delta(\lambda_1)$ is an integer, after (m_0-1) subtracting steps such that

$$\delta(\lambda_{m_0}) = \delta(\lambda_1) - (m_0-1) = 0$$

we have a m_0-th layer of w.v. with power equal to zero; here $m_0=\delta(\lambda_1)+1$.

If $\delta(\lambda_1)$ is an half integer, after $(m_{\frac{1}{2}}-1)$ subtracting steps such that

$$\delta(\lambda_{m_{\frac{1}{2}}}) = \delta(\lambda_1) - (m_{\frac{1}{2}}-1) = \frac{1}{2}$$

we have a $m_{\frac{1}{2}}$-th layer of w.v. with power equal to $\frac{1}{2}$; here $m_{\frac{1}{2}} = \delta(\lambda_1) + \frac{1}{2}$.

In both cases due to the symmetry of the process the total number T of layers (called the height of the w.v. system) is then

$$T = 2\delta(\lambda_1) + 1 .$$

As we shall see the power degeneracy cannot diminish as the number of subtracting steps grows (up to m-1 steps) and consequently the degeneracy is maximum either for the m_0-th layer if $\delta(\lambda_1)$ is integer, say q_{m_0},

or for the $m_{\frac{1}{2}}$-th layer if $\delta(\lambda_1)$ is half integer, say $q_{m_{\frac{1}{2}}}$.

This maximum power degeneracy q_m is called the width of the w.v. system.

So that finally we have for the dimension N of the representation (counting each w.v. with its multiplicity)

if $\delta(\lambda_1)$ is integer $N = 2(q_1+\ldots+q_i+\ldots+q_{m_0-1}) + q_{m_0}$

if $\delta(\lambda_1)$ is half integer $N = 2(q_1+\ldots+q_i+\ldots+q_{m_{\frac{1}{2}}})$

(with $q_1 = 1$ and $q_{i+1} \geqslant q_i$).

In both cases we have $T = 2\delta(\lambda_1)+1 \leqslant N$ the equal sign corresponding to the case of no degeneracy.

2.b. Effective determination of o.w.v.

The first layer being occupied by the unique h.w.v. $\lambda_1 = L$ let us look for the w.v.'s of the second layer.

According to rule (I) since $\lambda_1 \in \{\text{w.v.}\}$

$$\lambda_1 + \alpha_i \notin \{\text{w.v.}\}$$

we have $Q(\lambda_1, \alpha_i) = 0$ for $i = 1, 2, \ldots, \ell$.

As $\frac{2(\lambda_1, \alpha_i)}{(\alpha_i, \alpha_i)} = m_i$, for $\lambda_1 - \alpha_i$ to be a w.v. we have the condition

$$\frac{2(\lambda_1, \alpha_i)}{(\alpha_i, \alpha_i)} + Q(\lambda_1, \alpha_i) = m_i > 0 \tag{31}$$

If there are q_2 values of $m_i \neq 0$, we obtain a second layer of q_2 different w.v. $\lambda_2 = \{\lambda_2^{(1)}, \ldots\ldots, \lambda_2^{(q_2)}\}$ with the same power $\delta(\lambda_2) = \delta(\lambda_1) - 1$.

Similarly the w.v. of the third layer are obtained by determining first $Q(\lambda_2^{(i)}, \alpha_j)$:

$$\lambda_2^{(i)} + \alpha_j = \lambda_1 - \alpha_i + \alpha_j \in \{\text{w.v.}\} \text{ if and only if } \alpha_i = \alpha_j$$

$$\lambda_2 + 2\alpha_j = \lambda_1 - \alpha_i + \alpha_j + \alpha_j \notin \{\text{w.v.}\}$$

so that $Q(\lambda_2^{(i)}, \alpha_j) = \delta_{i,j}$ and the condition for $\lambda_2^{(i)} - \alpha_j$ to be a w.v. is

$$\frac{2(\lambda_2, \alpha_j)}{(\alpha_j, \alpha_j)} + Q(\lambda_2^{(i)}, \alpha_j) = \frac{2(\lambda_1 - \alpha_i, \alpha_j)}{\alpha \quad \alpha} + \delta_{i,j} > 0$$

$$= m_j - \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} + \delta_{i,j} > 0$$

If $j = i$ we get: $$m_i > 1; \tag{32}$$

so that for $m_i \geq 2$, $\lambda_1 - 2\alpha_i$ is a w.v. of power $\delta(\lambda_1) - 2$.

If $j \neq i$ we get: in the case where α_j and α_i are not connected

$$m_j > 0, \tag{33,a}$$

(so that for $m_j \geq 1$, $\lambda_1 - \alpha_i - \alpha_j$ (with $|i-j| \geq 2$) is a w.v. with power $\delta(\lambda_1) - 2$).

in the case where α_j and α_i are connected i.e. $j = i \pm 1$ then $(\alpha_i, \alpha_j) = -1$ and the condition

$$m_j + \frac{2}{(\alpha_j, \alpha_j)} > 0 \tag{33,b}$$

is fulfilled even if $m_j = 0$; then $\lambda_2 - \alpha_j = \lambda_1 - \alpha_i - \alpha_j$ with $|i-j| = 1$ is a w.v. of power $\delta(\lambda_1) - 2$. So that with each w.v. of the second layer $\lambda_2^{(i)}$ we get at least one w.v. of the third layer.

To study the r-th layer let us write now a w.v. of the r-1-th layer as

$$\lambda_{r-1}^{(i)} = \lambda_1 - S_{r-1}^{(i)} \qquad (34)$$

where $S_{r-1}^{(i)} = \sum_{j=1}^{\ell} i_{r-1}^{j}\alpha_j$ (35)

with $\delta(S_{r-1}^{(i)}) = r-2$ and $\delta(\lambda_{r-1}^{(i)}) = \delta(\lambda_1) - r+2$;

(all i_{r-1}^{j} are positive or nul integers; $j = 1, 2, \ldots\ldots, \ell$).

For $\lambda_r^{(i)} = \lambda_{r-1}^{(i)} - \alpha_s$ to be a weight vector of the r-th layer we determine $Q(\lambda_{r-1}^{(i)}, \alpha$

$$\lambda_{r-1}^{(i)} + Q\alpha_s \in \{w.v.\}$$

$$\lambda_{r-1}^{(i)} + (Q+1)\alpha_s \notin \text{w.v.}$$

So that $Q = \sum_{j=1}^{\ell} i_{r-1}^{j}\delta_{j,s}$ and the condition for $\lambda_{r-1}^{(i)} - \alpha_s$ to be a w.v. is

$$\frac{2(\lambda_{r-1}^{(i)}, \alpha_s)}{(\alpha_s, \alpha_s)} + Q = \frac{2(\lambda_1 - S_{r-1}^{(i)}, \alpha_s)}{(\alpha_s, \alpha_s)} + Q > 0$$

$$m_s - \frac{2(S_{r-1}^{(i)}, \alpha_s)}{(\alpha_s, \alpha_s)} + \sum_{j=1}^{\ell} i_{r-1}^{j}\delta_{j,s} > 0$$

If $s \neq j$ and $|s-j| \geq 2$ for all j's such that $i_{r-1}^{j} \neq 0$, that is to say if α_s is none of the α_j involved in $S_{r-1}^{(i)}$ and if α_s is not connected with any one of them, then rule (I) gives

$$m_s > 0, \quad \text{i.e. } m_s \geq 1 \qquad (36)$$

If $s \neq j$ and α_s is connected with at least one of the α_j's involved in $S_{r-1}^{(i)}$ then for that value of j, $(\alpha_j, \alpha_s) = -1$; rule (I) is fulfilled even if $m_s = 0$.

(Notice that this conclusion remains true if α_s is connected with two α_j's, or exceptionally three α_j's in D_ℓ or in E_ℓ).

If $\alpha_s = \alpha_j$ i.e. if α_s is a particular α_j then rule (I) gives

$$m_j - i_{r-1}^{j} > 0 \quad \text{i.e. } m_j \geq i_{r-1}^{j} + 1 \qquad (37)$$

In particular among the q_{r-1} solutions of equation (30) applied to the (r-1)-th layer there is the maximal one $i_{r-1}^{s} = r-2$ (with $i_{r-1}^{j} = 0$ for all other j's) and correspondingly $\lambda_1 - (r-1)\alpha_s$ will be a w.v. of the r-th layer with power

$$\delta(\lambda_r) = \delta(\lambda_1) - r+1 \text{ if } m_s \geq r-1.$$

The conditions $m_s \geq 1$ for the second layer, $m_s \geq 2$ for the third layer, etc... $m_s \geq r-1$ for the r-th layer become obvious in terms of Young diagrams;

also we can see that power degeneracy cannot diminish as the number of subtracting steps grows as stated previously.

Due to the action of the Weyl group the w.v. system takes a spindle shape. Within a given layer $\{\lambda_r\}$ a certain w.v. M can occur more than once as soon as r $\geqslant$ 3; indeed we have:

$$M = \lambda_r^{(i)} = \lambda_1 - \sum_{j=1}^{\ell} i_r^j \alpha_j = \lambda_{r-1}^{(i_1)} - \alpha_{s_1} = \lambda_{r-1}^{(i_2)} - \alpha_{s_2} = \cdots \tag{38}$$

For example the w.v. system of the representation $\overset{1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{1}{\circ}$ of A_2 is:

$$\{w.v.\} = \{\alpha_1 + \alpha_2 \; ; \; \alpha_1 \, , \; \alpha_2 \; ; \; 0 \, , \; 0 \; ; \; -\alpha_2 , \; -\alpha_1 ; \; -(\alpha_1 + \alpha_2)\} \tag{39}$$

and the nul w.v. of the third layer is obtained in two ways from the second one. layer; so that the nul w.v. is degenerate and its multiplicity is two.

In general if M appear n_M times then M is said to be degenerate and n_M is its multiplicity (or the dimension of the corresponding degenerate subspace of the w.v. space); it means that each w.v. such as M has to be counted n_M times to maintain the fact that the dimension N of the representation-space is equal to the total number of w.v..

Freudenthal's recursion formula[10] gives the multiplicity n_M of M as

$$\left[(L + R, L + R) - (M + R, M + R)\right] n_M = 2 \sum_{\rho > 0} \sum_{k=1}^{\infty} (M + k\rho, \rho) n_{M + k\rho} \tag{40}$$

where R as for the Weyl's formula is given by equation(7).

To calculate dimensions of representatios by Weyl's formula (eq.6) one does not need L + R but R. As roots and weights are dual forms[1 – 15] with respect to the fundamental Killing quadratic form of the algebra the power $\delta(L)$ of the h.w.v. in the weight space corresponds to R in the root-space

Theorem I: $\delta(L) = \sum_{i=1}^{\ell} \frac{\Lambda^i}{\Lambda} m_i$ and $R = \frac{1}{2} \sum_{\rho > 0} \rho = \frac{1}{2} \sum_{i=1}^{\ell} \rho^i \alpha_i$ are dual elements.

The ordering of the roots is important for the use of this theorem; for $W_{\ell p1}$ the order is given with Table V; for $W_{\ell zc}$ one has to interchange m_i and $\alpha_{\ell+1-i}$ (B_ℓ and C_ℓ also as being dual too). With these precautions R can be built up out of Tables III and IV for $W_{\ell p1}$ and $W_{\ell zc}$.

Let us give two examples easy to check in no time.

For G_2 Table IV gives $$\delta\big(L(G_2)\big) = 5m_1 + 3m_2 \qquad (41,a)$$

then Theorem I: $$R(G_2) = 3\alpha_1 + 5\alpha_2 \qquad (41,b)$$

For F_4 Table IV gives $$\delta\big(L(F_4)\big) = 11m_1 + 21m_2 + 15m_3 + 8m_4 \qquad (41,c)$$

then Theorem I: $$R(F_4) = 8\alpha_1 + 15\alpha_2 + 21\alpha_3 \;\; 11\alpha_4. \qquad (41,d)$$

Now that we have $L = \sum_{k=1}^{\ell} a_k\alpha_k$ (Tables I & II) and $R = \sum_{k=1}^{\ell} b_k\alpha_k$ (Tables III & IV) using universally adopted Racah's notations[17,b] it is easy to build $K = L + R$ and consequently K^2 which is involved in Freudenthal's formula as well as in the second order Casimir operator whose eigenvalues are $K^2 - R^2 = L(L+2R) = \mathscr{C}$ for the representation defined by a given Dynkin diagram.

Using eq.5 and properties of the Cartan matrix involved in eq.12 we obtain

$$\mathscr{C} = K^2 - R^2 = \sum_{k=1}^{\ell} (a_k + 2b_k)m_k\frac{(\alpha_k,\alpha_k)}{2} = \sum_{k=1}^{\ell} a_k(m_k+2)\frac{(\alpha_k,\alpha_k)}{2}; \qquad (42)$$

given in Table V for $W_{\ell p1}$ and in Table VI for $W_{\ell zc}$. (The trivial exercise of specialization to particular values of p,z,and c is left to the reader).

The width of the weight diagram can be deduced easily now from Freudenthal's formula. We have seen that this width is the degeneracy n_o of the null weight vector when $\delta(L)$ is integer and $n_{\frac{1}{2}\alpha_i}$ of the w.v. $M = \frac{1}{2}\alpha_i$ when $\delta(L)$ is half integer.

In the first case we get:

$$(K^2-R^2)n_o = 2\sum_{\mu>0}\sum_{k=1}^{\infty}(k\mu,\mu)n_{k\mu} \qquad (40,a)$$

and in the second case:

$$\left[(K^2-R^2-\tfrac{5}{4}(\alpha_i,\alpha_i)-2\right]n_{\frac{1}{2}\alpha_i} = 2\sum_{\mu>0}\sum_{k=1}^{\infty}(\tfrac{1}{2}\alpha_i + k\mu,\mu)n_{\frac{1}{2}\alpha_i + k\mu} \qquad (40,b)$$

where μ is a positive root and $\frac{1}{2}\alpha_i + k\mu$ must be a weight.

In the appendix examples of application of these formula are given.

Table V : Eigenvalues of Casimir operator for $W_{\ell p1}$

$$\mathscr{C}_z = \frac{1}{\Delta_p}\Big\{\sum_{k=1}^{p-1}\{\sum_{i=1}^{i=k}(\Delta_p + k\delta)im_i + \sum_{i=k+1}^{p-1}(\Delta_p+i\delta)km_i + \sum_{i=p}^{\ell-1}2k(\ell-i)m_i + k(\ell-p)m_\ell$$

$$+ k\Big[2(\ell+1)(\ell-p) + (p-k)\Delta_p\Big]\}m_k$$

$$+ \sum_{k=p}^{\ell-1}\{\sum_{i=1}^{p-1}(\ell-k)2im_i + \sum_{i=p}^{i=k} p^2 + (2-p)i\ (\ell-k)m_i$$

$$+ \sum_{i=k+1}^{\ell-1} p^2 + (2-p)k\ (\ell-i)m_i + (\ell-k)pm_\ell$$

$$+ (\ell-k)\Big[2(\ell+1)p + (k-p)\Delta_p\Big]\}m_k$$

$$+ \{\sum_{i=1}^{p-1}(\ell-p)im_i + \sum_{i=p}^{\ell-1}p(\ell-i)m_i + \ell m_\ell + \ell\Big[2(\ell+1) - \Delta_p\Big]\}m_\ell\Big\} .$$

Ordering of the roots for $W_{\ell p1}$.

For A_ℓ as the coefficients of $\delta(L)$ are symmetric in i and $\ell+1-i$ the interchange has no effect and it is just as well to not do it. (see ref. [15] p.27).

For D_ℓ from an orthonormal basis $\{e_i\}$ of R^ℓ all roots are defined as $\pm e_i \pm e_j$ $(i \neq j)$

As we notice that Table III gives the same coefficient for $i = 1$ and for $i = \ell$ we define the simple roots in the following order:

$$\alpha_1 = e_{\ell-1}-e_\ell, \dots\dots\dots, \alpha_{\ell-i} = e_i - e_{i+1}, \dots\dots\dots, \alpha_{\ell-1} = e_1 - e_2,$$

$$\alpha_\ell = e_{\ell-1}+e_\ell .$$

For E_ℓ from an orthonormal basis $\{e_i\}$ of $\mathbb{R}^8$ all roots are defined as $\pm e_i \pm e_j$ $(i \neq j)$ and the vectors $\frac{1}{2}\sum_{i=1}^{\ell}(-1)^{m(i)}e_i$, with $\sum m(i) = \text{even}$; we define the simple roots as:

$$\alpha_\ell = e_1+e_2 \qquad \ell = 6, 7, 8 \quad \text{only}.$$

$$\alpha_1 = \tfrac{1}{2}(e_1+e_8-\sum_{i=2}^{7}e_i),\ \alpha_2 = e_2-e_1,\ \alpha_3 = e_3-e_2,\ \alpha_4 = e_4-e_3, \dots\dots, \alpha_{\ell-1} = e_{\ell-1}-e_{\ell-2}.$$

With the above ordering of the simple roots of $W_{\ell p1}$, if using Table III we write $\delta(L) = \sum_{k=1}^{\ell} b_k m_k$ then we get simply $R = \frac{1}{2}\sum_{\rho>0}\rho = b_k\alpha_k$, with $b_k = \frac{\Lambda^k}{\Delta_p}$.

Table VI : Eigenvalues of Casimir operator for $W_{\ell zc}$

$$\mathcal{C}_p = \frac{1}{\Delta_z}\Big\{ \sum_{k=1}^{z} \{ \sum_{i=1}^{k} i\Delta_z(k)m_i + \sum_{i=k+1}^{z-1} k\Delta_z(i)m_i + \sum_{i=z}^{\ell} k(\ell+1-i)m_i + k\Big[(\ell-k-z)\Delta_z + (\ell+1)(z+1)\Big]\} m_k$$

$$+ \sum_{k=z+1}^{\ell-z+1} \{ \sum_{i=1}^{z-1} ic(\ell+1-k)m_i + \sum_{i=z}^{i=k} (\ell+1-k)\Big[i+(1-c)z(i-z-1)\Big] m_i + \sum_{i=k+1}^{\ell} (\ell+1-i)\Big[k+(i-c)z(k-z-1)\Big] m_i$$

$$+ k\Big[(\ell-k-z)\Delta_z + (\ell+1)(z+1)\Big]\} m_k \frac{(\alpha_K,\alpha_K)}{2}$$

$$+ \sum_{k=\ell-z+2}^{\ell} \{ \sum_{i=1}^{z-1} ic(\ell+1-k)m_i + \sum_{i=z}^{i=k} (\ell+1-k)\Big[i+(1-c)z(i-z-1)\Big] m_i$$

$$+ \sum_{i=k+1}^{\ell} (\ell+1-i)\Big[k+(1-c)z(k-z-1)\Big] m_i$$

$$+(\ell+1-k)\Big[(z-\ell+k)\Delta_z + c(\ell+1)(\ell-z)\Big]\} m_k \frac{(\alpha_k,\alpha_k)}{2} \Big\} .$$

(The order of the roots is as given in reference 15 chap.V pages 28-29).

Of course the use of Tables V & V I can be avoided if one use the second form of equation (42)that we write again

$$\mathcal{C} = L(L+2R) = \sum_{k=1}^{\ell} a_k (m_k+2) (\frac{\alpha_k,\alpha_k}{2}) \qquad (42)$$

where only the coefficients of the h.w.v. L given in Tables I & II are involved. Anyway the ordering of the roots is still necessary to go from $\delta(L)$ to R.

IV.3. Matrices of I.R. of semi-simple Lie algebras.

IV.3.1. On Weyl's formula and dimensions of I.R. of semi simple Lie algebras.

Weyl's formula[1,11] give the dimension N of an I.R. as

$$N = \prod_{\mu>0}\left[\frac{(L,\mu)}{(R,\mu)} + 1\right] \qquad (43)$$

This formula implies the knowledge of L and of all the positive roots μ, ($R = \frac{1}{2}\sum_{\mu>0}\mu$ being deduced either directly from the μ's, or from $\delta(L)$).

What follows shows that the knowledge of the positive roots is enough.

As $\mu = \sum_{i=1}^{\ell}\mu^i\alpha_i$ $(\mu^i \in \mathbb{Z}^+)$ using eq.(11) we have:

$$(L,\mu) = \sum_{i=1}^{\ell}\mu^i m_i \frac{(\alpha_i,\alpha_i)}{2}.$$

As $R = \frac{1}{2}\sum_{\mu>0}\mu$ is also the highest weight of the I.R. corresponding to the Dynkin diagram such that all $m_i=1$ $(i=1,2,\ldots\ldots\ldots,\ell)$ we have:

$$(R,\mu) = \sum_{i=1}^{\ell}\mu^i \frac{(\alpha_i,\alpha_i)}{2} = \delta(L,\mu)$$

where $\delta(L,\mu)$ is the power (i.e. the sum of the m_i's coefficients) of (L,μ).

Formula (43) becomes:

$$N = \prod_{\mu>0}\left[\frac{\sum_{i=1}^{\ell}\mu^i m_i \frac{(\alpha_i,\alpha_i)}{2}}{\sum_{i=1}^{\ell}\mu^i \frac{(\alpha_i,\alpha_i)}{2}} + 1\right] \qquad (44)$$

For $W_{\ell p1}$ for which $(\alpha_i,\alpha_i)/2 = 1$ we get:

$$N(W_{\ell p1}) = \prod_{\mu>0}\left[\frac{\sum_{i=1}^{\ell}\mu^i m_i}{\delta(\mu)} + 1\right] \qquad (44,a)$$

where $\sum_{i=1}^{\ell}\mu^i m_i$ is obtained from $\mu = \sum_{i=1}^{\ell}\mu^i\alpha_i$ by interchanging α_i and m_i and $\delta(\mu) = \sum_{i=1}^{\ell}\mu^i$ is the power of the positive root μ.

For $W_{\ell zc}$ using previous notations we have:

$$N(W_{\ell zc}) = \prod_{\mu>0}\left[\frac{\sum_{i=1}^{z}\mu^i m_i + \sum_{i=z+1}^{\ell}\mu^i \frac{m_i}{c}}{\sum_{i=1}^{z}\mu^i + \sum_{i=z+1}^{\ell}\mu^i \frac{1}{c}} + 1\right] \qquad (44,b)$$

The number of positive roots ν of a given Lie algebra being called n_p the Coxeter index h is then $h = \frac{2n_p}{\ell}$ and the maximum power $\delta(\nu_m)$ of the positive roots is $\delta(\nu_m) = h-1$.
When expliciting (44,a) and (44,b) it is useful to give the factors of N in increasing order of $\delta(\nu)$ with

$$1 \leq \delta(\nu) \leq \delta(\nu_m) = h - 1$$

where $\delta(\nu) = 1$ corresponds to simple roots.

It follows from eq.(44) that to write the dimension of the I.R. of a given Lie algebra corresponding to a Dynkin diagram (or to a Young diagram) only the positive roots of that algebra are needed. The families of positive roots are build up out of an orthonormal basis $\{e_i\}$ of a vector space E according to Tables VII and VIII for $W_{\ell p1}$ and $W_{\ell zc}$ respectively.
The dimensions are then deduced according to the above method and given in Tables IX and X for $W_{\ell p1}$ and $W_{\ell zc}$ respectively.

At that stage it is useful to establish (at least for A_ℓ, B_ℓ, C_ℓ) the connection between Young and Dynkin diagrams. For Young diagrams (as oppose to Dynkin diagrams) one has to say in which Lie algebra they have to be considered. Then if λ_i is the length of the i-th line one has for A_ℓ:

$$m_i = \lambda_i - \lambda_{i+1} \text{ (for } i = 1,2,\ldots,\ell \text{)};$$

however for B_ℓ (algebra of $SO(2\ell+1)$) one has

$$m_i = \lambda_i - \lambda_{i+1} \text{ (for } i = 1,2,\ldots,\ell-1 \text{) and } m_\ell = 2\lambda_\ell;$$

for C_ℓ (algebra of $Sp(2\ell)$) one has

$$m_i = \lambda_{\ell+1-i} - \lambda_{\ell+2-i} \text{ (for } i = 2,\ldots,\ell \text{) and } m_1 = \lambda_\ell.$$

These precautions been taken, Tables IX and X can be used for instance to help the reduction of the I.R. of a group w.r.t. its invariant subgroups as for the decomposition of SU(n) into representations of SO(3) and the studies of the chain $SU(2\ell+1) \supset SO(2\ell+1) \supset SO(3)$ for ℓ integer

and of the chain $SU(2j+1) \supset Sp(2j+1) \supset SO(3)$ for j half-integer which are the root of the seniority concept so widely used by physicists (cf. M. Hamermesh[18,a] Chapter 11).

Table VII. Families of positive roots for $W_{\ell p1}$ algebras.

p	$W_{\ell p1}$	E	o.n. basis $\{e_i\}$ of: E	n_p	$\delta(\mu_m)$	1st family $e_{i+p}-e_i=\alpha_{i+1}+\dots\dots+\alpha_{i+p}$;	Other families of positive roots.
$\ell-1$	A_ℓ	$\mathbb{R}^{\ell+1}$	$i=0,1,\dots\dots,\ell$	$\frac{\ell(\ell+1)}{2}$	ℓ	$i=0,1,2,\dots\dots,\ell-1$ $\mu_m=e_\ell-e_0=\alpha_1+\alpha_2+\dots\dots+\alpha_\ell$	
ℓ	D_ℓ	$\mathbb{R}^\ell$	$i=0,1,\dots\dots,\ell-1$	$\ell(\ell-1)$	$2\ell-3$	$i=0,1,2,\dots\dots,\ell-2.$	2nd family $e_{i+p}+e_i=\alpha_1(1-\delta_{i;0})+\alpha_\ell+\alpha_2+\alpha_3+\dots\dots+\alpha_i+\alpha_{i+1}+\dots\dots+\alpha_{i+p}$ $+(\alpha_2+\alpha_3+\dots+\alpha_i)(1-\delta_{i,0})(1-\delta_{i,1})$ yields $\alpha_\ell=e_1+e_0$ for $p=1,\ i=0$ as well as $\mu_m=e_{\ell-1}+e_{\ell-2}=\alpha_1+2\alpha_2+2\alpha_3+\dots\dots+2\alpha_{\ell-2}+\alpha_{\ell-1}+\alpha_\ell$
3	E_ℓ	$\mathbb{R}^8$	$i=1,2,\dots\dots,8$				
		$n_p=(\ell-1)(\ell-2)+(\ell-6)\left[6(\ell-7)+1\right]+2^{\ell-2}$					
	$\ell=6,7$					$i=1,2,\dots\dots\ell-2$	2nd family $e_{i+p}+e_i=\alpha_2(1-\delta_{i,1})+\alpha_\ell+\alpha_3+\alpha_4+\dots\dots+\alpha_i+\alpha_{i+1}+\dots\dots+\alpha_{i+p}$
	$\ell=8$					$i=1,2,\dots\dots\ell-1$	$+(\alpha_3+\alpha_4+\dots\dots+\alpha_i)(1-\delta_{i,1})(1-\delta_{i,2})$ 3rd family. $\mu(j)=\frac{1}{2}\sum_{i=1}^{8}(-1)^{m(i)}e_i$ with $m(i)=0$ or 1 and $\sum m(i)=$ even.
	$\ell=6$			$2\times10+16$	11		$\alpha_1=\mu(1)=\frac{1}{2}(e_1+e_8-e_7-e_6-e_5-e_4-e_3-e_2)$; $\mu(i)=\alpha_1+\alpha_2+\dots\dots+\alpha_i$ $1\leq i\leq4$ $\mu(5)=\mu(4)+\alpha_\ell$ $\mu(6)=\mu(5)+\alpha_3$ $\mu(7)=\mu(6)+\alpha_2$ $\mu(8)=\mu(6)+\alpha_4+\alpha_5$ $\mu(9)=\mu(3)+\alpha_\ell$; $\mu(6+i)=\mu(i)+\alpha_5$ $4\leq i\leq7$; $\mu(14)=\mu(13)+\alpha_4$ $\mu(15)=\mu(14)+\alpha_3$ $\mu_m(16)=\mu(15)+\alpha_\ell$
	$\ell=7$			$2\times15+32+1$	17		$\mu(16+i)=\mu(i)+\begin{cases}\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6 & \text{for } i=1,5,6,8,9,11,12,16.\\ \alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_\ell & \text{for } i=2,3,4,7,10,13,14,15.\end{cases}$ $\mu_m(E_7)=\mu_{33}=e_8-e_7=2\alpha_1+3\alpha_2+4\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+2\alpha_7.$
	$\ell=8$			$2\times28+64$	29		$\mu(32+i)=\mu(16+i)+\alpha_7$ $1\leq i\leq16$ $\mu(48+i)=\mu(i)+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_\ell.$ $1\leq i\leq16$

Table VIII. Families of positive roots for $W_{\ell zc}$ algebras.

c	z	$W_{\ell zc}$	E	o.n. basis{ei} of E	n_p	$\delta(\gamma_m)$	1st family.	2nd family.	3d family.
2	$\ell-1$	B_ℓ	$\mathbb{R}^\ell$	$i=1,2,\dots,\ell$	ℓ^2	$2\ell-1$	$e_i-e_{i+p}=\alpha_i+\alpha_{i+1}+\dots+\alpha_{i+p-1}$ $i=1,2,\dots,\ell-1$ yields $\alpha_1,\dots,\alpha_{\ell-1}$	$e_i=\alpha_i+\alpha_{i+1}+\dots+\alpha_\ell$ $i=1,2,\dots\ell$ yields $\gamma_\ell=e_\ell$	$e_i+e_{i+p}=\alpha_i+\alpha_{i+1}+\dots+\alpha_{i+p-1}$ $+2\alpha_{i+p}+\dots+2\alpha_\ell$ $\gamma_m=e_1+e_2=\alpha_1+2\alpha_2+\dots+2\alpha_\ell$
2	1	C_ℓ	$\mathbb{R}^\ell$	$i=1,2,\dots,\ell$	ℓ^2	$2\ell-1$	$e_{i+p}-e_i=\alpha_{i+1}+\dots+\alpha_{i+p}$ $i=1.2.\dots.\ell-1$ yields the simple roots: $\alpha_2,\dots,\alpha_\ell$	$2e_i=\alpha_1+2\alpha_2+\dots+2\alpha_i$ $i=1.2.\dots.\ell$ yields $\alpha_1=2e_1$ and $\gamma_m=2e_\ell=\alpha_1+2\alpha_2+\dots+2\alpha_\ell$	$e_{i+p}+e_i=\alpha_1+\alpha_2+\dots+\alpha_i+\alpha_{i+1}+\dots+\alpha_{i+p}$ $+(\alpha_2+\dots+\alpha_i)(1-\delta_{i,1})$
2	2	F_4	$\mathbb{R}^4$	$i=1,2,\dots4$	24	11	$e_2-e_3=\alpha_1 \quad e_3-e_4=\alpha_2$ $e_2-e_4=\alpha_1+\alpha_2$ $e_1-e_2=2\alpha_4+2\alpha_3+\alpha_2$ $e_1-e_3=2\alpha_4+2\alpha_3+\alpha_2+\alpha_1$ $e_1-e_4=2\alpha_4+2\alpha_3+2\alpha_2+\alpha_1$ 4 th family. $\frac{1}{2}(e_1\pm e_2\pm e_3\pm e_4)$ yields	$e_4=\alpha_3$ $e_3=\alpha_2+\alpha_3$ $e_2=\alpha_1+\alpha_2+\alpha_3$ $e_1=\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$ $\alpha_4=\frac{1}{2}(e_1-e_2-e_3-e_4)$ $\mu(i)=\alpha_4+\dots+\alpha_{5-i} \quad 1\leq i\leq 4$ $\mu(5)=\mu(3)+\alpha_3$; $\mu(6)=\mu(4)+\alpha_3$ $\mu(7)=\mu(6)+\alpha_2$; $\mu(8)=\mu(7)+\alpha_3$	$\gamma_m=e_1+e_2=2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$ $e_1+e_3=\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$ $e_1+e_4=\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4$ $e_2+e_3=\alpha_1+2\alpha_2+2\alpha_3$ $e_2+e_4=\alpha_1+\alpha_2+2\alpha_3$ $e_3+e_4=\alpha_2+2\alpha_3$
3	1	G_2	$\mathbb{R}^3$	$i=1,2,3$	6	5	$e_i-e_{i+p}=\alpha_i+\alpha_{i+1}+\dots+\alpha_{i+p-1}$ $i=1,2$ yields $\alpha_1,\alpha_2,\alpha_1+\alpha_2$	$e_1+e_2+e_i-3e_3$ $i=3,2,1$ yields respectively : $\alpha_1+2\alpha_2,\alpha_1+3\alpha_2$ et $\gamma_m=2\alpha_1+3\alpha_2$	

Table IX. Dimensions of I.R. of $W_{\ell p1}$ algebras.

p	
$\ell -1$	$N(A_\ell)= \prod_{i=1}^{\ell}(m_i+1)\ldots\prod_{i=1}^{\ell+1-k}\left[\frac{m_i+m_{i+1}+\ldots+m_{i+k-1}}{k}+1\right]\ldots\ldots\left[\frac{m_1+m_2+\ldots+m_\ell}{\ell}+1\right]$
2	$N(D_\ell)= \Pi_1(D_\ell)\,\Pi_2(D_\ell)$ corresponds to the two families of roots. $\Pi_1(D_\ell)=\Pi(A_{\ell-1})$ $\Pi_2(D_\ell)=(m_\ell+1)\prod_{i=2}^{\ell-1}\left[\frac{m_\ell+m_2+m_3+\ldots+m_i}{i}+1\right]\prod_{i=3}^{\ell}\left[\frac{m_\ell+m_1+m_2+\ldots+m_{i-1}}{i}+1\right]\ \mathrm{X}$ $\mathrm{X}\prod_{2i+k=5}^{2\ell-3}\left[\frac{m_\ell+m_1+2(m_2+\ldots+m_i)+m_{i+1}+\ldots+m_{i+k}}{2i+k}+1\right].$
3	$N(E_\ell)= \Pi_1(E_\ell)\,\Pi_2(E_\ell)\,\Pi_3(E_\ell)$ corresponds to the three families of roots. $\Pi_1(E_\ell)\,\Pi_2(E_\ell)=\begin{cases}\Pi_1\ (D_{\ell-1}) \text{ for } \ell=6,7 & \text{contains } (\ell-1)(\ell-2) \text{ factors.}\\ \Pi_1\ (D_\ell) \text{ for } \ell=8 & \text{contains } \ell(\ell-1)=56 \text{ factors.}\end{cases}$ $\Pi_3(E_\ell)$ for $\ell = 6,7,8$ contains respectively 16,(32+1),64 factors $\left[\frac{f(i)}{\delta(f(i))}+1\right]$ build up with the roots of the third family. $N(E_\ell)=\left\{\prod_{i=2}^{\ell-1}(m_i+1)\prod_{\substack{i=2\\k=2,3,4}}^{\ell-k}\left[\frac{m_i+\ldots+m_{i+k-1}}{k}+1\right]\right\}\left\{(m_\ell+1)\prod_{i=2,3}^{4\leq i+k\leq\ell}\left[\frac{m_i+\ldots+M_{i+k-1}+m_\ell}{k+1}+1\right]\right.$ for $\ell=6$ $\left.\left[\frac{m_2+2m_3+m_4+m_\ell}{5}+1\right]\left[\frac{m_2+2m_3+m_4+m_5+m_\ell}{6}+1\right]\left[\frac{m_2+2m_3+2m_4+m_5+m_\ell}{7}+1\right]\right\}$ $\left\{\prod_{i=1}^{4}\left[\frac{m_1+\ldots+m_i}{i}+1\right]\left[\frac{m_1+m_2+m_3+m_4+m_\ell}{5}+1\right]\left[\frac{m_1+m_2+2m_3+m_4+m_\ell}{6}+1\right]\right.$ $\left.\left[\frac{m_1+2m_2+2m_3+m_4+m_\ell}{7}+1\right]\left[\frac{m_1+m_2+2m_3+2m_4+m_5+m_\ell}{8}+1\right]\prod_{i=9}^{16}\left[\frac{f(i)}{\delta(f(i))}+1\right]\right\}$ where $f(9)=f(3)+m_\ell$; $f(6+i)=f(i)+m_5$ for $4\leq i\leq 7$; $f(14)=f(13)+m_4$ $f(15)=f(14)+m_3$; $f(16)=f(15)+m_\ell$ To build $\Pi_3(E_7)\begin{cases}f(16+i)=f(i)+m_2+m_3+m_4+m_5+m_6 & \text{for } i=1,5,6,8,9,11,12,16\\ f(16+i)=f(i)+m_3+m_4+m_5+m_6+m_\ell & \text{for } i=2,3,4,7,10,13,14,15\\ f_{33}=2m_1+3m_2+4m_3+3m_4+2m_5+m_6+2m_7 & \end{cases}\Bigg\}$ are needed. To build $\Pi_3(E_8)\begin{cases}f(32+i)=f(16+i)+m_7\\ f(48+i)=f(i)+m_2+2m_3+2m_4+2m_5+2m_6+m_7+m_8\end{cases}\Bigg\}\ 1\leq i\leq 16$ are needed

Table X. Dimensions of I.R. of $W_{\ell zc}$ algebras.

c	z	
2	$\ell-1$	$N(B_\ell)=\prod_{\mu>0}\left[\frac{2\sum_{i=1}^{\ell-1}\mu^i m_i+\mu^\ell m_\ell}{2\sum_{i=1}^{\ell-1}\mu^i+\mu^\ell}+1\right]=\Pi_1(B_\ell)\Pi_2(B_\ell)\Pi_3(B_\ell)$ $\Pi_1(B_\ell)=\prod_{i=1}^{\ell-1}(m_i+1)\ldots\prod_{i=1}^{\ell+1-k}\left[\frac{m_i+m_{i+1}+\ldots+m_{i+k-1}}{k}+1\right]\ldots\left[\frac{m_1+m_2+\ldots+m_{\ell-1}}{\ell-1}+1\right]$ $\Pi_2(B_\ell)=(m_\ell+1)\prod_{i=1}^{\ell-1}\left[\frac{2(m_i+\ldots+m_{\ell-1})+m_\ell}{2(\ell-i)+1}+1\right]$ $\Pi_3(B_\ell)=\prod_{i=1}^{\ell-1}\left[\frac{m_i+\ldots+m_{\ell-1}+m_\ell}{\ell-i+1}+1\right]\prod_{i=1}^{i=\ell-2}\left[\frac{m_i+\ldots+m_{i+k-1}+2(m_{i+k}+\ldots+m_{\ell-1})+m_\ell}{2(\ell-i)-k+1}+1\right]$
2	1	$N(C_\ell)=\prod_{\mu>0}\left[\frac{2\mu^1 m_1+\sum_{i=2}^{\ell}\mu^i m_i}{2\mu^1+\sum_{i=2}^{\ell}\mu^i}+1\right]=\Pi_1(C_\ell)\Pi_2(C_\ell)\Pi_3(C_\ell)$ $\Pi_1(C_\ell)=\prod_{i=2}^{\ell}(m_i+1)\ldots\prod_{i=2}^{\ell+1-k}\left[\frac{m_i+\ldots+m_{i+k-1}}{k}+1\right]\ldots\left[\frac{m_2+m_3+\ldots+m_\ell}{\ell-1}+1\right]$ $\Pi_2(C_\ell)=(m_1+1)\prod_{i=2}^{\ell}\left[\frac{m_1+m_2+\ldots+m_i}{i}+1\right]$ $\Pi_3(C_\ell)=\prod_{i=2}^{\ell}\left[\frac{2m_1+m_2+\ldots+m_i}{i+1}+1\right]\prod_{i=2}^{i=\ell-1}\left[\frac{2(m_1+\ldots+m_i)+m_{i+1}+\ldots+m_{i+k}}{2i+k}+1\right]$
2	2	$N(F_4)=\prod_{\mu>0}\left[\frac{2(\mu^1 m_1+\mu^2 m_2)+\mu^3 m_3+\mu^4 m_4}{2(\mu^1+\mu^2)+\mu^3+\mu^4}+1\right]$ $=\prod_{i=1}^{4}(m_i+1)\ldots\prod_{i=1}^{5-k}\left[\frac{m_i+m_{i+1}+\ldots+m_{i+k-1}}{k}+1\right]\left[\frac{2m_2+m_3}{3}+1\right]$ $\prod_{i,j,k=1,2}\left\{\left[\frac{im_1+2m_2+m_3}{3+i}+1\right]\left[\frac{m_1+jm_2+m_3+m_4}{3+j}+1\right]\left[\frac{2m_2+km_3+m_4}{3+k}+1\right]\right\}$ $\prod_{i=2,3}\left[\frac{m_1+im_2+2m_3+m_4}{4+i}+1\right]\prod_{j=3,4}\left[\frac{2m_1+jm_2+2m_3+m_4}{5+j}+1\right]$ $\prod_{i,j=1,2}\left\{\left[\frac{2m_1+2m_2+im_3+m_4}{5+i}+1\right]\left[\frac{2m_1+4m_2+3m_3+im_4}{9+j}+1\right]\right\}$
3	1	$N(G_2)=(m_1+1)(m_2+1)\left(\frac{m_1+m_2}{2}+1\right)\left(\frac{2m_1+m_2}{3}+1\right)\left(\frac{3m_1+m_2}{4}+1\right)\left(\frac{3m_1+2m_2}{5}+1\right)$

An I.R. is called <u>basic</u> if all components m_j of the h.w.v. are zero except one $m_i = \delta_{i,j}$ for $j = 1,2,\ldots,i,\ldots,\ell$; such a representation is denoted $(W_\ell)\mathcal{B}_i$ and its dimension $N(W_\ell)\mathcal{B}_i$ is obtained by doing $m_j = \delta_{i,j}$ in Tables IX (for $W_{\ell p1}$) and X (for $W_{\ell zc}$). The results listed in Tables XI and XII have already been obtained[18,[illegible]] by a recursive method; they cannot be used to compute the dimension of any general I.R. since the dimension formulas are very far from been linear in the m_i's;they are only an example as well as a test of Tables IX and X.

The basic I.R. of smallest dimension will be called the <u>elementary</u> I.R. as it corresponds to the dimension of the smallest vector space of representation and according to our coherent notation (cf.section III, Type I and II) corresponds to a terminal simple root i.e. $i = 1, \ell$, or $\ell - 1$, this last value been specialy valid for D_ℓ and for E_ℓ (=6,7,8).

A representation of particular interest is also the one whose dimension is equal to the number r of parameters of the associated group; such a representation will be called the regular (or adjoint) representation and denoted R.R.;we have

$$r = 2n_p + \ell = \ell(\frac{2n}{\ell}p+1) = \ell(h+1) = \ell(\delta(\rho_m) + 2).$$

In general the regular (or adjoint) representation is a basic one except for the cases of

A_ℓ for which the R.R. is the I.R. $m_1 = m_\ell = 1$, $m_i = 0$ for $i = 2,3,\ldots,\ell-1$;

C_ℓ for which the R.R. is the following reducible representation:

$$\text{R.R.}(C_\ell) = (C_\ell)\mathcal{B}_{\ell-1} \oplus (C_\ell)\mathcal{B}_\ell \oplus (C_\ell)\mathcal{B}_o$$

where $(C_\ell)\mathcal{B}_o$ is the scalar identity representation for which all $m_i = 0$.
As for E_ℓ one has $n_p = (\ell-1)(\ell-2) + (\ell-6)[6(\ell-7)+1] + 2^{\ell-2}$,
which is not a simple function of ℓ to handle all the basic representations are computed directly using Tables IX and X.

The process of alternation.

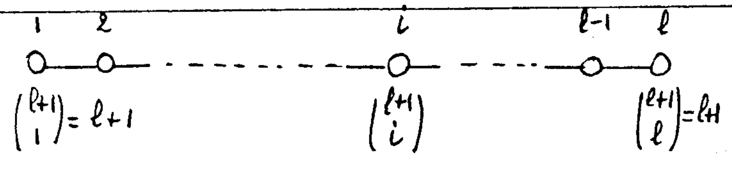
Starting from the representation space of the elementary I.R. of dimension say n for A_ℓ, D_ℓ, B_ℓ we can represent the Dynkin diagram given by $m_1 = 1$, $m_i = 0$ for $i = 2,3,\dots,\ell$ by a Young diagram consisting of a single box. Then the Young diagram corresponding to the Dynkin diagram given by $m_i = \delta_{i,j}$ is a column of i boxes i.e. a skew-symmetric tensor of rank i in $E^{\wedge n}$ and the number of linearly independant components of that tensor is equal to $\binom{n}{i}$; consequently the dimension of the I.R. given by the Dynkin diagram of the basic representation $m_i = \delta_{i,j}$ is also $\binom{n}{i}$ as a direct calculation using Tables IX and X yields (see Tables XI and XII). To make the above reasoning obvious for D_ℓ a relabeling of the roots interchanging i and ℓ-i has been used so that $\binom{2\ell}{\ell-i}$ becomes $\binom{2\ell}{i}$. The alternation process applied to E_6, E_7, E_8, C_ℓ, F_4, G_2 yields reducible representations (except for few cases of E_6). In the following tables whenever possible Dynkin diagrams have been displayed whith the dimension of the basic representation written below the corresponding simple root; possible reduction of the alternation process have also been expanded.

Table XI. Dimensions of particular representations of $W_{\ell p+}$.

$r = 2n_p + \ell = \ell(h+1)$ Comments.

A_ℓ

$\ell \geq 1$

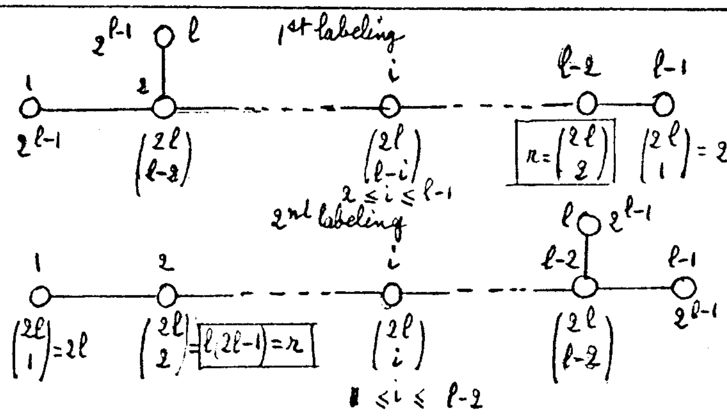


$r = \ell(\ell+1)+\ell = \ell(\ell+2)$

The R.R. corresponds to the non basic I. R. $\boxed{m_1 = 1, m_\ell = 1, \text{ all other } m_i \text{ being zero}}$

D_ℓ

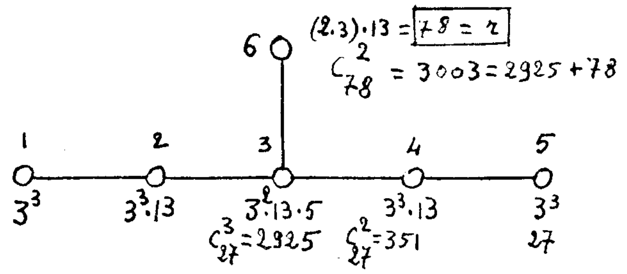
$\ell \geq 2$



$r = 2\ell(\ell-1)+\ell = \ell(2\ell-1) = \binom{2\ell}{2}$

The ordering of the roots can be reversed because we have seen (IV.1) that we can have $p = 2$ or $p = \ell-2$.

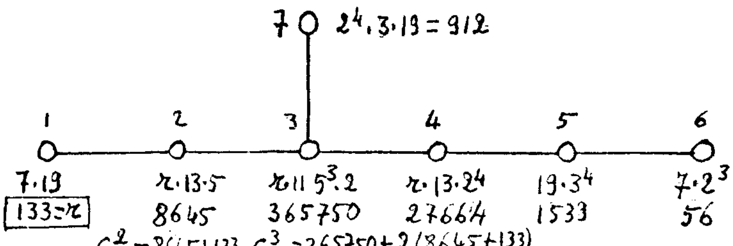
E_6



$2n_p = 72 \quad h = 12 \quad r = 78$

For E_ℓ as for D_ℓ (cf IV.1) the ordering of the roots can be reversed because we can have $p = 3$ or $p = \ell-3$. Notice the symmetry of E_6.

E_7

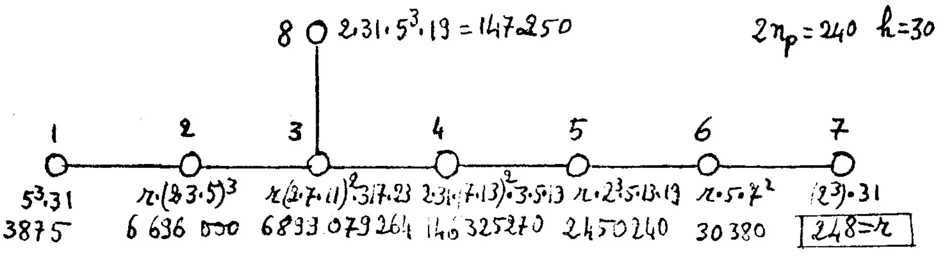


$C^2_{133} = 8645+133 \quad C^3_{133} = 365750 + 2(8645+133)$

$C^4_{56} = 365750+1540 \quad C^3_{56} = 27664+56 \quad C^2_{56} = 1539+1$

$2n_p = 126 \quad h = 18 \quad r = 133$

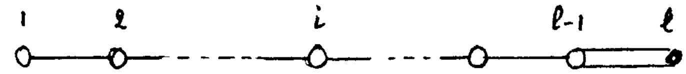
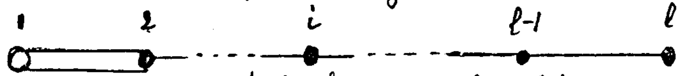
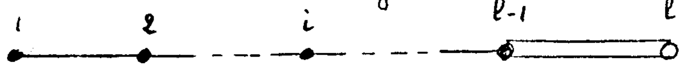
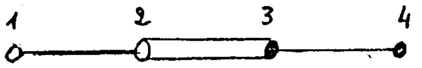
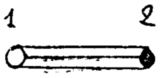
E_8



$2n_p = 240 \quad h = 30 \quad r = 248$

For E_8 (as shown above for E_7) the alternation process yields reducible representations.

Table XII. Dimensions of particular representations of W_{lzc}.

		$r = 2n_p + l = l(l+1)$	Comments
B_l $l \geq 2$	 $\binom{2l+1}{1}$ $\binom{2l+1}{2} = r$ $\binom{2l+1}{i}$ $\binom{2l+1}{l-1}$ 2^l $1 \leq i \leq l-1$	$r = 2l^2 + l = l(2l+1) = \binom{2l+1}{2}$	The R.R. corresponds to the basic I.R. $m_j = \delta_{l,j}$.
C_l $l \geq 2$	1st labeling  $2\left(\frac{1}{l+2}\right)\binom{2l+1}{l}$ $2\left(\frac{i}{l+i+1}\right)\binom{2l+1}{l-i+1}$ $2\left(\frac{l-1}{2l}\right)\binom{2l+1}{2}$ $2l$ $1 = i \leq l$ 2nd labeling  $2l$ $(l-1)(2l+1)$ $2\left(\frac{l-i+1}{2l-i+2}\right)\binom{2l+1}{i}$ $2\left(\frac{1}{l+2}\right)\binom{2l+1}{l}$ $1 \leq i \leq l$	$r = 2l^2 + l = l(2l+1)$	As well known $N(C_l)$ is always even (e.g. the factor 2 in $N(C_l)\mathcal{B}_i$ for all i's.). When l is odd, r is odd and no basic I.R. can be a regular one. For C_l the R.R. is reducible and using the scalar identity representation denoted $(C_l)\mathcal{B}_0$ one has $(C_l)R.R = (C_l)\mathcal{B}_{l-1} \oplus (C_l)\mathcal{B}_l \oplus (C_l)\mathcal{B}_0$.
F_4	 $(2^2) \cdot 13$ $2 \cdot 13 \cdot 7^2$ $13 \cdot 7 \cdot 3$ $13 \cdot 2$ $52 = r$ 1274 273 26 $C_{52}^2 = 1274 + 52$ $C_{26}^2 = 273 + 26 + 26$	$r = 2 \cdot 24 + 4 = 52$	The R.R. corresponds to the basic I.R. $m_j = \delta_{1,j}$.
G_2	 $2 \cdot 7$ 7 $14 = r$ $C_7^2 = 14 + 7$	$r = 2 \cdot 6 + 2 = 14$	The R.R. corresponds to the basic I.R. $m_j = \delta_{1,j}$

IV.3.2 Construction of the representation matrices of s.s.L.a.

IV3.2.1 Diagonal matrices.

To each weight vector $\lambda_r^{(i)}$, $\left(\begin{matrix} i = 1, \ldots\ldots, q_r \\ r = 1, \ldots\ldots, T \end{matrix}\right)$, corresponds a unique vector $v_r^{(i)}$ in the representation space E_N such that[1-15]

$$H_\mu v_r^{(i)} = (\mu, \lambda_r^{(i)}) v_r^{(i)} \tag{43,a}$$

with $\mu = \sum_{k=1}^{\ell} \mu^k \alpha_k$ being a positive root (all $\mu^k \in \mathbb{Z}^+$). (43,b)

Hence
$$(H_\mu)_{r,i}^{r,i} = (\mu, \lambda_r^{(i)}) = \sum_{k=1}^{\ell} \mu^k (\alpha_k, \lambda_r^{(i)}) \tag{44}$$

and $(H_\mu)_{r,i}^{r,i}$ is known when the $(H_{\alpha_k})_{r,i}^{r,i} = (\alpha_k, \lambda_r^{(i)}), k = 1, \ldots\ldots, \ell$ are known. Due to the symmetry of the weight vector system it suffices to write down its positive part only i.e. the $\delta(\lambda_1)$ first layers if $\delta(\lambda_1)$ is an integer (as the following one gives the degeneracy of the nul w.v.) or the $\delta(\lambda_1) + \frac{1}{2}$ first layers if $\delta(\lambda_1)$ is an half integer. The complete matrix of order N can then be filled up with the opposite numbers (to get a zero trace matrix as expected).

From equation (34), using (5,b) and (11) we get first

$$(S_r^{(i)}, \alpha_k) = \sum_{j=1}^{\ell} \left[i_r^j \, (\alpha_j, \alpha_k) - i_r^{j-1} - i_r^{j+1} \right] \delta_{j,k}; \tag{45}$$

hence for $W_{\ell zc}$ with $z + 1 \leqslant k \leqslant \ell$:

$$(H_{\alpha_k})_{r,i}^{r,i} = \frac{m_k}{c} - (\frac{2i_r^k}{c} - i_r^{k-1} - i_r^{k+1}); \tag{46}$$

for $W_{\ell zc}$ with $1 \leqslant k \leqslant z$, or for $W_{\ell p1}$ one has to make $c = 1$ in eq.(46).

In case the w.v. $M = \lambda_r^{(i)}$ presents a degeneracy of order n_M we get just as many identical diagonal elements.

A relatively general exemple of application of the formula (46) is given in appendix.

IV. 3.2. Non diagonal matrices.

The relation[11]

$$E_{-\mu} = -{}^{t}E_{\mu} \tag{47}$$

allows the study of E_μ for μ being only a positive root.

As $E_\mu v_s \propto v_r$, $(v_s, v_r \in E_N)$ we have $\lambda_r = \lambda_s + \mu \in \{w.v.\}$

and the non nul elements $(E_\mu)^r_s$ are such that $\lambda_r = \lambda_s + \mu$; (48,a)

i.e. are situated in the lower half of the matrix (E_μ) and connect w.v. of layers whose power differ by $\delta(\mu)$; in other words $r = s - \delta(\mu)$. (48,b)

(Of course if μ is a simple root $\delta(\mu) = 1$ and $r = s - 1$).

The proof of (48,a) is well known; for any other positive root ν the commutation relation:

$$[H_\nu, E_\mu] = (\nu,\mu)E_\mu \tag{48,c}$$

yields

$$(H_\nu)^r_r (E_\mu)^r_s - (E_\mu)^r_s (H_\nu)^s_s = (\nu,\mu)(E_\mu)^r_s$$

or

$$(\nu, \lambda_r - \lambda_s - \mu)(E_\mu)^r_s = 0$$

hence (48,a).

If there is no degeneracy of the w.v. system one has:

$$(E_\mu)^r_s = \pm\sqrt{(H_\mu)^s_s + \left[(E_\mu)^s_t\right]^2} \tag{49,a}$$

Indeed the commutation relation:

$$[E_\mu, E_{-\mu}] = H_\mu \tag{49,b}$$

yields

$$(E_\mu)^s_t (E_{-\mu})^t_s - (E_{-\mu})^s_r (E_\mu)^r_s = (H_\mu)^s_s \tag{49,c}$$

and using (47) we get

$$\left[(E_\mu)^r_s\right]^2 - \left[(E_\mu)^s_t\right]^2 = (H_\mu)^s_s \tag{49,d}$$

hence (49,a) which gives the elements of the non diagonal matrices in terms of the elements of the diagonal matrices given by formula (46).

Notice that (49,a) being not linear one can not expect to get (E_μ) as linear combination of the (E_α) with α as a simple root. Each matrix has to be calculated for its own sake.

From (48,b) we see that $r = s - \delta(\mu) = (t - \delta(\mu)) - \delta(\mu)$; so the calculation starts from $\lambda_s = - \lambda_1$ (the lowest w.v.) which yields $\lambda_t = 0$ i.e. $(E_\mu)_t^s = 0$; then $(E_\mu)_s^r = \pm \sqrt{(H\ )_s^s}$ is known from section 3.a eq.(46) and the procedure is carried over by ascending along a prallel to the diagonal as $\delta(\mu)$ is fixed.

The commutation relation

$$\left[E_\mu, E_\nu\right] = N_{\mu,\nu} E_{\mu+\nu} \quad , \qquad \mu,\nu,\mu+\nu \in \{\text{positive roots}\} \quad (50)$$

is used to obtain some coherence in signs.

If there is a degeneracy of the w.v. system i.e. if in the same layer a certain w.v. M occurs with the multiplicity n_M then the terms of the left hand side of equation (49,c) would be summed over the repeated indices t and r respectively.

Furthermore as we have now n_M values $(H_\nu)_s^s$ which are identical (for $s = 1,2,\ldots,n_M$) the number of independent equations is no more sufficient to determine all the matrix elements; the last commutation relation(eq.50) is then a useful complement. One can also choose arbitrarily the values of the relevant matrix elements of one of the operators which is tantamount to choosing arbitrarily a basis in the degenerate subspace (of the w.v. system) of dimension n_M;but the values so obtained will depend on this choice.

Degeneracy is often met and complicates apparently simple problems as for instance the study [18] of the chain $G_2 \supset A_2$.

Conclusions: The following results have been obtained in three steps:

I. In contrast to the point of view recently discussed in [16,17] consisting in breaking a given algebra into subalgebras we have considered here the building of two classes of algebras out of known algebras[20]:

$$W_{\ell pc=1}=\{A_\ell, D_\ell, E_\ell \text{ for } \ell= 6,7,8\}$$

$$W_{\ell zc\neq 1}=\{B_\ell, C_\ell, F_4, G_2\}.$$

This classification is based on equation (I) and on Chevalley's theorem[7,14] stating that the classification of Dynkin diagrams is equivalent to that of simple algebraic groups over closed fields of characteristic zero.

2. A study of the w.v. system has been performed using the results of Tables I & II of the first part. For the highest weight vector L we have calculated its power $\delta(L)$ and shown, for $W_{\ell pc=1}$ (Table III) as well as for $W_{\ell zc\neq 1}$ (Table IV), that $\delta(L)$ is either integer or half integer in agreement wit the fact that $2\delta(L)+I=T$ is the integral number of layers (or shells) of the w.v. system whether this system is degenerate or not.

In case of degeneracy of a particular weight vector M the Freudenthal's recursion formula gives the multiplicity n_M of M. In that formula as in Weyl's formula (eq.6) comes in the form $R=\frac{1}{2}\sum_{\rho>0}\rho$ which can be deduced from Tables 111&1V according to Theorem 1; hence the eigen values of the Casimir operator (given in Tables V&VI) and width of weight diagrams are deduced.

3. The results obtained above have been used to build up the matrices of zero trace (diagonal and non diagonal) representations for the two classes of algebras.

In appendix two examples are briefly studied to illustrate this paper.

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APPENDIX

Example 1.. $\underset{m_1}{0}\overset{c}{- - - - -}\underset{m_2}{\otimes}$

$$L = \lambda_1 = \frac{1}{4-c}\left\{(2m_1 + m_2)\alpha_1 + (cm_1 + 2m_2)\alpha_2\right\};$$

$$\delta(\lambda_1) = \frac{1}{4-c}\left\{(2 + c)m_1 + 3m_2\right\}; \qquad R = \frac{1}{4-c}\left\{3\alpha_1 + (2+c)\alpha_2\right\};$$

$$\mathcal{C} = L(L + 2R) = \frac{1}{4-c}\left\{(2m_1 + m_2 + 6)m_1 + \left((cm_1 + 2m_2) + 2(2+c)\right)\frac{m_2}{c}\right\}$$

or equivalently $\mathcal{C} = \frac{1}{4-c}\left\{(2m_1 + m_2)(m_1 + 2) + (cm_1 + 2m_2)\left(\frac{m_2 + 2}{c}\right)\right\}$.

According to section 2.b. we can write:

for the second layer:

$\lambda_2^{(1)} = \lambda_1 - \alpha_1 \in \{\text{w.v.}\}$ if and only if $m_1 \geqslant 1$.

$\lambda_2^{(2)} = \lambda_1 - \alpha_2 \in \{\text{w.v.}\}$ if and only if $m_2 \geqslant 1$.

If $m_1 m_2 \neq 0$ then $\lambda_1^{(1)}$ and $\lambda_2^{(2)} \in \{\text{w.v.}\}$ with the same power $\delta(\lambda_2) = \delta(\lambda_1) - 1$.

If $m_i = 0$ $(i,j=1,2)$ then $\lambda_2^{(j)} \in \{\text{w.v.}\}$ but $\lambda_2^{(i)} \notin$ w.v. $(j \neq i)$.

for the third layer:

$\lambda_3^{(1)} = \lambda_2^{(1)} - \alpha_1 = \lambda_1 - 2\alpha_1 \in \{\text{w.v.}\}$ if and only if $m_1 \geqslant 2$.

$\lambda_3^{(2)} = \lambda_2^{(1)} - \alpha_2 = \lambda_1 - \alpha_1 - \alpha_2$ w.v. even if $m_2 \equiv 0$

$\lambda_3^{(3)} = \lambda_2^{(2)} - \alpha_2 = \lambda_1 - 2\alpha_2 \in \{\text{w.v.}\}$ if and only if $m_2 \geqslant 2$

$\lambda_3^{(4)} = \lambda_2^{(2)} - \alpha_1 = \lambda_1 - \alpha_2 - \alpha_1 = \lambda_3^{(2)} \in \{\text{w.v.}\}$

As $\ell = 2$, there are no disconnected roots and the third layer contain at least the degenerated w.v. $\{\lambda_3^{(2)} = \lambda_3^{(4)}\}$ and at most the 4 above w.v. with the same power $\delta(\lambda_3) = \delta(\lambda_1) - 2$.

Particular cases can be considered:

for $c = 1$, take $m_1 = 0$, $m_2 = 1$

or $m_1 = 1$, $m_2 = 0$

or $m_1 = 1$, $m_2 = 1$ corresponding to the Young diagram  of SU(3) with $\delta(\lambda_1) = 2$ and $\{\text{w.v}\} = \{\alpha_1 + \alpha_2, \alpha_1, \alpha_2, 0, 0, -\alpha_2, -\alpha_1, -\alpha_2 - \alpha_1\}$

so that the dimension of the representation is 8 as forseen by Weyl's formula (6, 6_2

for $c = 2$, L and R are obvious and $L(L + 2R) = \frac{1}{2}\left\{(2m_1 + m_2 + 6)m_1 + (2m_1 + 2m_2 + 8)\frac{m_2}{2}\right\}$.

for $c = 3$, Weyl's formula (6) gives using (41,b) for the dimension N

$$N(G_2) = (m_1 + 1)(m_2 + 1)\left(\frac{m_1 + m_2}{2} + 1\right)\left(\frac{2m_1 + m_2}{3} + 1\right)\left(\frac{3m_1 + m_2}{4} + 1\right)\left(\frac{3m_1 + 2m_2}{5} + 1\right). \qquad (6, G_2)$$

For $m_1 = 0$, $m_2 = 1$ we have $N(G_2) = 7$ and Freudenthal's formula gives $n_0 = 1$.

According to section 3.a. and summarizing what we know from before we have:

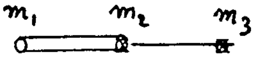
$\{w.v.\} = \{\ \lambda_1;\ \lambda_1 - \alpha_1,\ \lambda_1 - \alpha_2;\ \lambda_1 - \alpha_1 - \alpha_2,\ \lambda_1 - \alpha_2 - \alpha_1,\ \lambda_1 - 2\alpha_1,\ \lambda_1 - 2\alpha_2;\ldots\}$

$$H_{\alpha_1} = \begin{bmatrix} m_1 & & & & & & & \\ & m_1 - 2 & & & & & & \\ & & m_1 + 1 & & & & & \\ & & & m_1 - 1 & & & & \\ & & & & m_1 - 1 & & & \\ & & & & & m_1 - 4 & & \\ & & & & & & m_1 + 2 & \\ & & & & & & & \ddots \end{bmatrix}$$

$$H_{\alpha_2} = \begin{bmatrix} \frac{m_2}{c} & & & & & & & \\ & \frac{m_2}{c} + 1 & & & & & & \\ & & \frac{m_2 - 2}{c} & & & & & \\ & & & \frac{m_2}{c} + 1 - \frac{2}{c} & & & & \\ & & & & \frac{m_2}{c} - \frac{2}{c} + 1 & & & \\ & & & & & \frac{m_2}{c} + 2 & & \\ & & & & & & \frac{m_2 - 4}{c} & \\ & & & & & & & \ddots \end{bmatrix}$$

If μ is a positive root such that $\mu = \sum_{k=1}^{\ell} \mu^k \alpha_k$ we get for this example

$$H_\mu = \mu^1 H_{\alpha_1} + \mu^2 H_{\alpha_2}$$

Example II: Representations of C_3-algebra of group Sp(6): 

Using Table II we get:

$$\lambda_1 = L_{C_3} = \tfrac{1}{2}(3m_1+2m_2+m_3)\alpha_1 + (2m_1+2m_2+m_3)\alpha_2 + (m_1+m_2+m_3)\alpha_3 \;;$$

$$\delta\left(L_{C_3}\right) = \tfrac{1}{2}(9m_1+8m_2+5m_3); \qquad \delta\left(L_{B_3}\right) = 3m_1+5m_2+3m_3$$

$$R(C_3) = 3\alpha_1+5\alpha_2+3\alpha_3;$$

$$\mathcal{G} = \tfrac{1}{2}(3m_1+2m_2+m_3)(m_1+2) + (2m_1+2m_2+m_3)\left(\frac{m_2+2}{2}\right) + (m_1+m_2+m_3)\left(\frac{m_3+2}{2}\right)$$

Dimension:
$$N(C_3) = (m_1+1)(m_2+1)(m_3+1)\left(\frac{m_1+m_2}{2}+1\right)\left(\frac{m_2+m_3}{2}+1\right)\left(\frac{2m_1+m_2}{3}+1\right)$$
$$\times\left(\frac{m_1+m_2+m_3}{3}+1\right)\left(\frac{2m_1+m_2+m_3}{4}+1\right)\left(\frac{2m_1+2m_2+m_3}{5}+1\right). \qquad (6,C_3)$$

Second layer: conditions for $\lambda_2^{(i)}$ to be a weight vector:

$$\lambda_2^{(i)} = \lambda_1 - \alpha_i \quad \text{if and only if } m_i \geqslant 1, \text{ for } i = 1,2,3.$$

$$\delta(\lambda_2) = \delta(\lambda_1) - 1$$

third layer; conditions for the following vectors to be w.v. provided $\lambda_2^{(i)} \in \{\text{w.v.}\}$

$\lambda_3^{(1)} = \lambda_2^{(1)} - \alpha_1 = \lambda_1 - 2\alpha_1$ if and only if $m_1 \geqslant 2$

$\lambda_3^{(2)} = \lambda_2^{(1)} - \alpha_2 = \lambda_1 - \alpha_1 - \alpha_2$ even if $m_2 = 0$

$\lambda_3^{(3)} = \lambda_2^{(1)} - \alpha_3 = \lambda_1 - \alpha_1 - \alpha_3$ if and only if $m_3 \geqslant 1$

$\lambda_3^{(4)} = \lambda_2^{(2)} - \alpha_1 = \lambda_1 - \alpha_2 - \alpha_1 = \lambda_3^{(2)}$ even if $m_1 = 0$

$\lambda_3^{(5)} = \lambda_2^{(2)} - \alpha_2 = \lambda_1 - 2\alpha_2$ if and only if $m_2 \geqslant 2$

$\lambda_3^{(6)} = \lambda_2^{(2)} - \alpha_3 = \lambda_1 - \alpha_2 - \alpha_3$ even if $m_3 = 0$

$\lambda_3^{(7)} = \lambda_2^{(3)} - \alpha_1 = \lambda_1 - \alpha_3 - \alpha_1 = \lambda_3^{(3)}$ if and only if $m_1 \geqslant 1$

$\lambda_3^{(8)} = \lambda_2^{(3)} - \alpha_2 = \lambda_1 - \alpha_3 - \alpha_2 = \lambda_3^{(6)}$ even if $m_2 = 0$

$\lambda_3^{(9)} = \lambda_2^{(3)} - \alpha_3 = \lambda_1 - 2\alpha_3$ if and only if $m_3 \geqslant 2$

All with power $\delta(\lambda_3) = \delta(\lambda_1) - 2$.

Suppose $m_1 = m_2 = 0$, $m_3 = 1$, then $\delta(\lambda_1) = 5/2$ and we are left with the non degenerate w.v. system:

$$\{\text{w.v.}\} = \{\tfrac{1}{2}\alpha_1 + \alpha_2 + \alpha_3;\ \tfrac{1}{2}\alpha_1 + \alpha_2;\ \tfrac{1}{2}\alpha_1;\ -\tfrac{1}{2}\alpha_1;\ -\tfrac{1}{2}\alpha_1 - \alpha_2;\ -\tfrac{1}{2}\alpha_1 - \alpha_2 - \alpha_3\}$$

so that the dimension of the corresponding representation is 6, as forseen by Weyl's formula $(6,C_3)$.

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N.B. The 7 first references are given in order to justify the initials used to designate the two classes of algebras with possibilities of alternative interpretation.

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