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NON LINEAR RELATIVISTIC FIELD EQUATIONS

Giorgio VELO *

0. INTRODUCTION

In this talk I will be concerned with the existence and uniqueness of the solutions of the Cauchy problem for a partial differential equation (or system of partial differential equations) of the type

$$\Box \phi(t,x) + F(t,x,\phi(t,x)) = 0 \tag{0.1}$$

and with some properties of its solutions which might be relevant from the physical point of view. The ideas I will expose have been developed in collaboration with C. Parenti and F. Strocchi(1)(2)(3)(4)(5), and find their main motivations in theoretical physics. In fact in the last three-four years theorists of high energy physics have become more and more interested in quantum theories constructed around simple, stable (genuinely non-linear) solutions of some classical relativistic field equations. Examples of such equations and of their solutions are given by the two following one space-one time lagrangian theories.

1)
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}(\phi^2 - 1)^2 . \tag{0.2}$$

The Lagrange equation is

$$\Box \phi + 2(\phi^2 - 1)\phi = 0 \tag{0.3}$$

and a relevant solution of the type mentioned above is the following static one

$$\phi = \operatorname{tgh} x . \tag{0.4}$$

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2) $$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \cos\phi \qquad (0.5)$$

The Lagrange equation (Sine-Gordon equation) is

$$\Box\phi + \sin\phi = 0 \qquad (0.6)$$

and a relevant solution of the type mentioned above is the following static one

$$\phi = 4\,\mathrm{arctg}(\exp x) \qquad (0.7)$$

Solutions like (0.4) and (0.7) are usually called solitons or solitary waves.

All the quantization procedures, the Feynman path integral method, the WKB method, the canonical method seem to require a good knowledge of the properties of the classical solutions.(6) Furthermore these classical solutions should be reached by taking the limit as $\hbar \to 0$ of suitable expectation values of the quantum fields.(7) As a consequence the discussion of the classical aspects of the equation (0.1) seems to be a necessary preliminary step to the understanding of the quantum field theory based on that equation. Besides in theoretical physics, some equations of the type (0.1) are also employed in various areas of applied physics such as solid state and non linear optics. From the mathematical point of view the general method used to treat the equation (0.1) can be extended to a whole class of suitable non linear PDE.

In the time at my disposal I will discuss the following topics:

1) Initial value problem.
2) Classification of solutions: Hilbert sectors.
3) Dynamical charges, energy functionals and energy sectors.

Proofs will not be presented, the main emphasis will be on guiding ideas and motivations.

1. INITIAL VALUE PROBLEM.

The specific form of the equations I'm interested in is

$$\left(\frac{\partial}{\partial t}\right)^2 \phi(t,x) - \Delta \phi(t,x) + \nabla_\phi U(x, \phi(t,x)) = 0 \tag{1.1}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^s$ and Δ is the Laplacian in $\mathbb{R}^s$ (I will be mainly concerned with the cases $s = 1,2,3$). The symbol ϕ will denote a map from $\mathbb{R}^s$ to $\mathbb{R}^n$ (ϕ is an n-component field), $\mathbb{R} \ni t \longrightarrow \phi(t,\cdot)$ will denote a family of such maps, the potential U will denote a map from $\mathbb{R}^s \times \mathbb{R}^n$ to $\mathbb{R}$ differentiable in the ϕ variables. Part of the theory could accomodate some time dependence in U , here for simplicity I shall just consider time independent potentials.

The first proof of existence and uniqueness of the Cauchy problem for the case $U = \frac{1}{2} m^2 \phi^2 + \phi^4$ with $m \neq 0$ in $\mathbb{R}^3$ (or similar) goes back to Jörgens [(8)], and has been generalized subsequently by Segal. [(9)] In this approach one looks for solutions having initial data $\phi, \dot{\phi} \left(\equiv \frac{\partial \phi}{\partial t} \right)$ with finite "kinetic energy"

$$\frac{1}{2} \int_{\mathbb{R}^3} \left[(\nabla \phi)^2 + \phi^2 + \dot{\phi}^2 \right] dx < \infty , \tag{1.2}$$

i.e. one solves the Cauchy problem in the space of ϕ and $\dot{\phi}$ belonging to the Sobolev space $H^1(\mathbb{R}^3)$ and $L^2(\mathbb{R}^3)$ respectively (the ϕ considered here is a one component field).

However the results exclude the possibility of treating physically interesting situations, like field theories with symmetry breaking solutions, and more generally theories with soliton-like solutions, because these solutions do not decrease fast enough at infinity to satisfy the condition (1.2). In order to be able to analyze these theories the Cauchy problem has to be reconsidered in a space of initial data larger than the finite "kinetic energy" space. A guide of the direction in which to move is given by the constant and the soliton solutions. They violate condition (1.2) because of their behaviour at large x , whereas locally they are quite regular. Therefore it seems natural to replace (1.2) by the condition that for all open bounded regions

$\Omega \subset \mathbb{R}^s$

$$\int_{\Omega} \left[(\nabla\phi)^2 + \phi^2 + \dot{\phi}^2 \right] dx < \infty \tag{1.3}$$

and try to solve the equation (1.1) in the space of initial data satisying (1.3). This means that one wants to solve the Cauchy problem in the space of $\phi \in H^1_{loc}(\mathbb{R}^s)$ and $\dot{\phi} \in L^2_{loc}(\mathbb{R}^s)$. This local point of view has some physical motivations because any experiment and observation is necessarily bounded in space. Therefore one expects that any physically interesting solution should yield finite results for localized observations such as the "local kinetic energy". From the quantum field theoretic point of view the analog of having finite the local kinetic energy is for the system to be locally Fock, the analog of having finite the total kinetic energy is for the system to be globally Fock.

Equation (1.1) can be given the form of an integral equation in which the initial data are automatically incorporated:

$$u(t) = W(t)\, u_0 + \int_0^t W(t-s)\, f(u(s))\, ds \tag{1.4}$$

where

$$u = \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \psi_1 \\ \vdots \\ \phi_n \\ \psi_n \end{pmatrix}, \tag{1.5}$$

$$f(u) = \begin{pmatrix} 0 \\ \nabla_{\phi_1} U(x,\phi) \\ \vdots \\ 0 \\ \nabla_{\phi_n} U(x,\phi) \end{pmatrix} \tag{1.6}$$

and

$$\mathbb{R} \ni t \longrightarrow W(t) \tag{1.7}$$

is the one parameter group generated by $\begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \otimes 1_n$. (I recall that ψ is nothing else but $\frac{\partial \phi}{\partial t}$).

It is convenient to introduce some more notation. If Ω is an open sphere of radius $r(\Omega)$, for any $t \in \mathbb{R}$ with $|t| < r(\Omega)$, $\Omega(t)$ will denote the sphere concentric to Ω of radius $r(\Omega) - |t|$. The space $\left(H^1_{loc}(\mathbb{R}^s) \oplus L^2_{loc}(\mathbb{R}^s) \right) \otimes 1_n$ will be denoted by X . When equipped with the topology generated by the family of seminorms defined by

$$\| u \|^2_\Omega = \| \phi \|^2_\Omega + \| \psi \|^2_\Omega = \sum_{j=1}^{n} \left(\int_\Omega [(\nabla \phi_j)^2 + \phi_j^2] dx + \int_\Omega \psi_j^2 dx \right) \tag{1.8}$$

it becomes a Fréchet space (u is related to ϕ and ψ by (1.5)).

I can now state the following result on the free evolution of the equation (1.1), which result is a basic ingredient for the treatment of the non linear part.

Lemma 1. The collection of maps $W(t)$, $t \in \mathbb{R}$ (see eq.(1.7)) is a strongly continuous group of linear bounded operators on X . Furthermore, for any open sphere Ω and for any t such that $|t| < r(\Omega)$, the following estimate holds

$$\| W(t) u \|_{\Omega(t)} \leq \exp\left(\frac{|t|}{2}\right) \| u \|_\Omega . \tag{1.9}$$

The property (1.9) could be taken as the starting point of a definition of a group of hyperbolic type. It can be interpreted in the sense that the values of u outside Ω at $t = 0$ will never influence the region $\Omega(t)$ within the time t , namely that information travels at finite speed (taken conventionally equal to one). Such a behaviour is required by special relativity.

In order to state precisely the properties of the interaction term I need the following preliminary definitions. A map f from X to X is said to be locally Lipschitz if, for any open sphere $\Omega \subset \mathbb{R}^s$ and for any $\rho > 0$, there exists a $C(\Omega, \rho) > 0$ such that

a) $$\| f(u_1) - f(u_2) \|_{\Omega} \leq C(\Omega, \rho) \| u_1 - u_2 \|_{\Omega}$$

for all $u_j \in X$ with $\| u_j \|_{\Omega} \leq \rho$, $j = 1,2$,

b) $$\sup_{t \in [0, r(\Omega)/2]} C(\Omega(t), \rho) < \infty .$$

A potential U is then said to satisfy P1 (property 1) if the map f in (1.6) is locally Lipschitz. A potential U is said to satisfy P2 (property 2) if there exists two constants α and β such that

$$U(x,\phi) \geq -\alpha - \beta \, \phi \cdot \phi$$

for all $x \in \mathbb{R}^s$, $\phi \in \mathbb{R}^n$.

Theorem 1. Let U satisfy P1 and P2, let $u_0 \in X$. Then the equation (1.4) has unique solution $u \in \mathcal{C}^0(\mathbb{R}, X)$.

A few comments are in order. The assumption P2 of the theorem is easy to check. The assumption P1 is less immediate, but it is satisfied by a large class of physically interesting potentials. In particular, due to the Sobolev inequalities, it is implied by the following property 3 (P3). Such a property is written separately for $s = 1,2,3$:

$s=1$, $U(x,\phi) = \sum_{|\alpha| \geq 0} a_\alpha(x) \phi^\alpha$ with $a_\alpha(x) \in L^\infty(\mathbb{R})$ and

$$\sum_{|\alpha| \geq 0} \| a_\alpha \|_\infty \sigma^{|\alpha|} < \infty \quad \text{for any} \quad \sigma > 0 .$$

$s=2$, $U(x,\phi) = \sum_{|\alpha| \geq 0} a_\alpha(x) \phi^\alpha$ with $a_\alpha(x) \in L^\infty(\mathbb{R}^2)$ and

$$\sum_{|\alpha| \geq 0} \| a_\alpha \|_\infty |\alpha|^{|\alpha|/2} \sigma^{|\alpha|} < \infty \quad \text{for any} \quad \sigma > 0 .$$

$s=3$, $U(x,\phi)$ of class $\mathcal{C}^2$ in the ϕ variables with $U(x,0) \in L^\infty(\mathbb{R}^3)$,

$\nabla_{\phi_j} U(x,0) \in L^\infty(\mathbb{R}^3)$ and $\sup_{x \in \mathbb{R}^3} | \nabla_{\phi_i} \nabla_{\phi_j} U(x,\phi) | \leq$ Const.$(1 + \phi \cdot \phi)$ for all $i,j = 1,2 \ldots n$.

Concerning the proof of the theorem, P1 and P2 play a different role. P1 is used to prove existence and uniqueness for small time intervals, P2 is used to make the solution global in time. The hyperbolicity of the group generating the free motion (see Lemma 1) is crucial in the whole argument. One first solves the problem by inserting a space cut off both in the initial data and in part of the interaction. Then the cut off is removed by using the finite propagation speed.

The detailed proof of this theorem under more general assumptions as well as a discussion of the regularity properties of the solutions may be found in (3).

2. CLASSIFICATION OF SOLUTIONS.

In what follows I will consider a fixed potential V which satisfies P1 and P2. Under these conditions I recall that the equation (1.4) has a unique $\mathcal{C}^0(\mathbb{R},X)$ solution provided the initial data are in X. The family of all such solutions will be denoted by $\mathfrak{F}$. This is a quite large set in the sense that presumably physics is interested only in a subset of $\mathfrak{F}$.

One could try to compare these solutions depending on what they might have in common from the point of view of their global properties. To further discuss this idea it is convenient to introduce the notation Y for the space $\left(H^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)\right)\otimes\mathcal{H}_u$ which becomes in an obvious way a Hilbert space and represents the global analog of the space X.

Definition 1. Two elements u and u' of $\mathfrak{F}$ are said to be relatively small if $u-u'\in\mathcal{C}^0(\mathbb{R},Y)$.

Expressed in other words Definition 1 means that u and u' are near to each other from the global point of view in a continuous way with respect to the time variable. This condition of "relative smallness" is an equivalence relation and induces a partition of $\mathfrak{F}$ in classes of equivalence. The quotient set constructed in this way will be denoted by $\tilde{\mathfrak{F}}$ and, for any $u\in\mathfrak{F}$, C_u will denote the element of $\tilde{\mathfrak{F}}$ containing u. Among the elements of $\tilde{\mathfrak{F}}$ there are some which are more interesting, namely those invariant under time translation. To see more precisely what this means, I recall that, since U is taken to be time independent, if u belongs to $\mathcal{F}$ also u_τ, defined by $u_\tau(t) = u(t+\tau)$, belongs to $\mathcal{F}$ for any $\tau \in \mathbb{R}$. It is clear that the time translation by τ is compatible with the above partition of $\mathcal{F}$ into classes and therefore it generates a map from $\widetilde{\mathcal{F}}$ to $\widetilde{\mathcal{F}}$, in such a way that the τ translation of C_u is C_{u_τ}.

Definition 2. An element C_u of $\widetilde{\mathcal{F}}$ is said to be continuously invariant under time translation if

a) $C_u = C_{u_\tau}$ for all $\tau \in \mathbb{R}$.

b) For all $t \in \mathbb{R}$ $\quad Y\text{-}\lim\limits_{\tau \to 0} \left(u_\tau(t) - u(t) \right) = 0.$

The above definition makes sense since, if u satisfies b), then any $v \in C_u$ satisfies b). Furthermore to satisfy b) it is sufficient to require it only for some $t_0 \in \mathbb{R}$. We restrict our attention to the elements $\widetilde{\mathcal{F}}_0$ of $\widetilde{\mathcal{F}}$ satisfying the Definition 2.

The partition of $\mathcal{F}$ into classes of equivalence according to Definition 1 induces in an obvious way a partition of X into classes of equivalence, X being considered as the space of initial data at time $t=0$. The classes of X induced by the elements of $\widetilde{\mathcal{F}}_0$ will be called Hilbert sectors. A characterization of the Hilbert sectors is expressed by the following lemma whose proof is immediate.

Lemma 2. Let u_0 and u_0' belong to X, and let u and u' be the solutions of the equation (1.4) with $u(0)=u_0$ and $u'(0) = u_0'$. Then u_0 and u_0' belong to the same Hilbert sector iff $u_0 - u_0' \in Y$ and both $u(t) - u_0$ and $u'(t) - u_0'$ belong to $\mathcal{C}^0(\mathbb{R}, Y)$.

Each Hilbert sector is uniquely determined by any of its elements $u_0 = \begin{pmatrix} \phi_0 \\ \psi_0 \end{pmatrix}$ (see eq.(1.5)) and will be denoted by $\mathcal{H}_{\binom{\phi_0}{\psi_0}}$. The following theorem shows that the notion of Hilbert sectors is neither empty nor trivial.

Theorem 2. Let the potential U satisfy P2 and P3, let $\binom{\phi_0}{\psi_0} \in X$ with $\phi_0 \in \bigoplus_n L^\infty(\mathbb{R}^s)$. If

$$\psi_0 \in \bigoplus_n L^2(\mathbb{R}^s) \tag{2.1}$$

and

$$\Delta\phi_0 - \nabla_\phi U(x,\phi_0) \in \bigoplus_n H^{-1}(\mathbb{R}^s) , \tag{2.2}$$

then $\binom{\phi_0}{\psi_0}$ belongs to a Hilbert sector. Such a sector is formed by all $\binom{\phi'}{\psi'}$ for which $\phi' - \phi_0 \in \bigoplus_n H^1(\mathbb{R}^s)$ and $\psi' - \psi_0 \in \bigoplus_n L^2(\mathbb{R}^s)$. (I recall that a tempered distribution g belongs to $H^{-1}(\mathbb{R}^s)$ if its Fourier transform $\tilde{g}$ satisfies $\int |\tilde{g}(k)|^2(1+k^2)^{-1}dk < \infty$.)

In the statement of the above theorem the condition $\phi_0 \in \bigoplus_n L^\infty(\mathbb{R}^s)$ is technical and may be replaced by other conditions. It seems to be the most suitable for discussing concrete examples. The other two conditions seem to be more fundamental, in the sense that Theorem 2 has a kind of converse.

Theorem 3. Let the potential U satisfy P2 and P3, let $\binom{\phi_0}{\psi_0} \in X$ with $\phi_0 \in \bigoplus_n L^\infty(\mathbb{R}^s)$. If $\binom{\phi_0}{\psi_0}$ generates a Hilbert sector then conditions (2.1) and (2.2) are satisfied.

Detailed proofs of Theorems 2 and 3 in a more general framework can be found in [4]. Here I would only like to derive some immediate consequences from these theorems. From Theorem 1 it is clear that one can construct the Hilbert sector $\mathcal{H}_{\binom{\phi_0}{0}}$ for any $\phi_0 \in \left(\bigoplus_n H^1_{loc}(\mathbb{R}^s)\right) \cap \left(\bigoplus_n L^\infty(\mathbb{R}^s)\right)$ for which $\Delta\phi_0 - \nabla_\phi U(x,\phi_0) = 0$ (provided U satisfies P2 and P3). Examples of such ϕ_0's are the soliton type solutions (0.4) and (0.7) described in the introduction. Theorem 2 allows to find the Hilbert sectors generated by $\binom{c}{0}$ where c is a constant vector in $\mathbb{R}^n$. The c's for which $\mathcal{H}_{\binom{c}{0}}$ exists, are those for which $\nabla_\phi U(x,c) \in H^{-1}(\mathbb{R}^s)$. This condition is particularly interesting when U does not depend explicitely on the space variables. In this case the above condition becomes

$\nabla_\phi V(c) = 0$, namely the constant c must be a stationary point of the potential. This last remark is part of the so called Goldstone theorem.

3. DYNAMICAL CHARGES, ENERGY FUNCTIONALS, ENERGY SECTORS.

In this last part, I would like to show how the framework developed till now permits a rigorous discussion of some interesting physical concepts such as those of dynamical charge , of energy functional and of energy sector.

I will first discuss the notion of dynamical charge. The standard way of introducing a (conserved) charge is based on an invariance property of the Lagrangian density (or more precisely of the action) with respect to a Lie group of transformations. Via Noether's theorem one then finds as many conserved quantities (charges) as are the parameters of the group. The structure in sectors just described allows one to introduce conserved quantities in a way quite different from the above, in the sense that this new type of charge depends on the dynamics of the theory and not on kinematical symmetries imposed from the outside. Loosely speaking, for any Hilbert sector, the dynamical charge is represented by the "behaviour at ∞ " of any of its elements. Such a behaviour is the same for all elements belonging to the same Hilbert sector, as expressed by the following theorem.

Theorem 4. Let $\binom{\phi}{\psi}$ belong to the Hilbert sector $\mathcal{H}_{\binom{\phi}{\psi}}$, let $\lim_{r\to\infty} \phi(r\Omega) \equiv a(\Omega)$ exists in almost all directions $\Omega \in S^{s-1}$. Then, for all $\binom{\phi'}{\psi'} \in \mathcal{H}_{\binom{\phi}{\psi}}$, $\lim_{r\to\infty} \phi'(r\Omega) = a(\Omega)$ in almost all directions $\Omega \in S^{s-1}$.

This theorem is an obvious consequence of the definition of Hilbert sector and of the following technical result (for a proof see [(4)]).

Lemma 3. Let $\phi \in H^1(\mathbb{R}^s)$. Then $\lim_{r\to\infty} \phi(r\Omega) = 0$ in almost all directions $\Omega \in S^{s-1}$.

Therefore $a(\Omega)$, when it exists, is the same for all the elements ϕ of the same Hilbert sector; in particular it is conserved in time. These results justify why $a(\Omega)$ can be called a dynamical charge. To establish that $\lim_{r\to\infty} \phi(r\Omega)$ exists (almost everywhere for $\Omega \in S^{s-1}$) one usually needs some extra assumptions. In this connection it may be worth mentioning the following lemma.

Lemma 4. Let $s \geq 3$, let $\phi \in H^1_{loc}(\mathbb{R}^s)$ with $\nabla\phi \in L^2(\mathbb{R}^s)$. Then $\lim_{r\to\infty} \phi(r\Omega)$ exists in almost all directions $\Omega \in S^{s-1}$.

I turn now to energy considerations. The theory described by the equation (1.1) has conventionally for the energy density the expression

$$K(\phi,\psi) \equiv \frac{1}{2}\sum_{j=1}^{n}\left\{(\nabla\phi_j)^2 + \psi_j^2\right\} + U(x,\phi)\ , \tag{3.1}$$

which integrated over the whole space yields the total energy. However in general the expression at the r.h.s. of (3.1) is not integrable. Moreover there is an intrinsec ambiguity present in the definition of the energy density in the sense that one could add to the energy density (3.1) any function of the space variables without destroying the property of being formally conserved of the total energy. This ambiguity is resolved by observing that what one usually measures are energy differences. In the same way one could hope that, even in cases in which the expression (3.1) is not integrable, the difference between the energy densities corresponding to suitably chosen states might be integrable. In this connection the following results holds (see [4]).

Theorem 5. Let the potential U satisfy P2 and P3, let $\binom{\phi_0}{\psi_0} \in X$ with $\phi_0 \in \oplus_n L^\infty(\mathbb{R}^s)$ and let ψ_0 and ϕ_0 satisfy (2.1) and (2.2) respectively. Then for all $\binom{\phi}{\psi} \in \mathcal{H}_{\binom{\phi_0}{\psi_0}}$ (the Hilbert sector $\mathcal{H}_{\binom{\phi_0}{\psi_0}}$ is known to exist by theorem 2) with $\mathrm{supp}(\phi - \phi_0)$ compact, the function $K(\phi,\psi) - K(\phi_0,\psi_0)$ belongs to $L^1(\mathbb{R}^s)$ and the energy functional

$$\binom{\phi}{\psi} \longrightarrow E_{(\phi_0,\psi_0)}(\phi,\psi) := \int_{\mathbb{R}^s}\left[K(\phi,\psi) - K(\phi_0,\psi_0)\right]dx \tag{3.2}$$

defined for $\operatorname{supp}(\phi - \phi_0)$ compact has a unique continuous extension to the whole $\mathcal{H}\binom{\phi_0}{\psi_0}$. (Here the continuity is expressed in terms of the natural topology of $\mathcal{H}\binom{\phi_0}{\psi_0}$ considered as an affine variety based on Y). Furthermore the energy functional (3.2) is conserved in time.

In analogy to statistical mechanics and to quantum field theory, it would be desirable to reach the energy functional by taking the limit of the difference of the energy densities integrated over a finite volume as this volume invades the whole space. This seems to require some extra assumptions.

Theorem 6. Let the hypotheses of Theorem 5 be satisfied, with the additional assumption that $\nabla\phi_0 \in \underset{n}{\oplus} L^2(\mathbb{R}^s)$. Then for all $x_0 \in \mathbb{R}^s$, for all $\omega \in \mathcal{C}_0^\infty(\mathbb{R}^s)$ equal to 1 in a neighbourhood of the origin one has

$$E_{(\phi_0,\psi_0)}(\phi,\psi) = \lim_{R\to\infty} \int \left[K(\phi,\psi) - K(\phi_0,\psi_0)\right] \omega\left(\frac{x-x_0}{R}\right) dx . \tag{3.3}$$

Theorems 5 and 6 show that any two states within the same Hilbert sector have finite energy with respect to each other. However even states belonging to different Hilbert sectors could have this property. This happens for instance in the case of degeneracy. It is therefore convenient to group together those Hilbert sectors, whose relative energy is finite. Such a family of Hilbert sectors will be called an energy sector. This is a useful concept: in fact, even if time evolution makes each sector a closed world, small (finite energy) perturbations or quantum effects could cause transitions between Hilbert sectors which have relatively finite energy.

Precise definitions of energy sectors as well as their usefulness in discussing the symmetries of the theory may be found in (4) and especially in (5).

REFERENCES

1) C. Parenti, F. Strocchi and G. Velo: Phys. Lett., 59B, 157 (1975).
2) C. Parenti, F. Strocchi and G. Velo: Phys. Lett., 62B, 83 (1976).
3) C. Parenti, F. Strocchi and G. Velo: Ann. Scuola Norm. Sup. Pisa, 3, 443, (1976).
4) C. Parenti, F. Strocchi and G. Velo: Commun. Math. Phys., 53, 65 (1977).
5) C. Parenti, F. Strocchi and G. Velo: Nuovo Cimento, 39B, 147 (1977).
6) J.L. Gervais and A. Neveu, Editors: Extended Systems in Field Theory, Phys. Rep., 23C, 237 (1976).
7) K. Hepp: Commun. Math. Phys., 35, 265 (1974).
8) K. Jörgens: Math. Zeits., 77, 295 (1961).
9) I. Segal: Ann. Math., 78, 339 (1963).