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A Bound on the Total Number of Bound States in a Potential

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ABSTRACT

We prove that the total number of bound states in a potential is less than

$$\frac{1}{2\pi}\sqrt{\int |V^-(x)d^3x \int |V^-(x)|^2 d^3x}$$

in units where $2m/\hbar^2 = 1$; V^- is the attractive part of the potential. No assumption on the symmetry of the potential is needed. In the Appendix, a proof of the Hardy-Littlewood-Sobolev inequality with optimal constants is sketched.

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1. INTRODUCTION.

Recently, new conditions for the absence of bound states in a potential have been obtained [1]. In addition, a bound on the total number of bound states in a spherically symmetric potential has been obtained in GGMT :

$$N \leq I[1 + \tfrac{1}{4}\ln I] \qquad (1)$$

with

$$I = \frac{4}{3\sqrt{3}\pi^2} \int |V^-(x)|^{\frac{3}{2}} d^3x \qquad (2)$$

(we take $2m/\hbar^2 = 1$).

However, the bound (1) is not satisfactory in two respects. First it contains a logarithmic factor which, as we shall see, is probably spurious. Second, spherical symmetry of the potential is assumed. This is very unpleasant, because it is now widely recognized that Nature likes at least as much broken symmetry as symmetry ; therefore, even in the lowest energy configuration, a particle may experience a non-spherically symmetric potential.

If we decide not to assume spherical symmetry, what kind of bounds are at our disposal, if we exclude those trivially obtained by majorization of an arbitrary potential by a spherically symmetric one ? (Such a bound would have the defect of not being invariant under translations.) The oldest and most remarkable bound has been obtained by Schwinger [2]. It is

$$N_\alpha < \frac{1}{(4\pi)^2} \int d^3x\, d^3x' \, |V^-(x)|\,|V^-(x')| \, \frac{\exp - 2\alpha|x-x'|}{|x-x'|^2} \qquad (3)$$

N_α is the number of bound states with energy less than $-\alpha^2$. In particular, the total number of bound states for a short-range potential with no positive energy bound states is

$$N < \frac{1}{(4\pi)^2} \int d^3x\, d^3x' \, \frac{|V^-(x)|\,|V^-(x')|}{|x-x'|^2} \qquad (4)$$

which implies, by the Hardy-Littlewood-Sobolev theorem [3]

$$N < \left(\frac{\pi}{2}\right)^{\frac{2}{3}} \left[\int \frac{d^3x}{4\pi} \, |V^-(x)|^{\frac{3}{2}}\right]^{\frac{4}{3}} \qquad (5)$$

In the Appendix we give a proof of inequality (5) with a somewhat larger constant. We obtain the constant in (5) from a variation argument, which would be completely rigorous if we could prove strict uniqueness of the solution of the variation equation (we have local uniqueness).

The defect of (4) and (5) is that they give a poor upper bound in the strong coupling limit. Indeed if we replace V by λV, we have, in the limit $\lambda \to \infty$ [4],

$$N \sim \lambda^{\frac{3}{2}} \frac{1}{6\pi^2} \int d^3x \, |V^-(x)|^{\frac{3}{2}} \qquad (6)$$

while (4) and (5) give a λ^2 behaviour. In reference [4] corrective terms to (5), for finite λ, can be found but they depend on smoothness properties of the potential. In fact, in GGMT, it was proposed that $N \leq I$, where I is defined by Eq. (2), might be a bound. It was even argued that there could not be any bound with better numerical constants. Unfortunately, this conjecture was only checked for $I < 3$.

Here we shall establish a bound which has the correct dependence with respect to the coupling constant. However, so far, we have not been able to obtain a bound depending only on I. In the last section, we treat the special case of spherically symmetric potentials where additional results, following from GGMT, can be obtained. In a recent preprint, Barry Simon has also obtained a bound behaving like $\lambda^{\frac{3}{2}}$ [5].

2. THE GENERAL BOUND.

The shall follow very closely the elegant method of Schwinger [2]. We take a potential $\lambda V(x)$, in order to see the explicit dependence on the coupling. We can replace $V(x)$ by $-V^-(x)$, its attractive part.

This will necessarily increase the number of bound states. Then it is possible to symmetrize the kernel of the Schrödinger equation in integral form for a bound state of energy $-\alpha^2$:

$$\Phi(x) = \lambda \int K_\alpha(x,x') \Phi(x') \, d^3x' \tag{7}$$

where

$$\Phi(x) = \sqrt{V^-(x)}\psi(x) ,$$

ψ being the wave function,

$$K_\alpha(x,x') = \sqrt{V^-(x)} \, \frac{\exp - \alpha |x-x'|}{|x-x'|} \sqrt{V^-(x')} \tag{8}$$

The operator K_α is <u>positive</u>, as can be seen from its expression in momentum space

$$\tilde{K}_\alpha(p,p') = \int d^3k \, \frac{w^*(p-k)w(k-\mathring{p})}{k^2+\alpha^2} \tag{9}$$

w is the Fourier transform of $\sqrt{V^-}$.

Schwinger notices that the number of bound states of energy less than $-\alpha^2$ inside the potential λV, is equal to the number of characteristic values $\lambda_n < \lambda$ of Eq. (7) . In particular, if we define $N_\alpha(\lambda)$, number of bound states with energy less than α^2 in the potential λV and

$$n_\alpha(\lambda) = \sum_{\lambda_n<\lambda} \frac{1}{\lambda_n} = \sum_{\lambda_n<\lambda} < \Phi_n |K_\alpha| \Phi_n > , \tag{10}$$

where the Φ_n 's are the eigenstates of K_α with characteristic values λ_n, we have

$$N_\alpha(\lambda) \leq \lambda \, n_\alpha(\lambda) \tag{11}$$

An upper bound of $n_\alpha(\lambda)$ could be obtained by summing over all n's , i.e., by taking the trace of K_α . Unfortunately, this trace is divergent, as can be seen from (8).

This is the reason why Schwinger uses the iterated kernel instead of K_α ; then, unavoidably, the bound on the number of states is quadratic in λ . We want to avoid this and to do so we use an approximate kernel with a finite trace. A possible choice is

$$K_\alpha - K_\mu \qquad \mu > \alpha \, . \tag{12}$$

For $\mu \to \infty$, K_μ goes to zero in norm. On the other hand, the operator $K_\alpha - K_\mu$, as can be seen from its momentum space expression,

$$\int w^*(p-k)\left(\frac{1}{k^2+\alpha^2} - \frac{1}{k^2+\mu^2}\right) w(k-p')d^3K$$

is also positive.

We can rewrite $n_\alpha(\lambda)$ as

$$n_\alpha(\lambda) = \sum_{\lambda_n<\lambda} \langle\Phi_n|K_\alpha - K_\mu|\Phi_n\rangle + \sum_{\lambda_n<\lambda} \langle\Phi_n|K_\mu|\Phi_n\rangle$$

$$\leq \sum_{\lambda_n<\lambda} \langle\Phi_n|K_\alpha - K_\mu|\Phi_n\rangle + \sum_{\lambda_n<\lambda} \frac{\lambda}{\lambda_n}\langle\Phi_n|K_\mu|\Phi_n\rangle$$

and using the positivity of $K_\alpha - K_\mu$, K_μ , and K_α , we can extend the summation to infinity and get

$$n_\alpha(\lambda) < \mathrm{Tr}(K_\alpha - K_\mu) + \lambda\, \mathrm{Tr}(K_\alpha K_\mu) \tag{13}$$

which is our main result.

One can evaluate (13) quite easily :

$$n_\alpha(\lambda) <$$

$$\frac{\mu-\alpha}{4\pi}\int d^3x\, V^-(x)$$

$$+ \frac{\lambda}{(4\pi)^2}\int d^3x\, d^3x' V^-(x)\, \frac{\exp-(\mu+\alpha)|x-x'|}{|x-x'|^2} V^-(x') \tag{14}$$

By using Schwarz' or Yung's inequality, one can majorize the second term in (14) to get

$$n_\alpha(\lambda) < \frac{\mu-\alpha}{4\pi}\int d^3x\, V^-(x) + \frac{\lambda}{4\pi(\mu+\alpha)}\int d^3x\,|V^-(x)|^2 \qquad (15)$$

After taking the limit $\alpha \to 0$, we minimize, with respec to μ , and get the bound :

$$N < \lambda\, n_o(\lambda) < \frac{\lambda^{3/2}}{2\pi}\sqrt{\int V^-(x)d^3x\int |V^-(x)|^2 d^3x}\ . \qquad (16)$$

This bound has the correct $\lambda^{\frac{3}{2}}$ dependance, but requires more local regularity than the Schwinger bound. For instance, it diverges if the potential has a local singularity of the type $|x-x_o|^{-\frac{3}{2}}$. Numerically the bound is not too bad. In fact, one cannot improve the constant by more than a factor 2.9 .

Indeed, assume (as is the case) $N \simeq c\,\lambda^{\frac{3}{2}}$ for <u>large</u> λ . Then

$$n_o(\lambda) = \int_o^\lambda \frac{dN(\lambda)}{\lambda} \simeq 3c\lambda^{\frac{1}{2}} \qquad (17)$$

and we get

$$c < \frac{1}{6\pi}\sqrt{\int d^3x\, V^-(x)\int d^3x (V^-(x))^2}\ ,$$

to be compared with the asymptotic estimate (6) . Taking V^- to be a constant over a finite region of space, we see that the two bounds differ by a factor π . For small values of λ we lose the factor 3 appearing in (17) because it may be that the potential has no bound states up to a critical value λ and many immediately above. This is the case in the example mentioned in GGMT of a series of isolated potential wells far from one another. We know that an optimal condition for the absence of bound state is :

$$\frac{4\lambda^{3/2}}{3\sqrt{3}\,\pi^2}\int d^3x\,|V(x)|^{3/2} < 1$$

which is saturated by

$$V = -\,C(1+r^2)^{-2}$$

If we compare the two conditions, we get

$$\frac{\frac{1}{2\pi}\sqrt{\int |V|d^3x \int |V|^2 d^3x}}{\frac{4}{3\sqrt{3}\pi^2}\int |V|^{3/2}\, d^3x} = \frac{3\sqrt{3}}{4\sqrt{2}}\pi \simeq 2.88 \tag{18}$$

Infact, by a limited optimization with respect to X we can replace $\frac{1}{2\pi}$ by $\frac{1.114}{\pi^2}$ in (16) .

3. THE SPHERICALLY SYMMETRIC CASE

This is just an addendum to GGMT. In GGMT it was established for spherically symmetric potentials that ν_ℓ , the number of bound states with angular momentum ℓ , counted without the $(2\ell+1)$ degeneracy factor, was such that

$$(2\ell+1)\,\nu_\ell < \frac{1}{(2\ell+1)^{2(p-1)}}\, C_p \lambda^p \int r^{2p-1} V_-^p(r)\,dr \tag{19}$$

with

$$C_p = \frac{(p-1)^{p-1}\Gamma(2p)}{p^p\Gamma^2(p)}$$

A bound on the total number of bound states can be obtained by taking

$$N < \Sigma(2\ell+1)\nu_\ell$$

$$< \sum_{o}^{L} \frac{1}{(2\ell+1)^{2(p-1)}} C_p \lambda^p \int r^{2p-1} (V^-(r))^p dr$$

$$+ \sum_{L+1}^{\infty} \frac{1}{(2\ell+1)^{2(p'-1)}} C_{p'} \lambda^{p'} \int r^{2p'-1} (V^-(r))^{p'} dr$$

with $p' > p$. The old bound of Simon [6] can be considered as a limited case with $p = 1$, $p' = \infty$. One can also take $p = \frac{3}{2} - \epsilon$, $p' = \frac{3}{2} + \epsilon$. Replacing sums by integrals for brevity, we get

$$N \lesssim \frac{1}{2} \frac{1}{2\epsilon} (2L+1)^{2\epsilon} C_{3/2-\epsilon} \lambda^{3/2-\epsilon} \int (r^2 V)^{-\epsilon} V^{3/2} r^2 dr$$

$$+ \frac{1}{2} \frac{1}{2\epsilon} (2L+1)^{-2\epsilon} C_{3/2+\epsilon} \lambda^{3/2+\epsilon} \int (r^2 V)^{\epsilon} V^{3/2} r^2 dr$$

and minimizing with respect to $2L+1$

$$N \lesssim \frac{1}{2\epsilon} \sqrt{C_{3/2-\epsilon} C_{3/2+\epsilon}} \lambda^{3/2}$$

$$\times \sqrt{\int (r^2 V)^{-\epsilon} V^{3/2} r^2 dr \int (r^2 V)^{\epsilon} V^{3/2} r^2 dr} \qquad (20)$$

Again we recover the $\lambda^{\frac{3}{2}}$ dependance, but in addition we have a bound for potentials which have singularities in $r^{-2+\eta}$, η positive arbitrarily small. This suggests that inequality (16) , for non-spherically symmetric potentials, is not final.

This result is analogous to the one obtained by Bary SIMON in ref [5] which is

$$N < C_\epsilon \lambda^{3/2} [\|V\|_{3/2-\epsilon} + \|V\|_{3/2+\epsilon}]^{3/2} \quad .$$

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APPENDIX

A PROOF OF INEQUALITY (5)

The problem is to prove

$$\int \frac{V(x)V(x')d^3x d^3x'}{|x-x'|^2} < C\left[\int (V(x))^{3/2} d^3x\right]^{4/3} \qquad (A.1)$$

and to find the best possible constant C. This inequality is well known by a small group of specialists, but it is irritating to find no simple proof in the literature, and the existing proofs do not give numerical factors.

We remark first that according to Luttinger and Friedberg [7]

$$\int \frac{V(x)V(x')}{|x-x'|^2} d^3x\, d^3x' < \int \frac{V^*(x)V^*(x')}{|x-x'|^2} d^3x\, d^3x' \qquad (A.2)$$

where $V^*(x)$ is the spherical decreasing rearrangement of $V(x)$. Since the right-hand-side of (A.1) is invariant under rearrangement, we can always assume that V is spherically symmetric. Then we can carry angular integrations :

$$\int \frac{d^3x}{4\pi} \frac{d^3x'}{4\pi} \frac{V(r)V(r')}{|x-x'|^2}$$

$$= \frac{1}{2} \int r^2 dr\, r'^2\, dr' V(r)V(r') \frac{\log\left|\frac{r+r'}{r-r'}\right|}{r\, r'} \qquad (A.3)$$

Then using the Hölder inequality, we find that this is less than

$$\frac{1}{2}\left[\int r^2 dr\, r'^2\, dr' (V(r))^{3/2} (V(r'))^{3/2}\right]^{1/3}$$

$$\times\left[\int r^2 dr\, r'^2\, dr' (V(r))^{3/4} (V(r'))^{3/4} \left(\frac{\log\left|\frac{r+r'}{r-r'}\right|}{r\,r'}\right)^{3/2}\right]^{2/3} \tag{A.4}$$

Next we apply Schwarz' inequality to the second bracket :

$$\left[\int r^2 dr\, r'^2\, dr' (V(r))^{3/4} (V(r'))^{3/4} \left(\frac{\log\left|\frac{r+r'}{r-r'}\right|}{r\,r'}\right)^{3/2}\right]^2$$

$$< \left[\int r^2 dr (V(r))^{3/2}\ r'^2 dr' \left(\frac{r}{r'}\right)^{3/2} \left(\frac{\log\left|\frac{r+r'}{r-r'}\right|}{r\,r'}\right)^{3/2}\right]$$

$$\times \left[\int r'^2 dr' (V(r'))^{3/2}\ r^2 dr \left(\frac{r'}{r}\right)^{3/2} \left(\frac{\log\left|\frac{r+r'}{r-r'}\right|}{r\,r'}\right)^{3/2}\right] . \tag{A.5}$$

The integral

$$\int_0^\infty r^2 dr \left(\frac{r'}{r}\right)^{3/2} \left(\frac{\log\left|\frac{r+r'}{r-r'}\right|}{r\,r'}\right)^{3/2}$$

is convergent and dimensionless.

Hence

$$\int \frac{d^3x}{4\pi}\frac{d^3x'}{4\pi}\ \frac{V(x)V(x')}{|x-x'|^2}$$

$$< \frac{1}{2}\left[\int \frac{d^3x}{4\pi} |V(x)^{3/2}\right]^{4/3} \left[\int_0^\infty \frac{dx}{x}\left(\log\left|\frac{1+x}{1-x}\right|\right)^{3/2}\right]^{2/3} \tag{A.6}$$

with

$$\int_0^\infty \frac{dx}{x}\left(\log\left|\frac{1+x}{1-x}\right|\right)^{3/2} = 4\int_0^1 \frac{dx}{1-x^2}\left(\log\frac{1}{x}\right)^{3/2}$$

$$= \Gamma(5/2)\left[1+\frac{1}{3^{5/2}}+\frac{1}{5^{5/2}}+\ldots\right] \simeq 5.87$$

Therefore

$$\frac{1}{(4\pi)^2}\int \frac{d^3x\, d^3x'\, V(x)V(x')}{|x-x'|^2} < 1.63\left[\int \frac{d^3x}{4\pi}|V(x)|^{3/2}\right]^{4/3} \tag{A.7}$$

A best value for the constant can be obtained by exact minimization. Let us assume provisionally that is given by the solution of the variation equation :

$$(V(x))^{\frac{1}{2}} = C\int \frac{d^3x'}{|x-x'|^2}\, V(x') \tag{A.8}$$

We can restrict ourselves to spherically symmetric solutions, with $V(r)$ monotonous decreasing

$$(V(r))^{\frac{1}{2}} = C'\int \frac{r'dr'}{r}\log\left|\frac{r+r'}{r-r'}\right| V(r') \tag{A.9}$$

We have found two families of solutions of this equation :

(1) $$V(r) = \text{const } r^{-2} \tag{A.10}$$

This solution is not acceptable.

(2) $$V(r) = \lambda(a^2+r^2)^{-2} \tag{A.11}$$

At the present time we have no strict proof that solutions of type (2) are the only possible solutions. However, we have been able to prove that if $V(r)r^2$ does not tend to zero for $r \to \infty$, it is a constant. If $r^2V(r)$ tends to zero, it necessarily decreases like r^{-2}.

We leave aside question of uniqueness, and we proceed first to prove that there exists indeed a minimum for

$$K = \left[\int \frac{d^3x}{4\pi}(V(x))^{3/2}\right]^{4/3} \Bigg/ \int \frac{d^3x}{4\pi}\,\frac{d^3x'}{4\pi}\,\frac{V(x)V(x')}{|x-x'|^2} \qquad \text{(A.12)}$$

We have seen that we can restrict ourselves to spherically symmetric, decreassing $V(r)$ to find the infimum of K. Then, introducing $\rho(r) = r(V/r)^{\frac{1}{2}}$ we rewrite K as

$$K = \left(\int \frac{dr}{r}(\rho(r))^3\right)^{4/3} \Bigg/ \frac{1}{2}\int \frac{dr}{r}\,\frac{dr'}{r'}\,\log\left|\frac{r+r'}{r-r'}\right| |\rho(r)|^2|\rho(r')|^2$$

Changing variables to $\sigma(z) = \rho(r)$, $z = \log r$, we get

$$K = \left[\int_{-\infty}^{+\infty} dz\,|\sigma(z)|^3\right]^{4/3} \Bigg/ \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dz\,dz'\log\left|\frac{1+\exp(z-z')}{1-\exp(z-z')}\right| (\sigma(z)|^2|\sigma(z')|^2$$

Now we can use the theorem of Hardy, Littlewood and Polya on symmetric rearrangements. Replacing $\sigma(z)$ by its symmetric rearrangement around the origin, leaves the numerator of K invariant, but increases the denominator because

$$\log\left|\frac{1+\exp z}{1-\exp z}\right|$$

is symmetric, and decreasing for $z > 0$. If one looks carefully at the proofs, one sees that the rearrangement by a non-zero amount the denominator, unless $\sigma(z)$ is deduced from a symmetric decreasing function by a change of the origin. We conclude that to find the infimum of K, we can restrict ourselves to a class in which both rearrangement conditions are fulfilled :

1) $V(r)$ decreasing, i.e., $\exp(-z)\,\sigma(z)$ decreasing for $-\infty < z < \infty$;

2) $\sigma(z) = \sigma(-z)$ and $\sigma(z)$ decreasing for $z > 0$.

It then follows that $\exp(z)\,\sigma(z)$ is increasing. Hence, we consider functions $\sigma(z)$ such that

i) $$\sigma(z) = \sigma(-z)$$

ii) $$0 \geq \frac{d\sigma}{dz} \geq -\sigma(z) \qquad z > 0 \, .$$

Further, we may assume, since we are looking for the infimum, $K \leq C$; for instance, using $V = (1+r^2)^{-2}$, we find

$$K = \left(\frac{2}{\pi}\right)^{2/3} \simeq \frac{1}{1.35}$$

(to be compared with $1/1.63$) and so we can take

$$C = \left(\frac{2}{\pi}\right)^{2/3} \, . \qquad \text{(A.13)}$$

Therefore we have

$$\left[\int \frac{dr}{r}\,|\rho(r)|^3\right]^{4/3} \leq \left(\frac{2}{\pi}\right)^{2/3} \times \frac{1}{2}\int \frac{dr}{r}\,\frac{dr'}{r'}\,\log\left|\frac{r+r'}{r-r'}\right|\,(\rho(r))^{3/2}\,(\rho(r'))^{3/2} \times (\sup\rho)$$

$$\leq \left(\frac{2}{\pi}\right)^{2/3} \times \frac{1}{2}\,(\sup\rho) \times \int_0^\infty \frac{dr}{r}(\rho(r))^3 \int_0^\infty \frac{dx}{x}\,\log\left|\frac{1+x}{1-x}\right|$$

$$= \left(\frac{\pi}{2}\right)^{4/3} (\sup\rho) \int_0^\infty \frac{dr}{r}(\rho(r))^3 \qquad \text{(A.14)}$$

Hence for all ρ's , such that

$$K \leq \left(\frac{2}{\pi}\right)^{2/3}$$

we have

$$\int \frac{dr}{r}(\rho(r))^3 < \left(\frac{\pi}{2}\right)^4 (\sup \rho)^3 \qquad \text{(A.15)}$$

Now since K has an infimum, there is a sequence ρ_n such that $K(\rho_n)$ approaches this infimum. We may assume $\sup \rho_n = \rho_n(r = 1) = \sigma_n(z = 0) = 1$. Then

$$\left| \frac{d\sigma_n}{dz} \right| < \sigma_n < 1 \qquad \text{(A.16)}$$

The σ_n's are therefore equicontinuous. From the Ascoli-Arzela theorem, there is a subsequence of σ_n's approaching a limit which we call σ_∞ . We have

$$\sigma_\infty(0) = 1 \ .$$

$\sigma_\infty(z)$ is continuous and such that

$$\int_{-\infty}^{+\infty} dz \, |\sigma_\infty(z)|^3 = \int_0^\infty \frac{dr}{r} |\rho_\infty(r)|^3 < \left(\frac{\pi}{2}\right)^4 \qquad \text{(A.17)}$$

It is also easy to see that

$$\int dz dz' \log \left| \frac{1+\exp(z-z')}{1-\exp(z-z')} \right| |\sigma(z)|^2 |\sigma(z')|^2$$

approaches a limit. Indeed the integral restricted to $|z| < M |z'| < M$ approaches a limit, and the rest is bounded by

$$C \times \sup_{|z|>M} |\sigma(z)|^{\frac{1}{2}} \int_{-\infty}^{+\infty} |\sigma(z')|^3 dz'$$

but we have from the monotony of ρ

$$|\sigma(z)| < \left[\frac{\int dz' |\sigma(z')|^3}{2z} \right]^{1/3} < C\, z^{-\frac{1}{3}}$$

The conclusion is that $\sigma_\infty(z)$ indeed minimizes K . Then it follows easily that $\sigma_\infty(z)$ or $\rho_\infty(r)$ must satisfy the variation equation.

Let us prove that, if we exclude dilatations and impose that $r^2 V(r)$ is invariant in the change $r \to 1/r$, the solution $V(r) = 1/(1+r^2)^2$ of (A.9) is locally unique. We use the fact that we know a particular family of solutions :

$$\frac{\pi}{2a(a^2+r^2)} = \int \frac{r'dr'}{r} \log \left| \frac{r+r'}{r-r'} \right| \left(\frac{1}{a^2+r'^2}\right)^2 \qquad (A.18)$$

and, differentiating with respect to a^2 ,

$$\left(\frac{d}{da^2}\right)^n \left[\frac{\pi}{2a} \frac{1}{a^2+r^2} \right] = \int \frac{r'dr'}{r} \log \left| \frac{r+r'}{r-r'} \right| \left(\frac{d}{da}\right)^n \left(\frac{1}{a^2+r'^2}\right)^2 . \qquad (A.19)$$

Equation (A.19) shows that the kernel

$$H(r,r') = (1+r^2)\, \frac{r'}{r} \log \left| \frac{r+r'}{r-r'} \right| \frac{1}{(1+r'^2)^2}$$

applied to a polynomial of degree n in $1/(1+r^2)$, produces a polynomial of degree n . Its eigenvectors π_n satisfy the orthogonality relation :

$$\delta_{nm} = \int \pi_n\left(\frac{1}{1+r^2}\right) \pi_m\left(\frac{1}{1+r^2}\right) \frac{r^2 dr}{(1+r^2)^3}$$

This shows that the π_n are Jacobi polynomials which form a complete set :

$$\pi_n \sim P_n^{\frac{1}{2}\frac{1}{2}}\left(\frac{1-r^2}{1+r^2}\right)$$

[i.e., Tchebytcheff polynomials [8]]. The eigenvalues can be obtained by looking at the highest degree terms in $(a^2+r^2)^{-1}$:

$$P_n^{\frac{1}{2}\frac{1}{2}}\left(\frac{1-r^2}{1+r^2}\right) = \frac{\pi}{2(n+1)} \int H(r,r')\, P_n^{\frac{1}{2}\frac{1}{2}}\left(\frac{1-r'^2}{1+r'^2}\right)dr'$$

Let us now consider variations of V around the basic solution :

$$V = \left(\frac{1}{1+r^2}\right)^2 (1+\epsilon(r))$$

To prove that the solution $(1/1+r^2)^2$ is isolated, we insert this expression, retaining the lowest order in ϵ , excluding dilatations [$\epsilon(r)$ = const.] , and keeping the symmetry of $r^2V(r)$ by imposing $\epsilon(r) = \epsilon(1/r)$.
We expand

$$\epsilon = \Sigma\, C_n \pi_n$$

with

$$C_o = 0 \quad C_{2n+1} = 0$$

Inserting to lowest order in the integral equation with eigenvalue λ , we find

$$\pi_o + \tfrac{1}{2}\Sigma\, C_n \pi_n = \lambda \frac{\pi}{2}\left[\pi_o + \Sigma \frac{C_n}{n+1}\pi_n\right]$$

Since C_o and C_1 are zero, we see that the only eigenvalue close to $\lambda = 1$ is $\lambda = 1$, with eigenvector π_o .

This procedure also allows us to prove that $V_o = (1+r^2)^{-2}$ produces a local minimum. With

$$V = (1+r^2)^{-2}(1+\epsilon(r))$$

we compute

$$A = \left(\int |V|^{3/2} \frac{d^3x}{4\pi}\right)^{4/3} - \left(\frac{2}{\pi}\right)^{2/3} \int \frac{d^3x}{4\pi} \frac{d^3x'}{4\pi} \frac{V(x)V(x')}{|x-x'|^2} . \qquad (A.20)$$

In the ϵ expansion, the constant and linear terms in ϵ cancel by definition. We get

$$A = \tfrac{1}{2} \left(\int |V_o|^{3/2} (\epsilon(r))^2 \frac{d^3x}{4\pi}\right) \left(\int |V_o|^{3/2} \frac{d^3x}{4\pi}\right)^{1/3}$$

$$+ \tfrac{1}{2} \left(\int |V_o|^{3/2} \epsilon(r) \frac{d^3x}{4\pi}\right)^2 \left(\int |V_o|^{3/2} \frac{d^3x}{4\pi}\right)^{-2/3}$$

$$- \left(\frac{2}{\pi}\right)^{2/3} \int \frac{d^3x}{4\pi} \frac{d^3x'}{4\pi} \epsilon(r) V_o(x) \frac{1}{|x-x'|^2} \epsilon(r') V_o(x')$$

$$+ O(\epsilon^3)$$

with

$$\epsilon(r) = \sum_2^\infty C_n \pi_n$$

we see from the orthogonality condition that

$$\int |V_o|^{3/2} \epsilon(r) \frac{d^3x}{4\pi} = 0$$

and the whole expression reduces to

$$\tfrac{1}{2} \left(\frac{\pi}{2}\right)^{V_3} \left[\sum_{n=2}^\infty |C_n|^2 \left(\tfrac{1}{2} - 1/n+1\right) \right], \qquad (A.21)$$

which is strictly positive.

In fact the proof holds also for finite ϵ . Using the fact that

$$\frac{(1+\varepsilon)^{3/2} - 1 - \frac{3\varepsilon}{2}}{\varepsilon^2}$$

decreases for $\varepsilon > 0$, and that $(1+x)^{4/3} > 1+4/3(x)$ one easily finds that A is strictly positive for

$$-1 < \varepsilon(r) < \varepsilon_o$$

where ε_o is the solution of

$$(1+\varepsilon_o)^{3/2} - 1 - \frac{3}{2}\varepsilon_o = \frac{\varepsilon_o^2}{4} \qquad \text{(A.22)}$$

ε_o is strictly larger than 3. However, the proof that the V_o gives the absolute minimum is still missing.

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