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I. GENERAL REVIEW OF PERTURBATIVE RENORMALIZATION

There exist at present several versions[1] of renormalized perturbation theory, which develop in various directions some of the ideas contained in the work of N.N. Bogoliubov[2] and coworkers. The general set up, as recently described by H. Epstein and V. Glaser[1, 3] is as follows:

$\mathcal{F}$: Fock space of a family of free fields $\underline{\phi}_0(x)$

$U(a,A)$: the corresponding Fock representation of the covering of the Poincaré group

Ω : the vacuum state

$\underline{\mathcal{L}}_0(x)$: the "interaction" Lagrangian, a family of Wick monomials in $\underline{\phi}$ and its derivatives which, together with a given monomial, contains all its submonomials.

$\underline{g}(x)$ a corresponding family of smooth space time dependent coupling constants, with fast decrease at infinity.

One can construct by recursion an operator valued formal power series in g:

$$S(\underline{g}) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(i)^n}{n!} \int dx_1 \dots dx_n \, T(\underline{x}_1, \dots \underline{x}_n) \underline{g}(x_1) \dots \underline{g}(x_n)$$

whose coefficients $T(\underline{x}_1, \dots \underline{x}_n) \overset{\text{Def.}}{=} T(\underline{\mathcal{L}}_0(x_1) \dots \underline{\mathcal{L}}_0(x_n))$ are properly defined "time ordered" products of the Poincaré covariant interaction Lagrangians, densely defined in $\mathcal{F}$ as operator valued distributions. The antitime ordered products are defined as coefficients of the formal power series for $S^{-1}(g)$:

$$S^{-1}(\underline{g}) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int dx_1 \dots dx_n \, \overline{T}(\underline{x}_1, \dots, \underline{x}_n) \underline{g}(x_1) \dots \underline{g}(x_n)$$

The recursive construction can be so carried out that the following causal factorization property holds:

$$S(\underline{g}_1 + \underline{g}_2) = S(\underline{g}_1) S(\underline{g}_2)$$

$$\text{if } \operatorname{supp} \underline{g}_1 \gtrsim \operatorname{supp} \underline{g}_2$$

$$\text{i.e. } (\operatorname{supp} \underline{g}_1 + \overline{V}_+) \cap \operatorname{supp} \underline{g}_2 = \emptyset$$

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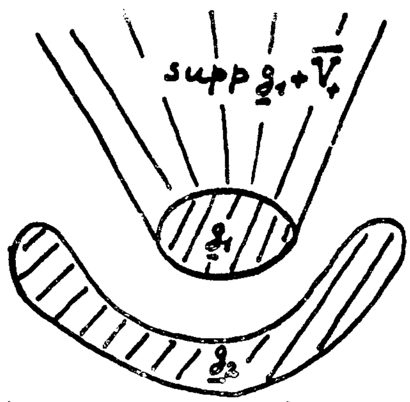

Fig. I Supp. $g_1 \gtrsim$ Supp. g_2

At each step of the recursive construction, there is an ambiguity whose minimality can be discussed via a precise theory of power counting. This ambiguity arises solely from the fact that causality and spectrum force the fields, and therefore their correlation functions, to be distributions. The problem to be solved concerns a certain "discontinuity" C_n which, in view of the causality assumption, has the union of two opposite closed cones $\Gamma_n^{\pm}$ as x-space support. One has to express C_n as the difference of a retarded and advanced distribution with respective supports Γ_n^{+}, Γ_n^{-}. This can be done, but not by mere multiplication by some Heaviside step function, and the solution is in general non-unique. At the nth step there arises an ambiguity in the definition of the operator valued distribution $T(x_1, \ldots, x_n)$ which is of the type

$$\Delta T(x_1, \ldots x_n) = \sum_i P_i(\partial)\, \delta(x_1 - x_2) \ldots \delta(x_{n-1} - x_n) : Q_{oi} :$$

where Q_{oi} is a Wick monomial in ϕ_0 and its derivatives, $P_i(\partial)$ a monomial of derivatives, and the sum ranges over all possible terms compatible with Wick's theorem such that

$$\deg. P_i + \omega(Q_{oi}) \geqslant \omega_n = \sum_{i=1}^{n} \left(\omega(\mathcal{L}_i) - 4 \right) + 4 .$$

The index ω assigned to a Wick monomial adds up the number of derivatives and the number of fields each of which weighted by its naive dimension ($\frac{2j+2}{2}$ for a field carrying spin j if no better estimate holds due to the presence of an indefinite metric in $\mathcal{F}$).

Given two solutions, S and $\hat{S}$, characterized by coefficients $T(x_1 \ldots x_n)$, $\hat{T}(x_1 \ldots x_n) = T(x_1 \ldots x_n) + \Delta T(x_1 \ldots x_n)$, $\hat{S}$ is obtained by applying the prescription "T" to the interaction term

$$\int \hat{\mathcal{L}}_0(x,g)\,dx = \int \underline{\mathcal{L}}_0(x)\,\underline{g}(x)\,dx + \sum_{n=2}^{\infty} \frac{(i)^n}{n!} \int dx_1 \ldots dx_n \quad \ldots$$

$$\ldots\ \Delta T(\underline{x}_1, \ldots \underline{x}_n)\ \underline{g}(x_1) \ldots \underline{g}(x_n)$$

Each operator time ordered product is expressed in terms of c-number distribution coefficients by means of Wick's theorem:

$$T\left(\mathcal{L}_{0,a_1}(x_1) \ldots \mathcal{L}_{0,a_n}(x_n)\right) = \sum \left(\Omega, T\left(\mathcal{L}^{r_1}_{0,a_1}(x_1), \ldots \mathcal{L}^{r_n}_{0,a_n}(x_n)\right)\Omega\right) \ldots$$

$$\ldots :Q^{r_1}_{0,a_1}(x_1) \ldots Q^{r_n}_{0,a_n}: C_{r_1, \ldots r_n} \quad .$$

where $:\mathcal{L}^{r_i}_{0,a_i}\ Q^{r_i}_{0,a_i}: = \mathcal{L}_{0,a_i}$ and $C_{r_1 \ldots r_n}$ are some combinatorial factors. Thus, in p-space, the vacuum expectation values of time ordered products are ambiguous up to polynomials of degrees determined by power counting. If all components of ϕ_0 carry a non-vanishing mass, a "central" solution can be defined, which vanishes at zero momentum together with its ω first derivatives, ω being the power counting index. Non minimal solutions can be defined by assigning to some components of $\mathcal{L}_0$ power counting indices larger than the naive index previously defined.

Physics usually requires to define an adiabatic limit which we now describe. Let us separate the components of $\underline{g}$ into two groups

$$\underline{\mathcal{L}}_0 \cdot \underline{g} = \underline{Q}_0 \circ \underline{J} + \underline{\mathcal{L}}_0^{int} \cdot \underline{\lambda}$$

and let us inquire in what sense we can let the coupling constants $\underline{\lambda}$, whereas the space time varying source functions $\underline{J}$ will be used to generate fields and composite operators, such as currents, etc. The first term of $\underline{Q}$ as power series in $\underline{\lambda}$ is given by $\underline{Q}_0$. One can see that as $\underline{\lambda}(x) \to \underline{\lambda}$ the expression

$$(\Omega, S(J,\underline{\lambda})\Omega) \,/\, (\Omega, S(0,\underline{\lambda})\Omega)$$

has a limit whose coefficients of powers of $\underline{J}$ are the Green's functions of the interacting operators. It is, however, only when suitable normalization conditions are satisfied - whereby the vacuum to vacuum transition is so normalized that $(\Omega, S(0,\underline{\lambda})\Omega) \to 0$ as $\underline{\lambda}(x) \to \underline{\lambda}$, and the two point function is so defined that the one particle pole is fixed at its Fock space value[3] - that $S(\underline{J},\underline{\lambda})$ has a limit in the operator sense as $\lambda(x) \to \lambda$. In many cases one does not want to do this, but rather constructs a theory of Green's functions which, after the adiabatic limit is taken, leads to an interpretation in a Fock space $\mathcal{F}_{phys.}$ different from the one given initially. This point of view is often to be taken in the description of systems with a broken symmetry as we shall see later.

The connection with conventional methods [2] which use regularization procedures is the following: Let ϕ_o^{reg} be a regularized free field defined in a larger Fock space $\mathcal{F}^{reg.} \supset \mathcal{F}$ and $\mathcal{L}_o^{reg.}(\phi_o^{reg})$ a regularized form of the interaction Lagrangian. Under certain conditions, (Pauli Villars regularization : $\mathcal{L}_o^{reg}(\phi_o^{reg}) = \mathcal{L}_o(\phi_o^{reg})$), it is possible to show that the central solution of the regularized theory goes over to the central solution of the non regularized theory in the sense of distributions, as the regularization is removed. It would be worthwhile to complete such a proof in more general cases, including that of the n-dimension regularization [4]

A particular solution of the regularized theory can easily be constructed in terms of Feynman graphs by mere multiplication of regularized Feynman propagators, and the corresponding central solution is uniquely determined by direct calculation of the regulator dependent Taylor expansions around $p = 0$ which can be gathered together into a redefinition of the interaction Lagrangian modulo regulator dependent ("infinite") counter terms.

Of particular interest are renormalizable theories for which $\omega(\mathcal{L}_o^{int}) \leq 4$ in which case possible counter terms occur in finite number, with coefficients formal power series in the coupling constants λ . In all the following, we shall limit ourselves to the study of renormalizable models. In particular, the next chapter will be devoted to a detailed description and comparison of the various, possibly non central as well as non minimal, solutions, in the adiabatic limit. The combinatorial identities which will be established will be the main tools in the study of the models treated in the later chapters.

CHAPTER I: REFERENCES AND FOOTNOTES

This list of references is by no means exhaustive, but contains reviews where extensive reference to originals can be found.

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[2] N.N. Bogoliubov, D.V. Shikov, "Introduction to the theory of quantized Fields", Interscience Pub. New York (1960).

[3] H. Epstein, V. Glaser: The role of locality in Perturbation Theory, CERN TH 1400, 16 September 1971, to appear in Ann. Institut Poincaré.
H. Epstein, V. Glaser, "Adiabatic Limit in Perturbation Theory" CERN TH 1344, 10 June 1971, in Meeting on Renormalization Theory, CNRS Marseille (1971).

[4] For a description of the n-dimensional regularization see, for instance, the forthcoming CERN report by G. 't Hooft, M. Veltman, also E.R. Speer, to appear.

II. DYNAMICS OF RENORMALIZABLE MODELS IN PERTURBATION THEORY

Within the framework of perturbation theory we will discuss the dynamical properties of renormalizable models based on the classical Lagrangian

$$\mathcal{L}_{cl.} = \mathcal{L}_{cl.0} + \mathcal{L}_{cl.}^{int} \tag{II.1}$$

$\mathcal{L}_{cl.0}$ denotes the free Lagrangian. The interaction part $\mathcal{L}_{cl.}^{int}$ is written as

$$\mathcal{L}_{cl.}^{int} = \sum_{j=1}^{J} \lambda_j \mathcal{L}_{cl.j} \tag{II.2}$$

where the coupling terms $\mathcal{L}_{cl.j}$ are monomials in the field components and their first derivatives. The Lagrangian is renormalizable if the naive dimension ω_j of each coupling term $\mathcal{L}_{cl.j}$ satisfies

$$\omega_j \leq 4$$

The components of all fields will be collected by a single field vector

$$\underline{\phi} = (\phi_1, \dots, \phi_A)$$

The construction of field operators and Green's functions by the method of Epstein and Glaser (I.1) was described in the previous Chapter. The advantage of this method is that with mathematical rigour the field operators are constructed recursively in perturbation theory such as to satisfy the fundamental principles of quantum field theory. For practical reasons, however, other methods of renormalization are more convenient which work in the adiabatic limit right from the beginning. In the work that follows we will use such an alternative method which is based on a renormalized version of the Gell-Mann Low formula [1]. First, we assign a degree δ_j satisfying

$$\omega_j \leq \delta_j \leq 4$$

to each coupling term $\mathcal{L}_{cl.j}$. This assignment will determine the number of subtractions to be used for separating the finite part of a Feynman integral. The time ordered Green's functions are constructed by

$$\begin{aligned} &\langle T\, \phi_{\alpha_1}(x_1), \dots \phi_{\alpha_m}(x_m) \rangle = \\ &\langle T\, e^{i\int \mathcal{L}_{eff.0}^{int}(z)dz}\, \phi_{\alpha_1.0}(x_1) \dots \phi_{\alpha_m.0}(x_m) \rangle^{norm.} \end{aligned} \tag{II.3}$$

The superscript "norm" indicates that vacuum diagrams (disconnected closed loops) should be omitted. $\mathcal{L}^{int}_{eff,o}$ denotes the effective interaction part of the Lagrangian and is given by

$$\mathcal{L}^{int}_{eff,o} = \sum \lambda_j \, N_{d_j} [\mathcal{L}_{cl.j\,o}] \tag{II.4}$$

The subscript $^{(o)}$ indicates that the free field propagators pertaining to $\mathcal{L}_{cl.o}$ are used to perform contractions. Time ordered functions involving the symbol N are defined as follows. The expansion of (II. 3) with respect to powers of λ_j leads to expressions of the form

$$\langle T \, \phi_{\alpha_1,o}(x_1) \dots \phi_{\alpha_m,o}(x_m) \; N_{\rho_1}[M_{1,o}(u_1)] \dots N_{\rho_n}[M_{n,o}(u_n)] \rangle \tag{II.5}$$

where the M_k may be any of the coupling terms $\mathcal{L}_{ij}$. We may as well consider the more general case of arbitrary non-linear monomials $M_{j,o}$ in the free field components and their derivatives. The integers ρ_j are restricted by

$$\rho_j \geq d_j$$

where d_j is the naive dimension of the monomial $M_{j,o}$. ρ_j is called the degree assigned to the monomial $M_{j,o}$. (II.5) represents a time ordered Green's function of the Wick products

$$:M_{k,o}:(u_k)$$

While the Wightman functions of Wick products (II. 5) are unique, the time ordered functions are only defined up to contact terms. The symbols $N_{\rho_1} \dots N_{\rho_n}$ serve to specify a unique choice of time ordered functions, yet to be defined. As has been emphasized by Bogoliubov the renormalization of the Gell-Mann Low formula rests upon a proper definition of the time ordered functions of free fields and their Wick products.

Without going into details we give a rough sketch of the definition which is rather involved for arbitrary diagrams due to the phenomenon of overlapping divergencies. One first expands the formal expression

$$\langle T \, \phi_{\alpha_1,o}(x_1) \dots \phi_{\alpha_m,o}(x_m) \, M_{1,o}(u_1) \dots M_{n,o}(u_n) \rangle \tag{II.6}$$

with respect to Feynman diagrams using Wick's rules. Each diagram has m external lines labeled by $x_1, \ldots, x_m$ and n vertices at $u_1, \ldots, u_n$. To each diagram Γ belongs a Feynman integral which in general diverges. We amputate the external lines and denote the unrenormalized Feynman integral in momentum space by

$$\text{(II.7)} \qquad \lim_{\varepsilon \to +0} \int dk_1 \ldots dk_t \; I_\Gamma \, (k_1 \ldots k_t ; p_1, \ldots p_m ; q_1, \ldots q_n)$$

The external momenta p_a, q_b correspond to the coordinates x_a, u_b. The momenta $k_1, \ldots, k_t$ form a set of independent integration variables. The unrenormalized integrand I_Γ is given by the usual Feynman rules pertaining to the Lagrangian (II. 1). The time ordered function (II.6) is defined by the finite part of (II.7) summed over all possible diagrams. The finite part will be formed by taking subtractions of the integrand I_Γ. To this end we introduce the degree

$$\text{(II.8)} \qquad \delta(\Gamma) = 4 - B - \frac{3}{2} F + \sum_{j=1}^{n} (\delta_j - 4)$$

of proper diagrams Γ. B is the number of external boson lines, F the number of external fermion lines. We recall that δ_j is the degree assigned to the vertex at u_j. If the degrees δ_j are chosen to be the naive dimensions d_j

$$\text{(II.9)} \qquad \delta_j = d_j$$

$\delta(\Gamma)$ is just the superficial divergence $d(\Gamma)$ of the diagram

$$\text{(II.10)} \qquad d(\Gamma) = 4 - B - \frac{3}{2} F + \sum_{j=1}^{m} (d_j - 4)$$

One has always

$$\delta(\Gamma) \geq d(\Gamma)$$

A proper diagram is called primitive if all proper subdiagrams have negative degree. For a primitive diagram the finite part is given simply by

$$\text{(II.11)} \qquad \lim_{\varepsilon \to +0} \int dk_1 \ldots dk_t \left(1 - t_{p,q}^{\delta(\Gamma)}\right) . \; I_\Gamma \, (k_1, \ldots k_t ; p_1, \ldots p_m ; q_1, \ldots q_n)$$

Here $t_{p,q}^{\delta(\Gamma)}$ denotes the Taylor operator in p, q up to order $\delta(\Gamma)$. If $\delta(\Gamma) < 0$, we set $t_{p,q}^{\delta(\Gamma)} = 0$. If the degrees equal the dimensions (see (II.9-10)) we precisely have Dyson's prescription for separating the finite part. In the general case oversubtractions are made which would not be required to render the integral finite. For arbitrary proper diagrams overlapping divergencies may occur which can be removed by applying Bogoliubov's combinatorial method (I.2). The finite part then is of the form

$$\text{(II.12)} \quad \lim_{\varepsilon \to +0} \int dk_1 \ldots dk_t \left(1 - t_{p,q}^{\delta(\Gamma)}\right)(\bullet\bullet\bullet) \cdot \cdot I_\Gamma (k_1, \ldots k_t ; p_1, \ldots p_m ; q_1, \ldots q_n) \quad .$$

where (...) indicated subtractions for subdiagrams. For details we refer to ref. [1] . This method of separating the finite part from a Feynman integral is thus an extension of the original work of Dyson and Salam. It can further be shown that the finite parts (II.12) equal those derived by Bogoliubov and Hepp with a somewhat different method[3] . Their results in turn are contained in the general class of solutions which was obtained by Epstein and Glaser in the adiabatic limit [I.3] .

As physical parameters of the theory we distinguish mass constants and coupling constants. Mass constants occur as coefficients of the free Lagrangian and are related to the discrete spectrum of the mass operator $P_\mu P^\mu$. The coupling constants may suitably be defined through various Green's functions. Let $m_1, \ldots, m_A$ and $g_1, \ldots, g_B$ be independent sets of mass or coupling constants which completely characterize the physics of the model. A corresponding set of renormalization conditions is then imposed on the Green's functions.

The renormalization conditions recursively determine $\lambda_1, \ldots, \lambda_J$ as power series in $g_1, \ldots g_B$ with finite, mass dependent coefficients. If all parameters λ_j vanish in zero order the expansion of (II.3) with respect to Feynman diagrams provides the power series expansion in g_j. However, if some λ_j do not vanish in zero order a series has to be summed for obtaining a finite order of perturbation theory. This difficulty is characteristic for some models with broken symmetries. A generalization of the present treatment will be proposed below in this Chapter which allows to avoid the summation problem.

We next turn to the definition of composite operators[1]. Let M be a non-linear monomial in the field components and their derivatives. We will construct a sequence

$$\text{(II.13)} \qquad N_a [M(y)] \; , \quad a = d, d+1, d+2, \ldots$$

of composite operators which are associated with this monomial. d is the naive dimension of M. Green's functions involving composite operators are defined by the following extension of (II.3):

$$\text{(II.14)} \quad \langle T\, Z \rangle = \langle T\, e^{i\int \mathcal{L}^{int}_{eff,0}(z)dz}\, Z_0 \rangle^{norm.}$$

with the abbreviation

$$\text{(II.15)} \quad Z = \phi_{\alpha_1}(x_1)\dots\phi_{\alpha_m}(x_m)\, N_{a_1}[M_1(y_1)]\dots N_{a_n}[M_n(y_n)]$$

We now formulate a generalization of the formalism which is useful for models with broken symmetries[4]. In particular, it provides a simple remedy for the summation problem of such models. This will be demonstrated in Chapter IV on the Goldstone and the Higgs models. In this generalization we let the classical Lagrangian be a polynomial in a new parameter s

$$\text{(II.16)} \quad \mathcal{L}_{cl}(s) = \mathcal{J}_4 + s\mathcal{J}_3 + s^2\mathcal{J}_2 + s^3\mathcal{J}_1$$

The naive dimensions d of the monomials occurring in $\mathcal{J}_\alpha$ are required to be less than or equal to

$$d \leq \alpha$$

Accordingly $\mathcal{J}_1$ is linear in the fields. The parameter s is allowed to vary within

$$\text{(II.7)} \qquad 0 \leq s \leq 1$$

While s = 1 is the physically relevant case the dependence on s will play an important part in the subtraction procedure.

Before writing down the analogue of (II.14) we have to clarify the definition of the free Lagrangian which becomes less trivial in this generalization. The physical mass and coupling constants of the theory refer to the case s = 1. Accordingly, renormalization conditions are also imposed at s = 1. As usual, the free Lagrangian $\mathcal{L}_{cl,0}$ at s = 1 is taken to be the Lagrangian of the incoming and outgoing fields, apart from possible normalization factors. For general s we define the free Lagrangian by the sum of all bilinear terms of $\mathcal{L}_{cl}(s)$ with the coefficients replaced by their zero order values. The dependence of these coefficients on the coupling constants will be discussed shortly. For the moment we only note that the coefficients of $\mathcal{L}_{cl,0}(s)$ are quadratic functions of s, due to (II.16).

The Green's functions of the theory are constructed by

$$(II.18)\quad \langle TZ \rangle = \langle T_s\, e^{i\int N_4[\mathcal{L}^{int}_{d,0}(s,z)]dz}\, Z_0 \rangle^{norm.}$$

with

$$(II.19)\quad \mathcal{L}^{int}_{d}(s) = \mathcal{L}_{d}(s) - \mathcal{L}_{d,0}(s) = \sum \lambda_j\, \mathcal{L}_{dj}(s)$$

and

$$(II.20)\quad Z = \phi_{\alpha_1}(x_1)\dots\phi_{\alpha_m}(x_m)\, N_{a_1}[O_1(y_1)]\cdots N_{a_n}[O_n(y_n)]$$

The formula defines the time ordered Green's functions of the fields, as well as Green's functions of composite operators. The symbol T_s indicates a special time ordering which we are going to define now. The expansion of (II.18) leads to expressions of the form

$$(II.21)\quad \langle T_s\, \phi_{\alpha_1}(x_1)\dots\phi_{\alpha_m}(x_m)\, N_{\rho_1}[M_1(u_1)]\cdots N_{\rho_r}[M_r(u_r)]\rangle$$

In defining (II.21) we proceed as before but replace (II.12) by

$$(II.22)\quad \lim_{\varepsilon\to+0}\int dk_1\dots dk_\ell\,(1-t^{\delta(\Gamma)}_{p,q,s})(\dots)\cdot$$

$$\cdot\, I_\Gamma(s;k_1,\dots k_\ell;p_1,\dots p_m;q_1,\dots q_r)$$

The new feature is that now subtractions are made in p_k, q_l and $\underline{s}$ simultaneously. In other words, $\underline{s}$ is treated like an external momentum, as far as subtractions indicated by (...) are modified similarly[5]. Finally, the renormalization conditions imposed at s = 1 determine the parameters λ_j as power series in g_j with mass dependent coefficients. Let $m_1, \dots, m_A$, $g_1, \dots, g_B$ be complete sets of independent mass and coupling constants at s = 1[6]. Through the renormalization conditions the parameters $\lambda_1, \dots \lambda_J$, become power series in the g_i with finite coefficients which depend on the mass constants[7]. Without proof we will now state some dynamical properties of Green's functions and field operators which hold to any order in perturbation theory. The dynamics is governed by the effective Lagrangian

$$(II.23)\quad \mathcal{L}_{eff}(s) = N_4(\mathcal{J}_4) + s\, N_3[\mathcal{J}_3] + s^2 N_2(\mathcal{J}_2) + s^3\mathcal{J}_1\ .$$

The principal dynamical laws can be stated as follows.

I. Action principle [2, 8]

$$\frac{\partial}{\partial \xi} \langle T \prod_j \phi_{\nu_j}(x_j) \rangle = i \langle T \int \frac{\partial \mathcal{L}_{eff}(\xi, z)}{\partial \xi} dz \prod_j \phi_{\nu_j}(x_j) \rangle \qquad \text{(II.24)}$$

ξ is an arbitrary parameter. In $\partial \mathcal{L}_{eff} / \partial \xi$ the differential operator acts only on the coefficients of the field monomials.

II. Space-time differentiation of normal products [2].

$$\partial_\mu N_\delta [M] = N_{\delta+1} [\partial_\mu M] \qquad \text{(II.25)}$$

$$\partial_\mu^z \langle T N_\delta [M(z)] \prod_j \phi_{\nu_j}(x_j) \rangle = \langle T N_{\delta+1} [\partial_\mu^z M(z)] \prod_j \phi_{\nu_j}(x_j) \rangle \qquad \text{(II.26)}$$

III. Linear relations among normal products [1, 2]

If $\varphi < \delta$, a given normal product of degree φ and its Green's functions may be written as linear combinations

$$N_\varphi [M] = \sum a_i N_\delta [M_i] \qquad \text{(II.27)}$$

$$\langle T N_\varphi [M(z)] \prod_j \phi_{\nu_j}(x_j) \rangle = \sum_i a_i \langle T N_\delta [M_i(z) \prod_j \phi_{\nu_j}(x_j) \rangle \qquad \text{(II.28)}$$

where the sum extends over all monomials M_i of dimensions $d \leq \delta$ with appropriate quantum numbers and invariance properties.

IV. Equations of motion

$$\frac{\delta \mathcal{L}_{eff}}{\delta \phi_k} - \partial^\mu \frac{\delta \mathcal{L}_{eff}}{\delta \partial^\mu \phi_k} = 0 \qquad \text{(II.29)}$$

(II.30)
$$\langle T\left[\frac{\delta\mathcal{L}_{eff}}{\delta\phi_k}(z)-\partial^\mu\frac{\delta\mathcal{L}_{eff}}{\delta\partial^\mu\phi_k}(z)\right]\prod_j\phi_{v_j}(x_j)\rangle = i\sum_j\delta_{k,j}\,\delta(z-x_j)\langle T\prod_{j\neq k}\phi_{v_j}(x_j)\rangle$$

In $\delta\mathcal{L}_{eff}/\delta\phi_k$, $\delta\mathcal{L}_{eff}/\delta\partial^\mu\phi_k$ the action of the differential operators on normal products is defined by

(II.31)
$$\frac{\delta}{\delta\phi_k}N_a[M] = N_{a-d_k}\left[\frac{\delta M}{\delta\phi_k}\right]$$

$$\frac{\delta}{\delta\partial^\mu\phi_k}N_a[M] = N_{a-d_k-1}\left[\frac{\delta M}{\delta\partial^\mu\phi_k}\right]$$

V. Equations of motion multiplied by a field component[8]

(II.32)
$$\phi_\ell\left[\frac{\delta\mathcal{L}_{eff}}{\delta\phi_k}-\partial^\mu\frac{\delta\mathcal{L}_{eff}}{\delta\partial^\mu\phi_k}\right]=0$$

(II.33)
$$\langle T\,\phi_\ell\left[\frac{\delta\mathcal{L}_{eff}}{\delta\phi_k}-\partial^\mu\frac{\delta\mathcal{L}_{eff}}{\delta\partial^\mu\phi_k}\right](z),\prod_j\phi_{v_j}(x_j)\rangle = i\sum_j\delta_{kj}\,\delta(z-x_j)\langle T\,\phi_\ell(x_j)\prod_{j\neq k}\phi_{v_j}(x_j)\rangle$$

Here multiplication of a normal product by a field component at the same point is defined by

(II.34)
$$\phi_\ell(z)\,N_a[M(z)] = N_{a+d_\ell}[\phi_\ell(z)M(z)]$$

VI. Equations of motion multiplied by a field monomial[9]

Let Q(x) be a monomial of dimension d in the fields and their derivatives. Then

(II.35)
$$\{Q\}\cdot\left[\frac{\delta\mathcal{L}_{eff}}{\delta\phi_k}-\partial^\mu\frac{\delta\mathcal{L}_{eff}}{\delta\partial^\mu\phi_k}\right]=0$$

$$\langle T \{Q\} \cdot \left[\frac{\delta \mathcal{L}_{eff}}{\delta \phi_k} - \partial^\mu \frac{\delta \mathcal{L}_{eff}}{\delta \partial^\mu \phi_k} \right](z) \prod_j \phi_{\nu_j}(x_j) \rangle = \tag{II.36}$$

$$= \sum_j \delta_{kj}\, \delta(z - x_j) \langle T\, N_d [Q(x_j) \prod_{l \neq j} \phi_{\nu_l}(x_l) \rangle$$

$\{Q\}$ acting on a normal product of a monomial M denotes the anisotropic normal prodct

$$\{Q\} \cdot N_a [M] = N_{a+d} [\{Q\} M] \tag{II.37}$$

which is defined in the following way. Let dim M denote the dimension of M. The Green's functions of (II.37) are again constructed by (II.22) with the subtraction degree $\delta(\gamma)$ assigned to each proper subgraph γ given by (II.10). However, the degree δ assigned to the normal product vertex is defined in an anisotropic manner. δ is equal to its minimal value, d + d(M), if γ has no internal lines arising from contractions with fields in M, and is equal to d + a otherwise. It should be noted that $N_{a+d} [\{Q\} M]$ can always be expanded in terms of isotropic normal products of degree a + d using relations similar to those of III.

Finally we discuss the connection between formulations involving the parameter $\underline{s}$ and the conventional treatment given in the beginning of this chapter. We start with the simplest case that the free Lagrangian of the $\underline{s}$-formulation is independent of $\underline{s}$. Then the rules for constructing Feynman integrals involve $\underline{s}$ only in form of powers assigned to some vertices of the diagram. $\underline{s}$ never occurs in the masses of the Feynman denominators. Analyzing the subtraction rules one finds that the model based on the s-dependent Lagrangian

$$\begin{aligned} &\mathcal{L}_{cl}(s) = \mathcal{L}_{cl,o}(s) + \mathcal{L}_{cl}^{int}(s) \\ &\mathcal{L}_{cl,o}(s) \text{ independent of } s \\ &\mathcal{L}_{cl}^{int}(s) = \mathcal{G}_4 + s\,\mathcal{G}_3 + s^2 \mathcal{G}_2 + s^3 \mathcal{G}_1 \end{aligned} \tag{II.38}$$

is equivalent to the model based on the conventional interaction Lagrangian

$$\mathcal{L}_{eff}^{int} = N_4[\mathcal{G}_4] + N_3[\mathcal{G}_3] + N_2[\mathcal{G}_2] + \mathcal{G}_1 \tag{II.39}$$

Stated more precisely, the Green's functions evaluated from (II.18) with N_4 applied to the interaction part (II.38) are at $s = 1$ identical to the Green's functions defined through (II.14) with interaction part (II.39). Thus the generalization is not expected to give any new information for such cases.

If the unperturbed mass parameters depend on s the situation is quite different. Let $\mathcal{L}_{cl,o}(s)$ and $\mathcal{L}_{cl}^{int}(s)$ be of the form

(II.40) $$\mathcal{L}_{cl.,o}(s) = \mathcal{F}_4 + s\,\mathcal{F}_3 + s^2\,\mathcal{F}_2$$

(II.41) $$\mathcal{L}_{cl.}^{int}(s) = \mathcal{G}_4 + s\,\mathcal{G}_3 + s^2\,\mathcal{G}_2 + s^3\,\mathcal{G}_1$$

with the coefficients of the monomials in $\mathcal{F}_\alpha$ independent of s and the coupling constants g_i. For the coefficients of the $\mathcal{G}_4$ it is assumed that they vanish in zero order of g_i. Again there is a formal equivalence between the s-dependent Lagrangian

(II.42) $$\begin{aligned}\mathcal{L}_{cl}(s) &= \mathcal{L}_{cl.,o}(s) + \mathcal{L}_{cl.}^{int}(s)\\ &= \mathcal{J}_4 + s\,\mathcal{J}_3 + s^2\,\mathcal{J}_2 + s^3\,\mathcal{J}_1\end{aligned}$$

and the conventional Lagrangian

(II.43) $$\mathcal{L}_{eff} = N_4[\mathcal{J}_4] + N_3[\mathcal{J}_3] + N_2[\mathcal{J}_2] + \mathcal{J}_1$$

But it turns out that the appropriate form of the free Lagrangian is

(II.44) $$\mathcal{L}_{eff,o} = N_4\,[\mathcal{F}_4 + \mathcal{F}_3 + \mathcal{F}_2]$$

so that the interaction part becomes

(II.45) $$\begin{aligned}\mathcal{L}_{eff.}^{int} = {} & N_4[\mathcal{G}_4] + N_3[\mathcal{G}_3] + N_2[\mathcal{G}_2] + \mathcal{G}_1\\ & + N_3[\mathcal{F}_3] - N_4[\mathcal{F}_3]\\ & + N_2[\mathcal{F}_2] - N_4[\mathcal{F}_2]\end{aligned}$$

The coefficients of $\mathcal{F}_2$ and $\mathcal{F}_3$ are independent of the coupling constants g_j. As was discussed above this leads to a summation problem since an infinite number of Feynman integrals contributes to a finite order in g_j. Since $\mathcal{F}_3$ and $\mathcal{F}_4$ are bilinear in the fields one deals with geometric series in momentum space which diverge in parts of the integration domains. Thus the formalism based on the Lagrangian (II.44 - 45) is not satisfactory. With the original s-formulation, however, each order in the g_j is represented by a finite number of Feynman integrals.

Despite of this it is always possible to construct a Lagrangian of the conventional type which is equivalent to (II.42), but in a non-trivial way. The disadvantage of this Lagrangian is that symmetries which the s-dependent Lagrangian may display could be completely distorted for the equivalent conventional Lagrangian. This happens to be the case for the Goldstone and the Higgs model.

We briefly state and prove this equivalence theorem[10].

EQUIVALENCE THEOREM

Let

$$\begin{aligned} \mathcal{L}'_{cl}(s) &= \mathcal{L}'_{cl,0}(s) + \mathcal{L}'^{\,int}_{cl}(s) \\ &= \mathcal{J}'_4 + (s-1)\,\mathcal{J}'_3 + (s^2-1)\,\mathcal{J}'_2 + (s^3-1)\,\mathcal{J}'_1 \end{aligned} \qquad \text{(II.46)}$$

$\mathcal{J}'_\alpha$ are polynomials of their fields and their first derivatives with s-independent coefficients, the naive dimensions of the monomials in $\mathcal{J}'_4$ be less than or equal to four, the dimensions of monomials in $\mathcal{J}'_{1,2,3}$ be less than or equal to 1,2,3. Let mass and coupling constants be defined through renormalization conditions on the time ordered Green's functions of fields. Then there exists a Lagrangian

$$\begin{aligned} \mathcal{L}''_{cl} &= \mathcal{L}''_{cl.0} + \mathcal{L}''^{\,int}_{cl.} \\ \mathcal{L}''^{\,int}_{eff} &= N_4\left[\mathcal{L}''^{\,int}_{cl.}\right] \end{aligned} \qquad \text{(II.47)}$$

with the same renormalization conditions such that the models based on (II.46) and (II.47) are identical. More precisely, the time ordered functions of the fields are identical whether constructed by (II.18) with the Lagrangian (II.46) or by (II.3) with (II.47). If the coefficients of the interaction part of (II.46) vanish in zero order of the coupling constants the same is true for (II.47). Hence no summation problem arises with (II.47).

Remark: Green's functions involving normal products may change. This does not impair the equivalence of the models but only indicates that similarly constructed normal products must not be identified.

Proof: In order to eliminate the s-dependent terms we will construct a family of Lagrangians

(II.48) $$\mathcal{L}_{cl}(s) = \mathcal{J}_4 + (s-1)\mathcal{J}_3 + (s^2-1)\mathcal{J}_2 + (s^3-1)\mathcal{J}_1$$

which includes the Lagrangian (II. 46) with

$$\mathcal{J}_4 = \mathcal{J}_4' = \mathcal{L}_{cl}(1) \quad , \quad \mathcal{J}_\alpha = \mathcal{J}_\alpha' \quad \alpha = 1,2,3 \; .$$

and a Lagrangian

$$\mathcal{L}_{cl}''(s) = \mathcal{J}_4'' \quad , \quad \mathcal{J}_1'' = \mathcal{J}_2'' = \mathcal{J}_3'' = 0$$

with no s-dependent terms. For this family it will be shown that the Green's functions of fields do not change.

Let

(II.49) $$M_{d1} , \ldots \; M_{d\,A(d)}$$

be all monomials of naive dimension d which can be formed out of the field components and their derivatives. We express $\mathcal{J}_4', \mathcal{J}_\alpha'$ as linear combinations of these monomials

$$\mathcal{J}_4' = \sum_{d=1}^{4} \sum_{j=1}^{A(d)} c_{dj}' \, M_{dj}$$

$$\mathcal{J}_\alpha' = \sum_{d=1}^{\alpha} \sum_{j=1}^{A(d)} t_{\alpha dj}' \, M_{dj} \qquad \alpha = 1,2,3$$

With this notation the original Lagrangian becomes

$$\mathcal{L}_{cl}'(s) = \sum_{d=1}^{4} \sum_{j=1}^{A(d)} c_{dj}' \, M_{dj} + \sum_{\alpha=1}^{3} (s^{4-\alpha}-1) \sum_{d=1}^{\alpha} \sum_{j=1}^{A(d)} t_{\alpha dj} \, M_{dj}$$

We now study the class

(II.50) $$\mathcal{L}_{d.}(s) = \sum_{d=1}^{4}\sum_{j=1}^{A(d)} c_{dj} M_{dj} + \sum_{\alpha=1}^{3} (s^{4-\alpha}-1) \sum_{d=1}^{\alpha}\sum_{j=1}^{A(d)} t_{\alpha dj} M_{dj}$$

of Lagrangians where the c_{dj} are functions of arbitrary parameters $t_{\alpha dj}$. Our intention is to place suitable restrictions on the functions c_{dj} such that at s = 1 the Green's functions remain independent of $t_{\alpha dj}$. To this end we form the derivative

$$\frac{\partial \langle T X\rangle}{\partial t_{\beta ek}} = i\sum_{d=1}^{a}\sum_{j=1}^{A(d)} \frac{\partial c_{dj}}{\partial t_{\beta ek}} \int dz \langle T N_4[M_{dj}(z)] X\rangle + i\int dz \langle T(N_\beta[M_{ek}(z)] - N_4[M_{ek}(z)])X\rangle$$

(II.28) then implies that $\langle T N_\beta[M_{ek}]X\rangle$ is a linear combination of Green's functions involving N_4-products

$$\langle T(N_\beta[M_{ek}(z)] - N_4[M_{ek}(z)])X\rangle = = -\sum r_{\beta ekdj} \langle T N_4[M_{dj}(z)] X\rangle$$

Thus

$$\frac{\partial\langle T X\rangle}{\partial t_{\beta ek}} = 0$$

holds provided

$$\frac{\partial c_{dj}}{\partial t_{\beta ek}} = r_{\beta ekdj}$$

The $r_{\beta ekdj}$ depend on $t_{\beta dk}$ directly and through the functionals c_{dj}. A more detailed discussion shows that a solution exists with the initial values

$$c_{dj} = c'_{dj} \quad at \quad t_{\beta dj} = t'_{\beta dj}$$

With these solutions as coefficients C_{dj} the family (II.50) of Lagrangians has all desired properties. We finally set $t_{\beta dj} = 0$ and find

$$\mathcal{L}''_{cl} = \sum_{j=1}^{A(d)} C''_{dj} M_{dj}$$

where C''_{dj} denotes the values at $t_{\beta dj} = 0$. Since the Green's functions of the basic fields do not change the same normalization conditions are satisfied throughout. In zero order of the coupling constants the coefficients $r_{\beta \epsilon k d j}$ vanish. Hence the zero order values of the c_{dj} do not change. Accordingly, the free part of the Lagrangian (II.50) remains the same at $s = 1$ and the coefficients of the interaction part $\mathcal{L}''_{eff}$ vanish in zero order. This completes the proof.

CHAPTER II: REFERENCES AND FOOTNOTES

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III. EXACT AND BROKEN SYMMETRIES: RENORMALIZATION EFFECTS.

In the following, we shall exclusively consider renormalizable effective Lagrangians whose dimension four kinematical parts involve non vanishing mass terms. Superrenormalizable couplings may otherwise be present and will be weighted by a power of the previously introduced parameter g which measures their degree of superrenormalizability. The presence of superrenormalizable interactions is, as we shall see, essential to the definition of broken symmetries [1]. In all cases, the symmetry or lack of symmetry is best described by the effective Noether theorem, which follows from the various effective equations of motion described in chapter II and is summarized by a set of Ward identities. We shall first, for the purpose of orientation, look at the simple case of space time symmetries, and then will go into the subject of internal symmetries.

1. Space time symmetries. The energy momentum tensor.

We recall the effective equations of motion:

$$\langle T\left(\frac{\delta \mathcal{L}_{eff}}{\delta \Phi} - \partial_\mu \frac{\delta \mathcal{L}_{eff}}{\delta \partial_\mu \Phi}\right)(x) X\rangle = i \sum_{k\in X} \delta(x-x_k)\langle T X_{\hat{k}}\rangle$$

where, for each term in $\mathcal{L}_{eff}$, the corresponding derivative with respect to φ has to be counted with a power counting degree diminished by dim φ. The Noether current associated with space time translations:

$$\varphi(x) \to \varphi(x+a) \simeq \varphi(x) + a^\mu \partial_\mu \varphi(x)$$

is

$$j_\mu^{Noeth.\,\nu} = \Pi_\mu \partial^\nu \varphi$$

with

$$\Pi_\mu = \frac{\delta \mathcal{L}_{eff}}{\delta \partial_\mu \varphi}$$

According to the preceding rules $j_\mu^{Noeth.\,\nu}$ has to be counted with dimension four. Using the effective equations of motion under the quadratic combination $\Pi_\mu \partial_\nu \varphi$, we have:

$$\partial^\mu \langle T\ \Pi_\mu \partial^\nu \varphi X\rangle = \langle T \partial^\mu \Pi_\mu \partial^\nu \varphi X\rangle + \langle T \Pi_\mu \partial^\mu \partial^\nu \varphi X\rangle$$

$$= \langle T \frac{\delta \mathcal{L}_{eff}}{\delta \varphi} \partial^\nu \varphi X \rangle + \langle T \frac{\delta \mathcal{L}_{eff}}{\delta \partial_\mu \varphi} \partial^\nu \partial_\mu X \rangle$$
$$-i \sum_{k \in X} \delta(x-x_k)\, \partial^\nu_{(k)} \langle T X \rangle$$
$$= \partial^\nu \langle T \mathcal{L}_{eff}(x) X \rangle + \sum_{k \in X} \delta(x-x_k) \frac{1}{i} \partial^\nu_{(k)} \langle T X \rangle$$

Hence, the canonical energy momentum tensor [2]

$$\Theta^{\nu,c}_\mu = \pi_\mu \partial^\nu \varphi - g^\nu_\mu \mathcal{L}_{eff}$$

fulfills the Ward identity

$$\partial^\mu \langle T \Theta^{\nu,c}_\mu(x) X \rangle = \sum_{k \in X} \delta(x-x_k) \frac{1}{i} \partial^\nu_{(k)} \langle T X \rangle$$

The Ward identity associated with Lorentz invariance usually requires the use of the Belinfante energy momentum tensor, of the form

$$\Theta^{\nu,B}_\mu = \Theta^{\nu,c}_\mu + \partial^\lambda f^\nu_{\mu\lambda}$$

where $f^\nu_{\mu\lambda}$ is given by the usual formula [3], is formally antisymmetric in μ, λ, so that it does not spoil the Ward identity and has to be assigned dimension 3 . The symmetry of $\Theta^B_{\mu\nu}$ in μ and ν being a consequence of the equations of motion, results into the asymmetry identity

$$\langle T \Theta^B_{\mu\nu}(x) X \rangle - \langle T \Theta^B_{\nu\mu}(x) X \rangle = \sum_{k \in X} \delta(x-x_k) S^{(k)}_{\mu\nu} \langle T X \rangle$$

where the $S^{(k)}_{\mu\nu}$ are the relevant spin matrices.

The asymmetry identity, and the Ward identity put together, result into the Ward identity for the Lorentz current:

$$M_{\lambda,\mu\nu} = x_\mu \Theta^B_{\lambda\nu} - x_\nu \Theta^B_{\lambda\mu}$$

$$\partial^\lambda \langle T M_{\lambda,\mu\nu}(x) X \rangle = \sum_{k \in X} \delta(x-x_k) \left[\frac{1}{i} \left(x^{(k)}_\mu \partial^{(k)}_\nu - x^{(k)}_\nu \partial^{(k)}_\mu \right) + S^k_{\mu\nu} \right] \cdot \langle T X \rangle$$

Finally, the dilation current which generates broken scale invariance can be conveniently written as

$$D_\lambda = x^\mu \theta^{c.c.J.}_{\lambda\mu}$$

where the "improved" energy momentum tensor [4] of Callan, Coleman, Jackiw is given by

$$\theta^{c.c.J.}_{\lambda\mu} = \theta^{B}_{\lambda\mu} + \sum_s \frac{1+b_s}{6} \left(\Box g_{\mu\nu} - \partial_\mu \partial_\nu \right) \phi_s^2$$

where the sum ranges over scalar fields and the factors $(1+b_s)$ are the finite wave function renormalization factors occurring in $\mathcal{L}_{eff}$ in front of the quadratic terms $\frac{1}{2}\partial_\mu \phi_s \partial^\mu \phi_s$

The corresponding Ward identity

$$\partial^\lambda \langle T\, D_\lambda(x)\, X \rangle = 2 \langle T M^2(x) X \rangle + \sum_{k \in X} \delta(x - x_{(k)}) \frac{1}{i} \cdot$$
$$\cdot \left(x^k_{(k)} \partial^{(k)} + d^{(k)} \right) \langle T X \rangle$$

where $M^2(x)$ denotes the set of mass terms occurring in $\mathcal{L}_{eff}$, and thus contains dimension four terms, and $d^{(k)}$ is the canonical dimension of field $\varphi^{(k)}$, is a consequence of the Ward identity together with the trace identity:

$$\langle T\, \theta^{\mu\, c.c.J.}_{\mu}(x)\, X \rangle = 2 \langle T M^2(x) X \rangle + \sum_{k \in X} \delta(x - x_k) \frac{1}{i} d^{(k)} \langle T X \rangle$$

The improvement thus does not yield a "soft" trace, and one can show that no other choice can be made so that the trace becomes "soft", at the expense of changing the canonical dimensions into abnormal dimensions [2][5]. By integrating the Ward identity for the dilation current or combining the trace identity with the Ward identity for the energy momentum tensor at zero momentum, one obtains the so called Callan Symanzik equation [2] after full use of the reduction formulae of chapter II.

2. Internal Symmetries.

Let $\mathcal{L}_{eff}$ be of the form

$$\mathcal{L}_{eff} = \mathcal{L}_{sym.} + \mathcal{L}_{break.}$$

where the terms in $\mathcal{L}_{break}$ have dimensions strictly less than four and all mass terms of dimension four in $\mathcal{L}_{sym.}$ are non vanishing. Symmetry and symmetry breaking are understood with respect to a transformation of the type

$$\underline{\varphi} \to {}^{g}\underline{\varphi} = \mathcal{D}(g)\underline{\varphi} \quad , g \in G \quad .$$

where G denotes a compact Lie group. $\mathcal{D}$ a finite dimensional unitary representation of G. The infinitesimal transformation law is of the type

$$\underline{\varphi} \to \underline{\varphi} + i\,\vec{\theta}.\delta\vec{\omega}\,\underline{\varphi}$$

where $\vec{\theta}$ represents the Lie algebra $\mathcal{G}$ of G in $\mathcal{D}$. When $\mathcal{D}$ is the adjoint representation, we shall denote by $\vec{t}$ the representatives of $\mathcal{G}$. Defining

$$\vec{j}_\mu^{\,Noeth.} = i N_3 \; \pi_\mu \vec{\theta}\, \varphi$$

and using the effective equations of motion, we obtain the Ward identity

$$\partial^\mu \langle T\, \vec{j}_\mu^{\,Noeth.}(x) \; X \rangle = \langle T \frac{\delta \mathcal{L}_{break}}{\delta\vec{\omega}}(x) X \rangle + \sum_{k\in X} \delta(x-x_k)\vec{\theta}_k \langle TX \rangle$$

which has to be understood, as the whole theory in the sense of multiple formal series in all the parameters of $\mathcal{L}_{break}$ and some of the parameters of $\mathcal{L}_{sym}$. One could similarly establish current algebra Ward identities[6] which assume a normal form provided a suitable time ordered product involving two currents is defined, which differs from the time ordered products we have used so far by allowed ambiguities.

It is clear that the treatment given so far calls for resummation procedures which allow a decent interpretation within the physical Fock space alluded to in chapter I. This can be done as follows. First of all, in general $\mathcal{L}_{break}$ will induce nonvanishing field vacuum expectation values $\langle \underline{\varphi} \rangle = \underline{F}$. One can first perform the change of variable

$$\underline{\varphi}' = \underline{\varphi} - \underline{F} \qquad \langle \varphi' \rangle = 0$$

One can show[7] that the Green functions for $\underline{\varphi}'$ can be computed from the Lagrangian

$$\mathcal{L}'(\underline{\varphi}') = \mathcal{L}(\underline{\varphi}' + \underline{F})$$

with the dimension assignment given by

$$N_s\,(\varphi'+F)^p = \sum_q \binom{p}{q} F^q\, N_{s-|q|}\,\varphi'^{\,p-q}$$

and the Ward identity reads

$$\partial^\mu \langle T J'_\mu(x)\, X'\rangle = \langle T \vec{D}'(x)\, X'\rangle + \sum_{k\in X} \delta(x-x_k)\, \vec{\theta}_k \langle T X'\rangle + \sum_k \delta(x-x_k)\,(\vec{\theta} F)_k \langle T X'_{\hat{k}}\rangle$$

where $\vec{J}'_\mu$ and $\vec{D}'$ are respectively deduced from $\vec{J}_\mu^{\,\mathrm{Noeth}}$, $\frac{\delta \mathcal{L}^{\mathrm{break}}}{\delta \omega}$ by the substitution rule $\varphi \to \varphi' + F$ according to the above formula. The introduction of the s parameter and its use as indicated in Chapter II allow to perform the necessary resummations. On the other hand, the equivalence theorem described in chapter II then asserts the existence of a dimension four effective Lagrangian (without a linear term) which yields the Green's functions $\langle T X'\rangle$. One should however be careful that composite operators $\vec{J}_\mu$, $\vec{D}$ such that

$$\langle T \vec{J}_\mu(x), X'\rangle_{\mathcal{L}_4} = \langle T J'_\mu(x)\, X'\rangle_{(\mathcal{L}_{sym}+\mathcal{L}_{break})(\varphi'+F)}$$

$$\langle T \vec{D}(x)\, X'\rangle_{\mathcal{L}_4} = \langle T \vec{D}'(x)\, X'\rangle_{(\mathcal{L}_{sym}+\mathcal{L}_{break})(\varphi'+F)}$$

are only known to have the same maximum dimension as $\vec{J}'_\mu$, $\vec{D}'$ respectively[7].

The determination of $\mathcal{L}_4$ by the requirement that Ward identities be fulfilled is thus in general incomplete at this stage, since only the dimension of $\vec{D}$ goes unchanged through the change of effective Lagrangian, except in the case of a linear symmetry breaking. In the case of a non linear symmetry breaking, the structure of $\vec{D}$ is not arbitrary. Together with a term of given dimension and covariance under G, it must contain all terms corresponding to the ambiguities compatible with covariance and power counting[1]. The question then arises to recover in the $\mathcal{L}_4$ version the information which characterizes the covariance of the breaking.

It is conjectured that this information can be recovered by studying the high momentum behaviour of two point and three point Green's functions via e.g. the Callan Symanzik equations.

Alternatively, let $\mathcal{L}_i$ be an irreducible tensorial term in $\mathcal{L}_{break}$, $\frac{\delta \mathcal{L}_i}{\delta \omega_\alpha} = \mathcal{L}_i^\alpha = \tau_{ij}^\alpha \mathcal{L}_j$ for some representation $\{\tau\}$ of $\mathcal{G}$.
Then, the following Ward identity holds:

$$\partial^\mu < T j_\mu^{Noeth.\beta}(x) \mathcal{L}_i^\alpha(y) X > = < T \frac{\delta \mathcal{L}_{break}}{\delta \omega_\beta}(x) \mathcal{L}_i^\alpha(y) X >$$
$$+ \delta(x-y) t_\gamma^{\beta\alpha} < T \mathcal{L}_i^\gamma(y) X > + \delta(x-y) \tau_{ij}^\beta < T \mathcal{L}_j^\alpha(y) X >$$
$$+ \sum_{k \in X} \delta(x-x_k) \Theta_{(k)}^\beta < T \mathcal{L}_i^\alpha(y) X >$$

One may thus look for D^α in the form $D^\alpha = \sum D_i^\alpha$
if $\mathcal{L}_{break}$ was of the form $\sum \mathcal{L}_i$

and determine the covariant operators D_i^α characterized by their maximum dimension and the fulfillment of Ward identities of the type[8]

$$\partial^\mu < T j_\mu^\beta(x) D_i^\alpha(y) X' > = < T D^\beta(x) D_i^\alpha(y) X' >$$
$$+ \delta(x-y) t_\gamma^{\beta\alpha} < T D_i^\gamma(y) X' > + \delta(x-y) \tau_{ij}^\beta < T D_j^\alpha X' >$$
$$+ \sum_{k \in X} \delta(x-x_k) \theta_k^\beta < T D_i^\alpha(y) X' >$$
$$+ \sum_{k \in X} \delta(x-x_k) (\theta^\beta F)_k < T D_i^\alpha(y) X'_{\hat{k}} >$$

where the time ordered products involving pairs of composite operators may eventually not be the conventional ones.

Once the characteristics of the initial Lagrangian have been recovered, through highly non linear, recursively soluble relations between the coefficients of $\mathcal{L}_4$ (values of vertex functions at zero momentum) two kinds of situations may occur. If enough parameters are left undetermined so that all physical masses of stable particles can be chosen as free parameters, one has a theory in a physical Fock space $\mathcal{F}_{phys}$ the perturbation parameter being $\hbar$, which counts the number of loops in Feynman diagrams. If not, no perturbative treatment known at present can describe the situation in terms of the correct physical Fock space.

The algebraic complexity of the general situation looks at the moment forbidding, and the more symmetric treatment indicated in Chapter IV is by far more attractive since there all operators retain all the symmetric aspects which are completely hidden in the $\mathcal{L}_4$ formalism. The only reason, other than computational, which calls for a study of the $\mathcal{L}_4$ formalism is the present lack of treatment in the present framework of systems for which, because of the group structure, some symmetric mass parameters vanish. This is admittedly a weakness of the formalism and calls for a more complete study of the interrelations between the various solutions to the decomposition problems posed by the causality requirement. In these more difficult cases, one may either first suitably approximate the theory by one in which no vanishing mass parameter is involved and study some zero mass limit. This is the method examplified in Chapter IV. Alternatively one may modify the starting point as will be shown in Chapter V and directly define the desired theory.

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A. Becchi: "Absence of strong interactions to the axial anomaly in the σ model", CERN TH 1611. January (1973). Comm. Math. Phys. to appear.

IV. Gauge Invariant Quantization of the Goldstone and Higgs Model

The general formulation of the previous sections will now be applied to the examples of the Goldstone and the Higgs model. We begin with the discussion of the Goldstone model. B. Lee and K. Symanzik developed two alternative methods of quantizing the Goldstone model[1,2] . In B. Lee's work the model is regularized and quantized in a gauge invariant manner. It is then shown that the regularization can be removed for the renormalized Feynman amplitudes. In the treatment that follows we use B. Lee's method of gauge invariant quantization but without introducing a regularization. Instead we will deal directly with the unregularized, but properly renormalized, Feynman amplitudes. It will be shown that the desired properties of the model follow easily as an application of the general theorems given in Section II and III. The connection to Symanzik's method will be discussed later.

As classical Lagrangian we propose

$$\text{(IV.1)} \quad \dot{\mathcal{L}}_{cl.}(s) = \partial_\mu \varphi^* \partial^\mu \varphi - (\eta_0^2 - s^2 w^2)\varphi^*\varphi - h_0(\varphi^*\varphi)^2 - \frac{\delta_0(s)}{\sqrt{2}}(\varphi + \varphi^*)$$

with

$$\text{(IV.2)} \quad \varphi = \frac{1}{\sqrt{2}}\left(\tau_1 + s v_0 + i\tau_2\right)$$

$$\tau_1 = Z_2^{1/2}\psi \ , \ \tau_2 = Z_2^{1/2}\chi \ , \ v_0 = Z_2^{1/2} v$$

ψ and χ denote the properly normalized fields. Their vacuum expectation values are required to vanish

$$\text{(IV.3)} \quad \langle \psi \rangle = \langle \chi \rangle = 0$$

while it is assumed that

$$\text{(IV.4)} \quad v_0 \neq 0$$

Expressed in terms of the fields ψ and χ (IV.1) is to be interpreted as an $\underline{s}$-dependent Lagrangian in the sense of Chapter II. We recall that the parameter $\underline{s}$ only serves to specify the subtraction procedure and is set equal to one finally.

$\delta_0(s)$ is a polynomial in $\underline{s}$ with

$$\delta_0(0) = 0$$

of which only the value

$$\delta_0(1) = \delta_0$$

at s = 1 is relevant.

The parameter w is restricted to a permissable range which will be given later. It can be shown that the Green's functions of the theory do not depend on the value of w [4].

For $\delta_0 = 0$ (IV.1) is the Lagrangian of the Goldstone model. Formally it is gauge invariant, but the gauge symmetry is spontaneously broken due to (IV.4). The model describes two particles σ and π which are associated with the fields ψ or χ resp. The π-particle has zero mass and represents the Goldstone particle. The σ-particle is unstable since it can decay into π-particles by the interaction term $\psi\chi^2$ present in the Lagrangian (IV.1).

For $\delta_0 \neq 0$ the gauge symmetry is explicitly broken by the term proportional to $\varphi + \varphi^*$. In this case the model is called the explicitly broken Goldstone model. It describes two massive particles σ and π of mass M and μ associated with the fields ψ or χ. We always assume $\mu < M$. The σ-particle is stable only for $M < 2\mu$. In the unstable case M denotes an appropriate mass parameter related to the complex pole of the ψ-propagator [5]. In the Goldstone limit $\delta_0 \to 0$ the π-particle becomes massless, i.e. $\mu \to 0$.

Unfortunately, the Lagrangian (IV.1) is not meaningful for $\delta_0 = 0$ since some of its coefficients are infrared divergent. In order to bypass infrared problems the case $\delta_0 \neq 0$, or $\mu \neq 0$, is considered first. Then the perturbation expansion is well-defined with finite coefficients of the Lagrangian (IV.1). Eventually the Goldstone limit $\mu \to 0$ is applied yielding Green's functions and S-matrix of the Goldstone model.

Apart from $\underline{s}$ and w the independent parameters of the theory will be the σ-mass M (or an appropriate mass parameter M in the unstable case), the π-mass μ and a suitably defined coupling constant g. Five renormalization conditions are required to hold at s = 1 [6]. These conditions uniquely determine the parameters η_0, h_0, δ_0, v_0, z_2 as power series in g with coefficients depending on M, μ and w, but independent of $\underline{s}$.

We now determine the free part of the Lagrangian (IV.1) following the instructions of Chapter II. Let $\mathcal{L}_{eff\,Q}$ denote the sum of all terms in $\mathcal{L}_{eff}$ which are quadratic in ψ and χ or their derivatives.

$$\begin{aligned} \mathcal{L}_{eff\,Q} = {} & \frac{1}{2} z_2 (\partial_\mu \psi \partial^\mu \psi) + \frac{1}{2} z_2 (\partial_\mu \chi \partial^\mu \chi) \\ & - \frac{1}{2} \left(\eta_0^2 - s^2 w^2 + 3 h_0 s^2 v_0^2 \right) \psi^2 \\ & - \frac{1}{2} \left(\eta_0^2 - s^2 w^2 + h_0 s^2 v_0^2 \right) \chi^2 \end{aligned} \qquad \text{(IV.5)}$$

The free part of the effective Lagrangian is defined by (IV.5) with the coefficients replaced by their zero order values:

$$\text{(IV.6)}\quad \mathcal{L}_{eff\,0} = \frac{1}{2}\,\partial_\mu\psi\,\partial^\mu\psi + \frac{1}{2}\,\partial_\mu\chi\,\partial^\mu\chi - \frac{1}{2}\,M^2(s)\,\psi^2 - \frac{1}{2}\,\mu^2(s)\,\chi^2\ ,$$

$$\lim_{g\to 0} Z_2 = 1,$$

$$\text{(IV.7)}\quad M^2(s) = \lim_{g\to 0}\left(\eta_0^2 - s^2 w^2 + 3\,h_0\,s^2\,v_0^2\right),$$

$$\mu^2(s) = \lim_{g\to 0}\left(\eta_0^2 - s^2 w^2 + h_0\,s^2\,v_0^2\right)\ .$$

The mass parameters M^2, μ^2 equal the values of (IV.7) at $s = 1$

$$M^2 = M^2(1)\quad ,\quad \mu^2 = \mu^2(1)$$

Since η_0, w, h_0, v_0 are independent of s we obtain

$$\text{(IV.8)}\quad \lim_{g\to 0} h_0\,v_0^2 = \frac{M^2 - \mu^2}{2}$$

$$\text{(IV.9)}\quad \begin{aligned} M^2(s) &= \frac{3}{2}\,\mu^2 - \frac{1}{2}\,M^2 + w_0^2 + s^2\left(\frac{3}{2}\,(M^2 - \mu^2) - w_0^2\right) \\ \mu^2(s) &= \frac{3}{2}\mu^2 - \frac{1}{2}\,M^2 + w_0^2 + s^2\left(\frac{1}{2}\,(M^2 - \mu^2) - w_0^2\right) \end{aligned}$$

w_0 denotes the zero order value of w.

Since we need the theory in the range $0 \leqslant s \leqslant 1$ we impose the consistency condition

$$M^2(s) \geqslant 0 \qquad \mu^2(s) \geqslant 0$$

As long as $M > 0$, $\mu > 0$ the model is not expected to suffer from infrared problems. One should therefore avoid vanishing mass values in the range $0 \leq s \leq 1$ by imposing the stronger condition

$$M^2(s) > 0\,,\ \mu^2(s) > 0 \quad \text{if} \quad M^2 > 0,\ \mu^2 > 0. \tag{IV.10}$$

This restricts the permissable values of w by

$$w_0^2 > \frac{1}{2} M^2 - \frac{3}{2} \mu^2 \tag{IV.11}$$

Particular convenient is a choice of w for which

$$w_0^2 = \lim_{g\to 0} h_0 v_0^2 = \frac{1}{2}\left(M^2 - \mu^2\right) > \frac{1}{2} M^2 - \frac{3}{2} \mu^2 \tag{IV.12}$$

With this the s-dependence (IV.9) of the masses becomes

$$\begin{aligned} M^2(s) &= \mu^2 + s^2\left(M^2 - \mu^2\right) \\ \mu^2(s) &= \mu^2 \end{aligned} \tag{IV.13}$$

Hence the π -mass is independent of s while the σ -mass has the simple form (IV.13).

With the free Lagrangian (IV.6) involving the s-dependent masses (IV.7) the perturbation expansion (II.14) of the Green's functions is completely determined.

The main advantage of this gauge invariant formulation is that partial current conservation follows quite naturally, essentially by following the classical derivation. The current operator is defined by taking the minimal normal product of the classical Noether current

$$j_\mu = i\, N\left[\varphi\, \partial_\mu \varphi^* - \varphi^* \partial_\mu \varphi\right] \tag{IV.14}$$

More precisely, the symbol N means that the minimal normal product should be applied to each monomial of the current expressed in terms of ψ and χ . According to Chapter III the Green's functions of this current operator satisfy the Ward identities

$$\begin{aligned} \partial_\mu \langle T\, j^\mu(x)\, \varphi(u_1) \ldots \varphi^*(v_1) \ldots \rangle &= i \sum_k \delta(x-u_k) \langle T \varphi(u_1) \ldots \varphi^*(v_1) \ldots \rangle \\ &- i \sum_k \delta(x - v_k) \langle T \varphi(u_1) \ldots \varphi^*(v_1) \ldots \rangle \\ &+ i \frac{\delta_0(s)}{\sqrt{2}} \langle T \left(\varphi(x) - \varphi^*(x)\right) \varphi(u_1) \ldots \varphi^*(v_1) \ldots \rangle \end{aligned}$$

$$\partial_\mu \langle T\, j^\mu(x)\, \varphi(u_1)\ldots \varphi^*(v_1)\ldots\rangle =$$
$$\sum_k \delta(x-u_k)\langle T\, \varphi(u_1)\ldots\varphi^*(v_1)\ldots\rangle$$
$$\text{(IV.15)} \quad - \sum_k \delta(x-v_k)\langle T\, \varphi(u_1)\ldots\varphi^*(v_1)\ldots\rangle$$
$$+\, i\, \frac{\delta_0(s)}{\sqrt{2}}\langle T\left(\varphi(x)-\varphi^*(x)\right)\varphi(u_1)\ldots\varphi^*(v_1)\ldots\rangle$$

In operator form the law of partial current conservation becomes

$$\text{(IV.16)} \quad \partial_\mu j^\mu(x) = i\, \frac{\delta_0(s)}{\sqrt{2}}\left(\varphi(x)-\varphi^*(x)\right)$$

(IV.16) follows from (IV.15) by applying the reduction formulae to the fields ψ and χ of arguments u_i and v_i.

It can be shown that in the Goldstone limit $\mu \to 0$ the time-ordered Green's function and the S-matrix exist [7]. The current of the Goldstone model is conserved.

It is characteristic for the gauge invariant approach that - apart from the linear term - the fields ψ and χ only appear in the gauge invariant combinations of the Lagrangian (IV.1). Without destroying the gauge invariant form of the non-linear part the Lagrangian (IV.1) may be replaced by an $\underline{s}$-independent Lagrangian of type (II.43-45). As was discussed in Chapter II such Lagrangians suffer from a summation problem in finite order of perturbation theory. In the present case the Lagrangian contains terms of the form

$$\text{(IV.17)} \quad N_4\,\psi^2 - N_2\,\psi^2\,, \quad N_4\,\chi^2 - N_2\,(\chi^2)$$

where the coefficients do not vanish in zero order [8]. Thus an infinite number of Feynman diagrams appears in any given order of perturbation theory.

On the other hand the equivalence theorem (equ. (II.46-47)) allows to construct an equivalent Lagrangian which contains only N_4-products and does not involve a summation problem in finite order. The Lagrangian is of the form

$$\text{(IV.18)} \quad \mathcal{L}_{eff} = \sum_{j=1}^{9} C_j\, N_4\left(M_j\right)$$

where the M_j denote the monomials

$$\psi^2, \psi^3, \psi^4, \chi^2, \chi^4, \psi\chi^2, \psi^2\chi^2, \partial_\mu\psi\partial^\mu\psi, \partial_\mu\chi\partial^\mu\chi$$

The perturbation expansion based on (IV.18) represents Symanzik's method of renormalizing the explicitly broken Goldstone model [3]. While the Lagrangian is not manifestly gauge invariant, the coefficients of the coupling terms are correlated in such a way that the Ward identities hold in the desired form.

The renormalization of the Higgs model was first developed by B. Lee by applying Symanzik's method to a regularized version [9]. B. Lee and Zinn-Justin extended the method of gauge invariant quantization to Higgs-Kibble models including the non-Abelian case [10]. In the remainder of this chapter we use the approach of gauge invariant quantization to renormalize the Abelian Higgs model by applying the general methods of Chapter II and III. As s-dependent Lagrangian we propose

$$\text{(IV.19)} \quad \begin{aligned} \mathcal{L}_{cl} = {} & (D_\mu\varphi)^*(D^\mu\varphi) - (\eta_0^2 - s^2 w^2)\varphi^*\varphi - h_0(\varphi^*\varphi)^2 \\ & - \frac{1}{4} B_{\mu\nu}B^{\mu\nu} + \frac{1}{2} m_0^2 B_\mu B^\mu - \frac{1}{2\alpha_0}(\partial_\mu B^\mu)^2 \\ & + \frac{\delta_0(s)}{\sqrt{2}}(\varphi + \varphi^*) \end{aligned}$$

with

$$D_\mu = \partial_\mu - ie_0 B_\mu \; ; \quad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$\text{(IV.20)} \quad B_\mu = Z_3^{1/2} A_\mu \, , \quad e_0 = Z_3^{-\frac{1}{2}} e \, , \quad \alpha_0 = Z_3 \alpha \, , \quad m_0 = Z_3^{-\frac{1}{2}} m$$

$$\varphi = \frac{1}{\sqrt{2}}(\tau_1 + s v_0 + i\tau_2)$$

$$\tau_1 = Z_2^{1/2}\psi \, , \quad \tau_2 = Z_2^{1/2}\chi \, , \quad v_0 = Z_2^{1/2} v$$

ψ, χ and A_μ denote the properly normalized fields. Their vacuum expectation values are required to vanish while it is assumed that

$$\text{(IV.21)} \qquad v_0 \neq 0$$

The gauge class used in (IV.19) is the analogue of the Gupta-Bleuler or Stueckelberg gauge in (massive) electrodynamics. The treatment of the Higgs model in the 't Hooft gauge will be discussed in Chapter V.

For $\delta_0 = 0$ and $m = 0$ (IV.19) is the Lagrangian of the Higgs model. The model described by (IV.19) for $\delta_0 = 0$, but non-vanishing original photon mass $m \neq 0$ will be called the pre-Higgs model. In both cases the Lagrangian (IV.19) is formally gauge invariant, but the gauge symmetry is spontaneously broken due to (IV.21).

For $\delta_0 \neq 0$ the gauge symmetry is explicitly broken by the term proportional to $\varphi + \varphi^*$. In this case the model is called the explicitly broken pre-Higgs model. No infrared problems occur for $\delta_0 \neq 0$ and the Lagrangian (IV.19) can be used for setting up renormalized perturbation theory without difficulties. For $\delta_0 = 0$ infrared divergencies occur for some coefficients of (IV.19), but the Goldstone limit $\delta_0 \to 0$ and the subsequent Higgs limit $m \to 0$ (in the Landau gauge) may be applied yielding the Green's functions of the pre-Higgs and the Higgs model.

The free Lagrangian of the explicitly broken Higgs model is

(IV.22)
$$\begin{aligned} \mathcal{L}_{eff\,0} = &-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_t^2(s) A_\mu A^\mu - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \\ &+ \frac{1}{2} \left(\partial_\mu \psi \partial^\mu \psi + \partial_\mu \chi \partial^\mu \chi \right) + w(s) A_\mu \partial^\mu \chi \\ &- \frac{1}{2} M^2(s) \psi^2 - \frac{1}{2} \mu^2(s) \chi^2 . \end{aligned}$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The s-dependence of the coefficients becomes (IV.9) and

(IV.23)
$$m_t^2(s) = m^2 + s^2 w^2$$
$$w(s) = s\, w$$

with

(IV.24)
$$w_{(0)} = \lim_{e \to 0} w \ , \quad W = \lim_{e \to 0} e v$$
$$\lim_{e \to 0} h_0 v^2 = \frac{M^2 - \mu^2}{2}$$

For M, m, μ, $w > 0$ the free theory can be shown to be consistent with positive masses in the range $0 \leqslant s \leqslant 1$ if μ is sufficiently small and w chosen to satisfy (IV.11).

The particles of the theory are determined from the free Lagrangian (IV.22) at $s = 1$. Following is a table of the particles, their masses and associated fields.

Particle	Mass [11]	Field
Vector meson	m_t	A_μ
[illegible]	M	ψ
π	$\varkappa$	$\left.\right\} \partial_\mu A^\mu, \chi$
Ghost particle	λ	

with

$$m_t^2 = m^2 + w^2$$

$$\text{(IV.25)} \quad \varkappa^2 = \frac{1}{2}\left(\alpha m^2 + \mu^2\right) - \frac{1}{2}\sqrt{\left(\alpha m_t^2 + \mu^2\right) - 4\alpha\mu^2 m_t^2}$$

$$\lambda^2 = \frac{1}{2}\left(\alpha m^2 + \mu^2\right) + \frac{1}{2}\sqrt{\left(\alpha m_t^2 + \mu^2\right) - 4\alpha\mu^2 m_t^2}$$

In the Goldstone limit $\delta_0 \to 0$, or equivalently $\mu \to 0$, the π-particle becomes massless ($\varkappa \to 0$) and represents the Goldstone particle of the massive Higgs model.

As usual an indefinite metric formulation is employed in order to quantize the Lagrangian (IV.19). In general, the S-matrix will not be unitary since the ghost particles of negative probabilities participate in the interaction. No physical interpretation of the model is possible then. In the Goldstone limit, however, the ghost particles are expected to decouple from the rest of the system. The argument proceeds as in electrodynamics. The ghost particles are described by the divergence $\partial_\mu A^\mu$ of the vector potential. The field equation (II.29) of the vector potential reads

$$\text{(IV.26)} \quad -\partial_\nu F^{\mu\nu} + \frac{1}{\alpha}\partial^\mu \partial_\nu A^\nu + m_t^2 A^\mu = j^\mu$$

with the current operator

$$\text{(IV.27)} \quad j^\mu = ie\, N\left[\varphi (D^\mu \varphi)^* - \varphi^* D^\mu \varphi\right] + (Z_3 - 1)\,\partial_\nu F^{\mu\nu}$$

The current operator is partially conserved

$$\text{(IV.28)} \quad \partial_\mu j^\mu = \frac{i}{\sqrt{2}}\,\delta e\,(\varphi - \varphi^*) = \delta e\, Z_2^{1/2}\chi$$

as follows from Ward identities similar to (IV.15). (IV.26) and (IV.28) yield

$$(\text{IV.29}) \quad (\Box + \alpha m^2)\,\partial_\mu A^\mu = i\alpha\delta e\, Z_2^{1/2}\chi$$

as field equation of $\partial_\mu A^\mu$. In the Goldstone limit $\delta_0 \to 0$ the divergence $\partial_\mu A^\mu$ becomes a free field

$$(\text{IV.30}) \quad (\Box + \alpha m^2)\,\partial_\mu A^\mu = 0$$

and the ghost particles decouple. Accordingly the S-matrix of the massive Higgs model is unitary. By using differential equations of the Callan-Symanzik type it can also be shown that the Green's functions are well defined in the Goldstone limit $\mu \to 0$ [12].

The only stable particles of the massive Higgs model are the π-particle and the free ghost particle.

In the limit $m \to 0$, the pre-Higgs model passes over into the Higgs model. The zero-mass π-particles decouple, with the massive spin-one unstable particles of the pre-Higgs model becoming stable in the limit. The massive spin-zero particles also become stable, provided $M < 2w$. The Green's functions of the A_μ, ψ and χ fields can be shown to exist in the Higgs limit, but only in Landau gauge ($\alpha = 0$) [12].

The equivalence theorem (II.46 - 47) can be applied to construct an equivalent Lagrangian consisting of N_4-products only. In contradistinction to (IV.19) the non-linear part of the N_4-Lagrangian is not manifestly gauge invariant. This N_4-version of the Higgs model represents B. Lee's original approach in the language of the normal product formalism.

Chapter IV: References and Footnotes

[1] The material of this section will be published in a series of papers by J,H Lowenstein, M. Weinstein, W. Zimmermann (part I and II), and J.H. Lowenstein, B. Schroer (part III).

[2] B. Lee, Nucl. Phys. B9, 649 (1969).

[3] K. Symanzik, Lett. Nuovo Cimento 2, 10 (1969) and Commun. Math. Phys. 16, 48 (1970).

[4] For the proof see part II of ref. [1].

[5] For problems concerning unstable particles in perturbation theory we refer to M. Veltman, Physica 29, 122 (1969) and part III of ref. [1].

[6] For the formulation of the renormalization conditions see ref. [1].

[7] K. Symanzik, Lett. Nuovo Cimento 2, 10 (1969) and Commun. Math. Phys. 16, 48 (1970).
F. Jegerlehner and B. Schroer, to be published.

[8] By appropriate choice of $\mathfrak{w}$ one of the coefficients can be made to vanish in zero order, but not both.

[9] B. Lee, Phys. Rev. D5, 823 (1972).

[10] B. Lee and J. Zinn-Justin, to be published.

[11] The values given are the masses in zero order. Only for stable particles may the zero order values be identified with the exact masses by suitable normalization conditions.

[12] See ref. [1], part III.

V. Models with vanishing Symmetric Mass Parameters

As stressed in chapter III, the combinatorics of intermediate renormalization comes into conflict with the possible occurrence of vanishing mass parameters, and, in particular, another description of symmetry breaking has to be found if the group structure implies the vanishing of some mass parameters. Typical examples of such symmetries are chiral symmetries, when spin 1/2 fields are involved, and gauge symmetries, although no difficulty should in principle arise if the physical masses are to be non vanishing. In such cases, one may first consider the classical theory as a limit of a theory where no vanishing mass parameter occurs, quantize the latter and let the spurious mass parameters go to zero. This is the road chosen in Chapter IV. The only alternative strategy which is known at present [1] is to investigate the structure of the classical Lagrangian which describes the tree approximation of the theory to be constructed, characterize the symmetry via the Ward identities which express the classical Noether theorem in presence of external sources, and look for an L_4 Lagrangian for which Ward identities of the type found in the tree approximation hold in finite renormalized form, in the sense that no composite operator different from those found in the tree approximation occur, with, however, possibly different coefficients. In case such a program cannot be completed, with composite operators occurring with their naive dimension, renormalized Ward identities are said to contain anomalies - e.g. of the type found in the trace identity for the energy momentum tensor cf. Ch. III. There exists at the moment, unfortunately, no general theorem which allows to predict from the group structure the presence of anomalies.

Since the present program is still at a very experimental stage, we shall content ourselves with the description of two examples: the σ model with nucleons and the Abelian Higgs Kibble model treated in 't Hooft's gauge.

1. The σ model involving nucleons [2]

Choosing for simplicity the chiral group to be $U(1) \times U(1)$, and denoting the meson field (π, σ), the nucleon field, ψ , the Lagrangian in the tree approximation is obtained from the formal Lagrangian

$$\mathcal{L}_{formal} = \bar{\psi} \gamma \frac{1}{i} \partial \psi + g\bar{\psi}(\sigma + i\pi\gamma_5 \psi) + \frac{1}{2}\left(\partial_\mu \pi \partial^\mu \pi + \partial_\mu \sigma \partial^\mu \sigma\right) - \frac{m^2}{2}\left(\pi^2 + \sigma^2\right) - \frac{\lambda}{4}\left(\pi^2 + \sigma^2\right)^2 + C\sigma$$

by the field translation $\sigma = \sigma' + F$, under the constraint that no term linear in σ' remains.
We thus obtain

$$\mathcal{L}_{tree} = \bar{\psi}\gamma\tfrac{1}{i}\partial\psi + gF\bar{\psi}\psi + g\bar{\psi}(\sigma' + i\pi\gamma_5)\psi$$
$$+ \tfrac{1}{2}\left(\partial_\mu\pi\partial^\mu\pi + \partial_\mu\sigma'\partial^\mu\sigma'\right) - \frac{m^2+\lambda F^2}{2}\pi^2 - \frac{m^2+3\lambda F^2}{2}\sigma'^2$$
$$- \frac{\lambda}{4}(\pi^2+\sigma'^2)^2 - \lambda F\sigma(\sigma'^2+\pi^2)$$

with the constraint $C = \lambda F^3 + m^2 F$
This Lagrangian may be written as follows

$$\mathcal{L}_{tree} = \bar{\psi}\left(\gamma\tfrac{1}{i}\partial + M\right)\psi + g\bar{\psi}(\sigma' + i\pi\gamma_5)\psi$$
$$+ \tfrac{1}{2}\partial_\mu\pi\partial^\mu\pi - \frac{m_\pi^2}{2}\pi^2 + \tfrac{1}{2}\partial_\mu\sigma'\partial^\mu\sigma' - \frac{m_\sigma^2}{2}\sigma'^2$$
$$- g^2\frac{m_\sigma^2 - m_\pi^2}{8M^2}(\pi^2+\sigma'^2)^2 - g\,\frac{m_\sigma^2 - m_\pi^2}{2M}\sigma'(\pi^2+\sigma'^2)$$

where the following change of parameters was made:

$$M = gF$$
$$m^2 + \lambda F^2 = m_\pi^2$$
$$m^2 + 3\lambda F^2 = m_\sigma^2$$

which implies $\lambda = g^2\,\dfrac{m_\sigma^2 - m_\pi^2}{2M^2}$

(Instead of M, m_π, m_σ, g we could have chosen as parameters M, m_π, m_σ, F).

The Ward identities are most easily obtained by applying Noether's theorem to $\mathcal{L}_{formal}$ + source term and performing the field translation afterwards. The result is

$$\partial^\mu \langle T j_\mu(x)\, X\rangle^{tree} = \sum_{k\in X} \delta(x-x_k)\,\theta_k \langle T X\rangle$$
$$+ \sum_{\pi\in X} \delta(x-x_\pi)\, F \langle T X_{\hat{\pi}}\rangle$$
$$+ C\langle T\,\pi(x)\, X\rangle$$

where θ_k represents an infinitesimal chiral transformation on field (k) ($\pi \to \sigma$, $\sigma \to -\pi$, $\psi \to i\gamma_5 \psi$ $\quad \bar{\psi} \to -\bar{\psi} i\gamma_5$)
and

One now looks for an effective Lagrangian

$$\begin{aligned}
\mathcal{L}_k = & (1+B)\bar{\psi}\gamma\frac{1}{i}\partial\psi + (M+A)\bar{\psi}\psi + g_\sigma(1+h_\sigma)\bar{\psi}\sigma\psi \\
& + g(1+h_\pi)\bar{\psi}\, i\pi\gamma_5\psi + \frac{1}{2}(1+b_\pi)\partial_\mu\pi\,\partial^\mu\pi - \frac{1}{2}(m_\pi^2+a_\pi)\pi^2 \\
& + \frac{1}{2}(1+b_\sigma)\partial_\mu\sigma\,\partial^\mu\sigma - \frac{1}{2}(m_\sigma^2+a_\sigma)\sigma^2 \\
& - g^2\,\frac{m_\sigma^2-m_\pi^2}{8M^2}\left[(1+l_\pi)\pi^4 + 2(1+l_{\pi\sigma})\pi^2\sigma^2 + (1+l_\sigma)\sigma^4\right] \\
& - g\,\frac{m_\sigma^2-m_\pi^2}{2M}\left[(1+k_\sigma)\sigma^3 + (1+k_\pi)\sigma\pi^2\right]
\end{aligned}$$

which depends on 13 parameters, and for a current

$$\begin{aligned}
J_\mu = & (1+\alpha_\pi)\pi\partial_\mu\sigma - (1+\alpha_\sigma)\sigma\partial_\mu\pi + F(1+\beta_\pi)\partial_\mu\pi \\
& + (1+\gamma)\bar{\psi}\gamma_\mu\gamma_5\psi
\end{aligned}$$

such that a Ward identity identical with that obtained in the tree approximation holds. This is possible and leaves freedom, without varying F, to fix the masses of π , σ, ψ at their zeroth order values m_π, m_σ, M and to set equal to unity the residues of the propagators of the π and the ψ fields at their respective poles. The number of free parameters is just the number of parameters occurring in the most general formal Lagrangian invariant under chiral transformations, except for a linear breaking term.

2. The Abelian Higgs Kibble model in the 't Hooft gauge.[3]

The interest of this model within the general class of spontaneously broken gauge models is that it exhibits most features characteristic of these models in so far as ultraviolet behaviour is concerned. Infrared difficulties, on the other hand, are avoided.

One starts from the formal Lagrangian

$$\mathcal{L}_{formal} = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + (D_\mu \varphi)^* (D^\mu \varphi) + \mu^2 \varphi^* \varphi - g(\varphi^*\varphi)^2 - \frac{1}{2\alpha}\left(\partial_\mu A^\mu + \rho\varphi_2\right)^2 + \frac{1}{\alpha}\bar{c}\,\mathcal{M}c$$

where, for the time being, the Faddeev-Popov term, $\bar{c}\mathcal{M}c$, whose convenience will be seen later, is ignored. The mode corresponding to broken symmetry is obtained by making the substitution

$$\varphi = \frac{\varphi_1 + v + i\varphi_2}{\sqrt{2}}$$

v being determined by the condition that no term linear in φ_1 survives:

$$\mu^2 = g v^2$$

A translation induced photon mass term yields a transverse photon mass

$$m^2 = e^2 v^2$$

The mass corresponding to the φ_1 field is

$$M^2 = -\mu^2 + 3 g v^2 = 2 g v^2$$

The mass matrix corresponding to the coupled $\varphi_2, \partial_\mu A^\mu$ quadratic form yields a degenerate eigenvalue at

$$\lambda^2 = \rho m$$

Before the introduction of the Faddeev Popov part of the Lagrangian the only non gauge invariant part of the Lagrangian is given by the gauge term

$$-\frac{G_\partial^2}{2\alpha} = -\frac{\left(\partial_\mu A^\mu + \rho\varphi_2\right)^2}{2\alpha}$$

In terms of the new variables, the Lagrangian reads:

$$\begin{aligned}\mathcal{L}_{tree} = {} & \frac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1 - \frac{M^2}{2}\varphi_1^2 + \frac{1}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2 - \frac{\rho^2}{2\alpha}\varphi_2^2 \\ & + \left(m - \frac{\rho}{\alpha}\right)\varphi_2\partial_\mu A^\mu - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu \\ & - e A_\mu\left(\varphi_1\partial^\mu\varphi_2 - \varphi_2\partial^\mu\varphi_1\right) + \frac{e^2}{2}A_\mu A^\mu\left(\varphi_1^2 + \varphi_2^2\right) \\ & + em A_\mu A^\mu \varphi_1 - \frac{e^2 M^2}{2m^2}\left(\varphi_1^2 + \varphi_2^2\right)^2 - \frac{2eM^2}{m}\varphi_1\left(\varphi_1^2 + \varphi_2^2\right)\end{aligned}$$

where g was eliminated against e through the relations

$$2 g v^2 = M^2, \quad e^2 v^2 = m^2, \quad \Rightarrow \quad \frac{2g}{e^2} = \frac{M^2}{m^2}$$

The restricted 't Hooft gauge is obtained for $\rho = \alpha m$ whereby the $\varphi_2 \partial_\mu A^\mu$ cross term vanishes. The gauge invariance is best expressed by applying the integrated Noether theorem corresponding to the gauge group to $\mathcal{L}_{formal}$+ source terms, and performing the field translation on the corresponding Ward identity, which assumes the form

$$0 = -\int \frac{1}{\alpha} \frac{\delta \mathcal{G}_x}{\delta \Lambda_y} \mathcal{G}_x + \text{source terms} = \frac{1}{\alpha} \int m_{yx} \mathcal{G}_x + \text{source terms}$$

On the other hand, the question of the gauge invariance of the physical scattering amplitudes first proceeds through the study of the variation of the connected Green's functional under the change of the gauge parameters γ (here, α, ρ):

$$\frac{\partial G^c(J)}{\partial \gamma} = -i \int \frac{\partial}{\partial \gamma} \left(\frac{\mathcal{G}^2}{2\alpha} \right) = -i \int \frac{\mathcal{G}}{\sqrt{\alpha}} \frac{\partial}{\partial \gamma} \left(\frac{\mathcal{G}}{\sqrt{\alpha}} \right)$$

where use has been made of the fact that the Green's functional is the Legendre transform of the Lagrangian.[1],[2] Gauge invariance is achieved if

$$\langle \mathcal{G} \rangle_{phys} = 0$$

for a suitable definition of physical states. In any event, it is desirable to convert the Ward identity into an identity of the Slavnov type:[3]

$$\mathcal{G}_x = \dots$$

which requires the inversion of the - in general field dependent - differential operator m. This is best achieved by introducing the Faddeev Propov fields with Lagrangian $\frac{1}{\alpha} \bar{C} m C$ and source terms $\bar{\xi} C + \bar{C} \xi$. The Ward identity then reads

$$0 = \int -\frac{1}{\alpha} m_{yx} \mathcal{G}_x + \frac{1}{\alpha} \bar{C}_z \frac{\delta m_{zt}}{\delta \Lambda_y} C_t + \textit{source terms}$$

and the Slavnov identity is obtained by integrating over y after multiplication by $\bar{C}_y$.[5] Using the equations of motion, the first term becomes

$$\int \bar{\xi}_x \mathcal{G}_x$$

whereas the second term vanishes if the Faddeev Propov ghost fields obey Fermi's statistics:

$$\int dy\, dz\, dt\ \bar{C}_y \bar{C}_z \frac{\delta \mathcal{G}_t}{\delta\Lambda_y \delta\Lambda_z} C_t = 0$$

by virtue of the abelianness of the gauge group. In the non abelian case,[3] this term does not vanish, but assumes a particularly simple form, independent of the gauge function: $\bar{C}\vec{t}\bar{C}\vec{\xi}$ where $\vec{t}$ denotes the infinitesimal generator of the internal Lie algebra of the gauge group, only in the case where the Faddeev Popov ghost fields obey Fermi's statistics.

Written in full, in terms of the translated field variables, the Slavnov identities read in the tree approximations

$$
\begin{aligned}
&\langle T\, (\partial_\mu A^\mu + \rho\varphi_2)(x)\ \varphi_1(x_1)\cdots\varphi_1(x_n)\ A_{\mu_1}(y_1)\cdots A_{\mu_m}(y_m)\ \varphi_2(z_1)\cdots\varphi_2(z_p) \\
&\qquad C(t_1)\ldots C(t_k)\ \bar{C}(u_1)\cdots\bar{C}(u_k)\rangle^c = \\
&- e\sum_{i=1}^{n}\langle T\, C(x)\ \varphi_1(x_1)\ldots(\varphi_2\bar{C})(x_i)\ldots\varphi_1(x_n)\ldots\bar{C}(u_k)\rangle^c \\
&+ \sum_{i=1}^{m}\langle T\, C(x)\ \varphi_1(x_1)\ldots\varphi_1(x_n)\ A_{\mu_1}(y_1)\ldots\partial_{\mu_i}\bar{C}(y_i)\ldots\bar{C}(u_k)\rangle^c \\
&+ m\sum_{i=1}^{p}\langle T\ C(x)\ \varphi_1(x_1)\ldots\varphi_1(x_n)\ A_{\mu_1}(y_1)\ldots A_{\mu_m}(y_m)\ \varphi_2(z_1)\ldots\bar{C}(z_i)\ldots\bar{C}(u_k)\rangle^c \\
&+ e\sum_{i=1}^{p}\langle T\, C(x)\varphi_1(x_1)\ldots\varphi_1(x_n)\ A_{\mu_1}(y_1)\ldots A_{\mu_m}(y_m)\varphi_2(z_1)\ldots(\varphi_2\bar{C})(z_i)\ldots\bar{C}(u_k)\rangle^c \\
&- \sum_{i=1}^{k}\langle T\, C(x)\ldots C(t_1)\ldots(\partial_\mu A^\mu + \rho\varphi_2)(t_i)\ldots C(t_k)\bar{C}(u_1)\ldots\bar{C}(u_k)\rangle^c
\end{aligned}
$$

The reason for including Faddeev Popov fields within the Slavnov identities is that among other things, it is believed that they will turn out to be relevant to the unitarity problem of the physical S operator for the fully renormalized theory.

One can show that conversely, if such identities are to hold in the tree approximation the Lagrangian must be of the form initially postulated except for the possible occurrence of wave function renormalization factors Z_A, Z_ϕ in front of the terms $-\frac{1}{4} G_{\mu\nu} G^{\mu\nu}$, $(D_\mu \varphi)^* (D_\mu \varphi)$, respectively. Written in full, the Faddeev Popov contribution to $\mathcal{L}_{tree}$ is

$$\frac{1}{\alpha}\left(\partial_\mu \bar{c}\, \partial^\mu c + \rho m\, \bar{c} c + \rho e \varphi_1 \bar{c} c \right)$$

The Lagrangian then depends on the following parameters:

$$Z_A , m ; Z_\phi , M , \rho , e , \alpha$$

which can be characterized as follows:

Z_A, m are related to the residue and pole of the transverse photon propagator

Z_ϕ, M are related to the residue and pole of the φ_1 field propagator

ρ is related to the common $[\partial_\mu A^\mu, \varphi_2, C]$ ghost propagator pole [4]

α is related to the residue of the Faddeev Popov ghost propagator

e is related to the $\varphi_1 \varphi_1 \varphi_1$ or $\varphi_1 \varphi_1 A^T$ scattering. Alternatively, g is related to $\varphi_1 \varphi_1 \varphi_1 \varphi_1$ scattering. In all the following, we shall assume m, M and the ghost mass to be restricted by inequalities which insure the stability of all three types of particles. In order that the physical scattering amplitudes be gauge invariant, one sees that, by virtue of the Slavnov identities, physical states should only contain φ_1, and A^T quanta.

The question is now to construct the most general dimension four effective Lagrangian whose zeroth order approximation in terms of the loop counting parameter coincides with $\mathcal{L}_{tree}$ and inquire whether constraints can be put on the coefficients in such a way that Slavnov type identities may hold.

Let then

$$
\begin{aligned}
\mathcal{L}^{(4)}_{eff} = & -\frac{1}{4}(1+b)G_{\mu\nu}G^{\mu\nu} + \frac{m^2+a}{2}A_\mu A^\mu - \frac{1}{2\alpha}(1+c)(\partial_\mu A^\mu)^2 \\
& + \frac{d}{4}(A_\mu A^\mu)^2 + \frac{1}{2}(1+B_1)\partial_\mu\varphi_1\partial^\mu\varphi_1 - \frac{M^2+A_1}{2}\varphi_1^2 \\
& + \frac{1}{2}(1+B_2)\partial_\mu\varphi_2\partial^\mu\varphi_2 - \frac{1}{2}\left(\frac{\rho^2}{\alpha}+A_2\right)\varphi_2^2 + \left(\frac{\rho}{\alpha}-m-\tilde{f}_2\right)A_\mu\partial^\mu\varphi_2 \\
& + e(1+f_1)A_\mu\varphi_2\partial^\mu\varphi_1 - e(1+f_2)A_\mu\varphi_1\partial^\mu\varphi_2 + \frac{e^2}{2}(1+g_1)A_\mu A^\mu\varphi_1^2 \\
& + \frac{e^2}{2}(1+g_2)A_\mu A^\mu\varphi_2^2 + em(1+\tilde{g}_1)A_\mu A^\mu\varphi_1 \\
& - \frac{e^2M^2}{2m^2}(1+h_1)\varphi_1^4 - \frac{e^2M^2}{2m^2}(1+h_2)\varphi_2^4 - \frac{e^2M^2}{m^2}(1+h_{12})\varphi_1^2\varphi_2^2 \\
& - \frac{2eM^2}{m}(1+k_1)\varphi_1^3 - \frac{2eM^2}{m}(1+k_2)\varphi_2^2\varphi_1 \\
& + \frac{1+H}{\alpha}\Big[\partial_\mu\bar{c}\,\partial^\mu c - (\rho m+A)\bar{c}c - \rho e(1+\ell)\bar{c}\varphi_1 c\Big] \\
& \qquad + r_1\bar{c}\varphi_1^2 c + r_2\bar{c}\varphi_2^2 c + s\,\bar{c}A_\mu A^\mu c \;\Big]
\end{aligned}
$$

We now look for an identity of the Slavnov type, recalling the conservation of the number of Faddeev Popov ghost pairs, and the invariance under charge conjugation:

$$A_\mu \to -A_\mu\,;\quad \varphi_2 \to -\varphi_2\,;\quad \varphi_1\to\varphi_1\,;\quad c\to c\,;\quad \bar{c}\to\bar{c}\,.$$

One can use the effective equations of motion for $\bar{c}$, $\partial_\mu A^\mu$, φ_1, φ_2 under the following combinations: [3]

$$
\begin{aligned}
\int N_5\,\bar{c}\,\Box\,\partial_\mu A^\mu &= cst\int \delta_{\bar{c}}\,\partial_\mu A^\mu + \int N_5\,\delta_{\bar{c}}\,\partial_\mu A^\mu \\
&= cst\int \bar{c}\,\partial_\mu\,\delta_{A_\mu} + \int \bar{c}\,\partial_\mu J^\mu
\end{aligned}
$$

$$\int N_4 \bar{c} \,\Box\, \varphi_2 = cst \int \varphi_2 \delta_{\bar{c}} + \int N_4 j_{\bar{c}} \varphi_2$$
$$= cst \int \bar{c}\, \delta_{\varphi_2} + \int N_4 \bar{c}\, j_2$$

$$\int N_5 \bar{c} \,\Box\, (\varphi_1 \varphi_2) = cst \int \delta_{\bar{c}} N_2 (\varphi_1 \varphi_2) + N_5 j_{\bar{c}} \{\varphi_1 \varphi_2\}$$
$$= \int 2 \bar{c}\, \partial_\mu \varphi_1 \partial^\mu \varphi_2 + cst \int \bar{c}\, \varphi_2 \delta_{\varphi_1}$$
$$- \int N_5 j_1 \{\bar{c} \varphi_2\} + cst \int \bar{c}\, \varphi_1 \delta_{\varphi_2} + \int N_5 j_2 \{\bar{c} \varphi_1\}$$

where the brackets $\{\cdot\}$ indicate anisotropic normal products and j_1, j_2, $j_{\bar{c}}$, J_μ are simply related (within the addition of mass terms and the multiplication through wave function renormalization coefficients) to the corresponding field sources.

The reduction of anisotropic to isotropic normal product, as well as the anisotropic use of the equations of motion for φ_1, φ_2 in the evaluation of $\partial^\mu J_\mu$ allows to cast these three identities into the form

$$\int \partial_\mu A^\mu \delta_{\bar{c}} = \text{wanted terms + unwanted terms}$$
$$\int \varphi_2 \delta_{\bar{c}} = \text{wanted terms + unwanted terms}$$
$$\int N_2 \varphi_1 \varphi_2 \delta_{\bar{c}} = \text{wanted terms + unwanted terms}$$

Wanted terms are of the form $\int \bar{c}\, \partial_\mu \delta_{A_\mu}$, $\int \bar{c}\, \delta_{\varphi_2}$, $\int N_2 \bar{c} \varphi_1 \delta_{\varphi_2}$, $\int N_2 \bar{c} \varphi_2 \delta_{\varphi_1}$

After repeated use of Zimmermann's identities unwanted terms can be cast into the form of itegrals of the following monomials, all counted with dimension 5 :

$\bar{c} \varphi_2$,
$\bar{c}\, \partial_\mu A^\mu$, $\bar{c} \varphi_1 \varphi_2$

$$\bar{c}\varphi_2\varphi_1^2,\ \bar{c}\varphi_2 A_\mu A^\mu,\ \bar{c}\varphi_2^3,\ \bar{c}\varphi_1\partial_\mu A^\mu,\ \bar{c}A_\mu\partial^\mu\varphi_1,$$
$$\bar{c}\varphi_2\varphi_1^3,\ \bar{c}\varphi_2^3\varphi_1,\ \bar{c}A_\mu A^\mu\varphi_1\varphi_2,\ \bar{c}\partial_\mu A^\mu\varphi_1^2,\ \bar{c}A_\mu\partial^\mu\varphi_1\varphi_1,$$
$$\bar{c}\partial_\mu A^\mu\varphi_2^2,\ \bar{c}\varphi_2\partial_\mu\varphi_2 A^\mu,\ \bar{c}\partial_\mu A^\mu A_\lambda A^\lambda,\ \bar{c}A_\lambda\partial^\lambda(A_\mu A^\mu),$$
$$\bar{c}\partial_\mu\varphi_1\partial^\mu\varphi_2,\ \bar{c}\partial_\mu\bar{c}A^\mu c\ .$$

There are 19 unwanted terms whereas the Lagrangian depends on 25 coefficients. One of which is connected with the normalization of the Faddeev Popov ghost field. It turns out that one can express the unwanted term $N_5\,\bar{c}\varphi_1\varphi_2$ in terms of $\int\varphi_2\,\delta_{\bar{c}}$, wanted terms, and other unwanted terms, and, similarly $\int N_5\bar{c}\,\partial_\mu\varphi_1\partial^\mu\varphi_2$ in terms of $\int N_2\,\varphi_1\varphi_2\,\delta_{\bar{c}}$, wanted terms and other unwanted terms. Imposing then the vanishing of the remaining 17 unwanted terms, one gets Slavnov identities of the form:

$$\begin{aligned}
&\langle T\left[\partial_\mu A^\mu + (\rho+\delta\rho)\varphi_2 + \delta\rho_{12}N_2(\varphi_1\varphi_2)\right]\varphi_1(x_1)\dots\varphi_1(x_n)\\
&\qquad A_{\mu_1}(y_1)\dots A_{\mu_m}(y_m)\,\varphi_2(z_1)\dots\varphi_2(z_p)\,c(t_1)\dots c(t_k)\,\bar{c}(u_1)\dots\bar{c}(u_k)\rangle^c\\
&= -e\left(\frac{1+\beta_1}{1+H}\right)\sum_{i=1}^{n}\langle T\,c(x)\,\varphi_1(x_1)\dots(\varphi_2\bar{c})(x_i)\dots\varphi_1(x_n)\dots\dots\bar{c}(u_k)\rangle^c\\
&\quad+\left(\frac{1+\beta_A}{1+H}\right)\sum_{i=1}^{m}\langle T\,c(x)\dots\dots A_{\mu_1}(y_1)\dots\partial_{\mu_i}\bar{c}(y_i)\dots\dots\bar{c}(u_k)\rangle^c\\
&\quad+m\left(\frac{1+\beta_{12}}{1+H}\right)\sum_{i=1}^{p}\langle T\,c(x)\dots\dots\varphi_2(z_1)\dots\bar{c}(z_i)\dots\varphi_2(z_p)\dots\dots\bar{c}(u_k)\rangle^c\\
&\quad+e\left(\frac{1+\beta_2}{1+H}\right)\sum_{i=1}^{p}\langle T\,c(x)\dots\dots\varphi_2(z_1)\dots(\varphi_1\bar{c})(z_i)\dots\varphi_2(z_p)\dots\dots\bar{c}(u_k)\rangle^c\\
&\quad-\sum_{i=1}^{k}\langle T\,c(x)\dots\dots c(t_1)\dots\left[\partial_\mu A^\mu+(\rho+\delta\rho)\varphi_2+\delta\rho_{12}N_2(\varphi_1\varphi_2)\right](t_i)\dots\\
&\qquad\qquad c(t_k)\dots\dots\bar{c}(u_k)\rangle^c\ .
\end{aligned}$$

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where the coefficients $\delta\rho$, $\delta\rho_{12}$, β_1, β_A, β_{12}, β_2 are finite formal power series in $\hbar$. Apart from the fact that α only occurs in the combination $\alpha/(1+H)$ it is not known at present whether simple relations automatically hold between these coefficients because of the complicated way in which they were obtained. In particular, there is not a priori reason why $\delta\rho_{12}$ should vanish. In other words, had we started from a gauge function of the form $B = \partial_\mu A^\mu + \rho\varphi_2 + \rho_{12}\varphi_1\varphi_2$ in the tree approximation- the most general expression of dimension 2 odd under charge conjugation - we would have obtained a Slavnov identity involving a renormalized gauge function of the form $B^{\text{Ren.}} = \partial_\mu A^\mu + (\rho+\delta\rho)\varphi_2 + (\rho_{12}+\delta\rho_{12}) N_2\, \varphi_1\varphi_2$ and even when ρ_{12} vanishes the induced $\delta\rho_{12} N_2(\varphi_1\varphi_2)$ term may survive. In fact, if we believe that the tree approximation property

$$\langle T\ B(x)\, B(y)\rangle \propto \delta(x-y)$$

survives through radiative corrections, one can see that the $N_2(\varphi_1\varphi_2)$ induced term has to be present.

Once the Slavnov identities have been obtained, there are, besides the combination $\alpha/(1+H)$, seven parameters left free which can be used to fulfill the following normalization conditions: position of the poles, residues being unity at these poles for the transverse photon and φ_1 propagator, mass shell value of the three or four φ_1 (or $\varphi_1\varphi_1 A^\mu$) vertices equal to the tree approximation value, and finally double vanishing of the inverse determinant of the coupled $\partial_\mu A^\mu$, φ_2 propagator matrix at $p^2 = \rho m^2$.[6] The last condition combined with the Slavnov identity implies that the Faddeev Popov ghost propagator has a simple pole at $p^2 = \rho m^2$, whose residue can be normalized to one by means of the parameter H, if one wishes to do so. The theory is then completely interpretable within a Fock space with indefinite metric whose structure will be reported elsewhere together with the appropriate asymptotic theory.

The crucial test for the correctness of this renormalization scheme of course relies on the check of unitarity and gauge invariance of the physical S operator which has not yet been attacked within the present framework, and, next the existence of local gauge invariant observables which leave the physical subspace invariant, in the same way as in quantum electrodynamics.[7]

Chapter V : References and Footnotes

[1] This is the strategy advocated by J. Schwinger in: J. Schwinger "Particles Sources and Fields", Addison Wesley Pub. Co., Reading, Mass. (1970).
J. Schwinger, "Particles and Sources", Gordon & Breach New York (1970) (Brandeis 1967). See also:
A. Rouet, R. Stora, Lectures given at the Universities of Geneva and Lausanne (1973).

[2] Ch. III, Refs. [1] and [8]. The determination of an L_4 Lagrangian such that Ward identities hold, performed in Ref. [7] of Ch. III requires one more constraint than is allowed by the number of parameters at disposal. That constraint, a relation between some r coefficients, can be shown to be automatically fulfilled by an argument concerning the high momentum behaviour of vertex functions. This argument is due to O. Piguet and similar to one used by him in the construction of an L_4 Lagrangian for the Higgs Kibble model with massive photons of Chapter IV .
(O. Piguet, private communication).

[3] More details can be found in A. Rouet, R. Stora, Ref. [1] , and A. Rouet, article in preparation.

[4] We wish to thank B.W. Lee for a discussion on this point.

[5] This way of obtaining the Slavnov identities can be found in : L. Quaranta, A. Rouet, R. Stora, E. Tirapegui "Spontaneously broken gauge invariance: Ward identities, Slavnov identities, gauge invariance", in "Renormalization of Yang Mills Fields and Applications to Particle Physics", CNRS Marseille, June 19-23 (1972), where however the treatment of renormalization is formal due to the a priori possible occurence of infrared difficulties which does not happen in the treatment given in this chapter. Concerning the renormalization of gauge field theories see G.'t Hooft and B.W. Lee's Lectures, in this volume, where the original work by G.'t Hooft, B.W. Lee, M. Veltman, J. Zinn-Justin is reported.

[6] Such a normalization condition was used in a version of the renormalization of the Higgs Kibble model in Stueckelberg's gauge, by J.H. Löwenstein, M. Weinstein, W. Zimmermann.

[7] J.H. Löwenstein, B. Schröer, PR. D $\underline{6}$, 1553 (1972) .