RECHERCHE COOPÉRATIVE SUR
PROGRAMME N° 25

H. J. BORCHERS

Decomposition of Families of Unbounded Operators

Les rencontres physiciens-mathématiciens de Strasbourg - RCP25, 1975, tome 22
« Exposés de : H. Araki, H.J. Borchers, J.P. Ferrier, P. Krée, J.F. Pommaret, D. Ruelle, R. Stora et A. Voros », , exp. n° 2, p. 1-28
<http://www.numdam.org/item?id=RCP25_1975__22__A2_0>

© Université Louis Pasteur (Strasbourg), 1975, tous droits réservés.

L'accès aux archives de la série « Recherche Coopérative sur Programme n° 25 » implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Decomposition of Families of Unbounded Operators

H. J. Borchers
Institut für Theoretische Physik
Universität Göttingen

Contents:

I. Introduction

In this seminar I will talk about a joint work with J. Yngvason, which is submitted to the Communications in Mathematical Physics in two papers with the titles:

On the Algebra of Field Operators. The Weak Commutant and Integral Decomposition of States.

and:

Integral Representations for Schwinger Functionals and the Moment Problem over Nuclear Spaces.

Our investigation was originated by the following question: Let $\mathcal{A}$ be a $*$-algebra with identity and let ω be a state on $\mathcal{A}$ this means ω is a normalized positive linear functional on $\mathcal{A}$. When can such a functional be decomposed into extremal ones? This leads first to the problem of characterising extremal states. For the case of one single symmetric operator, it is known that it is not at all necessary to investigate the algebra generated by this operator, but, one gets along by looking at this operator alone.

This leads us to the

I. 1. Definition A partial $*$-algebra is a complex vector space A together with an invalution $x \in \mathcal{A} \rightarrow x^* \in \mathcal{A}$ with the usual operation $(x+\lambda y)^* = x^* + \bar{\lambda} y^*$, $x^{**} = x$. And a subset $M \subset A \times A$ such that $(x, y) \in M$ implies $(y^*, x^*) \in M$, (x, y_1) and $(x, y_2) \in M$ implies $(x, y_1 + y_2) \in M$ and for every pair $(x,y) \in M$ exist an element $x \cdot y \in A$ fulfilling the usual operations.

I. 2. Definition Let A be a partial $*$-algebra, a pair $(\pi, \mathcal{D})$ is called a $*$-representation of A if the following conditions are fulfilled.

(i) $\mathcal{D}$ is a pre-Hilbert-space with completion $\mathcal{H}(\mathcal{D})$.

(ii) To every $x \in A$ exists a linear operator $\pi(x)$ defined on $\mathcal{D}$ with values in $\mathcal{H}(\mathcal{D})$ such that

(α) $\pi(x+\lambda y) = \pi(x) + \lambda \pi(y)$

(β) if $(x, y) \in M$ then we have $\pi(y)\mathcal{D} \subset \mathcal{D}$ and $\pi(xy) = \pi(x)\pi(y)$

(γ) for $f, g \in \mathcal{D}$ we have the relation $(f, \pi(x)\, g) = (\pi(x^*)\, f, g)$.

Remarks

1) The last condition implies, that every operator $\pi(x)$ is closable.
2) It is not assumed that the common domain $\mathcal{D}$ is closed in any topology, also not in the topology induced by the graph norms of all the operators $\pi(x)$.
3) If A is a $*$-algebra and ω a state on A, then one gets $(\pi, \mathcal{D})$ in the usual way by the G.N.S construction.

If we want to define extremal states, then we have to speak about the commutant of the representation. But, in the case of unbounded operators we have to distinguish between two different kinds of commutants.

I.3. Definition: Let $(\pi, \mathcal{D})$ be a $*$-representation of a partial $*$-algebra A then we define

(a) The strong commutant:

$$(\pi, \mathcal{D})'_s = \{ C \in \mathcal{B}(\mathcal{H}(\mathcal{D})),\ C\mathcal{D} \subset \mathcal{D} \text{ and } \pi(x)C = C\,\pi(x) \text{ on } \mathcal{D} \text{ for all } x \in A \}$$

(b) The weak commutant:

$$(\pi, \mathcal{D})'_w = \{ C \in \mathcal{B}(\mathcal{H}(\mathcal{D})),\ (Cf, \pi(x)g) = (\pi(x^*)f, C^* g) \text{ for all } f, g \in \mathcal{D} \text{ and all } x \in A \}$$

With this notation we can characterize extremal states, namely a state on a $*$-algebra A is extremal if and only if the weak commutant of the cyclic representation π_ω is trivial i.e, consists of scalar multiples of the identity. This statement follows trivially from the properties of the weak commutant which are listet in the following

I.4. Lemma: Let $(\pi, \mathcal{D})$ be a $*$-representation of a partial $*$-algebra then

(a) The strong commutant $(\pi, \mathcal{D})'_s$ is an algebra .

(b) The weak commutant $(\pi, \mathcal{D})'_w$ is

(i) weakly closed and contains the identity

(ii) invariant under the adjoint operation, i.e. $C \in (\pi, \mathcal{D})'_w$ implies $C^* \in (\pi, \mathcal{D})'_w$

(iii) is generated by its positive elements

(iv) $(\pi, \mathcal{D})'_s \subset (\pi, \mathcal{D})'_w$.

All these properties follow easily from the definitions, so that we will not give the proofs. But we will make some

Remarks

1) $(\pi, \mathcal{D})'_s$ is an algebra, but in the general situation it is not a $*$-algebra and is also not closed in any reasonable operator topology.

2) $(\pi, \mathcal{D})'_w$ is not an algebra in general.

If you call $\bar{\mathcal{D}} = \bigcap_{x \in A} \mathcal{D}_{\overline{\pi(x)}}$ then $(\pi, \mathcal{D})$ has an extension $(\tilde{\pi}, \bar{\mathcal{D}})$ by continuity. It is simple to show that $(\pi, \mathcal{D})$ and $(\tilde{\pi}, \bar{\mathcal{D}})$ have the same weak commutant.

3) If one deals with a family of bounded operators, then one can put $\mathcal{D} = \mathcal{H}(\mathcal{L})$ In this case, the strong and weak commutants coincide.

If you now look at the decomposition theory for bounded operators then it consists of two things

Step 1. In this case you have only one kind of commutant which is a von Neuman algebra. Pick a maximal abelian algebra M in this commutant.

Step 2. Try to define an integral decomposition with respect to M, which is always possible if the Hilbert space is separable.

This leads for unbounded operators to the following two questions

1) How can you find a maximal commuting algebra, since one wants to make a decomposition with respect to the weak commutant?

2) Assume you can construct a maximal commuting algebra in the weak commutant which additional information is needed in order to define an integral decomposition?

Both questions will be treated separately.

II. Extension theory

As a guide let us consider the case of one symmetric operator. In this case one would try to find a self-adjoint extension in order to find a maximal abelian subalgebra in the commutant. But in the case that this operator has non symmetric defect indices, we can define selfadjoint extensions only in some enlarged Hilbert space. So the first step will be to look for extensions of a family of unbounded operators.

II. 1. Definition: Let A be a partial $*$ -algebra and $(\pi, \mathcal{D})$ a representation of A. A representation $(\hat{\pi}, \hat{\mathcal{D}})$ will be called an extension of $(\pi, \mathcal{D})$ if

(i) $(\hat{\pi}, \hat{\mathcal{D}})$ is a $*$-representation of A.

(ii) $\hat{\mathcal{D}} \supset \mathcal{D}$ and the norm on $\hat{\mathcal{D}}$ coincides on $\mathcal{D}$ with the original norm on $\mathcal{D}$

(iii) For every $x \in A$ the restriction $\hat{\pi}(x) \restriction \mathcal{D}$ is equal to $\pi(x)$.

Remark :

It is not required in this definition that the two Hilbert spaces coincide but from $\mathcal{D} \subset \hat{\mathcal{D}}$ follows that $\mathcal{H}(\mathcal{D})$ is a subspace of $\mathcal{H}(\hat{\mathcal{D}})$.

If $(\hat{\pi}, \hat{\mathcal{D}})$ is an extension of $(\pi, \mathcal{D})$ and b is a bounded linear operator on $\mathcal{H}(\hat{\mathcal{D}})$ then we can define an operator on $\mathcal{H}(\mathcal{D})$ by E b E, where E denotes the projection onto $\mathcal{H}(\mathcal{D})$. Since this mapping occurs quite often in the following, we will introduce a separate notation for it.

II. 2. Definition: Let $(\hat{\pi}, \hat{\mathcal{D}})$ be an extension of $(\pi, \mathcal{D})$ and E be the projection onto $\mathcal{H}(\mathcal{D})$ then we define for any bounded operator b on $\mathcal{H}(\hat{\mathcal{D}})$

$$\rho(b) = EbE$$

II. 3. Lemma:

(i) ρ is linear, commutes with the involution and preserves ordering i.e.

$$\rho(x+\lambda y) = \rho(x) + \lambda \rho(y) \quad ; \quad \rho(x^*) = \rho(x)^*$$

and $x \geq y$ implies $\rho(x) \geq \rho(y)$.

(ii) ρ is weakly continuous.

(iii) If $b \in (\hat{\pi}, \hat{\mathcal{D}})'_w$ then follows $\rho(b) \in (\pi, \mathcal{D})'_w$.

These properties are all easy to verify and can be done by the reader.

Remark: ς maps the whole weak commutant of $(\hat{\pi}, \hat{\mathcal{D}})$ into the weak commutant of $(\pi, \mathcal{D})$, in particular it maps also the strong commutant of $(\hat{\pi}, \hat{\mathcal{D}})$ into the weak commutant of $(\pi, \mathcal{D})$. This means that if b commutes strongly with $(\hat{\pi}, \hat{\mathcal{D}})$ then ς(b) need not commute strongly any more with $(\pi, \mathcal{D})$. Therefore the plan of attack consists in the inverse procedure, namely to take elements b from the weak commutant and try to define an extension $(\hat{\pi}, \hat{\mathcal{D}})$ in such a way that $b = \varsigma(b')$ for some strongly commuting element $b' \in (\hat{\pi}, \hat{\mathcal{D}})'_s$. This plan leads naturally to the following

II. 4. Definition:

1) A triple $(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$ will be called an induced extension of $(\pi, \mathcal{D})$ if
   (i) $(\hat{\pi}, \hat{\mathcal{D}})$ is an extension of $(\pi, \mathcal{D})$
   (ii) $\hat{\mathcal{M}}$ is an abelian $*$-algebra of bounded operators on $\mathcal{H}(\hat{\mathcal{D}})$ with $1 \in \hat{\mathcal{M}}$ and $\hat{\mathcal{M}} \subset (\hat{\pi}, \hat{\mathcal{D}})'_s$.
   (iii) $\hat{\mathcal{D}}$ is the linear span of $\hat{\mathcal{M}}\mathcal{D}$ i.e. $\hat{\mathcal{D}} = \{ \sum m_i f_i \; ; \; m_i \in \hat{\mathcal{M}},\ f_i \in \mathcal{D} \}$

2) Denote by $\hat{\pi}(A) \vee \hat{\mathcal{M}}$ the linear span of $\{\hat{\pi}(x) \, ; \, x \in A\} \cup \hat{\mathcal{M}} \cup \{\hat{\pi}(x) m \, ; \, x \in A, m \in \hat{\mathcal{M}}\}$

and assume $(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$ and $(\hat{\hat{\pi}}, \hat{\mathcal{N}}, \hat{\hat{\mathcal{D}}})$ are two induced extension then we introduce a semiordering by

$$(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}}) \prec (\hat{\hat{\pi}}, \hat{\mathcal{N}}, \hat{\hat{\mathcal{D}}})$$

if

(i) $\hat{\mathcal{D}} \subset \hat{\hat{\mathcal{D}}}$

(ii) There exists a sub-algebra $\hat{\hat{\mathcal{M}}} \subset \hat{\mathcal{N}}$ isomorphic to $\hat{\mathcal{M}}$ such that

(iii) $(\hat{\hat{\pi}} \vee \hat{\hat{\mathcal{M}}}, \hat{\mathcal{N}}, \hat{\hat{\mathcal{D}}})$ is an induced extension of $(\hat{\pi} \vee \hat{\mathcal{M}}, \hat{\mathcal{D}})$.

The last definition points out that we want to construct induced extensions several times. Therefore we have to show that this concept is consistent.

II. 5. Lemma:

1) Let $(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$ be an induced extension of $(\pi, \mathcal{D})$ and assume there exist $x \in A$ such that $\pi(x)$ is bounded, then follows $\|\pi(x)\| = \|\hat{\pi}(x)\|$

2) Let $(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$ be an induced extension of $(\pi, \mathcal{D})$ and denote by $\hat{\mathcal{M}}^-$ the weak closure of $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}^-$ the linear span of $\hat{\mathcal{M}}^-\mathcal{D}$, then every operator $\hat{\pi}(x)$ has a continuous extension $\hat{\pi}^-(x)$ to $\hat{\mathcal{D}}^-$ and $(\hat{\pi}^-, \hat{\mathcal{M}}^-, \hat{\mathcal{D}}^-) \succ (\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$.

Proof: Assume $B \geq 0$ then follows:

$$\sum_{ij} (m_i f_i, \hat{B} m_j f_j) = \sum_{ij} (f_i, m_i^* m_j B f_j) = \sum_{ij} (B^{\frac{1}{2}} f_i, m_i^* m_j B^{\frac{1}{2}} f_j)$$

$$= \| \sum_i m_i B^{\frac{1}{2}} f_i \|^2 \geq 0$$

This shows $\hat{B}$ is positive. If now $\hat{B}$ is selfadjoint then follows the lower and upper bounds of $\hat{B}$ are the same as those of B. Hence $\|\hat{B}\| = \|B\|$. But this implies that 1) holds for arbitrary elements.

The second statement follows from the usual continuity since the elements

$$(m f, m \pi(x) f) \quad ; \quad m \in \mathcal{M}, \; f \in \mathcal{D} \qquad \text{belong}$$

to the graph of $\hat{\pi}(x)$.

Having an induced extension $(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$ then ς maps $\hat{\mathcal{M}}$ into the weak commutant of $(\pi, \mathcal{D})$. But we can hope to reconstruct $(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$ only if ς is a bijective mapping. Knowing $(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$ then we eventually want to construct an extension of this representation $(\hat{\hat{\pi}}, \hat{\hat{\mathcal{N}}}, \hat{\hat{\mathcal{D}}})$. This can be done hopefully by using operators which weakly commute with $\hat{\pi}$ and $\hat{\mathcal{M}}$. Therefore ς should also be unique on such elements. This leads us to the

II. 6. Definition

Let $(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$ be an induced extension of $(\pi, \mathcal{D})$ then

1) we define: $(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})'_w = (\hat{\pi}, \hat{\mathcal{D}})'_w \cap \hat{\mathcal{M}}'$

2) we say $(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$ is regular if the mapping ς restricted to $(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})'_w$ is injective.

II. 7. Lemma

1) Let $(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$ be a regular induced extension of $(\pi, \mathcal{D})$ and $(\hat{\hat{\pi}} \vee \hat{\hat{\mathcal{M}}}, \hat{\hat{\mathcal{L}}}, \hat{\hat{\mathcal{D}}})$ be a regular induced extension of $(\hat{\pi} \vee \hat{\mathcal{M}}, \hat{\mathcal{D}})$ then $(\hat{\hat{\pi}}, \hat{\hat{\mathcal{M}}} \vee \hat{\hat{\mathcal{L}}}, \hat{\hat{\mathcal{D}}})$ is a regular induced extension of $(\pi, \mathcal{D})$.

2) Every increasing family $(\pi^\lambda, \mathcal{M}^\lambda, \mathcal{D}^\lambda)$ of regular induced extensions is majorized by a regular induced extension $(\pi, \mathcal{M}, \mathcal{D})$.

Proof: The first statement follows easily from the fact that the product of two injective mappings is again injective.

For the second statement define $\mathcal{D} = \bigcup_\alpha \mathcal{D}^\alpha$ and $\pi(x) f = \pi^\alpha(x) f$ if $f \in \mathcal{D}^\alpha$ and $\mathcal{M} = \bigcup_\alpha \mathcal{M}^\alpha$ where $\mathcal{M}^\alpha$ is naturally imbedded in $\mathcal{M}^\beta$ for $\beta > \alpha$. This means $b \in \mathcal{M}^\alpha$ is defined on $\mathcal{D}$ and is bounded by lemma II. 5. Let $b \in (\pi, \mathcal{M}, \mathcal{D})'_w$, then $\rho(b) = 0$ implies by $\rho(b) = \rho^\alpha(\rho_\alpha(b))$, where ρ^α is the map from $\mathcal{B}(\mathcal{H}(\mathcal{D}^\alpha)) \to \mathcal{B}(\mathcal{H}(\mathcal{D}^0))$ and ρ_α the map from $\mathcal{B}(\mathcal{H}(\mathcal{D})) \to \mathcal{B}(\mathcal{H}(\mathcal{D}^\alpha))$, that $\rho_\alpha(b) = 0$ or $E_\alpha b \restriction \mathcal{H}(\mathcal{D}^\alpha) = 0$. Since $\bigcup_\alpha \mathcal{H}(\mathcal{D}^\alpha)$ is dense in $\mathcal{H}(\mathcal{D})$ it follows that b = 0 which implies the second statement.

This shows that the concept of regular induced extensions is very reasonable. If we can associate with it a set then we can use Zorn's Lemma. Therefore two problems remain, namely to associate a set with it and also to construct them. To answer these questions we assume that we have given a regular induced extension.

II. 8. Lemma

1) Let $(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$ be an induced extension of $(\pi, \mathcal{D})$ such that $\rho \restriction \hat{\mathcal{M}}$ is injective.

Let
$$\mathcal{M} = \rho(\hat{\mathcal{M}}) \subset (\pi, \mathcal{D})'_w$$
$$\mathcal{K} = \rho(\hat{\mathcal{M}}^+)$$
and define
$$\phi : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$$
by
$$\phi(m_1, m_2) = \rho(\rho^{-1}(m_1) \cdot \rho^{-1}(m_2))$$
then

(i) $\mathcal{M}$ is a selfadjoint subset of $(\pi, \mathcal{D})'_w$

(ii) $\mathcal{K}$ is a convex cone with

(α) $\mathcal{K}$ generates $\mathcal{M}$

(β) $1 \in \mathcal{K}$

(γ) if $m \in \mathcal{K}$ then exist $0 \leq \lambda(m) < \infty$ such that
$$\lambda(m) \cdot 1 - m \in \mathcal{K}$$

(iii) ϕ has the properties:

a) ϕ is an abelian product on $\mathcal{M}$ i.e.
$$\phi(m_1, m_2) = \phi(m_2, m_1)$$
$$\phi(m_1, \phi(m_2, m_3)) = \phi(\phi(m_1, m_2), m_3) := \phi(m_1, m_2, m_3)$$
$$\phi(1, m) = m \; ; \; \phi(m_1, m_2 + \lambda m_3) = \phi(m_1, m_2) + \lambda \phi(m_1, m_3)$$
$$\phi(m_1, m_2)^* = \phi(m_2^*, m_1^*).$$

b) ϕ is positive i.e. for $p \in \mathcal{K}$, $m_i \in \mathcal{M}$ and $f_i \in \mathcal{D}$ we have
$$\sum_{ij} (f_i, \phi(m_i^*, p, m_j) f_j) \geq 0$$

2.) Let $\mathcal{M}$, $\mathcal{K}$, ϕ satisfy (i), (ii) and (iii) then there exists an induced extension such that

a) $\rho \restriction \hat{\mathcal{M}}$ is injective

b) $\mathcal{M} = \rho(\hat{\mathcal{M}})$

c) $\mathcal{K} \subset \rho(\hat{\mathcal{M}}^+)$

d) $\phi(m_1, m_2) = \rho(\rho^{-1}(m_1) \cdot \rho^{-1}(m_2))$.

e) This extension is unique up to unitary equivalence.

3.) Let $(\hat{\pi}^i, \hat{\mathcal{M}}^i, \hat{\mathcal{D}}^i)$ $i = 1, 2$ be two induced extensions such that $\rho^i \restriction \hat{\mathcal{M}}^i$ is injective then $(\hat{\pi}^1, \hat{\mathcal{M}}^1, \hat{\mathcal{D}}^1) < (\hat{\pi}^2, \hat{\mathcal{M}}^2, \hat{\mathcal{D}}^2)$ (after some unitary transformation) if and only if $\mathcal{M}^1 \subset \mathcal{M}^2$ and $\phi^1 = \phi^2 \restriction \mathcal{M}^1 \times \mathcal{M}^1$.

Proof:

The first part is a simple application of Lemma II. 3. The existence of an induced extension is just as easy if we remark that by (iii) b) we have a scalar product on $\mathcal{M} \times \mathcal{D}$. If $\mathcal{N}$ is the null space under this scalar product then $\hat{\mathcal{D}} = \mathcal{M} \times \mathcal{D} / \mathcal{N}$. The rest ist only simple computation.

If we have two different induced extensions such that $\rho^1 \restriction \hat{\mathcal{M}}^1$ and $\rho^2 \restriction \hat{\mathcal{M}}^2$ are injective and such that ρ^i fulfill all the required properties then a simple computation shows that $U \sum_i \hat{m}_i^1 f_i = \sum_i \rho_2^{-1} \rho_1 \hat{m}_i^1 f_i$

defines a unitary operator mapping $\mathcal{H}(\hat{\mathcal{D}}^1)$ onto $\mathcal{H}(\hat{\mathcal{D}}^2)$ which also has all other required properties. The verification of the last part is then straightforward.

This last lemma shows that the unitary equivalence classes of regular induced extensions form a semi-ordered set characterized by subsets $\mathcal{M}$, $\mathcal{K}$ and functions ϕ from $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$. Therefore we get:

II. 9. Corollary: Let $(\pi, \mathcal{D})$ be a representation of a partial $*$-algebra, then there exist maximal regular induced extensions.

It remains only to give the explicit construction of a regular induced extension.

II. 10. Lemma: Let $x \in (\mathbb{T}, \mathfrak{D})'_w$ with $0 \leq x \leq 1$ and $x \neq \lambda 1$

Let $\mathcal{M} = \{\lambda x + \mu(1-x) ; \lambda, \mu \in \mathbb{C}\}$, $\mathcal{K} = \{\lambda x + \mu(1-x) ; \lambda, \mu \geq 0\}$

and $\phi(\lambda_1 x + \mu_1(1-x), \lambda_2 x + \mu_2(1-x)) = \lambda_1\lambda_2 x + \mu_1\mu_2(1-x)$

then the conditions of Lemma II. 8. are satisfied. $\hat{\mathcal{M}}$ is generated by 1 and a projector e with $\varrho(e) = x$.

II. 11. Lemma: Let x be as in the previous lemma, then the extension defined by x is regular, if and only if x is extremal in the weakly compact set

$((\mathbb{T}, \mathfrak{D})'_w)^+_1 = \{x \in (\mathbb{T}, \mathfrak{D})'_w ; 0 \leq x \leq 1\}$.

Proof: Let $0 \leq x \leq 1$ and x an extreme point of $((\mathbb{T}, \mathfrak{D})'_w)^+_1$, then extremality is equivalent to the following: The equations $0 \leq x + y \leq 1$ and $0 \leq x - y \leq 1$ implies y = o, or equivalently x is extremal if and only if $y \in (\mathbb{T}, \mathfrak{D})'_w$ and $-x \leq y \leq x$ and $-(1-x) \leq y \leq (1-x)$ implies y = o.

Let now $w \in (\hat{\mathbb{T}}, \{e, 1\} \hat{\mathfrak{D}})'_w$ with $w = w^*$ and $\|w\| = 1$ and $\varrho(w) = 0$ for some w or $\varrho(ew) + \varrho((1-e)w) = 0$.

Now $-e \leq ew \leq e$ and $-(1-e) \leq (1-e)w \leq 1-e$

implies $-x \leq \varrho(ew) \leq x$ and $-(1-x) \leq \varrho((1-e)w) \leq 1-x$.

From $\varrho(ew) = -\varrho((1-e)w)$ and extremality of x follows $\varrho(ew) = \varrho((1-e)w) = 0$. Since the extension $(\hat{\mathbb{T}}, \hat{\mathfrak{D}})$ is induced by e and (1-e) we have $\hat{\mathfrak{D}} = e\mathfrak{D} + (1-e)\mathfrak{D}$. Hence we get :

$(ef_1 + (1-e)g_1, w(ef_2 + (1-e)g_2)) = (f_1, ewf_2) + (g_1, (1-e)wg_2)$

$= (f_1, \varrho(ew)f_2) + (g_1, \varrho((1-e)w)g_2) = 0$. This shows $w = 0$.

Since ϱ commutes with the involution it follows from this that $\varrho \restriction (\hat{\mathbb{T}}, \{e, 1\}, \hat{\mathfrak{D}})'_w$ is injective.

From this we get:

II. 12. Theorem:

1) Every regular induced extension of $(\mathbb{T}, \mathfrak{D})$ is majorized by a maximal one.

2) A regular induced extension of $(\mathbb{T}, \mathfrak{D})$ is maximal if and only if

$(\hat{\mathbb{T}}, \hat{\mathcal{M}}, \hat{\mathfrak{D}})'_w = (\hat{\mathbb{T}}, \hat{\mathcal{M}}, \hat{\mathfrak{D}})'_s = \hat{\mathcal{M}}$.

3) To every extremal $x \in \{(\mathfrak{T},\mathfrak{D})'_w\}_1^+$ there exists a maximal induced extension $(\hat{\mathfrak{T}}, \hat{\mathcal{M}}, \hat{\mathfrak{D}})$ and a projection $e \in \hat{\mathcal{M}}$ such that $x = \varrho(e)$.

4) Assume that the Hilbert-space $\mathcal{H}(\mathfrak{D})$ is separable, then $\mathcal{H}(\hat{\mathfrak{D}})$ is also separable.

Proof: Statements 1, 2 and 3 are collections of the previous results. So only 4 needs some consideration. The unit ball $\hat{\mathcal{M}}_1$ is a weak compact set and is mapped by ϱ into the weak compact set $\{(\mathfrak{T},\mathfrak{D})'_w\}_1$. Since ϱ is continuous and injective it follows that ϱ^{-1} is also continuous. Since $\mathcal{H}(\mathfrak{D})$ is separable, it follows that the weak topology on $\{(\mathfrak{T},\mathfrak{D})'_w\}_1$ is countable and hence also the weak topology of $\hat{\mathcal{M}}_1$. Since $\hat{\mathcal{M}}_1 \cdot \mathcal{H}(\mathfrak{D})$ is total in $\mathcal{H}(\hat{\mathfrak{D}})$ it follows that $\mathcal{H}(\hat{\mathfrak{D}})$ is separable.

III. Extensions with positivity conditions, abelian algebras

In many applications we want that certain elements of the partial algebra A are represented by positve operators. In order to handle also this situation, we introduce the following notations:

III. 1. Definition:

1) Let A be a partial $*$-algebra and A_h its hermitian part. A cone $P \subset A_h$ is called a regular cone if
   (i) $P \ni 1$
   (ii) if $x \in P$ and $y \in A$ and if $y^* x y$ is defined then follows $y^* x y \in P$
   (iii) $P \cap -P = o$
2) Let P be a regular cone in A. A representation $(\pi, \mathcal{D})$ is called P-positive if $f \in \mathcal{D}$ and $x \in P$ implies $(f, \pi(x) f) \geq 0$.
3) Let $(\pi, \mathcal{D})$ be a P-positive representation of A. An extension $(\hat{\pi}, \hat{\mathcal{D}})$ is called a P-positive extension if $(\hat{\pi}, \hat{\mathcal{D}})$ is an extension of $(\pi, \mathcal{D})$ and $\hat{\pi}(x)$ is a positive operator for every $x \in P$.

If we have a P-positive representation of A then we can treat the extension theory in almost the same way as in the last section. The only change consists in introducing a different order amongst operators in the weak commutant.

III. 2. Definition:

1) Let $(\pi, \mathcal{D})$ be a P-positive representation and $x \in (\pi, \mathcal{D})'_w$ then we define $x \gg o$ if $(f, \pi(p) x f) \geq 0$ for all $f \in \mathcal{D}$ and all $p \in P$.
2) By $C_1^+(\pi, \mathcal{D}, P)$ we denote all $x \in (\pi, \mathcal{D})'_w$ with $o \ll x \ll 1$ and by $C(\pi, \mathcal{D}, P)$ the linear span of $C_1^+(\pi, \mathcal{D}, P)$.
3) Let $(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$ an induced P-positive extension then we define

$$C_1^+(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}}, P) = C_1^+(\hat{\pi}, \hat{\mathcal{M}}, P) \cap \hat{\mathcal{M}}'.$$

Replacing now the set $\{(\pi,\mathcal{D})'_w\}_1^+$ resp. $\{(\hat{\pi},\hat{\mathcal{U}},\hat{\mathcal{D}})'_w\}_1^+$ by the convex weakly compact sets $C_1^+(\pi,\mathcal{D},P)$ resp. $C_1^+(\hat{\pi},\hat{\mathcal{U}},\hat{\mathcal{D}},P)$ we can now proceed as in the last section. The outcome is the following

III. 3. Theorem:

1) Every P-positive representation $(\pi,\mathcal{D})$ of A is majorized by a maximal regular induced P-positive extension .

2) A regular induced P-positive extension $(\hat{\pi},\hat{\mathcal{U}},\hat{\mathcal{D}})$ is maximal if and only if

$$C_1^+(\hat{\pi},\hat{\mathcal{U}},\hat{\mathcal{D}},P) = \hat{\mathcal{U}}_1^+ .$$

The main field of application of this last theorem is the case of abelian algebras. If one has given a representation $(\pi,\mathcal{D})$ of an abelian $*$-algebra A one usually wants to know whether one can find an extension $(\hat{\pi},\hat{\mathcal{D}})$ such that all symmetric operators are essentially selfadjoint on $\hat{\mathcal{D}}$ and that there spectral projections commute. In order to formulate the results we need some notations

III. 4. Definition:

1) Let A be an abelian $*$-algebra, a representation $(\pi,\mathcal{D})$ is called standard (Powers [1]) if
   (i) If $x \in A_h$, then $\pi(x)$ is essentially selfadjoint on $\mathcal{D}$.
   (ii) for $x, y \in A_h$ the spectral projections of $\pi(x)$ and $\pi(y)$ commute.

2) Let $V \subset A_h$ be a linear subspace, then we say V generates A if for $x \in A$ exists a finite number of elements $v_1 \dots v_n \in V$ and a polynomial P such that $x = P(v_1, v_2, \dots v_n)$.

3) Denote by V^* the algebraic dual of V and let $Z \subset V^*$ be a subset then we denote by

$$P(Z) = \{ P(v_1, \cdots v_n) \; ; \; P(\omega(v_1), \omega(v_2) \cdots \omega(v_3)) \geq 0 \text{ for all } \omega \in Z \} .$$

With these notations we get the following result:

III. 5. <u>Theorem:</u> Let A be an abelian $*$-algebra generated by $V \subset A_h$ and Z a subset of V^*. Let π be a cyclic representation of A with cyclic vector Ω i.e. $\mathcal{D} = \pi(A)\Omega$. Then the following statements are equivalent:

1) $\Leftrightarrow$ 2) A) $\Leftrightarrow$ B)

1) The functional $T(x) = (\Omega, \pi(x)\Omega)$ is positive on $P(\mathbf{Z})$

2) The representation $(\pi, \pi(A)\Omega)$ is $P(Z)$ positive and has a maximal regular induced $P(Z)$-positive extension .

A) $(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$ is a maximal regular induced $P(\mathbf{Z})$-positive extension of $(\pi, \pi(A)\Omega)$.

B) $(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$ is a regular induced extension such that

(a) $\hat{\mathcal{M}}\Omega$ is dense in $\mathcal{H}(\hat{\mathcal{D}})$ and $\hat{\mathcal{M}}$ is a maximal abelian algebra .

(b) $\hat{\pi}$ is standard

(c) The joint spectrum of $\hat{\pi}(v_1), \dots \hat{\pi}(v_n), v_i \in V$ belongs to the closure of the set
$$\{\omega(v_1), \omega(v_2), \dots \omega(v_n) \; ; \; \omega \in Z\}$$
(which is a subset of $\mathbb{R}^n$).

<u>Proof:</u> 1 $\Leftrightarrow$ 2 Since $P(Z)$ is a regular cone it follows with $x \in P$ and $y \in A$ that also $y^* x y \in P$. Hence if T is positive on $P(\mathbf{Z})$ then follows $(\pi, \pi(A)\Omega)$ is a $P(\mathbf{Z})$-positive representation. The second part of 2) is then just the last theorem. The converse conclusion is trivial.

B $\Rightarrow$ A . Since $\hat{\pi}$ is standard it follows from c) and the spectral theory that $(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$ is $P(Z)$ positive. Since $\hat{\mathcal{M}}$ is maximal abelian it follows that $C_*^+(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}}, P(Z)) = \hat{\mathcal{M}}_*^+$ which shows that $(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$ is a maximal regular induced $P(\mathbf{Z})$ positive extension.

A $\Rightarrow$ B Let $Q(v_1, \dots v_n)$ be a real polynomial. Let S be a finite dimensional linear subspace of $V + \hat{\mathcal{M}}_h$ such that $v_1, \dots v_n \in S$. Denote by A_S the algebra generated by S and $\mathcal{H}_S = \overline{\hat{\pi}(A_S)\Omega}$. $T(x) = (\Omega, \hat{\pi}(x)\Omega)$ restricted to A_S defines a finite dimensional moment problem. Since the moments are $P_S = P(Z) \cap A_S$ positive, there exists a P positive solution defined on a Hilbert-space $\tilde{\mathcal{H}}_S \supset \mathcal{H}_S$. Denote by E_S the projection onto $\mathcal{H}_S$ and $\tilde{Q}_S$ be the representative of P in $\tilde{\mathcal{H}}_S$ which is selfadjoint.

$(\tilde{\mathbb{Q}}_S^2+1)^{-1}$ is a bounded operator such that $C = E_S\,(\tilde{\mathbb{Q}}_S^2+1)^{-1}\,E_S$ is an element of $C_1^+\,(A_S, \mathcal{D}_S, P_S)$ and $C\,(\tilde{\mathbb{Q}}_S^2+1)\,\Omega = \Omega$

For every S denote by $\mathcal{L}_S$ the set of bounded operators on $\mathcal{H}(\hat{\mathcal{D}})$ such that

(i) $C\,(\tilde{\mathbb{Q}}_S^2+1)\,\Omega = \Omega$

(ii) $\|C\| \leq 1$

(iii) $C\,\mathcal{H}_S \subset \mathcal{H}_S$ and

(iv) $C \restriction \mathcal{H}_S \in C_1^+\,(\hat{\pi}(A_S), \hat{\pi}(A_S)\Omega, P_S)$

By the above construction follows $\mathcal{L}_S \neq \emptyset$ and $\mathcal{L}_S$ is convex and weakly closed. For $S_1 \subset S_2$ follows $\mathcal{L}_{S_2} \subset \mathcal{L}_{S_1}$. This shows that the $\mathcal{L}_S$ have the finite intersection property. Since the unit ball is weakly compact follows $\bigcap\limits_S \mathcal{L}_S \neq \emptyset$

If C belongs to this intersection then follows $C \in C_1^+\,(\hat{\pi}(A) \vee \hat{\mathcal{M}}, \hat{\mathcal{M}}, \hat{\mathcal{D}}, P(Z))$ and hence by maximality follows $C \in \hat{\mathcal{M}}_1^+$. From $C\,\hat{\pi}\,(\mathbb{Q}^2+1)\,\Omega = \Omega$ follows $C = \hat{\pi}\,(\mathbb{Q}^2+1)^{-1}$ showing that $\hat{\pi}\,(\mathbb{Q}^2+1)$ is essentially self-adjoint on $\hat{\mathcal{D}}$ and is affiliated to $\hat{\mathcal{M}}$. Since now $\{\hat{\pi}\,(\mathbb{Q}^2+1)\,\Omega\,,\ \hat{\mathcal{M}}\,\Omega\}$ span all of $\hat{\mathcal{D}}$ it follows that $\hat{\mathcal{M}}$ is maximal abelian. Hence $\hat{\pi}$ is standard.

Remark:

It is also possible to treat the noncyclic case. To do this one has to extend the notation of positivity. If this is done, then the result is similar to the last theorem.

IV. Integral decomposition of states

In the following let us assume, that A is a $*$-algebra. If we have a representation $(\pi, \mathcal{D})$ of A then we have seen, that we can construct maximal regular induced extensions $(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$. Moreover, if $\mathcal{H}(\mathcal{D})$ was a separable Hilbert space then the same is true for $\mathcal{H}(\hat{\mathcal{D}})$. In this case we can make an integral decomposition of $\mathcal{H}(\hat{\mathcal{D}})$ with respect to $\hat{\mathcal{M}}$. This means there exists a locally compact space Λ and a finite positive Borel measure μ on it such that

$$\mathcal{H}(\hat{\mathcal{D}}) = \int_{\Lambda}^{\oplus} \mathcal{H}_\lambda \, d\mu(\lambda)$$

and such that $\hat{\mathcal{M}}$ consists of all bounded diagonal operators with respect to this decomposition.

Now it is natural to ask whether one can decompose also the representation $\hat{\pi}(A)$ Since $\hat{\pi}(x)$ is generally an unbounded operator, such a decomposition is not always possible. The only case which I know of which yields integral decompositions beyond von Neumans theory is the nuclear spectral theorem [2,3]. Therefore we will make the following

IV. 1. Assumptions

1) $\mathcal{D}$ is a nuclear vector space and the imbedding $\mathcal{D} \to \mathcal{H}(\mathcal{D})$ is continuous
2) A is a separable topological space, and π is a continuous representation
3) $\pi(A)\mathcal{D} \subset \mathcal{D}$ and the map $A \times \mathcal{D} \to \mathcal{D}$,

   $$(x, f) \to \pi(x) f$$

   is separately continuous as a map from $A \times \mathcal{D} \to \mathcal{H}(\mathcal{D})$

Under these conditions we get the following result:

IV. 2. Theorem: Assume the above assumptions and the integral decomposition of $\mathcal{H}(\hat{\mathcal{D}})$ with respect to $\hat{\mathcal{M}}$, then we get for almost all λ:

1) There exists a linear mapping $E_\lambda : \mathcal{D} \to \mathcal{H}_\lambda$ such that
   a) $\mathcal{D}_\lambda = E_\lambda \mathcal{D}$ is a nuclear space in the final topology continuously imbedded in $\mathcal{H}_\lambda$ and dense in $\mathcal{H}_\lambda$
   b) For all $f \in \mathcal{D}$; $\lambda \to E_\lambda f$ is a measurable field .
2) There exists a linear mapping $\vec{\pi}(x) \to \pi_\lambda(x)$ into the linear operators on $\mathcal{D}_\lambda$ such that :
   a) $E_\lambda \vec{\pi}(x) f = \pi_\lambda(x) E_\lambda f$
   b) $x \to \pi_\lambda(x)$ is a $*$-homomorphism and $\pi_\lambda(x)$ is a continuous representation when equipped with the final topology
   c) for $x \in A$ and $f \in \mathcal{D}$ the map $(x, f_\lambda) \to \pi_\lambda(x) f_\lambda$ is separately continuous .
3) If $g \in \hat{\mathcal{D}}$ and $\lambda \to g_\lambda$ is any measurable field representing g then
   a) $g_\lambda \in \mathcal{D}_\lambda$
   b) $\lambda \to \pi_\lambda(x) g_\lambda$ is a measurable field and
   $$\vec{\pi}(x) g = \int_\Lambda^\oplus \pi_\lambda(x) g_\lambda \, d\mu(\lambda) .$$
4) If $(\pi , \mathcal{D})$ is a cyclic representation i.e. $\mathcal{D} = \pi(A)\Omega$ then $\mathcal{D}_\lambda = \pi_\lambda(A)\Omega_\lambda$ with $\Omega_\lambda = E_\lambda \Omega$
5) $(\pi_\lambda , \mathcal{D}_\lambda)'_w = \mathbb{C} \cdot 1$
6) If $(\pi , \mathcal{D})$ is cyclic and $\pi(x)\Omega = 0$ then follows $\pi_\lambda(x)\Omega_\lambda = 0$

Proof:

1) The existence of $E_\lambda : \mathcal{D} \to \mathcal{H}_\lambda$ such that $f = \int E_\lambda f \, d\mu(\lambda)$ is the nuclear spectral theorem. Since E_λ is continuous follows $\mathcal{D}_\lambda \cong \mathcal{D}/_{\ker E_\lambda}$ is a nuclear space. The density of $\mathcal{D}_\lambda$ in $\mathcal{H}_\lambda$ will follow from 3 a).
2) Let $f, g \in \mathcal{D}$, $x \in A$ and
   $$T_\lambda(f,g,x) = (E_\lambda f, E_\lambda \pi(x) g) - (E_\lambda \pi(x^*) f, E_\lambda g)$$
   then follows for every bounded μ- measurable function $m(\lambda)$
   $$\int m(\lambda) T_\lambda(f,g,x) \, d\mu = (f, \hat{m}\, \vec{\pi}(x) g) - (\vec{\pi}(x^*) f, \hat{m} g) = 0$$
   and hence
   $$T_\lambda(f,g,x) = 0 \qquad a.e.$$

Define $\pi_\lambda(x)\, E_\lambda g \;:=\; E_\lambda\, \pi(x)\, g$,
then $\pi_\lambda(x)$ defines a $*$-representation a.e. since $T_\lambda = 0$ and A is separable. The rest follows again since Ker E_λ is closed.

3) If $f \in \hat{\mathcal{D}}$ then $f = \sum \hat{m}_i g_i$ with $\hat{m}_i \in \hat{\mathcal{M}}$ and $g_i \in \mathcal{D}$
Since $\hat{\mathcal{M}}$ is diagonalisable follows $\hat{m}_i \longrightarrow (\lambda \to m_i(\lambda) 1_\lambda)$
and hence f is represented by

$$\lambda \to \sum m_i(\lambda)\, E_\lambda g_i$$

which is measurable. Since $\hat{\mathcal{D}}$ is dense in $\mathcal{H}(\hat{\mathcal{D}})$ follows $E_\lambda \mathcal{D}$ is dense in $\mathcal{H}_\lambda$.

Now $\hat{\pi}(x) f = \hat{\pi}(x) \sum \hat{m}_i g_i = \sum \hat{m}_i\, \pi(x)\, g_i$

$$= \int \sum m_i(\lambda)\, \pi_\lambda(x)\, E_\lambda g_i \, d\mu(\lambda)$$

$$= \int \pi_\lambda(x) \sum m_i(\lambda)\, E_\lambda g_i \, d\mu(\lambda) .$$

4) follows from 2 and so does 6.

5) The proof of this is exactly as in the case of von Neumann algebras which can be carried over since only matrix elements are needed in the proof.

Now we shall apply the last result to the integral decomposition of states. We will assume that A is a nuclear separable $*$-algebra. The cases of physical interest are test function algebras over either $\mathcal{S}(\mathbb{R}^4)$ or $\mathcal{D}(\mathbb{R}^4)$. In both cases the algebra is separable. We will also assume that $1 \in$ A and that the product is separately continuous. Let now ω be a state on A, i.e. a continuous positive linear functional on A such that $\omega(1) = 1$.

By the G.N.S. construction we get a representation $(\pi_\omega , \pi_\omega \Omega)$. In this case $\mathcal{D} = \pi_\omega \Omega$ is automatically a separable nuclear space which is continuously imbedded in $\mathcal{H}(\mathcal{D})$.

Furtheron we will assume that the map $A \times \mathcal{D} \to \mathcal{H}(\mathcal{D})$ defined by $\pi(x) f$ is separately continuous. Since π is generally only weakly continuous, it is a separate assumption. But in the case where A is a barrelled topological space (as in the cases of interest) then $x \to \| \pi(x) f \|$ is automatically continuous (see e.g. [4]). With these assumptions we get:

IV. 3. Theorem:

Let A be a nuclear $*$ algebra with separately continuous product, and ω a state on A. Assume that for every $y \in A$, $x \to \omega(y^* x^* x y)^{\frac{1}{2}}$ is continuous.

Then exists a locally compact space Λ and a positive normalized Borel measure $\mu(\lambda)$ on Λ and states ω_λ on A such that

1) $\omega = \int_\Lambda \omega_\lambda \, d\mu(\lambda)$ is a weak integral decomposition and μ-almost everywhere
2) ω_λ is an extremal state
3) $x \to \omega_\lambda(y^* x^* x y)^{1/2}$ is continuous
4) If $L_\omega = \{ x \in A \, ; \, \omega(x^* x) = 0 \}$ then $L_\omega \subset L_{\omega_\lambda}$
5) $\pi_\omega(x)\Omega = \int_\Lambda \pi_{\omega_\lambda}(x)\Omega_\lambda \, d\mu(\lambda)$

   where π_ω and π_{ω_λ} fulfill the conditions of Theorem IV. 2.

Proof:

Let $(\pi_\omega, \mathcal{D}_\omega)$ be the G.N.S. representation of ω then $\mathcal{D}_\omega \cong A/L_\omega$ and hence a separable nuclear space.

Then we construct some maximal regular induced extension $(\hat{\pi}_\omega, \hat{\mathcal{H}}, \hat{\mathcal{D}})$ and apply Theorem IV. 2. to it. Define $\omega_\lambda(x) = (E_\lambda\Omega, E_\lambda \pi_\omega(x)\Omega) / (E_\lambda\Omega, E_\lambda\Omega)$ which gives the desired result.

V. The moment problem over nuclear spaces

We now want to apply our results to abelian $*$ algebras. If A is an abelian $*$-algebra of bounded operators then every extremal state is a character, it is a positive linear functional ω fulfilling

$$\omega(xy) = \omega(x)\,\omega(y)$$

But it is well known that on an algebra of unbounded operators not every extremal state is a character (non-solvability of the moment problem in more than one operator). Therefore the question arises to characterize those states which can be decomposed into characters.

V. 1. Assumptions:

1) A is an abelian nuclear $*$-algebra $1 \in A$ with a separetaly continuous product.
2) V is a linear subspace of A_h such that A(V) the algebra generated by $V \cup 1$ is dense in A.

   V is a real nuclear space in the topology induced by A.
3) V' denotes the (real) dual space of V and $Z \subset V'$ a subset

   P(Z) is the cone of polynomial $P(v_1 \ldots v_n)$ such that $P(\varphi(v_1), \varphi(v_2), \cdots \varphi(v_n)) \geq 0$ for all $\varphi \in Z$.

With these notations we get:

V. 2. Theorem: Let A, V, Z as above and let ω be a state then the following statements are equivalent

1) ω is positive on P(Z) and $x \to \omega(x^* x)^{1/2}$ is continuous
2) ω has a weak integral decomposition

$$\omega = \int_\Lambda \omega_\lambda \, d\mu(\lambda)$$

   where Λ is a locally compact space and μ is a positive normalized measure and the following holds for almost all λ :

   a) ω_λ is a character on A

   b) $\omega_\lambda \restriction V$ belongs to the weak closure of Z .

c) $L_{\omega_\lambda} \supset L_\omega$

d) There exists a continuous seminorm p on A and a function $C \in L_2(\Lambda, \mu)$ with $C \geq 0$ and $|\omega_\lambda(x)| \leq C(\lambda)\, p(x)$.

Proof:

The implication 2 ⇒ 1 is straightforward. The other direction we get by combining the results of the sections III and IV.
In the physical applications there appear special cases of algebras, namely symmetric tensor algebras over nuclear spaces. Since there are generally more than one way of defining a tensor algebra we want to be sufficiently specific.

V. 3. Assumptions:

1) Let V be a real linear nuclear vector space and V' its dual.
   Assume V is the strict inductive limit of a countable number of its subspaces V_k

$$V = \varinjlim V_k$$

2) We define the n th tensorial power of V as

$$V^{\hat{\otimes} n} = \varinjlim V_k \hat{\otimes} V_k \hat{\otimes} \cdots \hat{\otimes} V_k$$

   where $\hat{\otimes}$ denotes the completed π tensor product and

$$\underline{V} = \hat{\bigoplus_{n=0}^{\infty}} \left(V^{\hat{\otimes} n} + i\, V^{\hat{\otimes} n} \right)$$

   equipped with the direct sum topology.

3) S(V) the symmetric $\mathbb{C}$-tensor algebra which is derived from $\underline{V}$ in the standard way.

V. 4. Lemma:

1) If ω is a continuous positive linear functional on S(V) then $x \to \omega(x^* x)^{\frac{1}{2}}$ is continuous.

2) Let $Z \subset V'$, then the continuous characters on S(V) which are positive on P(Z) are in one to one correspondence with the elements of the weak closure Z

via the formula

$$\omega(P(v_1, \dots v_n)) = P(t(v_1), \dots t(v_n))$$

$v_i \in V$ and $\quad t = \omega \upharpoonright V \in \bar{Z}$.

The proof of these statements are fairly simple so that we can proceed in our investigation.

V. 5. Theorem: Let T be a linear functional on S(V) then the following concitions are equivalent.

1) T is continuous and positive on P(Z) with T(1) = 1

2) T has a weak integral decomposition

$$T(P(v_1, \dots v_n)) = \int_\Lambda P(\omega_\lambda(v_1), \dots, \omega_\lambda(v_n))\, d\mu(\lambda)$$

where (Λ, μ) is a standard measure space with $\mu \geq 0$ and $\int_\Lambda d\mu = 1$ and

a) $\omega_\lambda \in \bar{Z}$

b) $\lambda \to \omega_\lambda(v)$ is μ measurable for all $v \in V$ and there exists a function $C(\lambda) \geq 0$, $C(\lambda) \in L_2(\Lambda, \mu)$ and continuous seminorms P_n on V with

$$|\omega_\lambda(v)| \leq C(\lambda)^{1/n}\, P_n(v) \qquad n = 1, 2, \dots$$

3) $$T(P(v_1, v_2, \dots v_n)) = \int_{\bar{Z}} P(\omega(v_1), \omega(v_2), \dots \omega(v_n))\, d\nu_\omega$$

where ν is a measure on the σ-algebra generated by the weakly closed sets in V' and having the following property:

For any polynomially bounded continuous function f on $\mathbf{R}^n$ the integral

$$\int_{\bar{Z}} f(\omega(v_1), \dots \omega(v_n))\, d\nu_\omega$$

exists and is jointly continuous in $v_1 \dots v_2 \in V$.

Proof:

1) is implied by 2) or 3) in an obvious fashion. 1) implies 2) is theorem V. 2. except for the estimate. But we have from theorem V. 2.

$$|T_\lambda(v_1 \otimes v_2 \dots \otimes v_n)| = |\omega_\lambda(v_1) \cdot \omega_\lambda(v_2) \cdots \omega_\lambda(v_n)| \leq C(\lambda)\, P(v_1 \otimes \dots \otimes v_n)$$

where p is a continuous seminorm on S(V). But there exists a continuous seminorm q_n on V such that

$$p \restriction V^{\otimes n} < q_n^{\otimes n} \qquad \text{and therefore} \quad |\omega_\lambda(v)| < C(\lambda)^{\frac{1}{n}} \; q_n(v).$$

2) $\Rightarrow$ 3) By Lemma V. 4. we have a one-valued map F from Λ into V'. Define a set $M \subset V'$ be measurable if $F^{-1}(M)$ is measurable in Λ and $\nu(M) = \mu(F^{-1}(M))$. Measu rable functions can be transported in an analogue manner. It remains to show that all weakly closed sets of V' are measurable. Let $K \subset V'$ be weakly closed then $\omega_\lambda \in K$ if and only if the character χ_λ defined by ω_λ is positive on the cone P(K). Using Lemma V. 4. we see that all these characters are continuous with respect to a fixed seminorm p on S(V) and therefore P(K) is separable. If $P \in P(K)$ then the set $\{\lambda ; \chi_\lambda(P) \geq 0\}$ is μ-measurable and therefore also $\{\lambda ; \omega_\lambda \in K\}$ is measurable as countable intersection of measurable sets.

The continuity property remains then a consequence of a simple estimate.

VI. Application to quantum field theory

A Wightman functional W is usually defined as a state over the testfunction algebra $\underline{\mathcal{S}}$ fulfilling the conditions (see e g [5])

(α) W is translational invariant

(β) W anihilates the two-sided locality ideal I_c

(γ) W anihilates the left spectral ideal

If $(\pi_W, \mathcal{D})$ is the cyclic representation constructed from W then due to the spectrum condition follows that also every operator b belonging to the weak commutant of this representation commutes automatically with the unitary representation of the translation group. From this follows:

VI. 1. Lemma: Let $(\hat{\pi}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$ be a maximal regular extension of $(\hat{\pi}_W, \mathcal{D})$ then the unitary group representation U(a) in $\mathcal{H}(\mathcal{D})$ of the translation group extends to a unitary group representation $\hat{\mathcal{U}}$ (a) on $\mathcal{H}(\hat{\mathcal{D}})$ such that $\hat{\mathcal{M}}$ and $\hat{\mathcal{U}}$ (a) commute.

As a consequence of this we get

VI. 2. Theorem: Every Wightman-state W can be decomposed into a weak integral over extremal Wightman states

$$W = \int_{\Lambda} W_\lambda \, d\mu(\lambda) \quad .$$

In the framework of local algebras of bounded operators it is well known that the extremality of the state and the uniqueness of the vacuum in the representation space are equivalent [6]. But, due to pathologies associated with unbounded operators it is possible to show that this is no longer true for Wightman fields. We now want to investigate under what conditions a Wightman state can be decomposed into extremal ones with a unique vacuum. We start first with some

VI. 3. Notations and Remarks

1) Let A(f) be some Wightman field ($f \in \underline{\mathcal{S}}$) defined on some domain $\mathcal{D}$ in a Hilbert space $\mathcal{H}$ and let $\mathcal{U}$ (a) be the unitary representation of the translation group which is defined with it.

Denote by P_o the projection onto the subspace invariant under $U(a)$, and define $\mathcal{H}_0 = P_0 \mathcal{H}$.

2) Assume there is a cyclic subset $\mathfrak{Z}_0 \subset \mathcal{H}_0$ such that $\mathcal{D}$ = linear span $\{A(f)\mathfrak{Z}_0\}$.
Denote by $\bar{\mathcal{D}}$ the completion of $\mathcal{D}$ in the graph topology induced by all $A(f)$.
Define $\mathcal{D}_0 = \bar{\mathcal{D}} \cap \mathcal{H}_0$ and $\mathcal{D}$ = linear span $\{A(f)\mathcal{D}_0\}$.

3) Due to the spectrum condition, the following statements are true (see [7])
   a) $P_0 \mathcal{D} \subset \mathcal{D}_0$
   b) The operators $A_o = P_o A P_o$ are well defined and generate a commutative $*$-algebra on $\mathcal{D}_0$.
   c) $(A_0, \mathcal{D}_0)$ is a representation of $S(\underline{\mathcal{S}})$ i.e. the symmetric tensor algebra over $\underline{\mathcal{S}}$.

4) Let $\underline{\mathcal{S}}^+$ be the set of positive elements in $\underline{\mathcal{S}}$ i.e. $\{\sum_i f_i^* f_i$; the sum converges in $\underline{\mathcal{S}}\}$ and denote by $P(\underline{\mathcal{S}}^+) \subset S(\underline{\mathcal{S}})$ the set
$$\{\sum_i f_i^* p_i f_i \;;\; p \in \underline{\mathcal{S}}^+ ,\; f_i \in S(\underline{\mathcal{S}})\}.$$
With this notation we have more precisely $(A_o, \mathcal{D}_o)$ is a $P(\underline{\mathcal{S}}^+)$-positive representation of $S(\underline{\mathcal{S}})$.

With these notations we get

VI.4. Theorem:

1) There is a one to one correspondence between :
   1. unitary equivalence classes of induced regular extension $(\hat{A}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$ of $(A, \mathcal{D})$ and
   2. unitary equivalence classes of regular induced $P(\underline{\mathcal{S}}^+)$-positive extensions
$$(\hat{A}_0, \hat{\mathcal{M}}_0, \hat{\mathcal{D}}_0) \quad \text{of} \quad (A_0, \mathcal{D}_0).$$

2) If $(\hat{A}, \hat{\mathcal{M}}, \hat{\mathcal{D}})$ and $(\hat{A}_0, \hat{\mathcal{M}}_0, \hat{\mathcal{D}}_0)$ are corresponding extensions then the following diagram commutes :

$$\begin{tikzcd}
\hat{\mathcal{M}} \arrow[r, "\hat{r}"] \arrow[d, "\rho"'] & \hat{\mathcal{M}}_0 \arrow[d, "\rho_0"] \\
\mathcal{M} \arrow[r, "r"] & \mathcal{M}_0
\end{tikzcd}$$

where the horizontal arrows stand for the restriction of invariant operators to the invariant subspace. It is a normal isomorphism of $*$-algebras since $\hat{\mathcal{U}}$ belongs to the strong commutant of $\hat{A}$. ς and ς_0 are the weakly continuous mappings discussed earlier.

Proof: The details of the proof are easy computations after having answered the following question. Take a positive operator $b \in (A_0, \mathcal{D}_0)'_w$ when can it be lifted to a positive operator $\tilde{b}$ in $(A, \mathcal{D})'_w$ such that $b = P_0 \tilde{b}$?

Assume $\tilde{b} \in (A, \mathcal{D})'_w$, $1 \geq \tilde{b} \geq 0$ and $f \in \mathcal{D}_0$,

then we have

$$0 \leq (A(x)f, \tilde{b}\, A(x)f) = (A(x^*x)f, \tilde{b} f) = (A(x^*x)f, P_0 \tilde{b} f)$$

$$= (A_0(x^*x)f, P_0 \tilde{b} f)$$

This shows $P_0 \tilde{b} \in \mathcal{C}^+_1 (A_0, \mathcal{D}_0, \mathcal{P}(\underline{\mathcal{S}}^+))$.

If on the other hand $b \in \mathcal{C}^+_1 (A_0, \mathcal{D}_0, \mathcal{P}(\underline{\mathcal{S}}^+))$ then we can define $\tilde{b}$ by the same equation.

We now want to apply this result to Wightman functionals. To this end we need some

VI. 5. Notations:

1) Let W be a Wightman state and (A, $A(\underline{\mathcal{S}})\Omega$) the cyclic representation constructed from W.
   Denote by A_0 the associated representation of $S(\underline{\mathcal{S}})$ and $W_0(x) = (\Omega, A_0(x)\Omega)$ the corresponding state on $S(\underline{\mathcal{S}})$.
2) For every $x \in \underline{\mathcal{S}}^+$ we want that the spectrum of $A_0(x)$ is the positive halfline. Therefore the common spectral set is the dual cone of $\underline{\mathcal{S}}^+$ which we denote by $\underline{\mathcal{S}}'^+$
   Then $\mathcal{P}(\underline{\mathcal{S}}'^+)$ denotes the set of all polynomials

$$\Big\{ P(x_1, \dots x_n) \;\; ; \;\; x_i \in \underline{\mathcal{S}}_h \; , \text{ and for } \varphi \in \underline{\mathcal{S}}'^+ \text{ we have}$$

$$P(\varphi(x_1), \dots \varphi(x_n)) \geq 0 \Big\}$$

We now get:

VI. 6. Theorem:

Let W be a Wightman state, then the following conditions are equivalent:

1) W has a weak integral decomposition

$$W = \int_{\Lambda} W_{\lambda} \, d\mu(\lambda)$$

with Λ and μ as usual and W_{λ} has the cluster property a.e.

2) W_o has a weak integral decomposition

$$W_0 = \int_{\Lambda} W_{0\lambda} \, d\mu(\lambda)$$

with Λ, μ as above and $W_{0\lambda}$ is $\mathcal{P}(\underline{\mathcal{S}}'^{+})$ positive character a.e.

3) W_o is positive on $\mathcal{P}(\underline{\mathcal{S}}'^{+})$.

The proof is only a collection of previous results.

There is another area of applications, these are the so-called Schwinger functionals and which are used extensively in the constructive field theory. But in order to treat the problems which are connected with them, some further results are needed which are beyond this representation given here. The results which we have obtained in Göttingen so far will be presented at the conference in Marseille.

References

[1] Powers, R. T., Comm. Math. Phys. 21, 85 (1971)

[2] Maurin, K.: General Eigenfunction Expansions and Unitary Representation of Topological Groups. Warszawa Polish Scientific Publishers 1968

[3] Gel'fand, I. M., Vilenkin, N. Ya.: Generalized Functions, Vol. 4. New York, Academic Press 1964

[4] Lassner, G.: Rep. Math. Phys. 3, 179 (1972)

[5] Borchers, H. J.: Algebraic Aspects of Wightman Field Theory. In: R. N. Sen and C. Weil (Eds.) Statistical Mechanics and Field Theory (Haifa Summer School), New York, Halsted Press 1972

[6] Araki, H.: Progr. Theor. Phys. 32, 844 (1964)

[7] Borchers, H. J.: Comm. Math. Phys. 1, 49 (1965)