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INTRODUCTION

Our aim will be to show why some physical arguments have led us to mix together differential and algebraic theories that, a priori, had nothing to do one with each other. The later theories and their relations, sketched on the following picture, will be presented in a self contained way.

$$
\begin{tikzcd}
 & \text{Deformation theory of Lie algebras} \arrow[d, Rightarrow] & \\
\text{Algebraic geometry (linear algebraic groups)} \arrow[r] & \text{Deformation theory of structures} & \text{Formal theory of systems of partial differential equations} \arrow[l] \\
 & \text{Lie pseudogroups} \arrow[u, Rightarrow] &
\end{tikzcd}
$$

PHYSICAL BACKGROUND

The kind of situation to be met in physics is as follows: In order to describe a continous phenomenon, one has to introduce a tensor field, or, more generally, a <u>geometric object</u> ω, with local coordinates $\omega^\tau(x)$, over a convenient C^∞ paracompact manifold X of finite dimension n, mainly R^4. This fact agrees with a description of the universe by local observers, and the meaning is that, for any finite transformation $\bar{x} = \varphi(x)$ of the base space X or of its local coordinates, the ω^τ are transformed accordingly to the rules:

$$\bar{\omega} = \varphi(\omega) \quad : \quad \omega^\tau(x) = \Phi^\tau\left(\bar{\omega}(\varphi(x)), \partial_\mu \varphi^k\right)$$

$$1 \leq |\mu| = \mu_1 + \dots + \mu_n \leq q$$

We then express the relation between field and matter by a system of p.d.e. : $I^{\nu}(\omega^{\tau}, \partial_{\mu}\omega^{\tau}) = M^{\nu}(x)$ or simply $I(\omega \frac{\partial \omega}{\partial x}) = M(x)$.

EXAMPLES: 1) Maxwell equations for the electromagnetic field.

2) Einstein equations for the gravitational field.

We make the following important remarks:

1) The coordinates x^{i} appear only in the right members.As for the left members I ,they are only functions of the ω^{τ} and their derivatives of different orders,(quasi)linear in the top order ones, and rational,that is to say quotients of polynomials,in the other derivatives and in the ω^{τ} .

Thus,in vacuum,the equations $I(\omega, \frac{\partial \omega}{\partial x}) = 0$ can be taken as differential polynomials with constant coefficients (2 d).

2) The $M(x)$ have in general to satisfy some compatibility conditions. For the two examples given above,they are the well known divergence relations.

3) In vacuum,that is to say when $M(x) = 0$,their may exist,at least locally,a potential,that is to say a way to know the $\omega^{\tau}(x)$ as functions of some $A(x)$ and their derivatives,such that $I\left(\omega(A,\frac{\partial A}{\partial x}), \frac{\partial \omega}{\partial x}(A, \frac{\partial A}{\partial x})\right) \equiv 0$. This is of course well known for Maxwell equations,but it is still an open problem for Einstein equations.

4) People do not like to manipulate non linear systems of p.d.e..

As Maxwell equations are already linear,there is nothing to do.

On the contrary,one has to linearise Einstein equations,and the mechanism to apply will be described.

• One has to choose a standard or special field,solution of the equations in vacuum:for our purpose it is the Minkowski metric.

•• Then we have to introduce a small parameter t (say 1/c in our case) and take a Taylor expansion: $\omega_t(x) = \omega(x) + t\,\underset{1}{\Omega}(x) + \cdots$

... Finally, we introduce partial differential operators $\mathcal{D}$, $\mathcal{D}_1$, $\mathcal{D}_2$ and rewrite 3) :

When $\mathcal{D}_1 . \underset{1}{\Omega} \equiv 0$ there may exist functions $\xi(x)$ such that $\underset{1}{\Omega} \equiv \mathcal{D}.\xi$, and $\mathcal{D}_1 \circ \mathcal{D} \equiv 0$; moreover, in order that $\mathcal{D}_1 \underset{1}{\Omega} = \underset{1}{M}$, we must have $\mathcal{D}_2 . \underset{1}{M} \equiv 0$, and $\mathcal{D}_2 \circ \mathcal{D}_1 \equiv 0$.

For the reader not familiar with differential geometry, we recall some classical facts.

Let E be a vector bundle over X, with local coordinates (x^i, u^k) i=1,..,n , k=1,..,dim E . We introduce the vector bundle $J_q(E)$ over X , with local coordinates (x^i, u^k, u^k_μ) $1 \leqslant |\mu| \leqslant q$, where the u^k_μ are transformed like the derivatives $\partial_\mu u^k$ under a change of coordinates.

<u>DEFINITION</u>: A linear partial differential operator $E \xrightarrow{\mathcal{D}} F$ of order q is given in local coordinates by the formulas:

$$u^k(x) \rightsquigarrow v^l(x) = \sum_{|\mu| \leqslant q} A^{l\mu}_k(x) \, \partial_\mu u^k(x)$$

where l=1,..,dim F and we call Ⓗ the set of sections of E , solutions of the system of p.d.e. (Σ): $\mathcal{D} u = 0$

We associate with $\mathcal{D}$ the unique morphism $\phi : J_q(E) \longrightarrow F$, such that $\mathcal{D} = \phi \circ j_q$ and given in local coordinates by the formulas:

$$j_q : u^k(x) \rightsquigarrow \partial_\mu u^k(x) \quad , \quad \phi : u^k_\mu(x) \rightsquigarrow \sum_{|\mu| \leqslant q} A^{l\mu}_k(x) \, u^k_\mu(x)$$

We define the r^c prolongation of ϕ as a morphism:

$$p_r(\phi) : J_{q+r}(E) \longrightarrow J_r(F)$$

obtained by taking the derivatives of v^l up to order r , and we call $\sigma_r(\phi)$ the q+r-top order part of $p_r(\phi)$.

The kernel of $\sigma_r(\phi)$, called g_{q+r} , is defined by: $\sum_{|\mu|=q} A^{l\mu}_k(x) \, u^k_{\mu+\nu} = 0$ where $|\mu| = q, |\nu| = r$ and we easily check that g_{q+r} is uniquely determined by g_q .

DEFINITION: Let $E, F_0, F_1, \ldots, F_p, \ldots$ be vector bundles over X, and let $\mathcal{D}, \mathcal{D}_1, \ldots, \mathcal{D}_p, \ldots$ be linear partial differential operators. We say that the sequence:

$$0 \longrightarrow \Theta \longrightarrow E \xrightarrow{\mathcal{D}} F_0 \xrightarrow{\mathcal{D}_1} F_1 \longrightarrow \cdots \longrightarrow F_{p-1} \xrightarrow{\mathcal{D}_p} F_p \longrightarrow \cdots$$

is a differential complex of finite length if $\mathcal{D}_p \circ \mathcal{D}_{p-1} \equiv 0$ and $F_p \equiv 0$ if p is big enough.

EXAMPLE: Let T (T^*) be the tangent (cotangent) bundle of X, and let $\Lambda^p T^*$ be the vector bundle over X, the sections of which are exterior p-forms. We have the Poincare complex:

$$0 \longrightarrow \Theta \longrightarrow \Lambda^0 T^* \xrightarrow{d} \Lambda^1 T^* \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n T^* \longrightarrow 0$$

where d is just exterior differentiation and $\Lambda^0 T^*$ is written for $X \times R$.

Now the preceeding situation is just equivalent to exhibit a differential complex, and we are mainly interested in the initial part described by the following picture:

$$\begin{tikzcd}
\text{potential} \arrow[d, squiggly] & \text{field} \arrow[d, squiggly] & \text{matter} \arrow[d, squiggly] & \\
E \arrow[r, "\mathcal{D}"] & F_0 \arrow[r, "\mathcal{D}_1"] & F_1 \arrow[r, "\mathcal{D}_2"] & F_2 \\
& \text{field equations} \arrow[u] & \text{identities} \arrow[u] &
\end{tikzcd}$$

We are led to our first problem:

PROBLEM I: . Given any $\mathcal{D}$, does there exist such a complex and a reason to forget $\mathcal{D}$?

.. What kind of $\mathcal{D}$ must be taken in order to give a physical meaning to the truncated complex thus obtained ?

Now we have seen that the main system of p.d.e. to deal with is a non linear one,and we state our second problem:

PROBLEM II: Why are the field equations in vacuum given by a set of differential polynomials ?

Finally we will look for the different kinds of "deformations" that are used in physics.

Some methods of deformation have been introduced by Spencer (4,5) and others,since 1957,in order to describe structures on manifolds and their perturbations.For example,the deformation of a riemannian structure goes through the deformation of a metric tensor as in general relativity.

Now we show that it is possible to pass from the inhomogeneous Galilee group in two variables:

$$x' = x + vt + a_x \quad , \quad t' = t + a_t$$

to the inhomogeneous Lorentz group in two variables:

$$x' = x\,\mathrm{ch}\lambda + t\,\mathrm{sh}\lambda + b_x \quad , \quad t' = x\,\mathrm{sh}\lambda + t\,\mathrm{ch}\lambda + b_t$$

In fact we have just to introduce a parameter $1/c$ and consider the group with the infinitesimal generators:

$$I_x = \frac{\partial}{\partial x} \quad , \quad I_t = \frac{\partial}{\partial t} \quad , \quad I_r = t\frac{\partial}{\partial x} + \frac{1}{c}\,x\frac{\partial}{\partial t}$$

satisfying the following commutation relations:

$$[I_x , I_t] = 0 \quad , \quad [I_x , I_r] = \frac{1}{c}\, I_t \quad , \quad [I_t , I_r] = I_x$$

At this time,we only need to take first $c=\infty$,then $c=1$.

More generally (1),a finite dimensional Lie algebra $\mathfrak{g}$, with underlying vector space V ,is determined by a set of structure constants c^i_{jk} ,representing a map $V \wedge V \to V$,and satisfying the well known Jacobi relations:

$$(J) \qquad J(c) \equiv c^l_{ij} \cdot c^m_{kl} + c^l_{jk} \cdot c^m_{il} + c^l_{ki} \cdot c^m_{jl} = 0$$

Such an algebra can be considered as a point c on a convenient algebraic variety and a deformation c_t as an other point depending on a parameter t .

<u>DEFINITION</u>: Two Lie algebras with the same V are said <u>equivalent</u> if one can get from one to the other by a change of basis of V .

<u>DEFINITION</u>: A Lie algebra is called <u>rigid</u> if it is equivalent to any small deformation.

The main trick in the deformation theory of Lie algebras is to linearise the problem, setting: $c_t = c + t \underset{1}{C} + \ldots$ and looking at the infinitesimal (first order) deformation.

This theory has been developped since 1964 by Gerstenhaber and others.

We state now our third and last problem:

<u>PROBLEM III</u>: Is there a link between the deformation theory of geometric objects used for example in General Relativity and the later deformation theory of Lie algebras ?

We will now outline the solution of the three problems stated in the former pages.

MATHEMATICAL BACKGROUND

I / <u>FORMAL THEORY OF SYSTEMS OF P.D.E.</u> :

This powerful theory has been developped during the past ten years, mainly by Spencer, Quillen, Goldschmidt (4). In the linear case, the use of diagrams is a generalisation of the vector notations grad, div, rot and the exterior derivative d for exterior forms.

The meaning is that it allows one to look at p.d.e. systems in a deep and coherent way, without any local writing, and to give intrinsic proofs that would be tedious by direct computations.

Of course, at the same time, it is also a new kind of abstraction to get used to.

REMARK: A main point is that we do not transform a given system of p.d.e. into an exterior system and thus we never lose any information on the base space X .

We now briefly review the principal results (2a,4).

. The key idea is that of involution. As we do not want to detail the definition, we will just say that it is a purely algebraic condition that can be easily verified on g_{q+r} for r big enough.

.. We may now suppose that g_q is involutive and that the first prolongation of (Σ) does not bring equations of order q that are not linear combinations of equations already in (Σ).

DEFINITION: We say that (Σ) (or $\mathcal{D}$) is formally integrable, involutive.

... We may also suppose that $\mathcal{D} = \phi \circ j_1$ with ϕ surjective, and get:

THEOREM: When $\mathcal{D}$ is such a formally integrable, involutive linear partial differential operator of order q , there exists a differential complex, called P-sequence:

$$\mathcal{P}(\Theta) \qquad 0 \longrightarrow \Theta \longrightarrow E \xrightarrow{\mathcal{D}} F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} \cdots \xrightarrow{\mathcal{D}_n} F_n \longrightarrow 0$$

of finite length n+1 , with $\mathcal{D}_1, \mathcal{D}_2, \ldots$ first order, formally integrable, involutive linear partial differential operators (2a).

REMARK: In proving this theorem, we have, at one time, to forget $\mathcal{D}$ and consider only the truncated P-sequence:

$$\mathcal{P}(\Omega) \qquad 0 \longrightarrow \Omega \longrightarrow F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} \cdots \xrightarrow{\mathcal{D}_n} F_n \longrightarrow 0$$

made up only with first order operators.

The later remark gives an answer to the first part of problem I.

Now, what kind of $\mathcal{D}$ must be useful in physics ?

II / LIE PSEUDOGROUPS :

DEFINITION: A Lie pseudogroup Γ is a continous group of transformations $y = f(x)$ of R^n, solutions of a system of p.d.e. :

$$(\mathcal{E}) \qquad H^\tau(x^i, y^k, \partial_\mu y^k) = 0 \qquad 1 \leq |\mu| \leq q$$

the finite equations of Γ. (2 b, c)

REMARK: The Lie pseudogroups were formerly known as infinite groups, in contrast to the finite groups, today known as Lie groups.

Because $y^i = x^i$ must be a solution, we may linearise ($\mathcal{E}$).

Setting $y^i = x^i + t\,\xi^i(x) + \ldots$ we get a linear system of p.d.e. :

$$(\Sigma) \qquad \mathcal{D}.\xi = 0$$

the infinitesimal equations of Γ. (ξ is a section of T)

The operator $\mathcal{D}$ has the property (just think about the Lie derivative of a tensor field !) that, if $\mathcal{D}.\xi_1 = 0, \mathcal{D}.\xi_2 = 0$ then $\mathcal{D}.[\xi_1, \xi_2] = 0$.

DEFINITION: Such an operator is called a Lie operator.

Let us now introduce new coordinates y^k_μ, called jet-coordinates, and transform them by a change of target $\bar{y} = \varphi(y)$ in the same way as the corresponding $\partial_\mu y^k$, while keeping the source x unchanged.

DEFINITION: A function $U(y, y^k_\mu)$ is called a differential invariant of Γ if $U(\bar{y}, \bar{y}^k_\mu) \equiv U(y, y^k_\mu)$ for any transformation $\bar{y} = \varphi(y) \in \Gamma$.

THEOREM: There exists a fundamental set of differential invariants such that ($\mathcal{E}$) can be written:

$$(\mathcal{E}) \qquad U^\tau(y, y^k_\mu) = \omega^\tau(x) \qquad \text{(Lie form)}$$

Moreover, under an arbitrary change of source $\bar{x} = \varphi(x)$ the ω^τ behave like the components of a geometric object:

$$\omega \to \bar{\omega} = \varphi(\omega) \quad : \quad \omega^\tau(x) = \phi^\tau(\bar{\omega}(\varphi(x)), \partial_\mu \varphi^k) \qquad 1 \leq |\mu| \leq q$$

and we have: $\omega = \Gamma(\omega)$: $U^\tau(y, y^k_\mu) \equiv \phi^\tau(\omega(y), y^k_\mu)$

Now we do want to effect a perturbation: $\omega_t(x) = \omega(x) + t\,\underset{1}{\Omega}(x) + \cdots$

At this time, (Σ) can be written:

$$(\Sigma) \qquad \underset{1}{\Omega}^\tau \equiv -L^{\tau\mu}_k(\omega(x))\,\partial_\mu \xi^k + \xi^i \frac{\partial \omega^\tau(x)}{\partial x^i} = 0$$

REMARK: If Γ contains the translations, as it does usually in physics, then the $\omega^\tau(x)$ must be constants that can be choosen as o or 1 .

THEOREM: In order to get a formally integrable involutive operator $\mathcal{D}$, the $\omega^\tau(x)$ must satisfy the following compatibility conditions:

$$(\mathrm{I}) \qquad \left\{ \begin{array}{ll} I_*\left(\omega(x), \dfrac{\partial \omega(x)}{\partial x}\right) = 0 & \text{(first kind)} \\ I_{**}\left(\omega(x), \dfrac{\partial \omega(x)}{\partial x}\right) = c & \text{(second kind)} \end{array} \right.$$

DEFINITION: The constants c are called structure constants.

Linearising, we get $\mathcal{D}_1$ such that $\mathcal{D}_1 . \underset{1}{\Omega} = 0$ when $\underset{1}{\Omega} = \mathcal{D}.\xi$.

THEOREM: In order to get a formally integrable involutive system (I), the structure constants must satisfy the following set of algebraic relations, called (generalised) Jacobi relations:

$$(\mathrm{J}) \qquad J(c) = 0$$

where the J are polynomials of order $\leqslant 2$.

DEFINITION: An algebraic Lie pseudogroup is a Lie pseudogroup defined by a system of polynomials in $\partial_\mu y^k$, $1 \leq |\mu| \leq q$ with coefficients C^∞ in x, y.

THEOREM: The differential invariants of an algebraic Lie pseudogroup can be choosen as rational functions of the y^k_μ, $1 \leq |\mu| \leq q$ with coefficients C^∞ in y . The compatibility conditions (I) are given by a set of differential polynomials with constant coefficients.

REMARK: The proof uses methods of algebraic geometry (2d) and we have to suppose that Γ is transitive, that is to say (ξ) and (Σ) do not contain equations of order 0 .

EXAMPLE: 1) All the classical examples, in particular those in which tensor fields are involved .

2)Let $\tilde{\Gamma}$ be the normaliser of Γ ,that is to say the largest group of transformations of R^n in which Γ is normal.Then $\tilde{\Gamma}$ is an algebraic pseudogroup $\forall\ \Gamma$.

Taking the field equations as subsystems of (I),called structured systems,we have answered to the second part of problem I and to problem II ;problem III only remains unsolved.

Let us now introduce a (small) parameter t and consider the new structure constants $c_t = c + t\,C + \dots$ satisfying $J(c_t)=0$, $\forall\, t$.
Let $\omega_t(x)$ be solution of the system:

$$\begin{cases} I_*\left(\omega_t(x), \dfrac{\partial \omega_t(x)}{\partial x}\right) = 0 \\ I_{**}\left(\omega_t(x), \dfrac{\partial \omega_t(x)}{\partial x}\right) = c_t \end{cases}$$

For different choices of $\omega(x)$ we have a family of pseudo-groups of transformations of X .In fact we have a fiber bundle $\mathcal{U} \overset{\omega}{\leftrightarrows} X$ and a structure over X is just a section ω of the later bundle.In particular,taking $\omega_t(x)$ as above,we have the family Γ_t .

DEFINITION:We say that two structures ω and $\bar{\omega}$ are equivalent if they give rise to the same pseudogroup Γ .

This definition is of course extended to the corresponding c and $\bar{c}$ and it is easy to show that it is a generalisation of the equivalence of two Lie algebras with the same underlying vector space.

The idea is to transfer the perturbation c_t of the structure constants c of the Lie algebra $\mathfrak{g}$ of a Lie group G to a perturbation G_t of G ,by means of a well known theorem of Lie.In fact,the $\omega^\tau(x)$ are just local coordinates for the left (or right) invariant Maurer-Cartan 1-forms defined on G,and the deformation theory can be developped as above.This is the last answer we needed.

We give examples, increasing the order of $(\mathcal{E})$ or (Σ), and giving $(\mathcal{E})$ in its Lie form.

1) Action of a Lie group : $G \times R \longrightarrow R$:

$\bullet$) $\Gamma : y = x + a$, $(\mathcal{E})\ \dfrac{\partial y}{\partial x} = 1$, $(\Sigma)\ \dfrac{\partial \xi}{\partial x} = 0$

$\bullet\bullet$) $\Gamma : y = ax + b$, $(\mathcal{E})\ \dfrac{\partial^2 y}{\partial x^2} \Big/ \dfrac{\partial y}{\partial x} = 1$, $(\Sigma)\ \dfrac{\partial^2 \xi}{\partial x^2} = 0$

$\bullet\bullet\bullet$) $\Gamma : y = \dfrac{ax+b}{cx+d}$, $(\mathcal{E})\ \dfrac{\dfrac{\partial^3 y}{\partial x^3}}{\dfrac{\partial y}{\partial x}} - \dfrac{3}{2}\left(\dfrac{\dfrac{\partial^2 y}{\partial x^2}}{\dfrac{\partial y}{\partial x}}\right)^2 = 0$, $(\Sigma)\ \dfrac{\partial^3 \xi}{\partial x^3} = 0$

2) Γ : transformations of R^n with jacobian $= 1$.

$$(\mathcal{E})\quad \frac{\partial(y^1,\dots,y^n)}{\partial(x^1,\dots,x^n)} = 1 \qquad (\Sigma)\quad \sum_{i=1}^{n} \frac{\partial \xi^i}{\partial x^i} = 0 \quad (\operatorname{div} \xi = 0)$$

3) Γ : holomorphic transformations of the complex plane.

$(\mathcal{E})\ f(J) = J$, $(\Sigma)\ \mathcal{L}(\xi) J = 0$ where J is the mixed tensor $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$(\mathcal{E})\ \dfrac{\partial y^2}{\partial x^2} - \dfrac{\partial y^1}{\partial x^1} = 0,\ \dfrac{\partial y^1}{\partial x^2} + \dfrac{\partial y^2}{\partial x^1} = 0$ (Cauchy-Riemann) , $(\Sigma)\ \dfrac{\partial \xi^2}{\partial x^2} - \dfrac{\partial \xi^1}{\partial x^1} = 0,\ \dfrac{\partial \xi^1}{\partial x^2} - \dfrac{\partial \xi^2}{\partial x^1} = 0$

4) $\Gamma :\quad y^1 = x^1 + a \ ,\quad y^2 = x^2 + g(x^1)$

$(\mathcal{E}) :\quad \dfrac{\partial y^1}{\partial x^1} = 1 \ ,\ \dfrac{\partial y^1}{\partial x^2} = 0 \ ,\ \dfrac{\partial(y^1, y^2)}{\partial(x^1, x^2)} = 1$

$(\Sigma)\quad \dfrac{\partial \xi^1}{\partial x^1} = 0 \ ,\ \dfrac{\partial \xi^1}{\partial x^2} = 0 \ ,\ \dfrac{\partial \xi^1}{\partial x^1} + \dfrac{\partial \xi^2}{\partial x^2} = 0$

$\Gamma = \Gamma_0$ can be deformed in order to get $\Gamma_1 \begin{cases} y^1 = f(x^1) \\ y^2 = x^2 f'(x^1) \end{cases}$ which is rigid.

CONCLUSION

Unfortunately, physicists are dealing with Lie pseudogroups, though they do not know what they are dealing with, because of the lack of a convenient mathematical treatment.

They just make out their own cooking for the cases they meet: actions of Lie groups, pseudogroups related to symplectic structures, riemannian structures, analytic structures, contact structures, ...

The classical approach of Cartan, using Maurer-Cartan equations, has been generalised by Guillemin and Sternberg (Ref 6 in 5), in order to describe, when $\mathcal{D}$ is a Lie operator, two differential complexes introduced by Spencer in the general case.More recently,Spencer, Goldschmidt and Malgrange have built up a new formalism to describe the same differential complexes (5),but it seems very difficult to use it properly in physics.

An algebraic attempt has also been made with infinite Lie algebras,but the methods are not easy to put into practice (3).

We have shown how to introduce an other differential complex, the P-sequence,that arises in a natural way from the study of any linear partial differential operator,and to construct its initial part when $\mathcal{D}$ was a Lie operator.

We believe that those methods are to become a new powerful tool in mathematical physics.

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