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ANALYTICITY PROPERTIES OF SPIN SYSTEMS

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INTRODUCTION

In this lecture we wish to review and describe recent work on analyticity properties of spin - $\frac{1}{2}$ systems both classical and quantum. The method used to derive these analyticity properties is essentially due to Ruelle and consists of interpreting and analysing integral equations of the Kirkwood-Salzburg type as equations on a suitably chosen Banach space. Whilst Ruelle's original work was for continuous classical statistical mechanical systems with two-body forces the work we review on spin systems allows a large class of many body interactions. The discreteness of the configuration space of a spin system allows us to greatly improve the analyticity region obtainable for continuous systems and symmetry between "spin up" and "spin down" can be further used to extend this analyticity region.

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1. DEFINITION OF SPIN SYSTEMS.

Let us associate with each point x of a v-dimensional cubic lattice Z^ν a two-dimensional vector space $\mathcal{H}_x$ and with each $\Lambda \subset Z^\nu$ the direct product space $\mathcal{H}_\Lambda = \prod_{x \in \Lambda} \otimes \mathcal{H}_x$. The space $\mathcal{H}_\Lambda$ has the dimension $2^{N(\Lambda)}$ whose $N(\Lambda)$ is the number of points of the set Λ.

The operators $(2 \times 2$ matrices$)$ acting on each $\mathcal{H}_x$ are linearly generated by the unit operator 1_x and three Pauli matrices $\underline{\sigma}_x$. Alternatively these operators can be generated by annihilation and creation operators (spin-raising and -lowering operators) a_x, a_x^+ defined by

$$a_x = \tfrac{1}{2}(\sigma_x^{(1)} - i\,\sigma_x^{(2)}) \qquad a_x^+ = \tfrac{1}{2}(\sigma_x^{(1)} + i\,\sigma_x^{(2)})$$

satisfying the anti-commutation relations

$$a_x a_x^+ + a_x^+ a_x = 1 \qquad a_x a_x = 0 = a_x^+ a_x^+ \tag{1}$$

Similarly the bounded operators $\mathcal{B}(\mathcal{H}_\Lambda)$ acting on $\mathcal{H}_\Lambda$ are generated by annihilation and creation operators $\{a_x, a_x^+ \,;\, x \in \Lambda\}$ which satisfy (1) and

$$[a_x, a_y^+] = 0 = [a_x, a_y] \quad \text{for} \quad x \neq y \,,\; x, y \in \Lambda \tag{2}$$

Let us next introduce a basis in $\mathcal{H}_\Lambda$ in the following manner. We define $|\emptyset>_\Lambda$ to be a normalised vector such that

$$a_x |\emptyset>_\Lambda = 0 \qquad \text{for all} \quad x \in \Lambda \tag{3}$$

and then introduce the normalised vectors $|X>_\Lambda$ by

$$|X>_\Lambda = \prod_{x \in X} a_x^+ \,|>_\Lambda \qquad X \subset \Lambda \tag{4}$$

We are interested in two different algebras of operators acting on $\mathcal{H}_\Lambda$. The quantum algebra $\mathcal{A}_Q(\Lambda)$ is defined as the algebra $\mathcal{B}(\mathcal{H}_\Lambda)$, i.e. the algebra of annihilation and creation operators, and the classical algebra $\mathcal{O}\!\mathcal{l}_C(\Lambda) \subset \mathcal{O}\!\mathcal{l}_Q(\Lambda)$ is defined as the abelian subalgebra of $\mathcal{B}(\mathcal{H}_\Lambda)$ generated by the set $\{a_x^+ a_x, 1_x \; ; \; x \in \Lambda\}$.

2. REDUCED DENSITY MATRICES AND INTEGRAL EQUATIONS.

Physically we consider the points $X \in Z^\nu$ as particle sites and assume that these particles interact via a Hamiltonian $H_\Lambda \subset \mathcal{O}\!\mathcal{l}_Q(\Lambda)$ or $H_\Lambda \subset \mathcal{O}\!\mathcal{l}_C(\Lambda)$. At this point we will not further specify of H_Λ other than assuming that it is hermitien. In the following our attention will be concentrated upon the reduced density matrices $\rho_\Lambda(X \; ; \; Y)$ which are defined as follows :

$$\rho_\Lambda(X \; ; Y) = \operatorname{Tr}_{\mathcal{H}_\Lambda} \left(e^{-H_\Lambda} \prod_{x \in X} a_x^+ \prod_{y \in Y} a_y\right) \Big/ \operatorname{Tr}_{\mathcal{H}_\Lambda} \left(e^{-H_\Lambda}\right) \tag{5}$$

For economy introduce the notation :

$$Z_\Lambda = \operatorname{Tr}_{\mathcal{H}_\Lambda} \left(e^{-H_\Lambda}\right) \quad a^+(X) = \prod_{x \in X} a_x^+ \quad a(Y) = \prod_{y \in Y} a_y \tag{6}$$

and then

$$\rho_\Lambda(X,Y) = \frac{1}{Z_\Lambda} \operatorname{Tr}_{\mathcal{H}_\Lambda} \left(e^{-H_\Lambda} a^+(X) a(Y)\right)$$

$$= \frac{1}{Z_\Lambda} \sum_{\substack{S \subset \Lambda \\ S \cap (X \cup Y) = \emptyset}} \langle Y \cup S \,|\, e^{-H_\Lambda} \,|\, X \cup S \rangle \tag{7}$$

It is easily checked that (7) may be inverted to give the relation

$$\frac{1}{Z_\Lambda} \langle T \,|\, e^{-H_\Lambda} \,|\, S \rangle = \sum_{\substack{R \subset \Lambda \\ R \cap (S \cup T) = \emptyset}} (-1)^{N(R)} \rho_\Lambda(R \cup S \; ; \; R \cup T) \tag{8}$$

Now (7) and (8) can be combined to derive integral relations for the ρ_Λ as follows. If $X=Y=\emptyset$ then $\rho_\Lambda(X,Y) = 1$; assume $Y\neq\emptyset$ and define $Y^1 = Y/\{y_1\}$ where y_1 is any point in Y. Now

$$\rho_\Lambda(X\,;\,Y) = \frac{1}{Z_\Lambda} \sum_{S\cap(X\cup Y)=\emptyset} \langle Y^1\cup S \mid a_{y_1}\, e^{-H_\Lambda} | X\cup S\rangle$$

$$= \frac{1}{Z_\Lambda} \sum_{\substack{S,T\subset\Lambda \\ S\cap(X\cup Y)=\emptyset}} \langle Y^1\cup S \mid e^{-H_\Lambda} \mid T\rangle\langle T \mid e^{H_\Lambda} a_{y_1}\, e^{-H_\Lambda} | X\cup S\rangle$$

(9) $$= \sum_{\substack{R,S,T\subset\Lambda \\ S\cap(X\cup Y) = \emptyset \\ R\cap(R\cup T\cup Y^1 = \emptyset}} \rho(R\cup T;Y^1\cup R\cup S)(-1)^{N(R)} \langle T \mid e^{H_\Lambda} a_{y_1}\, e^{-H_\Lambda} | X\cup S\rangle$$

where the first step is obtained from inserting a complete set of intermediate states and the second step uses (8) . Next changing variables to $V=R\cup T$ and $W = R\cup S$ (9) takes the form

(10) $$\rho_\Lambda(X;Y) = \sum_{\substack{W\cap Y^1 = \emptyset \\ W\subset\Lambda}} \sum_{\substack{V\subset\Lambda \\ V\supset W\cap(X\cup\{y_1\})}} \rho_\Lambda(V,Y^1\cup W)K^1(X,Y;V,W)$$

where $$K^1(X,Y;V,W) = \sum_{W\cap V\supset R\supset W\cap(X\cup\{y_1\})} (-1)^{N(R)} \langle V/R | e^{H_\Lambda} a_{y_1}\, e^{-H_\Lambda} | X\cup(W/R)\rangle$$

These are the integral relations which we will use to obtain analyticity properties. Note that the above derivation, which is due to the present author, does not depend upon any detailed structure of the Hamiltonian ; this structure is important only for the analysis of the integral equations and not for their derivation. This method of derivation also generalises to the case of continuous systems quantum or classical. We will analyse these equations in the two different cases, classical and quantum, separately.

3. CLASSICAL SPIN SYSTEMS.

We begin by parametrizing the Hamiltonian H_Λ in terms of one-body, two-body and many-body interactions. We define an interaction Φ as a function from the finite subsets $\Lambda \subset Z^\nu$ to the algebra $\mathfrak{A} = \bigcup_\Lambda \mathfrak{A}_C(\Lambda)$ with the properties

1 - $\Phi(X) \subset \mathfrak{A}_C(X)$ is hermitien

2 - $\Phi(X) = \tau_a \Phi(X-a)$ for $a \in Z^\nu$

where the translation automorphisms $\mathfrak{A} \xrightarrow{\tau_a} \mathfrak{A}$ is defined by $a_{x+a} = \tau_a a_x$, etc .

3 - $\|\Phi\| = \sum_{0 \in X} \|\Phi(X)\| < +\infty$

In terms of such interactions we define the Hamiltonian $H_\Lambda (= U_\Phi(\Lambda))$ of the finite system Λ by

$$H_\Lambda = U_\Phi(\Lambda) = \sum_{X \subset \Lambda} \Phi(X)$$

Example : to illustrate these abstract definitions consider the following example

$$\Phi(\{x\}) = -\mu a_x^+ a_x , \Phi(\{x_1, x_2\}) = \varphi(x_1 - x_2) a_{x_1}^+ a_{x_1} a_{x_2}^+ a_{x_2} , \Phi(\{x_1 .. x_k\}) = 0$$

$$k > 2$$

Conditions 1,2 and 3 are satisfied if μ and $\varphi(x)$ are real and the latter satisfies

$$\|\Phi\| \sum_{x\in Z^\nu} |\varphi(x)| < +\infty$$

The Hamiltonian corresponding to this interaction is given by

$$H_\Lambda = -\mu N_\Lambda + \tfrac{1}{2} \sum_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in \Lambda}} \varphi(x_1 - x_2) a^+_{x_1} a_{x_1} a^+_{x_2} a_{x_2}$$

Where N_Λ is the number operator i.e. $N_\Lambda = \sum_{x\in\Lambda} a^+_x a_x$. With such an interaction the spin system can be viewed as a lattice gas, i.e. the sites $x \in \Lambda$ can be occupied by particles interacting through a two-body potential $\varphi(x)$ at chemical potential μ .

It is convenient at a later stage to write $\Phi = (\Phi^{(1)}, \Phi')$ where $\Phi^{(1)}$ is the one-body interaction derived from Φ and Φ' contains the two-and many-body interactions. We will also always take $U_{\Phi^1}(\Lambda) = -\mu N_\Lambda$ and introduce the fugacity $z = e^\mu$.

As $\mathfrak{U}_C$ is generated by the operators $\{a^+_x a_x, 1_x; x \in Z^\nu\}$ the states $|X>_\Lambda$ are eigenfunctions of $\Phi(x)$ and we introduce the eigenvalues $\varphi(x)$ by

$$\Phi(Y)|X>_\Lambda = \varphi(Y)\,|X>_\Lambda \qquad \text{for } Y \subset X$$

$$\Phi(Y)\,|X>_\Lambda = 0 \qquad \text{for } Y \not\subset X$$

and then
$$U_\Phi(\Lambda)\,|X>_\Lambda = \sum_{S\subset X} \varphi(S)|X>_\Lambda$$

$$= U_\varphi(X)\,|X>_\Lambda$$

where
$$U_\varphi(X) = \sum_{S\subset X} \varphi(S)$$

It follows from this structure that the classical reduced density matrices are such that

$$\rho_\Lambda(X\,;\,Y) = 0 \qquad\qquad \text{if } X \neq Y$$

Whilst from (10) we find for $\rho_\Lambda(Y) = \rho_\Lambda(Y\,;\,X)$ the simplified integral equations

$$\rho_\Lambda(Y) = \sum_{\substack{W\subset\Lambda \\ W\cap Y^1=\emptyset}} \rho_\Lambda(Y^1\cup W)K^1(Y;W) \tag{11}$$

where
$$K^1(Y;W) = \sum_{\substack{R\subset W \\ R\cap Y=\emptyset}} (-1)^{N(W/R)} \langle Y^1\cup R|\, e^{U_\Phi(\Lambda)}\, a_{y_1}\, e^{-U_\Phi(\Lambda)} |Y\cup R\rangle$$

$$= \sum_{\substack{R\subset W \\ R\cap Y=\emptyset}} (-1)^{N(W/R)} \exp\{-[U_\varphi(Y\cup R) - U_\varphi(Y^1\cup R)]\}$$

$$= z \sum_{\substack{R\subset W \\ R\cap Y=\emptyset}} (-1)^{N(W/R)} \exp\{-[U_{\varphi'}(Y\cup R) - U_{\varphi'}(Y^1\cup R)]\} \tag{12}$$

where we have explicitly exhibited the dependance of K^1 on the fugacity z (φ' indicates the eigenfunctions associated with Φ').

The method of utilising these equations that Ruelle invented is to introduce a Banach space $\mathcal{E}$ of complex functions ψ on the non-empty finite subsets of Z^ν with the norm

$$|\psi| = \sup_{X\subset Z^\nu} |\psi(X)|$$

We see immediately that $\rho_\Lambda \in \mathcal{E}$. The major point of this definition is the fact that K^1 is a uniformly bounded operator on $\mathcal{E}$.

LEMMA

For $Y \subset Z^\nu$ fixed, we have

$$\sum_{\substack{W \subset Z^\nu \\ W \cap Y = \emptyset}} |K^1(Y;W)| \leq |Z| \, [\exp\{e^{\|\Phi'\|}-1\}-1] e^{\|\varphi'\|}$$

PROOF.

We have the following Proof of Gallavotti and Miracle-Sole

$$U_{\varphi'}(Y \cup R) - U_{\varphi'}(Y^1 \cup R) = \sum_{S \subset Y \cup R} \varphi'(S) - \sum_{S \subset Y^1 \cup R} \varphi'(S)$$

$$= U^1_{\varphi'}(Y) + I_{\varphi'}(Y,R)$$

where $U^1_{\varphi'}(Y) = \sum_{y_1 \in S \subset Y} \varphi'(S)$ $\qquad I_{\varphi'}(Y,R) = \sum_{\substack{y_1 \in T \subset Y \\ \emptyset \neq S \subset R}} \varphi'(T \cup S)$

Further introducing $J_{\varphi'}(Y,S) = \sum_{y_1 \in T \subset Y} \varphi'(T \cup S)$

we have $I_{\varphi'}(Y,R) = \sum_{\emptyset \neq S \subset R} J_{\varphi'}(Y,S)$

and $K^1(Y,W) = z \, e^{-U^1_{\varphi'}(Y)} \sum_{n \geq 1} \sum_{\substack{S_1 \ldots S_n \\ \cup S_i = W}} \prod_{j=1}^{n} (e^{-J_{\varphi'}(Y,S_j)} - 1)$

thus $|K^1(Y,W)| \leq |z| e^{\|\varphi'\|} \sum_{n \geq 1} \sum_{\substack{S_1 \ldots S_2 \\ \cup S_i = W}} \prod_{j=1}^{n} |e^{-J_{\varphi'}(Y,S_j)} - 1)$

$$\leq |z| e^{\|\varphi'\|} \sum_{n \geq 1} \left(\frac{e^{\|\varphi'\|}-1}{\|\varphi'\|}\right)^n \sum_{\substack{S_1 \ldots S_n \\ \cup S_i = W}} \prod_{j=1}^{n} |J_{\varphi'}(Y, S_j)|$$

because $|U_{\varphi'}(Y)| \leq \|\varphi'\|$ and $|J_{\varphi'}(Y,S)| \leq \|\varphi'\|$

Thus finally

$$\sum_{\substack{W \subset Z^\nu \\ W \cap Y^1 = \emptyset}} |K^1(Y;W)| \leq |z| e^{\|\varphi'\|} \sum_{n \geq 0} \frac{1}{n!} \left(\frac{e^{\|\varphi'\|}-1}{\|\varphi'\|}\right)^n \left(\sum_{\substack{S \cap Y^1 = \emptyset \\ S \neq \emptyset}} |J_{\varphi'}(Y,S)| \right)^n$$

$$\leq |z| e^{\|\varphi'\|} \exp(e^{\|\varphi'\|}-1)$$

As a result of this bound we may interpret (11) as an integral equation on $\mathcal{E}$ of the form

$$\rho_\Lambda = z\, \chi_\Lambda \alpha + \chi_\Lambda \mathcal{K}_\Phi \rho_\Lambda$$

where $\alpha(X)=1$ if $N(X)=1$ and $\alpha(X)=0$ if $N(X)=0$; the inhomageneous term in the equation comes from the term with $Y=\{y_1\}$ and $W=\emptyset$ in (11) and the operator $\mathcal{K}_\Phi$ is defined in the natural manner from K^1. Combining (11) and the result of the lemma one immediatly finds that

$$\|\mathcal{K}_\Phi\| < 2|z| e^{\|\Phi'\|} \exp(e^{\|\Phi'\|}-1) .$$

If z and Φ' are such that

$$\|\mathcal{K}_\Phi\| < 2|z| e^{\|\Phi'\|} \exp(e^{\|\Phi'\|}-1) < 1$$

than Ruelle's methods allow us to conclude that the limit

$$\lim_{\Lambda \to \infty} \rho_\Lambda(X) = \rho(X)$$

exists and that $\rho = \frac{z}{1-\mathcal{K}_\Phi}\alpha$

is analytic in a small complex neighbourhood of (z,Φ') [we have used the notation ρ for the element of $\mathcal{E}$ with components $\rho(X)$].

It also follows immediately that the thermodynamic pressure

$$P(\Phi) = \lim_{\Lambda\to\infty} \frac{1}{N(\Lambda)} \log Z_\Lambda(\Phi)$$

exists and is analytic in the same domain.

Note that as $\|\Phi'\|\to 0$ (13) is only satisfied if $2|z|<1$ thus in this limit the analyticity in z is only in a finite region. This is due to the fact that $\|K_\Phi\|$ remains finite as $\|\Phi'\|\to 0$. However, K_Φ is defined in terms of $K^1(Y,W)$ and these latter functions tend to zero as $\|\Phi'\|\to 0$ for all values of W except $W=\emptyset$ or y_1. This remark motivates us to rewrite the integral relations in the following form due to Gallavotti and Miracle-Sole.

$$\rho_\Lambda(Y) = z\, e^{-U^1_{\emptyset,}(Y)}[\rho_\Lambda(Y^1)-\rho_\Lambda(Y)] + \sum_{\substack{\emptyset\neq W\subset\Lambda \\ W\cap Y=\emptyset}} [\rho_\Lambda(Y^1\cup W)-\rho_\Lambda(Y\cup W)]K^1(Y\cup W)$$

i.e. we explicitly separate out the terms $W=\emptyset$ and $W=y_1$. This last relation can then be rewritten as

$$\rho_\Lambda(Y) = \frac{ze^{-U^1_{\emptyset'}(Y)}}{1+ze^{-U^1_{\emptyset'}(Y)}}[\rho_\Lambda(Y^1) + \sum_{\substack{\emptyset\neq W\subset\Lambda \\ W\cap Y=\emptyset}} \rho_\Lambda(Y^1\cup W)-\rho_\Lambda(Y\cup W)]H^1(Y,W)] \qquad (14)$$

where
$$H^1(Y,W) = \frac{1}{ze^{-U^1_{\emptyset}(Y)}} K^1(Y,W)$$

Finally interpreting (14) as an integral equation on $\mathcal{E}$ of the form

$$\rho_\Lambda = \frac{z}{1+z}\chi_\Lambda\alpha + \chi_\Lambda \mathcal{H}_\Phi \rho_\Lambda$$

we have $\| \mathcal{K}_{\Phi} \| \le \sup_{Y} \left| \frac{ze^{-U^1(Y)}}{1+ze^{-U^1(Y)}} \right| (2 \exp(e^{\|\Phi'\|}-1)-1)$ (15)

As previously we may conclude the existence and analyticity of the infinite volume correlation functions $\rho(X) = \lim_{\Lambda\to\infty} \rho_\Lambda(X)$, together with the existence and analyticity of the thermodynamic pressure if (z,Φ) are such that $\| \mathcal{K}_{\Phi} \| < 1$ e.g. if

$$\sup_{Y} \left| \frac{ze^{-U^1_{\Phi}(Y)}}{1+ze^{-U^1_{\Phi}(Y)}} \right| (2 \exp(e^{\|\Phi'\|}-1)-1) < 1 .$$

In particular as $\Phi' \to 0$ we have analyticity for $\left|\frac{z}{1+z}\right| < 1$ i.e. for $z \ge 0$.

Before proceeding to the quantum case we note that the important feature in the foregoing analysis is the fact that the kernel K^1 leads to a uniformly bounded operator $\mathcal{K}_{\Phi}$ on $\mathcal{E}$. Although our method of estimation rather obscures the physical reason behind this property it essentially derives from locality and short range forces ; this will become clearer in the quantum case.

4. QUANTUM SPIN SYSTEMS

Let us again begin byparametrising our Hamiltonian H_Λ . We consider, as in the classical case, an interaction Φ to be a function from the finite sets $X \subset Z^\nu$ to the algebra $\mathcal{O}\!\mathcal{L} = \bigcup_\Lambda \mathcal{O}\!\mathcal{L}_Q(\Lambda)$ but now we assume

$$1 - \Phi(X) \subset \mathcal{O}\!\mathcal{L}_Q(X) \quad \text{is hermitian}$$

$$2 - \Phi(X) = \tau_a \Phi(X-a) \quad \text{for all} \quad a \in Z^\nu$$

$$3 - \|\Phi\|_\lambda = \sum_{0 \in X} \|\Phi(X)\| e^{\lambda(N(X)-1)} < +\infty$$

where $\lambda > 0$.

The Banach space norm $\|\Phi\|_\lambda$ is a generalization of that used in the classical case where we had $\lambda = 0$. The necessity of taking $\lambda \neq 0$ will appear due to our inability to make such precise estimates, as previously. We will again write $\Phi = (\Phi^{(1)} \Phi')$ and take Φ $\Phi^{(1)}(X) = -\mu a_x^+ a_x$. Hence

$$U_{\Phi^{(1)}} = -\mu N_\Lambda$$

We further assume $[U_{\Phi^{(1)}}(\Lambda), \Phi'(X)] = 0$ for $X \subset \Lambda$, i.e. we assume the interaction conserves particle number. Note that whilst

$$[\Phi'(X_1), \Phi'(X_2)] = 0$$

if $X_1 \cap X_2 = \emptyset$ this is no longer true in general. Now we still have the integral relations

$$\rho_\Lambda(X;Y) = \sum_{\substack{W \cap Y^1 = \emptyset \\ W \subset \Lambda}} \sum_{\substack{W \subset \Lambda \\ V \supset W \cap (X \cup \{y_1\})}} \rho_\Lambda(V; Y^1 \cup W) K^1(X,Y;V;W)$$

where $$K^1(X,Y;V,W) = z \sum_{W\cap V\supset R\supset W\cap(X\cup\{y_1\})} (-1)^{N(R)} < V/R|e^{U_{\Phi'}(\Lambda)} a_{y_1} e^{-U_{\Phi'}(\Lambda)} |X\cup(W)R) >$$

and we will try andinterpret these relations on a Banach space $\mathcal{B}$ of complex functions ψ on pairs of non-empty subsets of Z^ν with the norm

$$|\psi| = \sup_{X,Y\subset Z^\nu} |\psi(X,Y)|$$

The major difficulty is in proving the uniform boudedness of the operator determined by K^1. As a preliminary to this calculation we prove the following lemma

<u>LEMMA 2</u> : <u>if</u> Φ <u>is such that</u> $2\|\Phi\|_1 < 1$ <u>then</u>

$$\lim e^{U_\Phi(\Lambda)} a_{y_1} e^{-U_\Phi(\Lambda)} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{Y_n\subset Z^\nu} \cdots \sum_{Y_1\subset Z^\nu} [[\Phi(Y_n)[\ldots[\Phi(Y_1)a_{y_1}]]]$$

<u>Proof</u>.

The equality is formally true and all we have to do is prove convergence. But

$$C_n = \| \sum_{Y_n\subset Z^\nu} \cdots \sum_{Y_1\subset Z^\nu} [\Phi(Y_n)\ldots[\Phi(Y_1)a_{y_1}]] \|$$

$$\leq \sum_{Y_n\cap S_{n-1}\neq 0} \cdots \sum_{y_1\in Y_1} \|[\Phi(Y_n)\ldots[\Phi(Y_1)a_{y_1}]] \|$$

where $S_{n-1} = \bigcup_{i=1}^{n-1} Y_i$ and we have used local commutativity. Therefore

$$C_n \le 2^n \sum_{k_1 \dots k_n} \prod_{i=1}^{n} (k_1+\dots+k_i+1) \sum_{O \in Y_i N(Y_i)=k_i+1} \|\Phi(Y_i)\|$$

$$\le 2^n \sum_{k_1 \dots k_n} (1+k_1 \dots +k_n)^n \prod_{i=1}^{n} \sum_{O \in Y_i \; N(Y_i)=k_i+1} \|\Phi(Y_i)\|$$

$$\le 2^n \; n! \; e \sum_{k_1 \dots k_n} \prod_{i=1}^{n} \sum_{O \in Y_i \; N(Y_i)=k_i+1} \|\Phi(Y_i)\| e^{k_i}$$

$$\le 2^n \; n! \; \|\Phi\|_1^n$$

Thus the series converges if $2\|\Phi\|_1 < 1$.

The fact that this pertubation series for $e^{U_\Phi(\Lambda)} a_y, e^{-U_\Phi(\Lambda)}$ converge at least for weak interactions leads us to replace this operator by its perturbation expansion in K^1 . Thus we write

$$K^1(X,Y;V,W) = z \sum_{n \ge 0} \frac{K_n^1(X,Y;V,W)}{n\,!}$$

where $K_n^1(X,Y;V,W) = \sum_{W \cap V \supset R \supset W \cap (X \cup \{y_1\}) S \subset \Lambda} (-1)^{N(R)} \sum \dots$

$$\dots \sum_{\substack{Y_1 \dots Y_n \\ \cup Y_i = S}} < V/R \,|\, [\Phi(Y_n) \dots [\Phi(Y_1) a_{y_1}]] X \cup (X/R) >$$

Note that in the last definition all sums are finite.

As a preliminary to the study of this operator consider

$$H^1(X,Y;V,W_iS_1) \;=\; \sum_{W\cap V\supset R\supset W\cap(X\cup\{y_1\})} (-1)^{N(R)} \langle V/R\,|\,A(S)\,|\,X\cup(W/R)\rangle$$

where $y_1 \in S$ and $A(S)\subset \mathcal{U}_Q(S)$. Introduce disjoint sets F, G, H, by $F=W/V$, $G=V/W$, $H=W\cap V$. Then

$$H^1(X,Y;F,G,H;S) = \sum_{H\supset R\supset H\cap(X\cup\{y_1\})} (-1)^{N(R)} \langle G\cup(H/R)\,|\,A(S)\,|\,X\cup F\cup(H/R)\rangle$$

$$= (-1)^{N(H)} \sum_{\substack{R\subset H\\ R\cap(X\cup\{y_1\})=\emptyset}} (-1)^{N(R)} \langle G\cup R\,|\,A(S)\,|\,X\cup F\cup R\rangle$$

$$= (-1)^{N(H)} \sum_{\substack{R_1\subset H\cap S\\ R_1\cap(X\cup\{y_1\})=\emptyset}} (-1)^{N(R_1)} \langle G\cup R_1\,|\,A(S)\,|\,X\cup F\cup R_1\rangle$$

$$\sum_{\substack{R_2\subset H/S\\ R_2\cap(X\cup\{y_1\})=\emptyset}} (-1)^{N(R_2)} \quad .$$

But the latter sum vanishes unless the range of summation is empty i.e. unless $H\subset S\cup X\cup\{y_1\}$. If $H\subset S\cup X\cup\{y_1\}$ we have

$$H^1(X,Y;F,G,H;S) \;=\; (-1)^{N(H)} \sum_{\substack{R\subset H\cap S\\ R\cap(X\cup\{y_1\})=\emptyset}} (-1)^{N(R)} \langle (G\cap S)\cup R\,|\,A(S)\,|\,(X\cap S)\cup(F\cap S)\cup R\rangle\langle G/S\,|\ldots$$

$$\ldots(X/S)\cup F/S\rangle$$

and this last expression is zero unless

$$G/S = X/S \quad \text{and} \quad F/S = \emptyset \;.$$

Thus we must have $F \subset S$ and $G/S = X\cup\{y_1\}/S$ where we have used $y_1 \in S$. Thus finally we find

$$\sum_{W\cap Y^1=\emptyset} \;\; \sum_{V\supset W\cap(X\cup\{y_1\})} |H^1(X,Y;V,W;S)|$$

$$\leq \sum_{F\subset S} \sum_{G\subset S/F\cup H} \;\; \sum_{H\subset S/F} \;\; \sum_{R\subset H/(X\cup\{y_1\})} |\langle G\cup R|A(S)|(X\cap S)\cup F\cup R\rangle|$$

$$\leq \frac{3}{5}\, \|A(S)\|\; 5^{N(S)}$$

where the final estimate arises from replacing the matrix element by $\|A(S)\|$ and camping out the remaining summations. Using this estimate procedure which is due to Greenberg we find

<u>LEMMA</u>

$$\sum_{W\cap Y^1=\emptyset} \;\; \sum_{V\supset W\cap(X\cup\{y_1\})} |K_n^1(X,Y;V,W)| \quad \leq 3n!\; \Big(2 \sum_{0\in X} \|\Phi'(X)\|(5e)^{N(X)-1}\Big)^n$$

$$\leq 3n!\; (2\|\Phi'\|_\lambda)^n \quad \text{for} \quad \lambda \geq 1+\log_e 5$$

<u>PROOF</u>

The proof consists of combining (16) and the method used to prove the proceeding lemma.

Thus we now see that the kernel K^1 will lead to a uniformly bounded operator on $\mathcal{E}$ if $2\|\Phi\|_\lambda < 1$ for $\lambda \geq 1+\log e^5$ and in this case we can derive analyticity results in the same manner as we did for the classical system. We note however that the kernel K_0^1 can be explicitly evaluated and then the integral relations take the form

$$\rho_\Lambda(X,Y) = \frac{z}{1+z\delta(y_1;X)} [\delta(y_1;X)\rho_\Lambda(X/y_1;Y^1) + \sum_{V,W}{}' \sum_{n\geq 1} \frac{1}{n!} K_n^1(X,Y;V,W)\rho_\Lambda(V;Y^1 \cup W)]$$

where $\delta(y;X) = 1$ if $y \in X$ and zero otherwise and the restrictions on the summations over V and W have been summarised by the prime. With this partial inversion we now have relations analogous to (14) which may be interpreted as integral equations on $\mathcal{E}$ of the form

$$\rho_\Lambda = \frac{z}{1+z} \chi_\Lambda \alpha + \chi_\Lambda \mathcal{H}_\Phi \rho_\Lambda$$

where $\alpha(X,Y) = 1$ if $X=Y$ and $N(X)=1$ and zero otherwise. Further $\mathcal{H}_\Phi$ has the bound

$$\|\mathcal{H}_\Phi\| \leq \left|\frac{z}{1+z}\right| + \frac{|z|6\|\Phi'\|_\lambda}{1-2\|\Phi'\|_\lambda} \tag{17}$$

provided $2\|\Phi'\|_\lambda < 1$ where $\lambda = 1+\log_e 5$. If z and Φ' are such that

$$\left|\frac{z}{1+z}\right| + \frac{|z|6\|\Phi'\|_\lambda}{1-2\|\Phi'\|_\lambda} < 1 \quad \text{and} \quad 2\|\Phi'\|_\lambda < 1$$

for $\lambda = 1+\log_e 5$ we may conclude that the thermodynamic correlation functions and pressure exist, are analytic in a small complex neighbourhood of (z,Φ') and as a vector $\rho \in \mathcal{E}$ given by $\rho = \frac{z}{1+z} \frac{1}{1-\chi_\Phi} \alpha$. [Actually a little more work is required to establish the analyticity properties but the necessary continuity and differentiability of the kernel $\mathcal{H}_\Phi$ can be established by estimates similar to the above.] Note that in the limit $\|\Phi'\|_\lambda \to 0$ the above result coincides with that obtained in the classical case.

5. A SYMMETRY PRINCIPLE

To conclude this review we now indicate how the above results may be extended by the use of a symmetry principle. Consider the algebra M_2 of 2×2 matrices. There exists a mapping $A \in M_2 \to \hat{A} \in M_2$ defined by

$$A \to \hat{A} = - A + \mathrm{Tr}(A)$$

which has the properties

$$\widehat{A^*} = \hat{A}^* \quad , \quad \mathrm{Tr}(A) = \mathrm{Tr}(\hat{A}) \quad , \quad \widehat{AB} = \hat{B}\hat{A} \quad .$$

[Each $A \in M_2$ can be written as a linear combination of the identity and three Pauli matrices $\underline{\sigma}$; the above symmetry corresponds to the mapping $\underline{\sigma} \to -\underline{\sigma}$.]
Similarly there is a mapping of $\mathcal{L}(\mathcal{H}_\Lambda)$ defined by

$$A \in \mathcal{L}(\mathcal{H}_\Lambda) \to \hat{A} \in \mathcal{L}(\mathcal{H}_\Lambda) \quad ; \; \hat{A} = \sum_{S \subset \Lambda} (-1)^{N(S)} \mathrm{Tr}_{\mathcal{H}_{\Lambda / S}}(A)$$

which has the properties

$$\widehat{A^*} = \hat{A}^* \quad ; \quad \mathrm{Tr}_{\mathcal{H}_\Lambda}(A) = \mathrm{Tr}_{\mathcal{H}_\Lambda}(\hat{A}) \quad , \quad \widehat{AB} = \hat{B}\hat{A} \; .$$

Next let us introduce a mapping $\mathcal{L}$, on the subspace of interactions Φ which are such that $\Phi(X) = 0$ for $N(X) > N_\Phi$ where N_Φ is a real number, by the definition

$$(\mathcal{L}\Phi)(S) = (-1)^{N(S)} \sum_{X \supset S} \mathrm{Tr}_{\mathcal{H}_{X/S}}(\Phi(X))$$

We find

$$U_{\mathcal{L}\Phi}(\Lambda) = \sum_{\substack{S \subset \Lambda \\ S \neq \emptyset}} \sum_{T \supset S} (-1)^{N(S)} \mathrm{Tr}_{\mathcal{H}_{T/S}}(\Phi(T))$$

$$U_\Phi(\Lambda) - \sum_{T \subset \Lambda} \sum_{S \subset T} (-1)^{N(S)} \mathrm{Tr}_{T/S}(\Phi(T))$$

and therefore $\widehat{U_{\Phi}(\Lambda)} - U_{\mathcal{L}\Phi}(\Lambda) = N(\Lambda)E_{\Phi} + \sum(\Lambda)$

where

$$E_{\Phi} = \sum_{0 \in S} \mathrm{Tr}_{\mathcal{K}_S} \frac{(\Phi(S))}{N(S)} = - E_{\mathcal{L}\Phi}$$

and $\sum(\Lambda)$ is a surface terme i.e. $N(\Lambda)^{-1} \| \bar{\sum}(\Lambda) \| \underset{\Lambda = \infty}{\rightarrow} 0$

We thus have

LEMMA

<u>If</u> $\Phi \in B$ <u>is such that</u> $\Phi(X) = 0$ <u>for</u> $N(X) > N_{\Phi}$ <u>then the thermodynamic pressure satisfies the symmetry relation</u>

$$P(\Phi) + \tfrac{1}{2} E_{\Phi} = P(\mathcal{L}\Phi) + \tfrac{1}{2} E_{\mathcal{L}\Phi} \ .$$

The proof of the lemma is a consequence of the above definitions and the standard arguments establishing the existence of the thermodynamic pressure and its independence of surface terms.

The importance of this symmetry principle is that it allows us to extend the analyticity properties obtained previously from the integral equations. Originally from the integral equations one derives analyticity of the correlation functions ρ in some domain but then from this one may deduce analyticity of the thermodynamic pressure P as a functional of the interactions in the same domain. Now however we may use the symmetry principle derived above to extend the analyticity domain of P and then finally one deduces straightforwardly that in this extended domain the thermodynamic correlation functions ρ exist and are analytic.

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REFERENCES

The general derivation of the integral equations and the symmetry relations given in this lecture were obtained by the author in Summer 1967, but were not published. The vital estimates of the kernels in the quantum case were provided by Greenberg, Comm. Math. Phys. 11, 314, 1969 .

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Further references are given in these papers.

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