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NATURAL EXTENSIONS AND MIXING FOR SEMI-GROUP ACTIONS

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ABSTRACT. We construct the natural extension of a system $(X, \mathcal{B}, \mu, \Gamma)$ where Γ is a finitely generated abelian countable semi-group of endomorphsisms of $(X, \mathcal{B}, \mu)$. When the action is one by group epimorphisms of a compact abelian metrizable group G, we give necessary and sufficient algebraic conditions on the base system $(G, \mathcal{B}_G, \lambda_G, \Gamma)$ for the natural extension to be mixing. We give also algebraic necessary and sufficient conditions for Γ to be mixing, and apply these to limit evaluation of ergodic averages on groups.

I. INTRODUCTION.

Our basic object in this paper will be a system $(X, \mathcal{B}, \mu, \Gamma)$ where $(X, \mathcal{B}, \mu)$ is a probability space, and Γ *is a finitely generated abelian countable semi-group of endomorphisms of* $(X, \mathcal{B}, \mu)$, i.e. any $\gamma \in \Gamma$ is a map $\gamma : X \to X$ which is measurable, onto mod 0, $\mu \circ \gamma^{-1} = \mu$, and Γ is endowed with a semi-group commutative addition "+" such that $\gamma + \gamma' \in \Gamma$ whenever both $\gamma, \gamma' \in \Gamma$.

Since Γ is finitely generated, there exists $\gamma_1, \dots, \gamma_d \in \Gamma$ such that for any $\gamma \in \Gamma$, there exists $(n_1, \dots, n_d) \in \mathbb{N}^d$ with $\gamma = \sum_{i=1}^d n_i \gamma_i$. Define $\Psi : \mathbb{N}^d \to \Gamma$ by

$$\Psi(n_1, \dots, n_d) = \sum_{i=1}^d n_i \gamma_i.$$

Let $\underline{0} = (0, \dots, 0)$ and 0_Γ be the zero element of Γ (the identity map of X). We need, before we proceed to the description of the natural extension of $(X, \mathcal{B}, \mu, \Gamma)$, the following elementary lemma.

Lemma 1. *For any finite non-empty subset $F \subset \Gamma$, there exists a finite $F' \subset \Gamma$ containing F and an element $\gamma_{F'}$ such that $F' = \{\gamma_{F'} - \gamma : \gamma \in F'\}$, where $\gamma'' = \gamma - \gamma'$ if $\gamma'' + \gamma' = \gamma$.*

Proof. If $F \subset \Gamma$ is finite, there exists a rectangle $R = [0, r_1] \times \dots \times [0, r_d] \subset \mathbb{N}^d$ such that $F \subseteq \Psi(R)$. Then choose $F' = \Psi(R)$ and $\gamma_{F'} = \Psi(\underline{R})$ where $\underline{R} = (r_1, \dots, r_d)$.

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Then if $\gamma \in F'$, $\gamma = \Psi(\underline{n})$, let $\gamma' = \Psi(\underline{R} - \underline{n})$: it follows that $\gamma = \gamma_{F'} - \gamma'$, and $\gamma' \in F'$. $\square$

Natural extensions are well known when Γ has only one generator ([C-F-S, p. 240]). When instead of a single endomorphism T acting on $(X, \mathcal{B}, \mu)$ we let Γ do so, we define the natural extension $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu}, \tilde{\Gamma})$ of $(X, \mathcal{B}, \mu, \Gamma)$ as follows. First let

$$\tilde{X} := \{\tilde{x} = (x_\gamma)_{\gamma \in \Gamma} : \gamma'(x_{\gamma+\gamma'}) = x_\gamma,\ \gamma, \gamma' \in \Gamma\}. \tag{1}$$

Next for $\beta \in \Gamma$, define $\Pi_\beta : \tilde{X} \to X$ by $\Pi_\beta((x_\gamma)_{\gamma\in\Gamma}) = x_\beta$, $\tilde{x} = (x_\gamma)_{\gamma\in\Gamma} \in \tilde{X}$. Further, for $B \in \mathcal{B}$, let $\tilde{\mu}(\Pi_\beta^{-1}(B)) = \lambda_G(B)$. Then using Lemma 1, define, for a rectangle $R \subset \mathbb{N}^d$, and a collection $(B_\gamma)_{\gamma\in\Psi(R)} \in \mathcal{B}^R$,

$$\tilde{\mu}\left(\{\tilde{x} = (x_\gamma)_{\gamma\in\Gamma} : x_\gamma \in B_\gamma,\ \gamma \in \Psi(R)\}\right) = \mu(\cap_{\underline{n}\in R}\Psi(\underline{R} - \underline{n})^{-1}(B_{\Psi(\underline{n})})). \tag{2}$$

By Kolmogorov's extension theorem, (2) ensures that $\tilde{\mu}$ extends to a unique probability measure on the σ-algebra $\tilde{\mathcal{B}} := \sigma(\cup_{\gamma\in\Gamma}\Pi_\gamma^{-1}(\mathcal{B}))$.

Next if $\gamma \in \Gamma$, let $\tilde{\gamma} : \tilde{X} \to \tilde{X}$ be defined by

$$\tilde{\gamma}(\tilde{x}) = (\gamma(x_{\gamma'}))_{\gamma'\in\Gamma} \tag{3}$$

for $\tilde{x} = (x_{\gamma'})_{\gamma'\in\Gamma} \in \tilde{X}$.

In Section II, we describe some more elementary properties of the natural extension, and pay attention in Theorem 1 to the connection between the ergodicity (resp. mixing) of Γ and the ergodicity (resp. mixing) of the action of $\{\tilde{\gamma} : \gamma \in \Gamma\}$ on $\tilde{X}$.

In Section III we restrict our attention to the case where $X = G$ is a compact abelian metrizable group, $\mu = \lambda_G$ is its Haar probability measure, and Γ is finitely generated by onto continuous commuting homomorphisms of G. In that case we show that the natural extension is itself a compact abelian metrizable group. Then using Theorem 1, [K-S, Theorem 2.4] and [C-F-S, p. 107], we obtain in Theorem 2 necessary and sufficient conditions expressed in $(G, \mathcal{B}_G, \lambda_G, \Gamma)$ for its natural extension to be mixing.

Section IV is devoted to the study of necessary and sufficient conditions for the action of Γ to be mixing. These are stated in Theorem 3. In Remark 1 we point out the rather different behaviours of the cases $d = 1$ and $d \geq 2$; indeed when $d = 1$ the mixing of the original action is equivalent to the one of its natural extension, while for $d \geq 2$, as already Theorems 2 and 3 may indicate formaly, we give an example for diagonal matrix actions on the 2 dimensional torus $\mathbb{T}^2$ where the original action mixes but the natural extension does not.

Finally in Section V with give in Corollary 2 applications to limit evaluation of ergodic averages, Theorem 3 giving sufficient conditions to ensure the equality

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ (\sum_{i=1}^{d} n^i \gamma_i)(g) = \int_G f d\lambda_G, \qquad \lambda_G \ a.e., \tag{4}$$

for $f \in L^p(\lambda_G)$, $p > 1$, using results from [Bo2].

II. Ergodicity and mixing of the natural extension.

Before we go to the ergodicity and mixing, we list a few more properties (or definitions) of the natural extension that we shall need. If Δ is an abelian semi-group of maps, for $\delta \in \Delta$, we let δ^{-1} reffer to the inverse image map, while $-\delta$ reffers to the inverse map if it exists.

Lemma 2. *The following properties hold;*
(i): for any $\gamma \in \Gamma$, $\tilde{\gamma}$, defined by (3), *is an automorphism of $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu})$, and*

$$(-\tilde{\gamma})((x_{\gamma'})_{\gamma' \in \Gamma}) = (x_{\gamma+\gamma'})_{\gamma' \in \Gamma}.$$

(ii): let $\tilde{\Gamma}$ be the group generated by $\{\tilde{\gamma} : \gamma \in \Gamma\}$; then $\tilde{\Gamma}$ is a finitely generated countable abelian group, a set of generators of which is $\{\widetilde{\gamma_1}, \dots, \widetilde{\gamma_d}\}$.
(iii): for $\beta, \gamma \in \Gamma$, $\Pi_\beta \circ \tilde{\gamma} = \gamma \circ \Pi_\beta$.

Proof. Assertions (ii) and (iii) are obvious. Checking (i) we essentially need to prove that $\tilde{\gamma}$ satisfies $\tilde{\mu} \circ \tilde{\gamma} = \tilde{\mu}$. But this is obviously true on subsets as in (2), using $\mu\gamma^{-1} = \mu$ and (3). $\square$

For a discrete countable abelian semi-group (or group) Δ, one may give a meaning to "$\delta \to \infty$" as follows; let $\phi : \mathbb{N} \to \Delta$ be a bijection, and define the neighbourhoods at infinity in Δ as to be the images of those of $\mathbb{N}$ under ϕ. This definition is readily seen to be independent of the choice of ϕ.

We say that an action of a countable semi-group (group) Δ on a probability space $(Y, \mathcal{C}, \nu)$ is *ergodic (respectively mixing)* if for any Δ-invariant subset $C \in \mathcal{C}$, $\nu(C)\nu(Y \setminus C) = 0$ (resp. for any two $f, g \in L^2(\nu)$, $\lim_{\delta \to \infty} < \delta(f), g > = < f, 1 >< 1, g >$) (here $\delta(f) := f \circ \delta$). Note that from the definition of ergodicity, the action of Δ is ergodic when the action of some single element $\delta \in \Delta$ is.

Now for a finitely generated abelian countable group Δ, we are faced to the problem of elements of finite order in respect to the definition of mixing (could it happen that arbitrarily close to ∞ we could find elements of finite order?). But for a group as Δ, as the following lemma shows, there are at most finitely many such.

Lemma 3. *For a finitely generated abelian countable group Δ,*

$$\#\{\delta \in \Delta : \exists n \in \mathbb{N} \setminus \{0\},\ n\delta = 0_\Delta\} < \infty.$$

Proof. Let Δ be generated by $\{\delta_1, \dots, \delta_d\}$. The Δ is an epimorphic image of the $\mathbb{Z}$ module $\mathbb{Z}^d$ by the following map;

$$\Phi((m_1, \dots, m_d)) = \sum_{i=1}^{d} m_i \delta_i,$$

$\underline{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$. Let Δ_F be the set of elements of finite order in Δ (we include 0_Δ). It is clearly a subgroup of Δ, and hence $\Phi^{-1}(\Delta_F)$ is a sub-group of $\mathbb{Z}^d$; whence it is finitely generated by say $\{\underline{m}_1, \dots, \underline{m}_r\}$. Then let $\beta_i = \Phi(\underline{m}_i)$, $1 \le i \le r$. Obviously $\{\beta_1, \dots, \beta_r\}$ generates Δ_F. Let p_i be the order of β_i, and $p = \prod_{i=1}^{r} p_i$. Then it is easy to see that $\#\Delta_F \le 2^r p$. $\square$

Theorem 1. *The action of Γ is ergodic if and only if the one of $\{\tilde{\gamma} : \gamma \in \Gamma\}$ is. And the action of Γ is mixing if and only if the action of $\{\tilde{\gamma} : \gamma \in \Gamma\}$ is.*

Proof. The proof is based on the one dimensional one given in [C-F-S, p. 241]. Assume there is a $B \in \mathcal{B}$ such that $\gamma^{-1}(B) = B$ for any $\gamma \in \Gamma$, and $0 < \mu(B) < 1$. Then for $\gamma \in \Gamma$, $\tilde{\gamma}^{-1}(\Pi_{0_\Gamma}^{-1}(B)) = \Pi_{0_\Gamma}^{-1}(B)$, and $\tilde{\mu}(\Pi_{0_\Gamma}^{-1}(B)) = \mu(B) \notin \{0,1\}$: thus the action of $\{\tilde{\gamma} : \gamma \in \Gamma\}$ is not ergodic.

Now let $(N_t)_{t\geq 1}$ be a net of $\mathbb{N}^d$ subsets satisfying Tempelman's conditions ([K]). Assume the action of Γ on X to be ergodic. Then let $F(\tilde{x}) = F(x_{\Psi(\underline{n})} : \underline{n} \in R) \in L^1(\tilde{\mu})$ depend on finitely many coordinates. From Lemma 2, (*iii*), there exists an $f \in L^1(\mu)$ such that $F(\tilde{x}) = f(x_{\underline{R}})$. From the extension of μ to $\tilde{\mu}$, we see that $\int_{\tilde{X}} F(\tilde{x})d\tilde{\mu}(\tilde{x}) = \int_X f(x_{\underline{R}})d\mu(x_{\underline{R}})$. And clearly $F \circ \widetilde{\Psi(\underline{m})}(\tilde{x}) = f \circ \Psi(\underline{m})(x_{\underline{R}})$, for any $\underline{m} \in \mathbb{N}^d$. So by substitution, and assumption,

$$\lim_{t\to\infty} \frac{1}{N_t} \sum_{\underline{m}\in N_t} F \circ \widetilde{\Psi(\underline{m})}(\tilde{x}) = \int_{\tilde{X}} F d\tilde{\mu}.$$

We conclude to the ergodicity of $\{\tilde{\gamma} : \gamma \in \Gamma\}$ by a density argument, using Banach's principle ([K]).

For the second equivalence, we apply a density argument in L^2 once observed that for given rectangle R, collections of measurable subsets $(A_{\Psi(\underline{n})})_{\underline{n}\in R}$, $(B_{\Psi(\underline{n})})_{\underline{n}\in R} \in \mathcal{B}^R$, and $\underline{m} \in \mathbb{N}^d$,

$$\begin{aligned}&\tilde{\mu}\left(\{\tilde{x} : x_{\Psi(\underline{n})} \in A_{\Psi(\underline{n})},\ \underline{n} \in R\} \cap \widetilde{\Psi(\underline{m})}^{-1}\left(\{\tilde{x} : x_{\Psi(\underline{n})} \in B_{\Psi(\underline{n})},\ \underline{n} \in R\}\right)\right)\\ &= \mu\left(\left(\cap_{\underline{n}\in R}\Psi(\underline{R} - \underline{n})(A_{\Psi(\underline{n})})\right) \cap \Psi(\underline{m})^{-1}\left(\cap_{\underline{n}\in R}\Psi(\underline{R} - \underline{n})(B_{\Psi(\underline{n})})\right)\right),\end{aligned}$$

using Lemma 2, (2), and the definition of mixing. □

III. The case of a compact abelian group G.

From this section on, we let $X := G$ a compact abelian metrizable group, $\mu := \lambda_G$ its Haar probability measure, $\mathcal{B} = \mathcal{B}_G$ the associated Borel σ-algebra. Also, we let Γ be a finitely generated countable abelian semi-group of onto continuous homomorphisms acting on G. We let $Epi(G)$ denote the set of such homomorphisms.

Using (1) – (3), Lemma 2, and the fact that the generators $\gamma_1, \ldots, \gamma_d$ of Γ belong to $Epi(G)$, the following lemma is easily proved.

Lemma 4. *Properties (i) – (iv) hold;*
(i): $\tilde{G}$ is a compact abelian metrizable group, equipped with the structure induced by the compact abelian metrizable group G^Γ.
(ii): $\widetilde{\lambda_G}$ is the Haar probability measure of $\tilde{G}$.
(iii): for $\gamma \in \Gamma$, the map $\tilde{\gamma}$ is an automorphism (invertible epimorphism) of $\tilde{G}$.
(iv): let $\tilde{\Psi} : \mathbb{Z}^d \to \tilde{\Gamma}$ be such that

$$\tilde{\psi}(\underline{m}) = \sum_{i=1}^{d} m_i \tilde{\gamma}_i,$$

for $\underline{m} = (m_1, \dots, m_d)$. *Then if we let* $\underline{m}^+ := (\max\{0, m_i\})_{1 \le i \le d}$ *and* $\underline{m}^- := (\min\{0, m_i\})_{1 \le i \le d}$, *we have*

$$\tilde{\Psi}(\underline{m})((x_\gamma)_{\gamma \in \Gamma}) = (\Psi(\underline{m}^+)(x_{\gamma + \Psi(-\underline{m}^-)}))_{\gamma \in \Gamma},$$

and $\tilde{\Psi}(\underline{n}) = \widetilde{\Psi(\underline{n})}$ *for* $\underline{n} \in \mathbb{N}^d$.

When Γ is generated by a single invertible element ($d = 1$), a necessary and sufficient condition for Γ to be mixing ([H, p. 53]) is that it has infinite orbits on non trivial characters.

When $d \geq 2$, [K-S, Theorem 2.4] gives a necessary and sufficient condition for the mixing of Γ when it is a finitely generated countable abelian subgroup of the group of automorphisms of G. And we see from Lemma 2 and Lemma 4 that if $\tilde{\Gamma}$ is mixing (on $\tilde{G}$) then using the factor map Π_{0_Γ} it forces the action of Γ to be mixing on G. We shall see in the proof of the following theorem that requiring G to be compact abelian metrizable is usefull for downloading on $(G, \mathcal{B}_B, \lambda_G, \Gamma)$ explicit conditions equivalent to the mixing of $\tilde{\Gamma}$. For $(\underline{m}, \underline{n}) \in (\mathbb{N}^d)^2 \setminus \{(\underline{0}, \underline{0})\}$, we write $\underline{m} \perp \underline{n}$ if $\sum_{i=1}^d m_i n_i = 0$.

Theorem 2. *The action of* $\tilde{\Gamma}$ *on* $\tilde{G}$ *is mixing if and only if for any nontrivial character* $\chi \in \hat{G}$, *for any* $(\underline{m}, \underline{n}) \in (\mathbb{N}^d)^2 \setminus \{(\underline{0}, \underline{0})\}$ *such that* $\underline{m} \perp \underline{n}$ *and* $\tilde{\Psi}(\underline{m}) - \tilde{\Psi}(\underline{n})$ *is of infinite order,*

$$\chi \circ \Psi(\underline{m}) \neq \chi \circ \Psi(\underline{n}).$$

Proof. From [K-S, Theorem 2.4], the action $\tilde{\Gamma}$ is mixing if and only if for any element $\beta \in \tilde{\Gamma}$ of infinite order, the action of $\{n\beta : n \in \mathbb{Z}\}$ is mixing. Since by Lemma 4, $(i) - (iii)$, β is an automorphism of $\tilde{G}$, by [H, p. 53], any β of infinite order is a mixing automorphism of $\tilde{G}$ if and only if it has infinite orbits on non trivial characters of $\tilde{G}$.

From [H-R, p. 364-365 & Lemma (24.4) p. 377], the characters of $\tilde{G}$ may be written as

$$\Xi := \prod_{\underline{n} \in R} \widetilde{\chi_{\underline{n}}}$$

where each $\chi_{\underline{n}} \in \hat{G}$, and the map above is defined on $\tilde{G}$ by

$$\left(\prod_{\underline{n} \in R} \widetilde{\chi_{\underline{n}}} \right)(\tilde{g}) = \prod_{\underline{n} \in R} \chi_{\underline{n}}(g_{\Psi(\underline{n})}),$$

for $\tilde{g} = (g_{\Psi(\underline{n})})_{\underline{n} \in \mathbb{N}^d}$. Since the action of $\tilde{\Gamma}$ is one by automorphisms, and for $\beta \in \tilde{\Gamma}$ there exists $\underline{m} \in \mathbb{Z}^d \setminus \{\underline{0}\}$ such that $\tilde{\Psi}(\underline{m}) = \beta$, the equation

$$\Xi \circ \beta \neq \Xi$$

is equivalent to the equation

$$\Xi \circ \widetilde{\Psi(\underline{m}^+)} \neq \Xi \circ \widetilde{\Psi(-\underline{m}^-)}. \tag{5}$$

For $\underline{n} \in R$, we have $g_{\underline{n}} = \Psi(\underline{R}-\underline{n})(g_{\underline{R}})$, so equation (5) reads equivalently as (using Lemma 2, (iii))

$$\left(\prod_{\underline{n}\in R} \chi_{\underline{n}} \circ \Psi(\underline{R}-\underline{n})\right) \circ \Psi(\underline{m}^+) \neq \left(\prod_{\underline{n}\in R} \chi_{\underline{n}} \circ \Psi(\underline{R}-\underline{n})\right) \circ \Psi(-\underline{m}^-), \qquad (6)$$

where the paranthesis contain characters of G. Asking for (6) to hold for any $\underline{m} \in \mathbb{Z}^d$ such that $\tilde{\Psi}(\underline{m}^+) - \tilde{\Psi}(-\underline{m}^-)$ is not of finite order, and Ξ non trivial, is now clearly equivalent to the condition stated in Theorem 2. $\square$

When Γ is isomorphic to $\mathbb{N}^d$, we say that $\mathbb{N}^d$ *acts on* G (i.e. the map Ψ is an isomorphism of semi-groups). And we say that $\mathbb{Z}^d$ *acts on* $\tilde{G}$ *when* $\tilde{\Psi}$ *is a group isomorphism.*

Lemma 5. *The following equivalence holds;*

$$\mathbb{N}^d \textit{ acts on } G \iff \mathbb{Z}^d \textit{ acts on } \tilde{G}.$$

Proof. Certainly, via the projection map Π_{0_Γ}, the right to left implication stated above is obvious. Now suppose $\mathbb{Z}^d$ does not act on $\tilde{G}$. Then $Ker(\tilde{\Psi}) \neq \{\underline{0}\}$, so there exists $\underline{m} \in \mathbb{Z}^d \setminus \{\underline{0}\}$ such that $\tilde{\Psi}(\underline{m}) = 0_{\tilde{\Gamma}}$. Then using Π_{0_Γ} again, we see that $\Psi(\underline{m}^+) = \Psi(-\underline{m}^-)$, which implies ($\underline{m} \neq \underline{0}$) that Ψ is not one to one. $\square$

Corollary 1. *When* $\mathbb{N}^d$ *acts on* G*, a necessary and sufficient condition for* $(\tilde{G}, \widetilde{\mathcal{B}_G}, \widetilde{\lambda_G}, \tilde{\Gamma})$ *to be mixing is that for any* $(\underline{m}, \underline{n}) \in (\mathbb{N}^d)^2 \setminus \{(\underline{0}, \underline{0})\}$ *with* $\underline{m} \perp \underline{n}$*, any non trivial* $\chi \in \hat{G}$*,*

$$\chi \circ \Psi(\underline{m}) \neq \chi \circ \Psi(\underline{n}).$$

Proof. Since the action of $\tilde{\Gamma}$ has no elements of finite order, the corollary is an immediate consequence of Theorem 2 and Lemma 5. $\square$

IV. Mixing of the action Γ.

In this section we take $(G, \mathcal{B}_G, \lambda_G, \Gamma)$ as in section III. We want to describe conditions for the action Γ to be mixing. We first need two lemmas.

Lemma 6. *If* $\gamma \in \Gamma$ *is of finite order, then* $\tilde{\gamma}$ *is of finite order in* $\tilde{\Gamma}$*. As a consequence, the set of elements of finite order of* Γ *is finite.*

Proof. From the definition of $\tilde{\gamma}$, if $n\gamma = 0_\Gamma$ then $n\tilde{\gamma} = 0_{\tilde{\Gamma}}$. And if $\gamma \neq \gamma'$, both elements of Γ, it is obvious that $\tilde{\gamma} \neq \tilde{\gamma}'$. The conclusion then follows from Lemma 3. $\square$

We shall write $\gamma \leq \gamma'$ *if there exists* $\gamma'' \in \Gamma$ *such that* $\gamma' = \gamma + \gamma''$.

Lemma 7. *Assume $(\gamma_n)_{n\geq 0}$ is a sequence in Γ such that $\lim_{n\to\infty}\gamma_n = \infty$. Then there exists a sub-sequence $(\gamma_{n_k})_{k\geq 0}$ tending to ∞ such that for any $k \geq 0$, $\gamma_{n_k} < \gamma_{n_{k+1}}$ and $\lim_{p\to\infty}\gamma_{n_{k+p}} - \gamma_{n_k} = \infty$.*

Proof. Let $\gamma_n \to \infty$, and assume that each γ_n is of infinite order (use Lemma 6). Let $\underline{m}_n \in \mathbb{N}^d$ be such that $\Psi(\underline{m}_n) = \gamma_n$. Next suppose contrarily to the expected first conclusion (via an eventual reindexation of the initial sequence) that for any $p, p' \geq 0$, $p' > p \Rightarrow \underline{m}_p \not\leq \underline{m}_{p'}$, where $(n_1, \dots, n_d) \leq (m_1, \dots, m_d)$ means that $n_i \leq m_i$ for all $1 \leq i \leq d$.

Let $\underline{m}_p = (m_1^{(p)}, \dots, m_d^{(p)})$, $p \geq 0$ and define

$$\mathbb{N}^d(\underline{m}_p) = \{\underline{m} \in \mathbb{N}^d : \underline{m} \not\geq \underline{m}_p\} = \cup_{i=1}^d \{\underline{m} : m_i < m_i^{(p)}\}.$$

Then our assumption reads $p' > p \Rightarrow \underline{m}_{p'} \in \mathbb{N}^d(\underline{m}_p)$, this last formulation being itself equivalent to

$$\forall p > 0,\ \underline{m}_p \in \cup_{i=1}^d \{\underline{m} : m_i < \min_{0\leq k<p}(m_i^{(k)})\}.$$

For $p > 0$ chose $i(p) \in \{1, \dots, d\}$ such that $\underline{m}_p \in \{\underline{m} : m_{i(p)} < \min_{0\leq k<p}(m_{i(p)}^{(k)})\}$. Then since $\{1, \dots, d\}$ is finite, there exists a $i_0 \in \{1, \dots, d\}$ and a subsequence $(\underline{m}_{p_j})_{j\geq 0}$ such that $i(p_j) = i_0$ for all $j \geq 0$. This leads to a contradiction since $m_{i(p_j)}^{(p_j)} \in \mathbb{N}$.

So with a reindexation of (γ_n) if necessary, we may assume that (γ_n) tends to ∞ and that $n \leq m \Rightarrow \gamma_n \leq \gamma_m$. Extracting once more ensures the conclusions of the lemma, using Lemma 6 again (a subsequence of one going to ∞ must go itself to ∞). □

The following theorem gives a different characterization of mixing than the one obtained in Theorem 2. Also it improves to a simpler form [C-F-S, Theorem 2, p. 108].

Theorem 3. *A necessary and sufficient condition for Γ to be mixing is that for any $\gamma \in \Gamma$ of infinite order and any non trivial character $\chi \in \hat{G}$,*

$$\chi \neq \chi \circ \gamma.$$

Proof. By the definition of mixing, using a density argument in $L^2(\lambda_G)$, and the orthogonality of the characters in $L^2(\lambda_G)$, Γ is mixing if and only if for any $(\chi_1, \chi_2) \in \hat{G}^2 \setminus \{(\hat{e}, \hat{e})\}$, where $\hat{e}$ denotes the trivial character,

$$\lim_{\gamma\to\infty}\int_G \chi_1(g)\chi_2\circ\gamma(g)\,d\lambda_G(g) = 0.$$

The condition stated in Theorem 3 is obviously necessary. To prove it is sufficient too, assume Γ not to be mixing, so that there exists two characters $(\chi_1, \chi_2) \neq (\hat{e}, \hat{e})$ and a subsequence $(\gamma_n)_{n\geq 0}$ tending to infinity such that

$$\int_G \chi_1(g)\chi_2\circ\gamma_n(g)\,d\lambda_G(g) = 1, \tag{7}$$

for all $n \geq 0$. Using Lemma 6 and Lemma 7, via a reindexation we shall not mention, we may assume that the sequence $(\gamma_n)_{n\geq 0}$ satisfies the properties of the sub-sequence mentionned in Lemma 7, and that each γ_n is of infinite order. Then (7) implies that for any $n, n' \geq 0$, $\chi_2 \circ \gamma_n = \chi_2 \circ \gamma_{n'}$. And for given $n \geq 0$, (7) meaning that $\overline{\chi_1} = \chi_2 \circ \gamma_n$, we must have $\chi_2 \circ \gamma_n \neq \hat{e}$ since otherwise, γ_n being onto, we would have $\chi_1 = \chi_2 = \hat{e}$.

So for given $n \geq 0$, the chracter $\chi_2 \circ \gamma_n$ is non trivial and satisfies

$$(\chi_2 \circ \gamma_n) \circ (\gamma_{n'} - \gamma_n) = \chi_2 \circ \gamma_n,$$

for all $n' \geq n$. Since by the assumed property of Lemma 7, $\lim_{n' \to \infty} (\gamma_{n'} - \gamma_n) = \infty$, there should exist, by Lemma 6, an $n' > n$ such that $\gamma_{n'} - \gamma_n$ is of infinite order. The theorem is proved. □

Remark 1. *The conditions stated for mixing in Theorems 2 and 3 are not equivalent when $d \geq 2$, even in the case when $\mathbb{N}^d$ acts on G. Indeed, let*

$$A_1 = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

and let them act on the 2 dimensional torus $\mathbb{T}^2 = \mathbb{R}^2 \,/\, \mathbb{Z}^2$ by letting

$$A_1(x,y) = (4x \mod 1, 3y \mod 1) \text{ and } A_2(x,y) = (2x \mod 1, 2y \mod 1).$$

First it is easy to check that the action they generate on $\mathbb{T}^2$ is an $\mathbb{N}^2$ mixing action, by Theorem 3. Then let $\chi(x,y) = e^{2\pi i x} \in \hat{\mathbb{T}}^2 \setminus \{\hat{e}\}$; we see that $\chi \circ A_1 = \chi \circ A_2^2$, so by Theorem 2 the associated action on the natural extension is not mixing. This shows that though when $d = 1$ Theorems 2 and 3 clearly agree, together whith the corresponding results from [C-F-S, p. 107-108] or [H, p. 53], when $d \geq 2$, the situation is quite different. Probably the equivalence concerning mixing stated in Theorem 1 is the best possible for arbitrary spaces and finitely generated abelian actions.

V. Applications to limit evaluation of ergodic averages.

In this section we assume, for the sake of simplicity, that $\mathbb{N}^d$ acts on G.

In [Bo2] J. Bourgain remarks without proof that, using results from [Bo1] as justification, if $p > 1$, the sequence $(\underline{m}_n)_{n\geq 1}$ where $\underline{m}_n = (n, n^2, \ldots, n^d)$ is L^p good universal for a.e. convergence for $\mathbb{N}^d$ actions generated by d commuting measure preserving maps of a probability space $(X, \mathcal{B}, \mu)$. That is for any $f \in L^p(\mu)$, there exists a function l_f such that

$$l_f(x) := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T_1^n \ldots T_d^{n^d} x) \tag{8}$$

exists μ-a.s..

The identification of the limit $l_f(.)$ is not a straightforward matter, and it may occur in certain cases that $l_f(.) \neq \int_X f d\mu$ ([A-N]). In the mixing case, the following Lemma shows why the limit must a. s. be equal to the integral. We formulate it in the setting of the $\mathbb{N}^d$ action on G.

Lemma 8. *If the action of $\mathbb{N}^d$ on G is mixing, then assuming* (8), *for $f \in L^p(\lambda_G)$,*

$$l_f(g) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f \circ (\sum_{i=1}^{d} n^i \gamma_i)(g) = \int_G f d\lambda_G, \quad \lambda_G - a.s..$$

Proof. Let $S_N(f) := \frac{1}{N} \sum_{n=0}^{N-1} f \circ (n\gamma_1 + \ldots + n^d \gamma_d)$. Assume first that $f \in L^\infty(\lambda_G)$. Then since Γ is mixing,

$$< S_N(f), g > \to_{N\to\infty} < f, 1 >< 1, g >,$$

for any $g \in L^2(\lambda_G)$. Also obviously $l_f \in L^\infty(\lambda_G)$ and so by the dominated convergence theorem and the Cauchy-Schwartz inequality we obtain

$$< l_f, g >=< f, 1 >< 1, g >,$$

for any $g \in L^2(\lambda_G)$. This proves $l_f(g) = \int_G f d\lambda_G$ λ_G-a.s..

Next for $f \in L^p(\lambda_G)$, assuming (8), observing $\|S_N(f)\|_p \le \|f\|_p$, we see that $l_f \in L^p(\lambda_G)$ and that $\|S_N(f) - l_f\|_p \to 0$. Since $p > 1$, for any $\varepsilon > 0$, there exists an $h \in L^\infty(\lambda_G)$ such that $\|f - h\|_p < \varepsilon$. And there exists $N_0 = N_0(\varepsilon, f, h)$ such that for $N \ge N_0$, $\|S_N(f) - l_f\|_p < \varepsilon$ and $\|S_N(h) - \int_G h d\lambda_G\|_p < \varepsilon$. It follows that $\|l_f - \int_G f d\lambda_G\|_p < 3\varepsilon$. □

The following corollary uses Theorem 3 and Lemma 8 to ensure (4).

Corollary 2. *If $\mathbb{N}^d$ acts on G with generators $\gamma_1, \ldots, \gamma_d$, if $p > 1$ and $f \in L^p(\lambda_G)$, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ (n\gamma_1 + n^2\gamma_2 + \ldots + n^d \gamma_d)(g) = \int_G f d\lambda_G, \qquad \lambda_G - a.s.,$$

if for any $\underline{m} \in \mathbb{N}^d \setminus \{\underline{0}\}$, any non trivial character $\chi \in \hat{G}$,

$$\chi \circ (\sum_{i=1}^{d} m_i \gamma_i) \ne \chi,$$

where $\underline{m} = (m_1, \ldots, m_d)$.

The conditions o the above corollary are checked out quite easily when the group G is a k dimensional torus and the generators of Γ are commuting diagonalisable integer entry matrices, each defining of course an onto epimorphism of the torus (non zero determinants).

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