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ON THE SCATTERING MATRIX FOR N-PARTICLE HAMILTONIANS

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Our goal here is to derive stationary representations for the scattering matrix of the N-particle Schrödinger operator. We consider arbitrary incoming or outgoing channels of scattering. In particular, it is shown that the scattering matrix is a weakly continuous function of the spectral parameter (energy).

1. Similarly to the two-particle case, the scattering theory for $N, N \geq 3$, interacting quantum particles with a Hamiltonian H can be formulated as the assertion about existence and completeness of some wave operators $W^{\pm}(H, \hat{H}; J)$. All possible channels of scattering (determined by decompositions of the N-particle system into non-interacting clusters of particles) are naturally taken into account in the framework of an auxiliary "large" Hilbert space $\hat{\mathcal{H}}$ in which some "effective free" Hamiltonian $\hat{H}$ acts. The operator $\hat{H}$ is constructed in terms of the kinetic energy operators of relative motion of clusters and energies of bound states of these clusters. The "identification" J is constructed in terms of the corresponding eigenfunctions. Thus scattering theory in the N-particle case is naturally formulated in a couple of (different) spaces.

However, results of the abstract scattering theory are definitely not applicable to the triple $\{\hat{H}, H; J\}$ which impedes to obtain stationary representations for the scattering matrix $S = S(J)$. Actually, one of the ingredients of the problem is a correct choice of an auxiliary identification $\tilde{J}$ such that $\{\hat{H}, H; \tilde{J}\}$ satisfies already standard assumptions of the smooth scattering theory. In particular, this gives a representation for $\tilde{S} = S(\tilde{J})$ in terms of boundary values of the resolvent $R(z)$ of H and the spectral family of $\hat{H}$.

Let us explain a construction of $\tilde{J}$ on the example of the channel of scattering, where all particles are asymptotically free both before and and after interaction. In this channel we choose $\tilde{J}$ as a first order differential operator $M^{(0)}$ with coefficients which are homogeneous functions of degree zero. It is required that these coefficients vanish in neighbourhoods of "singular" directions, where pair potentials do not tend to zero.

Thus we first construct the scattering matrix $\tilde{S}$ corresponding to the auxiliary identification $\tilde{J}$ and then find its relation to the physical scattering matrix S. Our restriction on coefficients of $M^{(0)}$ implies that we have a reasonably good representation for the physical scattering amplitude of the free channel only away from singular directions.

Our representations for the scattering matrix require a detailed study of boundary values of the resolvent $R(z)$ as $z = \lambda \pm i\epsilon$, $\epsilon \to 0$. In complement to the limiting absorption principle we prove uniform boundedness of operators $G_a R(z) G_b^*$, where G_a, G_b (indices a, b label decompositions of N particles into clusters) are suitable first order differential operators with coefficients vanishing as $|x|^{-1/2}$ at infinity.

Note that a time-dependent definition determines the scattering matrix for almost all energies only. This is not quite satisfactory from the physics point of view. Using stationary representations we show that the scattering matrix is actually well-defined for all energies and is a continuous (in a weak operator sense) function of the energy.

We follow here a non-perturbative approach of [1]-[3] to the N-particle scattering theory. Actually, our constructions rely heavily on the paper [3] where it is shown that methods of the smooth scattering theory can be applied to the N-particle problem. In the framework of the Faddeev's approach [4] to the three-particle scattering theory stationary representations for the scattering matrix were obtained in [5]. Note that the representations of [5] and of this paper do not coincide.

2. Let us briefly recall some basic definitions of the scattering theory. Let H_j, $j = 1, 2$, be a couple of self-adjoint operators in Hilbert spaces $\mathcal{H}_j$ $j = 1, 2$, respectively, and let $J : \mathcal{H}_1 \to \mathcal{H}_2$ be an H_1-bounded operator. Denote by $E_j(\Lambda) = E(\Lambda; H_j)$ the spectral family of H_j. Suppose that an interval Λ is such that the operator H_1 is absolutely continuous on the subspace $\mathcal{H}_1(\Lambda) = E_1(\Lambda)\mathcal{H}_1$. Suppose also that $JE_1(\Lambda) \in \mathcal{B}$ ($\mathcal{B}$ is the class of bounded operators). The wave operator for the pair H_1, H_2, "identification" J and interval Λ is defined by the relation

$$W^{\pm} = W^{\pm}(H_2, H_1; J, \Lambda) = s - \lim_{t \to \pm\infty} \exp(iH_2 t) J \exp(-iH_1 t) E_1(\Lambda) \quad (1)$$

under the assumption that this strong limit exists.

Suppose now that both wave operators (1) exist. Then the scattering operator for the triple H_1, H_2, J and the interval Λ is defined as

$$\mathbf{S} = \mathbf{S}(H_2, H_1; J, \Lambda) = W^{+}(H_2, H_1; J, \Lambda)^* W^{-}(H_2, H_1; J, \Lambda). \quad (2)$$

To introduce the scattering matrix we consider a diagonalization of the operator H_1 on Λ. In view of applications we suppose that the operator H_1

has the absolutely continuous spectrum of constant multiplicity on Λ. Thus we consider a representation of the space $\mathcal{H}_1(\Lambda)$ as the L_2-space of vector-functions with values in some auxiliary Hilbert space $\mathbf{N}$ ($\dim \mathbf{N}$ equals the multiplicity of the spectrum):

$$\mathcal{H}_1(\Lambda) \leftrightarrow L_2(\Lambda; \mathbf{N}). \tag{3}$$

Let F_1 be the operator which maps unitarily the left-hand side onto its right-hand side. We require that $(F_1 H_1 f)(\lambda) = \lambda(F_1 f)(\lambda)$ for $f \in \mathcal{H}_1(\Lambda)$ and, consequently,

$$\|(F_1 f)(\lambda)\|_{\mathbf{N}}^2 = d(E_1(\lambda)f, f)/d\lambda, \quad \text{a.e. } \lambda \in \Lambda.$$

Since the operator (2) commutes with H_1, it acts in (3) as multiplication by an operator-function $S(\lambda) = S(\lambda; H_2, H_1; J) : \mathbf{N} \to \mathbf{N}$, called the scattering matrix. In other words,

$$(\mathbf{S}f, g) = \int_\Lambda (S(\lambda)(F_1 f)(\lambda), (F_1 g)(\lambda))_{\mathbf{N}}\, d\lambda$$

for any $f, g \in \mathcal{H}_1(\Lambda)$. The scattering matrix is defined for a.e. $\lambda \in \Lambda$ and is bounded by $\|J\|^2$.

To describe a stationary formula for the scattering matrix it is necessary to introduce auxiliary wave operators

$$W^\pm(H_1, H_1; E_1(\Lambda)J^* J E_1(\Lambda)). \tag{4}$$

They commute with H_1 and hence act as multiplication by operator-functions $w^\pm(\lambda) : \mathbf{N} \to \mathbf{N}$ in the representation (3). We put

$$\mathbf{B}_+(z) = J^*\mathbf{V} - \mathbf{V}^* R_2(z)\mathbf{V}, \quad \mathbf{B}_-(z) = \mathbf{V}^* J - \mathbf{V}^* R_2(z)\mathbf{V},$$

where

$$\mathbf{V} = H_2 J - J H_1. \tag{5}$$

Let a formal operator $Z_1(\lambda)$ be defined by $Z_1(\lambda)f = (F_1 f)(\lambda)$. The scattering matrix satisfies two equalities (for each of the signs "+" and "−")

$$S(\lambda) = w^\pm(\lambda) - 2\pi i Z_1(\lambda)\mathbf{B}_\pm(\lambda + i0)Z_1^*(\lambda), \tag{6}$$

which we call its stationary representations. If $J = I$, then $w^+ = w^-$, $\mathbf{B}_+ = \mathbf{B}_-$ and, consequently, two representations (6) coincide. In the general case they are different and we give both since neither of them is better than another.

Of course, representations (6) are only formal. Below we give assumptions on $\mathbf{V}$ which guarantee that the operators $Z_1(\lambda)\mathbf{B}_\pm(\lambda + i0)Z_1^*(\lambda)$ are correctly defined, bounded and depend continuously (in the weak sense) on $\lambda \in \Lambda$.

It turns out that though the notation $Z_1(\lambda)$ is only formal the product

$$Z_1(\lambda)K^* =: Z_1(\lambda; K), \tag{7}$$

where $K : \mathcal{H}_1 \to \mathcal{K}$ ($\mathcal{K}$ is an auxiliary Hilbert space), can be correctly defined. This requires, of course, some conditions on the operator K which can be verified in applications. Let us accept the following

Definition 1 *Let K be an H_1-bounded operator. Suppose that there exists a bounded weakly continuous in $\lambda \in \Lambda$ operator $Z_1(\lambda; K) : \mathcal{K} \to \mathbf{N}$ such that for every $f \in \mathcal{K}$ and a.e. $\lambda \in \Lambda$ the equality*

$$(F_1 K^* f)(\lambda) = Z_1(\lambda; K) f$$

holds. In this case we say that the operator (7) is correctly defined. We set $U_1(\lambda; K) = Z_1(\lambda; K)^*$.

Now we are able to rewrite representations (6) in a correct form. The proof of the following statement (under fairly more general assumptions) can be found in [8] or [9], Theorem 5.5.3.

Theorem 2 *Let the perturbation (5) satisfy $\mathbf{V} = K_2^* \mathcal{V} K_1$, where $K_j : \mathcal{H}_j \to \mathcal{K}$, K_1 is H_1-bounded, K_2 is $|H_2|^{1/2}$-bounded and $\mathcal{V} \in \mathcal{B}$. Assume that the operators $Z_1(\lambda; K_1)$, $Z_1(\lambda; K_2 J)$ are correctly defined and their adjoints $U_1(\lambda; K_1)$, $U_1(\lambda; K_2 J)$ are strongly continuous in $\lambda \in \Lambda$. Suppose also that the operator-function*

$$\mathcal{R}_2(z; K_2) := K_2 R_2(z) K_2^*$$

is weakly continuous in z, $\operatorname{Re} z \in \Lambda$, $\pm \operatorname{Im} z \geq 0$. Put

$$\mathcal{A}^+(\lambda) = Z_1(\lambda; K_2 J) \mathcal{V} U_1(\lambda; K_1), \quad \mathcal{A}^-(\lambda) = \mathcal{A}^+(\lambda)^*, \tag{8}$$

$$\mathbf{A}(\lambda) = Z_1(\lambda; K_1) \mathcal{V}^* \mathcal{R}_2(\lambda + i0; K_2) \mathcal{V} U_1(\lambda; K_1) \tag{9}$$

and

$$A^\pm(\lambda) = \mathcal{A}^\pm(\lambda) - \mathbf{A}(\lambda). \tag{10}$$

Then the wave operators (1) and (4) exist and the scattering matrix $S(\lambda)$ satisfies for $\lambda \in \Lambda$ the relations

$$S(\lambda) = w^\pm(\lambda) - 2\pi i A^\pm(\lambda).$$

In particular, the operator-functions $S(\lambda) - w^+(\lambda)$ and $S(\lambda) - w^-(\lambda)$ are weakly continuous in $\lambda \in \Lambda$.

Remark. Theorem 2 can be naturally extended to the case where $\mathbf{V}$ is a finite sum $\mathbf{V} = \sum_n K_{2,n}^* \mathcal{V}_n K_{1,n}$. This case can be reduced to the former one if one introduces "vector" operators $K_j = \bigoplus_n K_{j,n}$ and a diagonal matrix operator $\mathcal{V} = \mathrm{diag}\{\mathcal{V}_n\}$.

3. The N-particle Schrödinger operator is the self-adjoint operator $H = T+V$ in the Hilbert space $\mathcal{H} = L_2(\mathbf{R}^d)$, where $T = -\Delta$ and V is multiplication by a real function $V(x)$. Suppose that some finite number α_0 of subspaces X^α of $X := \mathbf{R}^d$ is given and let x^α, x_α be the orthogonal projections of $x \in X$ on X^α and its orthogonal complement $X_\alpha = X \ominus X^\alpha$, respectively. We assume that

$$V(x) = \sum_{\alpha=1}^{\alpha_0} V^\alpha(x^\alpha),$$

where V^α are decaying real functions of variables x^α. Clearly, $V^\alpha(x^\alpha)$ tends to zero as $|x| \to \infty$ outside of any conical neighbourhood of X_α. However $V^\alpha(x^\alpha)$ is constant on planes parallel to X_α. Due to this property the structure of the spectrum of H is much more complicated than in the two-particle case.

Let us introduce the set of all linear sums

$$X^a = X^{\alpha_1} + X^{\alpha_2} + \ldots + X^{\alpha_k}$$

of subspaces X^{α_j}. The zero subspace $X^0 = \{0\}$ is included in this set and X itself is excluded. We consider also the set of orthogonal complements $X_a := X \ominus X^a$, which consists of all intersections of subspaces X_α. Let x^a and x_a be the orthogonal projections of $x \in X$ on the subspaces X^a and X_a, respectively. In the multiparticle terminology, index a parametrizes decompositions of an N-particle system into noninteracting clusters; x^a is the set of "internal" coordinates of clusters, x_a describes their relative motion.

The scattering matrix is, of course, well-defined only for short-range potentials. However, our estimates on the resolvent hold as well true in the case, where potentials contain long-range parts. Therefore we distinguish two types of assumptions on functions V^α. The first of them is related to the short-range case and the second – to the general one. Let us define operators $T^\alpha = -\Delta^\alpha$ in the space $\mathcal{H}^\alpha = L_2(X^\alpha)$. Derivatives of V^α are understood below in the sense of distributions. We usually use the same notation for a function and the operator of multiplication by this function.

Assumption 3 *Operators $V^\alpha(T^\alpha + I)^{-1/2}$ are compact in the space $\mathcal{H}^\alpha$ and operators*

$$(|x^\alpha| + 1)^\rho V^\alpha(T^\alpha + I)^{-1/2}$$

are bounded in $\mathcal{H}^\alpha$ for some $\rho > 1$.

Assumption 4 *Functions V^α admit representations $V^\alpha = V_s^\alpha + V_l^\alpha$, where the short-range V_s^α and long-range V_l^α parts satisfy the following conditions. Operators $V_s^\alpha(T^\alpha + I)^{-1}$, $V_l^\alpha(T^\alpha + I)^{-1}$, $|\nabla V_l^\alpha|(T^\alpha + I)^{-1}$ are compact in the* *space $\mathcal{H}^\alpha$ and operators $(|x^\alpha|+1)^\rho V_s^\alpha(T^\alpha+I)^{-1}$, $(|x^\alpha|+1)^\rho|\nabla V_l^\alpha|(T^\alpha+I)^{-1}$ are bounded in $\mathcal{H}^\alpha$ for some $\rho>1$.*

Let us define for each a a potential

$$V^a=\sum_{X^\alpha\subset X^a}V^\alpha,$$

which does not depend on x_a, and the remaining part $V_a=V-V^a$, which tends to zero outside of any conical neighbourhood of X_a. Let the operator H^a be defined in the space $\mathcal{H}^a=L_2(X^a)$ by the equality $H^a=T^a+V^a$, $T^a=-\Delta^a$. The union over all a of their point spectra

$$\Upsilon_0=\bigcup\sigma^{(p)}(H^a)$$

is called the set of thresholds for the operator H. *Below Λ is always an arbitrary bounded interval such that* $\bar\Lambda\cap\Upsilon=\emptyset$, where $\bar\Lambda$ is the closure of Λ and $\Upsilon=\Upsilon_0\cup\sigma^{(p)}(H)$. Let Q be multiplication by $(x^2+1)^{1/2}$. The following basic result (see [6, 7]) of spectral theory of multiparticle Hamiltonians is called the limiting absorption principle.

Proposition 5 *Let Assumption 4 hold. Then for any $r>1/2$ the operator $Q^{-r}(T+I)^{1/2}R(z)(T+I)^{1/2}Q^{-r}$ is continuous in topology of the norm with respect to z for* $\operatorname{Re} z\in\Lambda$, $\pm\operatorname{Im} z\geq 0$.

Let us give the precise formulation of the scattering problem for N-particle Hamiltonians. It can be conveniently performed in terms of the scattering theory in a couple of spaces. Our considerations are localized on an interval Λ. Let us introduce eigenvalues $\lambda^{a,n}$ of the operator H^a lying below Λ:

$$\lambda^{a,n}<\inf\Lambda. \tag{11}$$

Denote by $\psi^{a,n}$ a normalized eigenvector of the operator H^a corresponding to an eigenvalue $\lambda^{a,n}$. If some eigenvalues coincide, the corresponding eigenvectors are supposed to be orthogonal. We usually write $\mathbf{a}$ instead of a couple $\{a,n\}$.

Let us introduce a space

$$\hat{\mathcal{H}}=\bigoplus_{\mathbf{a}}\mathcal{H}_{\mathbf{a}},\quad \mathcal{H}_{\mathbf{a}}=\mathcal{H}_a=L_2(X_a), \tag{12}$$

of all scattering channels and a "free" Hamiltonian

$$\hat{H}=\bigoplus_{\mathbf{a}}T_{\mathbf{a}},\quad T_{\mathbf{a}}=T_a+\lambda^{\mathbf{a}},\quad T_a=-\Delta_a, \tag{13}$$

in this space. Here and below the sums are taken over all $\mathbf{a}$ satisfying (11). We define an identification $J:\hat{\mathcal{H}}\to\mathcal{H}$ by the relations

$$J=\sum_{\mathbf{a}}J^{\mathbf{a}},\quad J^{\mathbf{a}}f_{\mathbf{a}}=f_{\mathbf{a}}\otimes\psi^{\mathbf{a}}, \tag{14}$$

where the tensor product is determined by the decomposition $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}^a$. In particular, $T_0 = T$, $\lambda^0 = 0$ and $J^0 = I$. Note that $H_a = T_a \otimes I + I \otimes H^a$ and hence

$$H_a J^{\mathbf{a}} = J^{\mathbf{a}} T_{\mathbf{a}}. \tag{15}$$

The basic result of the scattering theory for N-particle Schrödinger operators can be formulated as follows.

Theorem 6 *Let Assumption 3 hold. Then the wave operators $W^{\pm}(H, \hat{H}; \hat{J}, \Lambda)$ exist, are isometric and complete.*

This theorem implies, of course, that the scattering operator $\mathbf{S}$ defined by (2) is unitary in the space $\hat{\mathcal{H}}(\Lambda) = E(\Lambda; \hat{H})\hat{\mathcal{H}}$ and commutes with the operator $\hat{H}$. Clearly, $\mathbf{S}$ is the matrix operator with components

$$\mathbf{S}_{\mathbf{ab}} = (W_{\mathbf{a}}^{+})^{*} W_{\mathbf{b}}^{-} : \mathcal{H}_b \to \mathcal{H}_a, \quad W_{\mathbf{a}}^{\pm} = W^{\pm}(H, T_{\mathbf{a}}; J^{\mathbf{a}}, \Lambda).$$

To define the corresponding scattering matrix we consider a diagonal for $\hat{H}$ representation of the space $\hat{\mathcal{H}}(\Lambda)$ in a direct integral:

$$\hat{\mathcal{H}}(\Lambda) \leftrightarrow L_2(\Lambda; \mathbf{N}), \tag{16}$$

where

$$\mathbf{N} = \bigoplus_{\mathbf{a}} \mathbf{N}_{\mathbf{a}}, \quad \mathbf{N}_{\mathbf{a}} = \mathbf{N}_a = L_2(\mathbb{S}^{d_a - 1}).$$

Let us introduce a unitary mapping $\hat{F} = \bigoplus_{\mathbf{a}} F_{\mathbf{a}}$ of the left-hand side of (16) onto its right-hand side. Explicitly, $F_{\mathbf{a}}$ is given by the standard formula

$$(F_{\mathbf{a}} f)(\lambda; \omega_a) = (\kappa^{\mathbf{a}})^{-1 + d_a/2} 2^{-1/2} (\Phi_a f)(\kappa^{\mathbf{a}} \omega_a), \quad \kappa^{\mathbf{a}} = (\lambda - \lambda^{\mathbf{a}})^{1/2},$$

where

$$(\Phi_a f)(\xi_a) = (2\pi)^{-d_a/2} \int \exp(-i\langle \xi_a, x_a \rangle) f(x_a) dx_a$$

is the Fourier transform in the space $L_2(\mathbf{R}^{d_a})$ (the symbol $\langle \cdot, \cdot \rangle$ denotes the scalar product in different Euclidean spaces). Then $(\hat{F}\hat{H}f)(\lambda) = \lambda(\hat{F}f)(\lambda)$ for $f \in \hat{\mathcal{H}}(\Lambda)$.

Once the spectral representation of $\hat{H}$ is fixed, the scattering matrix $S(\lambda) : \mathbf{N} \to \mathbf{N}$ can be introduced by the general time-dependent definition of p.2. Now $S(\lambda)$ is a "matrix" operator with components $S_{\mathbf{a},\mathbf{b}}(\lambda)$ acting from the space $L_2(\mathbb{S}^{d_b - 1})$ into the space $L_2(\mathbb{S}^{d_a - 1})$. We emphasize that $S(\lambda)$ is defined and unitary for a.e. $\lambda \in \Lambda$ only. Our goal below is to obtain stationary representations for it. When considering $S_{\mathbf{a},\mathbf{b}}(\lambda)$ we can put $J = J_{\mathbf{a}} + J_{\mathbf{b}}$ if $\mathbf{a} \neq \mathbf{b}$ and $J = J_{\mathbf{a}}$ if $\mathbf{a} = \mathbf{b}$. Thus, without loss of generality, we may suppose that the sums (12) - (14) contain a finite number of terms only.

4. To justify the representations for the scattering matrix we need to reinforce the estimates of [3] called the radiation conditions-estimates there. In contrast to [3], where estimates were formulated in terms of the unitary group $\exp(-iHt)$, we study the resolvent of H here.

Let ∇_a be the gradient in the variable x_a (i.e. $\nabla_a u$ is the orthogonal projection of ∇u on the subspace X_a) and let $\nabla_a^{(s)}$,

$$(\nabla_a^{(s)} u)(x) = (\nabla_a u)(x) - |x_a|^{-2}\langle(\nabla_a u)(x), x_a\rangle x_a,$$

be its orthogonal projection in X_a on the plane orthogonal to the vector x_a. Let

$$\mathbf{X}_a(\varepsilon) = \{|x_a| > (1-\varepsilon)|x|\}, \quad \varepsilon \in (0,1),$$

be a conical neighbourhood of $X_a \setminus \{0\}$. We put

$$\mathbf{\Gamma}_a(\varepsilon) = X \setminus \bigcup_{X_a \not\subset X_b} \mathbf{X}_b(\varepsilon).$$

Clearly, $\mathbf{\Gamma}_a(\varepsilon)$ gets larger as ε decreases and every closed cone $\mathbf{\Gamma}_a$ such that $\mathbf{\Gamma}_a \cap X_b = \{0\}$ if $X_a \not\subset X_b$ belongs to a cone $\mathbf{\Gamma}_a(\varepsilon)$ for sufficiently small ε. In particular, $\mathbf{\Gamma}_0$ is the whole space with some neighbourhoods of all subspaces X_α removed from it. Denote by $\chi(\mathcal{Y})$ the characteristic function of a set $\mathcal{Y} \subset \mathbf{R}^d$. Let us introduce the operator

$$G_a(\varepsilon) = \chi(\mathbf{\Gamma}_a(\varepsilon))Q^{-1/2}\nabla_a^{(s)} \tag{17}$$

which acts from the space $\mathcal{H}$ into $\mathcal{H} \otimes \mathbb{C}^{d_a}$. Our resolvent estimates are formulated in the following

Theorem 7 *Let Assumption 4 hold. Choose any $\varepsilon > 0$ and set $G_a = G_a(\varepsilon)$. Then for arbitrary a and b*

$$\|G_a R(z) G_b^*\| \le C, \quad \operatorname{Re} z \in \Lambda.$$

Corollary 8 *Let $\operatorname{Re} z \in \Lambda$, $\pm \operatorname{Im} z \ge 0$ and $r > 1/2$. Then the operator-function $Q^{-r}(T+I)^{1/2}R(z)G_b^*$ is strongly continuous and operator-functions $G_a R(z)(T+I)^{1/2}Q^{-r}, G_a R(z) G_b^*$ are weakly continuous in z.*

We need also some results on strong continuity of some operator-functions related to the spectral family of the "free" operator $\hat{H}$. In some sense these results can be considered as generalizations of Sobolev theorems about traces of functions on a sphere or, more precisely, of the corresponding dual assertions.

Let us introduce operators

$$(U_{\mathbf{a}}(\lambda)u)(x_a) = \kappa^{(d_a-2)/2}2^{-1/2}(2\pi)^{-d_a/2}$$
$$\times \int \exp(i\kappa\langle\omega_a, x_a\rangle)u(\omega_a)d\omega_a, \quad \omega_a \in \mathbb{S}^{d_a-1}, \ \kappa = (\lambda - \lambda^{\mathbf{a}})^{1/2} > 0,$$

and

$$G_b(\varepsilon)J^{\mathbf{a}}U_{\mathbf{a}}(\lambda) : \mathbf{N}_a \to \mathcal{H}. \tag{18}$$

Theorem 9 *Let Assumption 4 hold. Then for arbitrary a, b and $\varepsilon > 0$ the operator (18) is bounded and strongly continuous in $\lambda > \lambda^{\mathbf{a}}$.*

5. In this section we construct an auxiliary identification $\tilde{J}$ such that the triple $\{\hat{H}, H, \tilde{J}\}$ satisfies the assumptions of Theorem 2. This gives immediately stationary representations for the scattering matrix $\tilde{S}(\lambda)$ corresponding to $\tilde{J}$. A verification of the assumptions of Theorem 2 is based on the estimates of p.4. We return back to the "physical" identification J in the next part.

Let us introduce a differential operator

$$M^{(a)} = 2^{-1} \sum_{j=1}^{d} (m_j^{(a)} D_j + D_j m_j^{(a)}), \quad m_j^{(a)} = \partial m^{(a)} / \partial x_j, \tag{19}$$

with a "generating" function $m^{(a)}$ obeying the following conditions:

1^0 $m^{(a)}(x)$ is a real C^∞-function, which is homogeneous of degree 1 for $|x| \geq 1$ and $m^{(a)}(x) = 0$ for $|x| \leq 1/2$.

2^0 Let b be arbitrary. If $x \in \mathbf{X}_b(\varepsilon)$ and $|x| \geq 1$, then $m^{(a)}(x) = m^{(a)}(x_b)$, i.e. $m^{(a)}(x)$ does not depend on x^b.

3^0 Let $X_a \not\subset X_b$. If $x \in \mathbf{X}_b(\varepsilon)$, then $m^{(a)}(x) = 0$.

Functions $m^{(a)}$ with such such properties were constructed in [3]. We emphasize that ε is a fixed positive number which can be chosen arbitrary small.

Let us consider the "effective perturbation" (cf. (5)) for the pair H_a, H relative to the identification $M^{(a)}$:

$$\mathbf{V}^{(a)} := H M^{(a)} - M^{(a)} H_a = [T, M^{(a)}] + [V^a, M^{(a)}] + V_a M^{(a)}, \tag{20}$$

Properties of different terms in the right-hand side were considered in [3]. They are formulated in the following three assertions.

Proposition 10 *Let Assumption 3 hold and let $m^{(a)}$ satisfy the conditions 1^0, 2^0. Then*

$$[V^a, M^{(a)}] = (T + I)^{1/2} Q^{-r} B_a Q^{-r} (T + I)^{1/2},$$

where $2r = \rho > 1$ and $B_a \in \mathcal{B}$.

Proposition 11 *Let Assumption 3 hold and let $m^{(a)}$ satisfy the conditions 1^0, 3^0. Then*

$$V_a M^{(a)} = (T + I)^{1/2} Q^{-r} \tilde{B}_a Q^{-r} (T + I)^{1/2},$$

where $2r = \rho > 1$ and $\tilde{B}_a \in \mathcal{B}$.

Proposition 12 *Let $m^{(a)}$ satisfy the conditions 1^0, 2^0. Then*

$$[T, M^{(a)}] = \sum_b G_b^* B_{a,b} G_b + Q^{-3/2} B_a^{(0)} Q^{-3/2}, \quad B_{a,b} \in \mathcal{B}, \quad B_a^{(0)} \in \mathcal{B},$$

where the operators G_b are defined by (17).

Recall that the operator $\hat{H}$ and its spectral representation were constructed in p.3. Let us introduce formal operators $Z_{\mathbf{a}}(\lambda), Z_{\mathbf{a}}(\lambda)f = (F_{\mathbf{a}}f)(\lambda)$. Then (also formally) $U_{\mathbf{a}}(\lambda) = Z_{\mathbf{a}}(\lambda)^*$. We set

$$\mathbf{Z}_{\mathbf{a}}(\lambda) = Z_{\mathbf{a}}(\lambda)(J^{\mathbf{a}})^*, \quad \mathbf{U}_{\mathbf{a}}(\lambda) = \mathbf{Z}_{\mathbf{a}}(\lambda)^* = J^{\mathbf{a}}U_{\mathbf{a}}(\lambda).$$

Let us define the identification $\tilde{J}$ by the relation (cf. (14))

$$\tilde{J} = \sum_{\mathbf{a}} \tilde{J}^{\mathbf{a}}, \quad \tilde{J}^{\mathbf{a}} = M^{(a)}J^{\mathbf{a}}. \tag{21}$$

We check that the triple $\{\hat{H}, H, \tilde{J}\}$ satisfies on Λ the assumptions of Theorem 2.

Let us consider $H\tilde{J} - \tilde{J}\hat{H}$. In virtue of (15)

$$\tilde{\mathbf{V}}^{(\mathbf{a})} := H\tilde{J}^{\mathbf{a}} - \tilde{J}^{\mathbf{a}}T_{\mathbf{a}} = (HM^{(a)} - M^{(a)}H_a)J^{\mathbf{a}} = \mathbf{V}^{(\mathbf{a})}J^{\mathbf{a}}.$$

According to the equality (20) and Propositions 10-12 this operator admits the representation

$$\tilde{\mathbf{V}}^{(\mathbf{a})} = (T+I)^{1/2}Q^{-r}B_aQ^{-r}(T+I)^{1/2}J^{\mathbf{a}} + \sum_b G_b^*B_{a,b}G_bJ^{\mathbf{a}}, \tag{22}$$

where $r > 1/2$, $B_a, B_{a,b}$ are bounded operators and the sum is taken over all b. The operator (22) can be factorized as $\tilde{\mathbf{V}}^{(\mathbf{a})} = K^*\mathcal{V}K_{\mathbf{a}}^{(0)}$, where $\mathcal{V} \in \mathcal{B}$, K is a "vector" operator with components $Q^{-r}(T+I)^{1/2}$ and G_b (all b), $K_{\mathbf{a}}^{(0)}$ is a "vector" operator with components $Q^{-r}(T+I)^{1/2}J^{\mathbf{a}}$ and $G_bJ^{\mathbf{a}}$.

The weak continuity of the operator $KR(z)K^*$ in z, $\operatorname{Re} z \in \Lambda, \pm \operatorname{Im} z \geq 0$, is a consequence of Proposition 5 and Corollary 8. The assumptions of Theorem 2 with respect to the "free" operator $T_{\mathbf{a}}$ require boundedness and strong continuity of the operators $K_{\mathbf{a}}^{(0)}U_{\mathbf{a}}(\lambda)$ and $K_{\mathbf{a}}\tilde{J}^{\mathbf{a}}U_{\mathbf{a}}(\lambda)$. This is equivalent to the same statements about the operators

$$Q^{-r}(T+I)^{1/2}J^{\mathbf{a}}U_{\mathbf{a}}(\lambda), \quad G_bJ^{\mathbf{a}}U_{\mathbf{a}}(\lambda),$$

$$Q^{-r}(T+I)^{1/2}M^{(a)}J^{\mathbf{a}}U_{\mathbf{a}}(\lambda), \quad G_bM^{(a)}J^{\mathbf{a}}U_{\mathbf{a}}(\lambda),$$

acting from the space $\mathbf{N}_{\mathbf{a}}$ into $\mathcal{H}$. Boundedness and strong continuity of these operators are consequences of the standard Sobolev (dual) trace theorem and of Theorem 9.

Thus we have obtained the following

Theorem 13 *Let Assumption 3 hold. Suppose that $\tilde{J}$ is defined by (19), (21), where $m^{(a)}$ obeys the conditions $1^0 - 3^0$. Then the triple $\{\hat{H}, H, \tilde{J}\}$ satisfies on Λ the assumptions of Theorem 2.*

Theorem 2 gives immediately stationary representations for the scattering matrix $\tilde{S}(\lambda) = S(\lambda; H, \hat{H}, \tilde{J})$ or for its components $\tilde{S}_{\mathbf{a},\mathbf{b}}(\lambda)$. Actually, following (8)-(10), we set

$$\mathcal{A}^+_{\mathbf{a},\mathbf{b}}(\lambda) = \mathbf{Z_a}(\lambda) M^{(a)} \mathbf{V}^{(b)} \mathbf{U_b}(\lambda), \ \mathcal{A}^-_{\mathbf{a},\mathbf{b}}(\lambda) = \mathbf{Z_a}(\lambda)(\mathbf{V}^{(a)})^* M^{(b)} \mathbf{U_b}(\lambda), \quad (23)$$

$$\mathbf{A_{a,b}}(\lambda) = \mathbf{Z_a}(\lambda)(\mathbf{V}^{(a)})^* R(\lambda + i0) \mathbf{V}^{(b)} \mathbf{U_b}(\lambda) \quad (24)$$

and

$$A^{\pm}_{\mathbf{a},\mathbf{b}}(\lambda) = \mathcal{A}^{\pm}_{\mathbf{a},\mathbf{b}}(\lambda) - \mathbf{A_{a,b}}(\lambda). \quad (25)$$

We note that the representation (20) together with analytical results of Propositions 10-12 allow us to rewrite the right-hand sides of (23), (24) as combinations of bounded operators. This gives a precise sense to these definitions. We emphasize that operators (23) are strongly and the operator (24) is weakly continuous in λ.

The auxiliary wave operators

$$\mathbf{w}^{\pm}_{\mathbf{a},\mathbf{b}} = W^{\pm}(T_{\mathbf{a}}, T_{\mathbf{b}}; E_{\mathbf{a}}(\Lambda)(J^{\mathbf{a}})^* M^{(a)} M^{(b)} J^{\mathbf{b}} E_{\mathbf{b}}(\Lambda)) \quad (26)$$

act as multiplications by operator-functions $w^{\pm}_{\mathbf{a},\mathbf{b}}(\lambda) : \mathbf{N_a} \to \mathbf{N_b}$. They can be calculated explicitly (see Proposition 17 below). Combining Theorems 2 and 13 we arrive at the following

Theorem 14 *Under the assumptions of Theorem 13 the scattering matrix* $\tilde{S}_{\mathbf{a},\mathbf{b}}(\lambda)$ *satisfies the following two equalities*

$$\tilde{S}_{\mathbf{a},\mathbf{b}}(\lambda) = w^{\pm}_{\mathbf{a},\mathbf{b}}(\lambda) - 2\pi i A^{\pm}_{\mathbf{a},\mathbf{b}}(\lambda). \quad (27)$$

6. The operator $S_{\mathbf{a},\mathbf{b}}(\lambda)$ can (at least formally) be represented by its kernel parametrized by angular variables $\omega_a \in \mathbb{S}^{d_a-1}$ and $\omega_b \in \mathbb{S}^{d_b-1}$. As it is physically natural to expect, these kernels acquire additional singularities as ω_a approaches some plane $X_{a'} \subset X_a$, $X_{a'} \neq X_a$ or ω_b approaches some plane $X_{b'} \subset X_b$, $X_{b'} \neq X_b$. We concentrate here on the simplest case when ω_a and ω_b are separated from all such "singular" planes $X_{a'}$ and $X_{b'}$, respectively.

To derive representations for components $S_{\mathbf{a},\mathbf{b}}(\lambda)$ of the scattering matrix $S(\lambda) = S(\lambda; H, \hat{H}, J)$ with the physical identification J we find a relation connecting $\tilde{S}(\lambda)$ and $S(\lambda)$. First, we get rid of operators $M^{(a)}$ in the definition of wave operators

$$W^{\pm}(H, H_a; M^{(a)}, \Lambda) = W^{\pm}(H, H_a) W^{\pm}(H_a, H_a; M^{(a)}, \Lambda) \quad (28)$$

(this equality is called the multiplication theorem for wave operators). It turns out that the operators $W^{\pm}(H_a, H_a; M^{(a)}, \Lambda) J^{\mathbf{a}}$ can be calculated explicitly. Let us define on X_a the function

$$\underline{m}^{(a)}(x_a) = |x_a| m^{(a)}(x_a |x_a|^{-1}).$$

This function coincides with the trace of $m^{(a)}(x)$ on the subspace X_a if $|x_a| \geq 1$ and it is extended by homogeneity (of degree 1) to all $x_a \neq 0$. Denote by χ_Λ the characteristic function of Λ.

We start with the case $a = 0$ and set

$$\Omega_0^\pm = \Omega_0^\pm(m^{(0)}) = W^\pm(H_0, H_0; M^{(0)}, \Lambda).$$

These operators commute with H_0. Recall that

$$(\exp(-iH_0 t)f)(x) \sim (2it)^{-d/2} \exp(ix^2/4t)\hat{f}(x/2t), \quad \hat{f} = \Phi_0 f,$$

as $t \to \pm\infty$ (the sign " $\sim$ " means that the difference between left and right sides tends to zero in $L_2(\mathbf{R}^d)$).

Lemma 15 *Let $m^{(0)}$ be an arbitrary smooth homogeneous function of degree 1. Then the operator $\Phi_0 \Omega_0^\pm \Phi_0^*$ is multiplication by the function $\pm\underline{m}^{(0)}(\pm\xi)\chi_\Lambda(\xi^2)$.*

In the general case we have

Proposition 16 *Let $m^{(a)}$ be an arbitrary smooth homogeneous function of degree 1 and $m^{(a)}(x) = m^{(a)}(x_a)$ in some conical neighbourhood of X_a for $|x| \geq 1$. Then*

$$W^\pm(H_a, H_a; M^{(a)}, \Lambda)J^{\mathbf{a}} = \Omega_{\mathbf{a}}^\pm(m^{(a)}) \otimes J^{\mathbf{a}},$$

where the operator $\Phi_a \Omega_{\mathbf{a}}^\pm(m^{(a)})\Phi_a^*$ acts as multiplication by the function

$$\pm\underline{m}^{(a)}(\pm\xi_a)\chi_\Lambda(\lambda^{\mathbf{a}} + \xi_a^2).$$

In particular, this assertion allows us to calculate the operators (26).

Proposition 17 *If $\mathbf{a} \neq \mathbf{b}$, then $\mathbf{w}_{\mathbf{a},\mathbf{b}}^\pm = 0$. If $\mathbf{a} = \mathbf{b}$, then $\mathbf{w}_{\mathbf{a},\mathbf{a}}^\pm = \Omega_{\mathbf{a}}^\pm(m^{(a)})^2$, where $\Phi_a \Omega_{\mathbf{a}}^\pm(m^{(a)})\Phi_a^*$ is the same as in Proposition 16.*

Furthermore, with a help of (28) we find a relation between the scattering operators

$$\Omega_{\mathbf{a}}^+(m^{(a)})\mathbf{S}_{\mathbf{a},\mathbf{b}}\Omega_{\mathbf{b}}^-(m^{(b)}) = \tilde{\mathbf{S}}_{\mathbf{a},\mathbf{b}},$$

corresponding to the identifications J and $\tilde{J}$. In terms of the scattering matrices it means that for a.e. $\lambda \in \Lambda$

$$m_{\mathbf{a}}^+(\lambda)S_{\mathbf{a},\mathbf{b}}(\lambda)m_{\mathbf{b}}^-(\lambda) = \tilde{S}_{\mathbf{a},\mathbf{b}}(\lambda), \tag{29}$$

where $m_{\mathbf{a}}^\pm(\lambda)$ is multiplication by $\pm(\lambda - \lambda^{\mathbf{a}})^{1/2} m^{(a)}(\pm\omega_a)$.

We can rewrite the representations (27) for $\tilde{S}_{\mathbf{a},\mathbf{b}}(\lambda)$ taking into account Proposition 17. According to (29) this gives representations for the "physical" scattering matrix $S_{\mathbf{a},\mathbf{b}}(\lambda)$ cut-off by the operators $m_{\mathbf{a}}^\pm(\lambda)$.

Theorem 18 *Under the assumptions of Theorem 13*

$$m_{\mathbf{a}}^{+}(\lambda)S_{\mathbf{a},\mathbf{b}}(\lambda)m_{\mathbf{b}}^{-}(\lambda) = m_{\mathbf{a}}^{\pm}(\lambda)^{2}\delta_{\mathbf{a},\mathbf{b}} - 2\pi i A_{\mathbf{a},\mathbf{b}}^{\pm}(\lambda), \quad a.e.\ \lambda \in \Lambda, \tag{30}$$

where $A_{\mathbf{a},\mathbf{b}}^{\pm}(\lambda)$ *is defined by equalities (23)-(25),* $\delta_{\mathbf{a},\mathbf{a}} = I$ *and* $\delta_{\mathbf{a},\mathbf{b}} = 0$ *if* $\mathbf{a} \neq \mathbf{b}$.

Because of the property 3^0 of functions $m^{(a)}$ and $m^{(b)}$, the equality (30) gives a representation for the sesquilinear form of $S_{\mathbf{a},\mathbf{b}}(\lambda)$ only on functions vanishing in neighbourhoods of singular directions (see the beginning of this part).

The weak continuity of $S_{\mathbf{a},\mathbf{b}}(\lambda)$ on dense sets together with the bound $\|S_{\mathbf{a},\mathbf{b}}(\lambda)\| \leq 1$ ensure the following

Theorem 19 *Let Assumption 3 hold. For any* $\mathbf{a}$, $\mathbf{b}$ *the operator* $S_{\mathbf{a},\mathbf{b}}(\lambda)$ *is weakly continuous in* $\lambda \in \Lambda$. *Thus the representation (30) holds for all* $\lambda \in \Lambda$.

Probably continuity of $S_{\mathbf{a},\mathbf{b}}(\lambda)$ holds in the strong sense as well. On the contrary, in contrast to the two-particle problem in the general case one can not expect continuity in the topology of the norm.

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