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Propagation of Singularities for Non Real Pseudo-Differential Operators

Publications de l'Institut de recherche mathématiques de Rennes, 1992-1993, fascicule 1
« Fascicule d'équations aux dérivées partielles », , exp. n° 5, p. 1-25

http://www.numdam.org/item?id=PSMIR_1992-1993__1_A5_0

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Propagation of singularities for non real pseudo-differential operators

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January 1993

Abstract

The purpose of this work is to prove a theorem of propagation of singularities for a class of non real pseudo-differential operator with multiple characteristics. The main tools are L^2 estimates on the time dependent Schrödinger equation related to P . We extend here the results of [6]; we improve the results announced by the second author in [7].

The second part of this work consists in an extension of the result of [5] to complex valued symbols.

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Part I

The General result.

We start by a general result which could not be optimal in all the cases scanned here. The approach is similar to [6] but we use also time dependent L^2 estimate and some informations on the parametrix constructed in [6]. The main difference with the proof in [7] is that we need to have an analysis of the microlocal structure of the parametrix of the time dependent Schrödinger equation associated with the self-adjoint part of our operator.

1 Introduction and main statement

Let $P(x, \lambda^{-1}D_x, \lambda)$ be a pseudo-differential operator depending on a large parameter λ , defined by the Weyl formula :

$$\begin{aligned}
P(x, \lambda^{-1} D_x, \lambda) u(x, \lambda) &= op_{1/2}((p(x, \xi/\lambda, \lambda))(u))(x) \\
&= \left(\frac{\lambda}{2\pi}\right)^n \int p\left(\frac{x+y}{2}, \xi, \lambda\right) e^{i\lambda(x-y)\xi} u(y, \lambda) dy d\xi
\end{aligned} \tag{1}$$

We shall write the operator given by formula 1 $(p(x, \xi, \lambda))^{\omega_\lambda}$.

The full symbol $p(x, \xi, \lambda)$ has an expansion as

$$p(x, \xi, \lambda) = p_1(x, \xi) + ip_2(x, \xi) + \lambda^{-1} p_0(x, \xi, \lambda) \tag{2}$$

where p_1 and p_2 are real and $p_2 \geq 0$. $p_0(x, \xi, \lambda)$ is a zero order symbol i.e. satisfies estimates :

$$\text{For all multi-indices } \alpha \text{ and } \beta \quad \left| D_x^\alpha D_\xi^\beta p_0(x, \xi, \lambda) \right| \leq C_{\alpha, \beta}.$$

It is a consequence of formulas 1 and 2 that

$$P(x, \lambda^{-1} D_x, \lambda) = P_1(x, \lambda^{-1} D_x, \lambda) + iP_2(x, \lambda^{-1} D_x, \lambda) \tag{3}$$

where P_1 and P_2 are self-adjoint pseudo-differential operators with symbols respectively

$$P_1 = (p_1(x, \xi) + \lambda^{-1} Re(p_0(x, \xi, \lambda)))^{\omega_\lambda} \tag{4}$$

$$P_2 = (p_2(x, \xi) + \lambda^{-1} Im(p_0(x, \xi, \lambda)))^{\omega_\lambda} \tag{5}$$

We shall make L^2 estimates on the solutions $u(t)$ of the Schrödinger equation $(D_t + P(x, \lambda^{-1} D_x, \lambda))u(t) = 0$. We shall therefore use some constructions made in [6]. We need to recall the hypotheses of this work.

- (H)₁ : Let Φ_t be the bicharacteristic flow of p_1 at the time t . Let ρ_0 in $T^*\mathbf{R}^n$ be a point near which we shall work. Let $h(t) \in o(t)$ when $t \rightarrow \infty$, $h \geq 0$ a function, and W a neighborhood of ρ_0 such that any bicharacteristic curve of p_1 with end points lying in

$$\Lambda(W, h) = \left\{ \begin{array}{l} (\rho_1, \rho_2) \in W^2 ; \exists t \geq 0 \text{ such that } \rho_1 = \Phi_t(\rho_2), \\ \text{and if } 0 \leq s \leq t \Phi_s(\rho_2) \in W \text{ and } |p_1(\rho_1)| \leq \exp(-h(t)) \end{array} \right\} \tag{6}$$

is N_0 admissible¹ for a function $\varepsilon \in]0, 1] \rightarrow N_0(\varepsilon) \in \mathbf{R}^+$. We refer to [6] pp 468-469 for a definition and for sufficient conditions which

¹A curve $t \in [0, T] \rightarrow \gamma(t) \in W$ is N_0 admissible if for any $\varepsilon > 0$, there exists a partition of $[0, T]$ in intervals of type $\mathcal{I}$ and $\mathcal{J}$, the number of these intervals is less than $N_0(\varepsilon)$. An interval of type $\mathcal{I}$ remains at a distance less than ε from a point in the double characteristic set of P_1 , an interval of type $\mathcal{J}$ has a length less than $N_0(\varepsilon)$.

imply this property. We shall not recall here the details, but we just mention that it is satisfied if the bicharacteristics of p_1 whose length is large enough leave a neighborhood of (ρ_0, ρ_0) .

Let $N = \{\rho \in T^*(\mathbb{R}^n); p_1(\rho) = dp_1(\rho) = 0\}$ be the set of double characteristics of P_1 .

- $(H)_2$: The main assumption is that on N , the dimension of the space spanned by the generalized eigenvectors associated with eigenvalues of positive imaginary part is constant.
- $(H)_3$: On N , $Imp_0(\rho) > 0$. This inequality means that Imp_0 has a positive lower bound with respect to ρ and λ .

Let us define

$$C(\overline{W}) = \bigcap_{h \in \omega} \overline{\Lambda(W, h)} \quad (7)$$

where ω is the set of all non negative increasing functions $h(t) \in o(t)$ when $t \rightarrow \infty$.

We consider $C(\overline{W})$ as a relation in $T^*(\mathbb{R}^n)$.

We note by $OF(u)$ the oscillatory front set of a bounded family of tempered distributions $u(x, \lambda)$.

Let us recall that we say that $(x_0, \xi_0) \in (OF(u))^c$ if there are neighborhoods V of x_0 and L of ξ_0 such that for any $\varphi \in C_0^\infty(V)$

$$\text{For all } N \in \mathbb{N}, \text{ for } \lambda \geq 1 \sup_{\xi \in L} |\widehat{\varphi u}|(\lambda \xi, \lambda) \leq C_N \lambda^{-N}.$$

The main result can now be stated.

Theorem 1 *Assume that the assumptions $(H)_1$, $(H)_2$, $(H)_3$ are satisfied for a suitable set W and for some function $h_0 \in \omega$. Let $u(x, \lambda)$ be a bounded family of tempered distributions. If $OF(Pu) \cap \overline{W} = \emptyset$ and $C(\overline{W})(\rho) \cap \partial W \cap OF(u) = \emptyset$, then $\rho \notin OF(u)$.*

We easily deduce :

Corollary 1 *Let $\rho_0 \in T^*(\mathbb{R}^n) \setminus 0$, $P(x, D_x)$ a pseudo-differential operator in the usual sense. Assume that $(H)_1$, $(H)_2$, $(H)_3$ are satisfied for a neighborhood W of ρ_0 and a function $h_0 \in \omega$. Let u be a distribution such that $WF(Pu) \cap \overline{W} = \emptyset$ and $C(\overline{W})(\rho) \cap \partial W \cap WF(u) = \emptyset$, then $\rho \notin WF(u)$.*

The main difference between the proof of this theorem and the corresponding result in [7] is the presence in the bicharacteristic flow of p_1 of expansive directions. This will make us to use fully the construction of the parametrix of [6] instead of using only microlocalisations in a semi-global L^2 inequality for the solutions of the time dependent Schrödinger equation associated with P_1 .

2 Estimates on solutions of the Schrödinger equation

We shall work with a family of solutions of the Schrödinger equation $(D_t + P(x, \lambda^{-1} D_x))u(t) = 0$, where $D_t = (1/i\lambda)\partial/\partial t$.

We need to make a Fourier-Bros-Iagolnitzer transformation (see [9] and [6]). Let

$$Tu(x, \lambda) = \left(\frac{\lambda}{2i\pi}\right)^{n/2} \int e^{-\lambda/2((x-y)^2 - x^2/2)} u(y, \lambda) dy \quad (8)$$

this is a unitary transformation from $L^2(\mathbb{R}^n)$ to the space $H_{\varphi_0}(\mathbb{C}^n)$ of entire functions in $L^2(\mathbb{C}^n, e^{-2\lambda\varphi_0} L(dx))$, where $L(dx)$ is the Lebesgue measure in $\mathbb{C}^n$, and $\varphi_0(x) = \frac{1}{4}|x|^2$.

We note by the same letter an operator and its conjugate by T .

We have a Bergman projector from $L^2(\mathbb{C}^n, e^{-2\lambda\varphi_0} L(dx))$ to $H_{\varphi_0}(\mathbb{C}^n)$ given by the formula

$$Su(x, \lambda) = \left(\frac{\lambda}{2i\pi}\right)^n \int e^{\lambda x \bar{y}/2} u(y, \lambda) e^{-2\lambda\varphi_0(y)} L(dy) \quad (9)$$

see [9] for these formulas. Let us say that the formula 9 is obtained by integrating the formal integral in TT^{-1} along a suitable contour.

In [6] we can find some constructions for an approximate solution $E_t u(x, \lambda)$ of the equation $(D_t + P_1(x, \lambda^{-1} D_x))(E_t u) \equiv 0$; $E_t u|_{t=0} \equiv u$, we shall make this more precise later.

$$E_t u(x, \lambda) = \left(\frac{\lambda}{2i\pi}\right)^n \int e^{i\lambda\varphi(t, x, \bar{y})} e(t, x, \bar{y}, \lambda) \chi(t, x, \bar{y}) u(y, \lambda) e^{-2\lambda\varphi_0(y)} L(dy) \quad (10)$$

where $\varphi(t, x, y)$ is a solution of the phase equation with value $\varphi(0, x, \bar{y}) = -ix\bar{y}/2$; $e(t, x, y, \lambda)$ is a solution of transport equations, $\chi(t, x, y)$ is a cut-off function.

Let

$$\Gamma_t(W, h) = \left\{ (x, y); \left(x, \frac{2}{i} \frac{\partial \varphi_0}{\partial x}, y, \frac{2}{i} \frac{\partial \varphi_0}{\partial y}\right) \in \Lambda'_t(W, h) \right\} \quad (11)$$

where $\Lambda'_t(W, h)$ is the image by the complex canonical transformation generated by $\varphi_0(x)$ of the Lagrangean sub manifold

$$\Lambda_t(W, h) = \left\{ (\rho_1, \rho'_2) \in W^2; \text{ such that } \rho_1 = \exp(tH_{p_1})(\rho_2) \text{ if } 0 \leq s \leq t \right. \\ \left. \exp(sH_{p_1})(\rho_2) \in W \text{ and } |p_1(\rho_1)| \leq \exp(-h(t)) \right\} \quad (12)$$

where ρ'_2 is the antipodal point of ρ_2 .

$\Gamma_t(W, h)$ is totally real in $\mathbb{C}^n \times \mathbb{C}^n$. We again refer to [6] for the construction of a convenient projection $\mathbb{C}^n \times \mathbb{C}^n \rightarrow \Gamma_t(W, h)$; $z \rightarrow m(t, z) \in \Gamma_t(W, h)$; $m(t, z)$ will be defined uniquely by the additional property that $z - m(t, z) \in iT_{m(t, z)}\Gamma_t(W, h)$.

Therefore functions on $\Gamma_t(W, h)$ give rise to almost analytic extension in $\mathbb{C}^n \times \mathbb{C}^n$. Appropriate controls with respect to t are obtained in [6]. From these controls it follows that all derivatives with respect to (t, x) of these maps are bounded by some $\exp(h(t))$ with a function $h(t) \in o(t)$ when $t \rightarrow \infty$. So $\varphi(t, x, y), \dots, \chi(t, x, y)$ will be almost analytic on Γ_t and there derivatives are bounded by some $\exp(h(t))$. A more precise decay in time for the amplitude $e(t, x, y)$ is obtained and this will be discussed later since this point is essential in our discussion.

Proposition 1 *There are two constants $C > 0$, $\gamma > 0$ such that*

$$\|E_t\| \leq C\lambda^n \exp(-1/2\gamma t) \quad (13)$$

γ being as close as we wish of the lower bound of

$$T^+(\rho) = \sum_{z_j \in \text{Spec}(F_{p_1}), \text{Re} z_j > 0} \text{Re} z_j.$$

We shall not use this result here, but it is worth mentioning since it is the key point of the proof in [6]. The proof will be derived from elements of the proof of the following Proposition.

Proposition 2 *There are constants $M > 0$, $C > 0$ such that*

$$\|E_t\| \leq C \quad (14)$$

if $\exp(Mt) \leq \lambda$, the norms are taken in $H_{\varphi_0}(\mathbb{C}^n)$.

Proof. Let us prove Proposition 2. Let E_t^* be the adjoint of E_t in $H_{\varphi_0}(\mathbb{C}^n)$. If we write $d\mu(x) = e^{-2\lambda\varphi_0(x)} L(dx)$ the kernel (with respect to μ) of E_t^* is given by $E_t^*(x, y) = \int S(x, z) \overline{E_t(y, z)} d\mu(z)$, the kernel of $E_t^* E_t$ is therefore $E_t^* E_t(x, y) = \int S(x, x_1) \overline{E_t(x_2, x_1)} E_t(x_2, y) d\mu(x_1) d\mu(x_2)$. We write this integral

$$E_t^* E_t(x, y) = \int e^{i\lambda H(t, x, y; x_1, x_2)} f(t, x, y; x_1, x_2, \lambda) d\mu(x_1) d\mu(x_2) \quad (15)$$

In 15 we have

$$H(t, x, y; x_1, x_2) = \psi(x, x_1) - \overline{\varphi(t, x_2, \overline{x_1})} + \varphi(t, x_2, \overline{y}) + 2i(\varphi_0(x_1) + \varphi_0(x_2)) \quad (16)$$

and

$$f(t, x, y; x_1, x_2, \lambda) = c\lambda^{3n} \overline{e(t, x_2, \overline{x_1})} e(t, x_2, \overline{y}) \overline{\chi(t, x_2, \overline{x_1})} \chi(t, x_2, \overline{y}) \quad (17)$$

where c is some absolute constant, $\psi(x, x_1) = -ix\overline{x_1}/2$. We first investigate the critical points of H with respect to (x_1, x_2) ; we estimate, using [6] page 505 (5.30)

$$\begin{aligned} \operatorname{Im} H(t, x, y; x_1, x_2) + \Phi(x, y) \geq \\ c(|x - x_1|^2 + |(x_2, \overline{x_1}) - m_t(x_2, \overline{x_1})|^2 + |(x_2, \overline{y}) - m_t(x_2, \overline{y})|^2) \end{aligned} \quad (18)$$

with the notation $\Phi(x, y) = \varphi_0(x) + \varphi_0(y)$. We have

$$\begin{aligned} H'_{x_1} &= -\overline{\varphi'_y(t, x_2, \overline{x_1})} - i/2\overline{x_1} \\ H'_{\overline{x_1}} &= -i/2x - \overline{\varphi'_y(t, x_2, \overline{x_1})} + i/2x_1 \\ H'_{x_2} &= -\overline{\varphi'_x(t, x_2, \overline{x_1})} + \varphi'_x(t, x_2, \overline{y}) + i/2\overline{x_2} \\ H'_{\overline{x_2}} &= -\overline{\varphi'_x(t, x_2, \overline{x_1})} + \varphi'_x(t, x_2, \overline{y}) + i/2x_2 \end{aligned} \quad (19)$$

It results from these relations that we have a "real" critical point when $x_1 = x$, $(x_2, \overline{x_1}) \in \Gamma_t$, $(x_2, \overline{y}) \in \Gamma_t$, i.e. when $x = y$ the critical point being $x_1 = x$, $x_2 = \Phi_t(x)$. Let $\varepsilon_1 > 0$ a small number to be chosen later.

In the integral 15, using 18 we can restrict the integration over the set of (x_1, x_2) such that $|(x_2, \overline{x_1}) - m_t(x_2, \overline{x_1})|^2 + |(x_2, \overline{y}) - m_t(x_2, \overline{y})|^2 \leq \varepsilon_0 \lambda^{-1+\varepsilon_1}$, $|x - x_1|^2 \leq \varepsilon_0 \lambda^{-1+\varepsilon_1}$ for some ε_0 to be chosen later; we neglect then a term $\mathcal{O}(\lambda^{-\infty})$. We deduce from the relations 19 that a zone

$|\varphi'_x(t, x_2, \bar{x}) - \varphi'_x(t, x_2, \bar{y})| \geq C\varepsilon_0^{1/2} \lambda^{(-1+\varepsilon_1)/2}$ give also a term $\mathcal{O}(\lambda^{-\infty})$. As $m_t(x_2, \bar{x})$ and $m_t(x_2, \bar{y}) \in \Gamma_t$, we have $\Phi_t(m_t(x_2, \bar{x})_y) = m_t(x_2, \bar{x})_x$ and $\Phi_t(m_t(x_2, \bar{y})_y) = m_t(x_2, \bar{y})_x$, Φ_t is a diffeomorphism whose first two derivatives are bounded by some e^{α} , we have then $|x - y| \leq C\varepsilon_0 e^{\alpha} \lambda^{(-1+\varepsilon_1)/2}$.

Let us estimate the Hessian of H at real critical points.

We first make a complexification and we write $\tilde{x}_1, \tilde{x}_2$ instead of $\bar{x}_1, \bar{x}_2$, we shall refer to the set $\{\tilde{x}_1 = \bar{x}_1, \tilde{x}_2 = \bar{x}_2\}$ as the real. Let $\varphi_1(t, \tilde{x}_2, x_1)$ be an almost analytic extension of $\varphi(t, \bar{x}_2, \bar{x}_1)$. Let $\tilde{H}$ be an almost analytic function on

$$\tilde{\Gamma}_t = \{(x, y, x_1, \tilde{x}_1, x_2, \tilde{x}_2); (\tilde{x}_2, \bar{x}_1) \in \Gamma_t, (x_2, \bar{y}) \in \Gamma_t\}$$

extending H . We compute $\nabla^2 \tilde{H}(t, x, x; x, \bar{x}, \Phi_t(\bar{x}), \overline{\Phi_t(\bar{x})})$ as the map

$$\begin{aligned} (\delta x_1, \delta \tilde{x}_1, \delta x_2, \delta \tilde{x}_2) \rightarrow \\ (i/2\delta \tilde{x}_1 - \varphi''_{1y\bar{y}} \delta x_1 - \varphi''_{1x\bar{y}} \delta \tilde{x}_2, i/2\delta x_1, \varphi''_{xx} \delta x_2 + i/2\delta \tilde{x}_2, \\ -\varphi''_{1xx} \delta \tilde{x}_2 - \varphi''_{1y\bar{x}} \delta x_1 + i/2\delta x_2) \end{aligned} \quad (20)$$

the computation being simplified by the fact that at such particular points $\nabla^2 \varphi$ is $\mathbb{C}$ -linear. This map is invertible and its inverse will have the same norms as the inverse of

$$(\delta x_2, \delta \tilde{x}_2) \rightarrow (\varphi''_{xx}(t, x_2^c, \bar{x}) \delta x_2 + i/2\delta \tilde{x}_2, -\overline{\varphi''_{xx}(t, x_2^c, \bar{x})} \delta \tilde{x}_2 + i/2\delta x_2) \quad (21)$$

with $x_2^c = \Phi_t(\bar{x})$. We have then to estimate the inverse of the map $\overline{\varphi''_{xx}(t, x_2^c, \bar{x})} \varphi''_{xx}(t, x_2^c, \bar{x}) - \frac{1}{4}$. We refer to ([6] Proposition 5.2 (5.28)) to obtain

$$\nabla^2 \varphi(t, z) = \begin{pmatrix} \frac{1}{4} \bar{b} a^{-1} & \frac{i}{2} \bar{a}^{-1} \\ -\frac{i}{2} a^{-1} & \frac{1}{4} b^t a^{-1} \end{pmatrix} \quad (22)$$

where $(t, z) \in \Gamma$, a and b are defined by the relation 23

$$(\delta x - i\delta \xi, \frac{\delta x + i\delta \xi}{2i}) = \begin{pmatrix} a & b \\ \frac{1}{4}\bar{b} & \bar{a} \end{pmatrix} (\delta y - i\delta \eta, \frac{\delta y + i\delta \eta}{2i}) \quad (23)$$

if $(\delta x, \delta \xi, \delta y, \delta \eta)$ is a tangent vector of Λ_t . We refer to [6] section 5, page 490. We have

$$(t, z) \in \Gamma, \quad \|\varphi''_{xy}^{-1}(t, z)\| \leq 2 \|a(t, z)\| \quad (24)$$

Moreover $\bar{\varphi}_{xx}''\varphi_{xx}'' - \frac{1}{4} = -\frac{1}{4}a^{-1*}a^{-1}$ as a consequence of the relation $a^*a - \frac{1}{4}b\bar{b} = I$. So the norm of the inverse of the map 21 is $|\det a(t, z)|^2$. As computed in [6] the module of $e(t, z)$ at a point $(t, z) \in \Gamma$ is precisely $|\det a(t, z)|^{-1/2}$.

This means that in the stationary phase expansion of integral 15 the powers of λ and the exponentials decays in time vanish, it remains only the normal λ^n . See [6] section 6.2, page 509-510, relations (6.4) and (6.5).

The condition $\exp(Mt) \leq \lambda$ will allow us to give sense to the application of stationary phase expansion with a complex phase function (see [4]) uniformly with respect to (t, x, y) .

We need to be more specific in the application of the stationary phase method. We check here some steps with uniform controls in t .

In a neighborhood $V = \{(x, y); |x - y| \leq \varepsilon_0 e^{-M_0 t}\}$ of the diagonal we have a C^∞ map $(t, x, y) \rightarrow Z_c(t, x, y) \in \mathbb{C}^{4n}$, where $Z = (x_1, \tilde{x}_1, x_2, \tilde{x}_2)$, such that $\partial_Z \tilde{H}(t, x, y; Z_c(t, x, y)) = 0$. The derivatives of this map satisfy some estimates like $|D_{t,x,y}^\alpha Z_c(t, x, y)| \leq C_\alpha \exp(ct|\alpha|)$ for some constants C_α and $c > 0$. Hence we have

$$|Z_c(t, x, y) - Z_c(t, x, x)| \leq C \exp(ct) |x - y| \quad (25)$$

We can define a symmetric complex matrix $Q(t, x, y)$ such that

$$\partial_{ZZ}^2 \tilde{H}(t, x, y; Z_c(t, x, y)) = iQ^2(t, x, y)$$

which is well defined and smooth since V is connected and simply connected.

We have $\|Q^{-1}(t, x, y)\| \leq C \exp(ct)$ for some constants c and $C > 0$. We have

$$\begin{aligned} \operatorname{Im} \tilde{H}(t, x, y; Z) &= \operatorname{Im}(\tilde{H}(t, x, y; Z_c(t, x, y)) + \\ &\quad 1/2 \operatorname{Im} \partial_{ZZ}^2 \tilde{H}(t, x, y; Z_c(t, x, y))(Z - Z_c(t, x, y))^2 + \\ &\quad O((\operatorname{dist}(Z_c(t, x, y), \operatorname{real}))^\infty + |Z - Z_c(t, x, y)|^3) \end{aligned} \quad (26)$$

we choose a point Z such that $Z \in \operatorname{real} = \{x_1 = \tilde{x}_1; x_2 = \tilde{x}_2\}$, then $\operatorname{Im}(\tilde{H}(t, x, y; Z) + \Phi(x, y)) \geq 0$, and $\operatorname{dist}(Z_c(t, x, y), \operatorname{real}) \leq |Z - Z_c(t, x, y)|$. We use then the estimate (2.6) of [4] and we derive

$$\operatorname{Im} \tilde{H}(t, x, y; Z_c(t, x, y)) + \Phi(x, y) \geq C e^{-at} \operatorname{dist}(Z_c(t, x, y), \operatorname{real})^2 \quad (27)$$

This relation shows that the stationary phase expansions are independent of the choice of particular almost analytic extensions but is inadequate to bound the L^2 norm.

We shall compute $H_c(t, x, y) = (t, x, y; Z_c(t, x, y))$ with a Taylor expansion on the diagonal

$$\begin{aligned} H_c(t, x, y) &= H_c(t, x, x) + \nabla_y \tilde{H}(t, x, x)(y - x) + 1/2 \nabla_{yy}^2 H(t, x, x)(y - x)^2 \\ &\quad - 1/2((\tilde{H}_{ZZ}^*)^{-1} \tilde{H}_{Zy}^*(y - x), \tilde{H}_{Zy}^*(y - x)) + \mathcal{O}(e^{\alpha t}(x - y)^3) \end{aligned} \quad (28)$$

The second term in 28 is $\varphi'_y(t, x_2^c, \bar{x})(\bar{y} - \bar{x}) = -i/2x(\bar{x} - \bar{y})$. The third term is given by $1/2\varphi_{yy}^*(\bar{y} - \bar{x})^2$. $\tilde{H}_{Zy}^*(y - x) = (0, 0, \varphi_{xy}^*(\bar{y} - \bar{x}), 0)$. The inverse $(\tilde{H}_{ZZ}^*)^{-1}\delta X = \delta Z$ is given by the relations

$$\varphi_{yx}^* \delta Z_2 - \varphi_{yy}^* \delta Z_1 - i/2\delta \tilde{Z}_1 = \delta X_1, \quad i/2\delta Z_1 = \delta \tilde{X}_1 \quad (29)$$

$$(\delta Z_2, \delta \tilde{Z}_2) = \begin{pmatrix} -(1/4 - \varphi_{xx}^* \varphi_{xx}^*)^{-1} \varphi_{xx}^* & -i/2(1/4 - \varphi_{xx}^* \varphi_{xx}^*)^{-1} \\ -i/2(1/4 - \varphi_{xx}^* \varphi_{xx}^*)^{-1} & (1/4 - \varphi_{xx}^* \varphi_{xx}^*)^{-1} \varphi_{xx}^* \end{pmatrix} (\delta X_2, \delta \tilde{X}_2) \quad (30)$$

We make $\delta X = (0, 0, \varphi_{xy}^*(\bar{y} - \bar{x}), 0)$. The fourth term in 28 is given by $1/2(\varphi_{xy}^*(\bar{y} - \bar{x}), (1/4 - \varphi_{xx}^* \varphi_{xx}^*)^{-1} \varphi_{xx}^* \varphi_{xy}^*(\bar{y} - \bar{x}))$. Adding these two terms we have to compute $1/2(\varphi_{yy}^* + \varphi_{yx}^*(1/4 - \varphi_{xx}^* \varphi_{xx}^*)^{-1} \varphi_{xx}^* \varphi_{xy}^*)$. We recall that $(1/4 - \varphi_{xx}^* \varphi_{xx}^*) = 1/4a^{-1*}a^{-1}$, so $\varphi_{yy}^* + \varphi_{yx}^*(1/4 - \varphi_{xx}^* \varphi_{xx}^*)^{-1} \varphi_{xx}^* \varphi_{xy}^* = 1/4^t b^t a^{-1} - 1/4a^{-1}aa^*b\bar{a}^{-1t}a^{-1} = 0$. We have therefore $H_c(t, x, y) = H_c(t, x, x) - i/2x\bar{y} + i/2x\bar{x}$.

We have now to compute $H_c(t, x, x)$, $(\partial/\partial t)H_c(t, x, x) = -\varphi'_t(t, x_2^c, \bar{x}) + \varphi'_t(t, x_2^c, \bar{x})$ where $x_2^c = \Phi_t(\bar{x})$; on Γ_t $\varphi'_t(t, x, y) + p_1(x, \varphi'_x(t, x, y)) = 0$ then $Im\varphi'_t(t, x, y) = 0$; so $H_c(t, x, x) = H_c(0, x, x) = \psi(x, x) + 2iIm\psi(x, x) + 4i\varphi_0(x) = -\frac{i}{2}|x|^2$.

We have obtained

$$H_c(t, x, y) = -\frac{i}{2}x\bar{y} + \mathcal{O}(e^{\alpha t}|x - y|^3) \quad (31)$$

We deduce from 31 that in V ,

$$ImH_c(t, x, y) + \Phi(x, y) \geq C^{-1}|x - y|^2 \quad (32)$$

From the usual estimate of L^2 norms, we obtain the result of Proposition 2. Moreover the relation $e^{Mt}\lambda^{-1} \leq 1$ shows that we have an convergent asymptotic development in term of uniform decay in λ . $\sharp$

Proposition 3 Let M and t satisfy $e^{Mt}\lambda^{-1} \leq 1$, then $E_t^* E_t$ is a pseudo-differential operator of order zero belonging to a class of $S(1, g_\epsilon)$ (see [3] Chapter 18) where $g_\epsilon = \lambda^{2\epsilon}(|dx|^2 + |d\xi|^2)$, where $\epsilon > 0$ depends on M and on the properties of the flow of H_{p_1} ; when $M \rightarrow \infty$, $\epsilon \rightarrow 0$.

Let us recall that a symbol $a \in S(1, g_\epsilon)$ satisfies uniform estimates

$$\text{For all multi-indices } \alpha, \beta, \left| D_x^\alpha D_\xi^\beta a(x, \xi, \lambda) \right| \leq C_{\alpha, \beta} \lambda^{\epsilon(|\alpha| + |\beta|)}.$$

Proof.

We shall derive this property from Proposition 2 and from the characterization of pseudo-differential operators due to Beals [1]. Let us estimate the L^2 norm of the first commutators $x^\alpha D_x^\beta E_t^* E_t - E_t^* E_t x^\alpha D_x^\beta$ for $|\alpha| + |\beta| = 1$. Using the computations of the proof above we express

$$\begin{aligned} x^\alpha D_x^\beta E_t^* E_t u(x) &= \int x^\alpha (H'_{xc}(t, x, y))^\beta f(t, x, y) e^{i\lambda H_c(t, x, y)} u(y) d\mu(y) \\ &\quad + \mathcal{O}(e^{at}\lambda^{-1}) \end{aligned} \quad (33)$$

in this formula $u \in H_{\varphi_0}(\mathbb{C}^n)$, the notation $\mathcal{O}$ means that the remainder has the same form but the order of the symbol is lowered. We have $E_t^* E_t(x^\alpha D_x^\beta u)(x) = \int f(t, x, y) e^{i\lambda H_c(t, x, y)} y^\alpha D_y^\beta u(y) d\mu(y)$, we integrate by part in this formula so

$$\begin{aligned} E_t^* E_t(x^\alpha D_x^\beta u)(x) &= (-1)^{|\beta|} \int f(t, x, y) e^{i\lambda H_c(t, x, y)} y^\alpha \\ &\quad (H'_{yc}(t, x, y) + i\bar{y}/2)^\beta u(y) d\mu(y) + \mathcal{O}(e^{at}\lambda^{-1}) \end{aligned} \quad (34)$$

We compare 33 and 34, using 31 we get $H'_{xc}(t, x, y) = -i\bar{y}/2 + \mathcal{O}(e^{at}(x-y)^2)$ and $H'_{yc}(t, x, y) = \mathcal{O}(e^{at}(x-y)^2)$, $x-y + \mathcal{O}(e^{at}(x-y)^2) = H'_{yc}(t, x, y) + i/2y$.

We need an extra notation to make these integration by parts (more) rigorous. Let $G = (\lambda^{-1/2} + |x-y|)^{-1} e^{Ct}(|dx|^2 + |dy|^2)$, $M_k = (\lambda^{-1/2} + |x-y|)^{-k}$, let $h = H_c(t, x, y) + 2i\varphi_0(y)$. Assume that an amplitude $f(t, x, y) \in S(M_k, G)$, we note $A(f)$ the integral operator with amplitude f and phase function h . Using an integration by parts with the operator $L = (|h'_y|^2 + \lambda^{-1})^{-1} (\bar{h}'_y \partial / \partial \bar{y} + 1)$ and the fact that $u(y)$ is holomorphic we can replace f by $\lambda^{-N} L^N(f)$; so the same operator is given with an amplitude in $S(M_{k+2N} \lambda^{-N} e^{CNt}, G)$. We have shown before that if $f \in S(M_k, G)$ $ad_X(A(f)) = A(f_1)$ with $f_1 \in S(M_k e^{at} M_{-2}, G)$, where X is either x_j or D_{x_k} . So $ad_{X_1} \dots ad_{X_k}(A(f)) = A(f_k)$ with $f_k \in S(e^{Ckt} \lambda^{-k}, G)$. $\#$

We deduce then that $E_t^* E_t$ is a pseudo-differential operator in the class $S(1, g_t)$ where g_t is the metric $g_t = e^{2Ct}(|dx|^2 + |d\xi|^2)$.

This result is then optimal with the restriction that we may not have the best constant C and that we consider here spatially homogeneous metrics. $\sharp$

3 Semi global L^2 estimates.

We shall work as in [7] with L^2 estimates for solutions of the Schrödinger equation $(D_t + P(x, D_x))u(t) = 0$. More precisely

$$\begin{aligned} 2\lambda \operatorname{Im} \int_T^{T_0} ((D_t + P(x, \lambda^{-1} D_x))u(t), \alpha(t)u(t)) dt = \\ (\alpha(T)u(T), u(T)) - (\alpha(T_0)u(T_0), u(T_0)) + \int_T^{T_0} (M(t)u(t), u(t)) dt \end{aligned} \quad (35)$$

with the notation

$$M(t) = (\partial\alpha(t)/\partial t) - i\lambda[P_1, \alpha(t)] + 2\lambda \operatorname{Re}(\alpha(t)P_2) \quad (36)$$

$\alpha(t)(x, \lambda^{-1} D_x)$ is a family of self-adjoint operators to be chosen later, T_0 will depend only on λ . We have to make $(\partial\alpha(t)/\partial t) - i\lambda[P_1, \alpha(t)] + 2\lambda \operatorname{Re}(\alpha(t)P_2)$ as large as possible. We choose $\alpha(t) = E_t \beta(t) E_t^*$ where E_t has been constructed in section 2. $\beta(t)$ will be chosen later.

If we note $R(t) = (\partial/\partial t)E_t - i\lambda P_1 E_t$, we obtain

$$\begin{aligned} (\partial\alpha(t)/\partial t) - i\lambda[P_1, \alpha(t)] + 2\lambda \operatorname{Re}(\alpha(t)P_2) = \\ E_t(\partial\beta(t)/\partial t)E_t^* + 2\lambda \operatorname{Re}(E_t \beta(t) E_t^* P_2) + 2\operatorname{Re}(R(t)\beta(t)E_t^*) \end{aligned} \quad (37)$$

we shall deal later with the last term in 37. We have written in 5 $P_2 = ((p_2(x, \xi) + \lambda^{-1} \operatorname{Im}(p_0(x, \xi, \lambda)))^w$. Let $\beta(t) = \exp(-2\gamma t)$ where $\operatorname{Im}(p_0(x, \xi, \lambda)) \geq \gamma$. If $Q = P_2 - \gamma\lambda^{-1}$ the Weyl symbol of Q is non negative. We have to estimate from below the operator $\operatorname{Re}(E_t E_t^* Q)$, we have proved in Proposition 3 that $E_t E_t^* \in S(1, g_\epsilon)$, it is then a consequence of the Fefferman-Phong inequality that $\operatorname{Re}(E_t E_t^* Q) \geq -C\lambda^{-2+2\epsilon}$. So from 37 we obtain

$$(M(t)u(t), u(t)) \geq -C\lambda^{-1+2\epsilon}e^{-2\gamma t}|u(t)|^2 + 2(\operatorname{Re}(R(t)\beta(t)E_t^*)u(t), u(t)) \quad (38)$$

We shall deal later with the last term in 38. In 35 we shall input $u(t) = u$, so the left hand-side of 35 is an $\mathcal{O}(\lambda^{-\infty})$ uniformly in time.

We make an induction. Let W be an open neighborhood of ρ_0 , we say that $u \in H^\sigma(W)$ if for any pseudo-differential operator $\varphi(x, \lambda^{-1}D_x)$ with $\text{supp } \varphi \subset W$ u satisfies $|\varphi u| \leq C\lambda^{-\sigma}$. Assume $u \in H^\sigma(W)$.

Let W_1, W_2 be two open sets such that $W_1 \subset\subset W_2 \subset\subset W$. We use the construction of [6] section 6.3 of a cut-off function $\chi(t, x, y)$ defined by taking an almost analytic extension of the restriction to Γ_t of the function

$$\tilde{\chi}(t, y) = \begin{cases} \zeta_1(y) \exp(-\int_0^t \frac{\psi}{\zeta^2}(\Phi_s(y)) ds) & \text{if } \int_0^t \frac{\psi}{\zeta^2}(\Phi_s(y)) ds < \infty \\ 0 & \text{if } \int_0^t \frac{\psi}{\zeta^2}(\Phi_s(y)) ds = \infty \end{cases} \quad (39)$$

with the notations $\zeta \in C_0^\infty(W_2)$, $\zeta \equiv 1$ in $\overline{W_1}$; $\psi \in C_0^\infty(W_2 \setminus \overline{W_1})$, $\psi \equiv 1$ in a neighborhood of $\partial \{x; \zeta(x) > 0\}$; $\zeta_1(y) \in C_0^\infty(\{x; \zeta(x) > 0\})$ is one on a neighborhood of $\overline{W_1}$. It was proved in [6] that such constructions give a smooth function whose derivatives are bounded by some $\exp(h_1(t))$ with $h_1 \in o(t)$ when $t \rightarrow \infty$. If we add a cut-off function in $\{y; |p_1(y)| \leq e^{-h(t)}\}$ the corresponding $\tilde{\chi}$ will be supported in

$$\Lambda_{W_2} = \left\{ (t, x, y); x = \Phi_t(y), \text{ for } 0 \leq s \leq t, \Phi_s(y) \in W_2, |p_1(y)| \leq c_2 e^{-h(t)} \right\} \quad (40)$$

with value 1 on Λ_{W_1} .

In view the lower bound

$$\text{Im} \varphi(t, x, y) + \Phi(x, y) \geq C^{-1} |(x, y) - m_t(x, y)|^2 \quad (41)$$

we shall remain as close to Γ_t as we wish.

We make $T = 0$ and $T_0 = 1/M \ln \lambda$. The condition in Proposition 3 is satisfied for $0 \leq t \leq T_0$. $\alpha(0) = E_0 E_0^*$ is elliptic in W_1 . $\beta(T_0) E_{T_0} E_{T_0}^*$ is a pseudo-differential operator with wave front set contained in W_2 , belonging to the class $S(e^{-2\gamma T_0}, g_\epsilon)$. We have therefore $(\alpha(T_0)u, u) \leq C\lambda^{-2\sigma-2\gamma/M}$ if $u \in H^\sigma(W)$. Using 38 and the fact that ϵ is close to 0, we shall conclude that $H^{\sigma+\gamma/M}(W)$ after taking care of the last term 35.

The operator $R(t)$ comes from that E_t is not an exact solution of the equation $(D_t + P_1)E_t = 0$, which is due to the presence of the χ . The function χ is itself necessary since we want to localize near $\Lambda(W, h)$ to get our theorem. This analysis has been carried out in [6] section 6.4. That we wish to say is that the assumption $\overline{\Lambda(W, h)}(\rho) \cap \partial W \cap OF(u) = \emptyset$ implies that $|R(t)^*u| = \mathcal{O}(\lambda^{-\infty})$. This kind of truncature are precisely what we need to derive a propagation of singularities theorem from an ordinary L^2 inequality; as we said above this makes all this machinery necessary. #

Remark 1 *The condition $(H)_3$ allows to prove a theorem of propagation of singularities with a loss of one derivative, in this sense this condition is sharp.*

Part II

A more precise result in a particular case.

We shall be able to get a sharper result in a symplectic case analogous to the case treated in [5].

4 Construction of the stable manifolds.

In this section we shall use some elements of [10] Appendix A, and [6] Section 4, we prefer to recall all this material in our proof than to use obscure references to these works.

Let $p(x, \xi)$ be an analytic complex function. Let $\rho_0 \in N \cap \mathbf{R}^{2n}$ where

$$N = \{(x, \xi) \in \mathbf{C}^{2n}; p(x, \xi) = dp(x, \xi) = 0\} \quad (42)$$

Let $H_p = p'_\xi \partial / \partial x - p'_x \partial / \partial \xi$ be the hamiltonian field, we mean by bicharacteristic of p the integral curves of the real vector field on $\mathbf{C}^{2n}$ $H_p + \overline{H}_p$.

Let

$$\Lambda_t = \{(\rho(t, \rho), \rho); \rho \in \mathbf{C}^{2n}, p(\rho) = 0\} \text{ and } \Lambda_{t, \mathbf{R}} = \{(\rho(t, \rho), \rho); \rho \in \mathbf{R}^{2n}, p(\rho) = 0\} \quad (43)$$

in 43 $t \rightarrow \rho(t, \rho)$ is a bicharacteristic curve starting at ρ . We shall assume

• (H_1) :

$$Imp(\rho) \geq 0 \text{ if } \rho \in \mathbf{R}^{2n}. \quad (44)$$

The fundamental matrix is $F_p(x, \xi) = dH_p(x, \xi) = \begin{pmatrix} p''_{\xi x} & p''_{\xi \xi} \\ -p''_{xx} & -p''_{x\xi} \end{pmatrix}$.

In the Jordan decomposition of $F_p(\rho)$, we note $W_+(\rho) = \oplus_{Re \lambda > 0, \lambda \in Spec(F_p)} V_\lambda$, $W_-(\rho) = \oplus_{Re \lambda < 0, \lambda \in Spec(F_p)} V_\lambda$. V_λ are the generalized eigenspaces.

- (H_2) : We assume that

$$\mathbb{C}^{2n} = W_+(\rho) \oplus W_-(\rho) \oplus W_0(\rho) \oplus \text{Ker} F_p \quad (45)$$

and the dimensions of these three spaces are constant along N .

The assumption (H_2) means there are no non zero eigenvalue in $i\mathbb{R}$, that there is also no generalized eigenspace relative to zero and that $\dim W_+(\rho) = r_+$, $\dim W_-(\rho) = r_-$ are constant. We note respectively by $P_+(\rho)$, $P_-(\rho)$ and $P_0(\rho)$ the corresponding projectors, it follows from our assumption that these maps are analytic.

As $\sigma(V_\lambda, V_\mu) = 0$ if $\lambda + \mu \neq 0$, $W_+(\rho) \oplus W_0(\rho) \subseteq W_+(\rho)^{\perp\sigma}$, then $r_+ \leq r_-$, then

$$r_+ = r_- = r \text{ and } W_\pm(\rho)^{\perp\sigma} = W_\pm(\rho) \oplus W_0(\rho) \quad (46)$$

- (H_3) : We also assume that in a neighborhood of ρ_0 , there is a constant C_0 such

$$|H_p(\rho)| \leq C_0 |(I - P_0(\rho_0))H_p(\rho)| \quad (47)$$

it is a consequence of the assumptions of constant ranks that 47 is independent of ρ_0 .

- (H_4) : We shall assume that on $N' = \{(x, \xi) \in \mathbb{R}^{2n}; \text{Rep}(x, \xi) = d\text{Rep}(x, \xi) = 0\}$, we have $\mathbb{C}^{2n} = W'_+ \oplus W'_- \oplus W'_{+i} \oplus W'_{-i} \oplus \text{Ker} F_{\text{Rep}}$ where $W'_\pm$ are the corresponding spaces for F_{Rep} and $W'_{\pm i} = \oplus_{i\lambda \in \text{Spec}(F_{\text{Rep}}), \pm\lambda > 0} V_\lambda$. We suppose also that the quadratic form $[v, \bar{v}] = \frac{1}{i}\sigma(v, \bar{v}) \leq 0$ on W'_{+i} . This means in fact simply that $V'_0 = \text{Ker} F_{\text{Rep}}$ and that there are no difference of harmonic oscillators in a spectral decomposition of F_{Rep} . In addition we assume that N' is a smooth manifold and that $\text{Ker} \tilde{F}_{\text{Rep}}(\rho) = T_\rho N'$.

The first step is to construct stable manifolds for the complex symbol p . Let (x, y) be coordinates such that $x \in W_+(\rho_0) \oplus W_0(\rho_0)$, $y \in W_-(\rho_0)$, we split again $x = (x', z)$ where $x' \in W_+(\rho_0)$, $z \in W_0(\rho_0)$. We note $W^+(\rho) = W_+(\rho) \oplus W_0(\rho)$. Let us note again by $P_+(\rho)$ an analytic extension of this function away from N .

When we split $\mathbb{C}^{2n} = W^+(\rho) \oplus W_-(\rho)$, we have a decomposition of $F_p(\rho) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(\rho)$. We shall split further $\alpha = \begin{pmatrix} \alpha_0 & \alpha_1 \\ \alpha_2 & \alpha_3 \end{pmatrix}$ along $W_+(\rho) \oplus W_0(\rho)$.

It is a consequence of the assumptions that in a close neighborhood of N ,

$$\begin{aligned} \text{Spec}(\alpha_0) &\subseteq \{z \in \mathbb{C}, \text{Re} z \geq c\}, \text{Spec}(\delta) \subseteq \{z \in \mathbb{C}, \text{Re} z \leq -c\}, \\ \|\beta\| &\leq \varepsilon, \|\gamma\| \leq \varepsilon \|\alpha_i\| \leq \varepsilon \text{ for } 1 \leq i \leq 3; c > 0, \varepsilon \text{ is small} \end{aligned} \quad (48)$$

Let ρ be a point close to ρ_0 and $t \rightarrow \rho(t, \rho)$ the bicharacteristic issued from ρ . The evolution of tangent vectors is given by the linear differential equation $\frac{d}{dt}v_t = F_p(\rho(t, \rho))v_t$, $v_t|_{t=0} = v_0$.

We find a linear map $\varphi_t(\rho)$ from $W^+(\rho_0)$ to $W_-(\rho_0)$ such that the evolution of the space $W^+(\rho_0)$ along the flow is given by $W_t^+(\rho) = \{(\delta x, \varphi_t(\rho)\delta x), \delta x \in W^+(\rho_0)\}$. This is achieved as in [6] by solving the equation

$$\dot{\varphi}_t + \varphi_t \alpha - \delta \varphi_t + \varphi_t \beta \varphi_t - \gamma = 0, \varphi|_{t=0} = 0. \quad (49)$$

In view of the relations 48, which imply that $\text{Spec}(\alpha) \subseteq \{z \in \mathbb{C}, \text{Re} z \geq -\varepsilon\}$ we know that the equation 49 can be solved for $t \geq 0$ and we have $\|\varphi_t\| \leq C\varepsilon$ for some small ε .

We define a suitable norm to construct regions stable under the flow. Let $\alpha_0 = \alpha(\rho_0)$ and

$$C_0 = \int_{-\infty}^0 \exp(t\alpha_0^*) \exp(t\alpha_0) dt, C_0 > 0, C_0\alpha_0^* + \alpha_0 C_0 = Id_{W_+(\rho_0)} \quad (50)$$

The restriction of $F_p(\rho_0)$ to $W(\rho_0)$ is expressed by $\delta y \rightarrow \delta_0(\delta y)$, $\text{Spec}(\delta_0) \subseteq \{z \in \mathbb{C}, \text{Re} z < -c\}$. We define $D_0 > 0$, $D_0\delta_0^* + \delta_0 D_0 = -Id_{W_-(\rho_0)}$.

We note $\|v_{x'}\|^* = (C_0 v_{x'}, v_{x'})^{1/2}$, $\|v_x\|^2 = \|v_{x'}\|^{*2} + |v_z|^2$, $\|v_y\|_* = (D_0 v_y, v_y)^{1/2}$ and $\|v\|^2 = \|v_{x'}\|^{*2} + \|v_y\|_*^2 + |v_z|^2$.

We expand

$$\frac{d}{dt}(\rho_t - \rho_0) = F_p(\rho_0)(\rho_t - \rho_0) + \mathcal{O}((\rho_t - \rho_0)^2) \text{ so}$$

$$\frac{d}{dt} \|\rho_t - \rho_0\|^2 = |(\rho_t - \rho_0)_{x'}|^2 - |(\rho_t - \rho_0)_y|^2 + \mathcal{O}((\rho_t - \rho_0)^3) \quad (51)$$

Proposition 4 Let $0 \leq f(x) \in C_0^\infty(\mathbb{R}^+)$ be a function $f \leq \eta$. Let

$$B(\rho_0, f) = \{x \in W^+(\rho_0); \|x'\|^* < f(|z|^2)\} \quad (52)$$

$E(\rho_0, T, f)$ be the region

$$E(\rho_0, T, f) = \left\{ \begin{array}{l} \rho; \rho - \rho_0 \in W^+(\rho_0); \rho_t - \rho_0 \in B(\rho_0, f) \\ \text{and } |\rho_t - \rho_0| < \varepsilon \text{ for } 0 \leq t \leq T \end{array} \right\} \quad (53)$$

There exist a bounded set of analytic functions $x \in B(\rho_0, f) \rightarrow \lambda(t, x) \in W_-(\rho_0)$ such that $E(\rho_0, T, f)$ can be identified with the set

$$E'(\rho_0, T, f) = \left\{ \begin{array}{l} \rho; \rho - \rho_0 \in W^+(\rho_0), \text{ for } 0 \leq t \leq T, |\rho_t - \rho_0| < \varepsilon \text{ and} \\ \exists x_t \in W^+(\rho_0), x_t \in B(\rho_0, f) \text{ such that } \rho_t - \rho_0 = (x_t, \lambda(t, x_t)) \end{array} \right\} \quad (54)$$

This is proved as in [6] by induction on T . Let us sketch the proof.

Assume that we have constructed the function $\lambda(t, x)$ for $t = T_0$. We shall prove that it can be extended for some amount in time. The curve $s \rightarrow \rho(s)$ defined by $\rho(s) = \exp(-T_0 H_p)(\rho_0 + (x_{T_0} + s\delta x, \lambda(T_0, x_{T_0} + s\delta x)))$, is a curve in $\rho_0 + W^+(\rho_0)$. It follows from the definition of φ_t that $(\partial/\partial x)\lambda(T_0, x) = \varphi_{T_0}(\rho(0))$, therefore $\|(\partial/\partial x)\lambda(T_0, x)\| \leq C\varepsilon$. We have also $\lambda(T_0, 0) = 0$. In view of the analyticity we derive further controls on all the derivatives.

We note $g(x) = \|x'\|^* - f(|z|^2)$, and $\tilde{f}(z) = f(|z|^2)$ with $f(0) > 0$.

We define $\psi_{t, T_0}(x) = P^+(\rho_0)(\exp((t - T_0)H_p))(\rho_0 + (x, \lambda(T_0, x)) - \rho_0)$, $\tilde{x} = \psi_{t, T_0}(x)$. The map ψ_{t, T_0} is close to the identity when t is close to T_0 . We have

$$|\psi_{\tau, T_0}(x) - x| \leq C|\tau - T_0||x| \quad (55)$$

We want to prove that $\psi_{t, T_0}(x) \in B(\rho_0, f)$ implies $x \in B(\rho_0, f)$. We assume first that $|x'| \geq C^{-1}|x|$.

$$\begin{aligned} \frac{d}{d\tau}g(\psi_{\tau, T_0}(x)) &= \|\psi_{\tau, T_0}(x)_{x'}\|^{*-1} \langle \psi_{\tau, T_0}(x)_{x'}, (\frac{d}{d\tau}\psi_{\tau, T_0}(x))_{x'} \rangle \\ &\quad - \nabla \tilde{f}((\psi_{\tau, T_0}(x))_z) \cdot (\frac{d}{d\tau}\psi_{\tau, T_0}(x))_z \end{aligned} \quad (56)$$

In view of relation 55 we can replace $\psi_{\tau, T_0}(x)$ by x in the first term of 56 modulo $\mathcal{O}((\tau - T_0))$.

We compute $\frac{d}{d\tau}\psi_{\tau, T_0}(x) = P^+(\rho_0)H_p(\mu(\tau, T_0, x))$ where $\mu(\tau, T_0, x) = \exp((\tau - T_0)H_p)(\rho_0 + (x, \lambda(T_0, x)))$.

Using the estimate $\|\mu(\tau, T_0, x) - (\rho_0 + (x, \lambda(T_0, x)))\| \leq C|\tau - T_0||x|$, we obtain $H_p(\mu(\tau, T_0, x)) = F_p(\rho_0)(x, \lambda(T_0, x)) + \mathcal{O}((\tau - T_0)|x| + |x|^2)$. Therefore $H_p(\mu(\tau, T_0, x)) = (\alpha_0(\rho_0)x', 0, \delta(\rho_0)\lambda(T_0, x)) + \mathcal{O}((\tau - T_0)|x| + |x|^2)$. Hence

$$\|\psi_{\tau, T_0}(x)_{x'}\|^{*-1} \langle \psi_{\tau, T_0}(x)_{x'}, (\frac{d}{d\tau}\psi_{\tau, T_0}(x))_{x'} \rangle \geq C^{-1}|x'| \quad (57)$$

We want now to estimate the second term in 56, $(\frac{d}{d\tau}\psi_{\tau, T_0}(x))_z = \mathcal{O}((\tau - T_0)|x| + |x|^2)$, $\nabla \tilde{f}(z) = \mathcal{O}(|z||\nabla f|)$. Therefore $\frac{d}{d\tau}g(\psi_{\tau, T_0}(x)) \geq C^{-1}|x'|$ when $|x'| \geq C^{-1}|x|$. Then $g(\psi_{\tau, T_0}(x)) \geq g(x)$, so $x \in B(\rho_0, f)$.

If on the contrary we have $|z| \geq C \|x'\|^*$, then the point x is interior to $B(\rho_0, f)$.[#]

We prove now :

Proposition 5 *There exist an involutive manifold $E(\rho_0, \infty)$ of codimension r , stable under H_p , contained in $p^{-1}(0)$, such that $\lim_{t \rightarrow \infty} \rho(-t, \rho)$ exist and belongs to $N = \{\rho; p(\rho) = H_p(\rho) = 0\}$ for any bicharacteristic curve issued from a point $\rho \in E(\rho_0, \infty)$.*

By the Ascoli's theorem, we know that there is a sequence $t_j \rightarrow \infty$, such that the functions $\lambda(t_j, x) \rightarrow \lambda(\infty, x)$.

Let $E^t(\rho_0, f) = \exp(tH_p)(E(\rho_0, t, f))$.

Let $t \rightarrow \rho(t, \rho)$ be a bicharacteristic curve such that $\rho_t \notin E^t(\rho_0, f)$, let $\gamma_t \in E^t(\rho_0, f)$ such that $\rho_t - \gamma_t \in (T_{\gamma_t} E^t(\rho_0, f))^\perp$, the orthogonality being relative to the $\| \cdot \|$ norm, the length of $\rho_t - \gamma_t$ measures the distance $d(\rho_t, E^t(\rho_0, f))$.

We compute $\frac{d}{dt}(\rho_t - \gamma_t) = H_p(\rho_t) - \frac{d}{dt}\gamma_t$. Let us write $\gamma_t = \rho_0 + (x_t, \lambda(t, x_t))$; and $\gamma_t = \exp(tH_p)(\delta_t)$, $\delta_t \in E(\rho_0, t, f)$, so $\frac{d}{dt}\gamma_t = H_p(\gamma_t) + d(\exp(tH_p)(\delta_t))\dot{\delta}_t$, $\dot{\delta}_t \in W^+(\rho_0)$. Then $d(\exp(tH_p)(\delta_t))\dot{\delta}_t = (\zeta_t, \varphi_t(\delta_t)\zeta_t)$ for some $\zeta_t \in W_+(\rho_0)$. We have proved above that $\varphi_t(\delta_t) = (\partial/\partial x)\lambda(t, y_t)$ where $\exp(-tH_p)((\rho_0 + (y_t, \lambda(t, y_t)))) = \delta_t$, so $y_t = x_t$ and $(\zeta_t, \varphi_t(\delta_t)\zeta_t) \in T_{\gamma_t} E^t(\rho_0, f)$.

Therefore $\frac{d}{dt} \|\rho_t - \gamma_t\|^2 = \langle H_p(\rho_t) - H_p(\gamma_t), \rho_t - \gamma_t \rangle$, where $\langle, \rangle$ is the scalar product for $\| \cdot \|$.

$H_p(\rho_t) - H_p(\gamma_t) = F_p(\rho_0)(\rho_t - \gamma_t) + \mathcal{O}(|(\rho_t - \gamma_t)|^2 + |(\rho_t - \gamma_t)| |\gamma_t - \rho_0|)$.

As $(\rho_t - \gamma_t) \in (T_{\gamma_t} E^t(\rho_0, f))^\perp$, we have the relation

$$\begin{pmatrix} C_0 & 0 \\ 0 & I \end{pmatrix} (\rho_t - \gamma_t)_x + \varphi_t(\delta_t)^* D_0 (\rho_t - \gamma_t)_y = 0.$$

We deduce that $|(\rho_t - \gamma_t)_x| \leq C\varepsilon |(\rho_t - \gamma_t)_y|$.

$\langle F_p(\rho_0)(\rho_t - \gamma_t), \rho_t - \gamma_t \rangle = |(\rho_t - \gamma_t)_x|^2 - |(\rho_t - \gamma_t)_y|^2$, therefore

$$d(\rho_t, E^t(\rho_0, f)) \leq C \exp(-C^{-1}t) \quad (58)$$

Let $\rho_t = (x, \lambda(t, x)) \in E^t(\rho_0, f)$, if $s \leq t$ we write $\rho_t = \rho_s(\rho_{t-s})$, then there exist $y \in W^+(\rho_0)$ such that $|(y, \lambda(s, y)) - (x, \lambda(t, x))| \leq C \exp(-C^{-1}s)$, so $|\lambda(t, x) - \lambda(s, x)| \leq C\varepsilon |x - y| + |\lambda(s, y) - \lambda(t, x)| \leq 2C \exp(-C^{-1}s)$. We have therefore proved that $\lambda(t, x) \rightarrow \lambda(\infty, x)$ in the space of holomorphic functions.

We define

$$E(\rho_0, \infty) = \{\rho; \rho = \rho_0 + (x, \lambda(\infty, x)) \text{ for some } x \in B(\rho_0, f)\} \quad (59)$$

A proof similar shows that

$$d(\rho_t, E(\rho_0, \infty)) \leq C \exp(-C^{-1}t). \quad (60)$$

Starting from a point $\rho \in E(\rho_0, \infty)$, we prove then that $\lim_{t \rightarrow \infty} \rho(-t, \rho)$ exist.

We prove first that $E(\rho_0, \infty)$ is H_p invariant. The tangent space $T_\rho E(\rho_0, \infty) = \left\{ (\delta x, \left(\frac{\partial}{\partial x}\right)\lambda(\infty, x)\delta x) \right\}$ is the limit of the spaces $\left\{ (\delta x, \left(\frac{\partial}{\partial x}\right)\lambda(t, x)\delta x) \right\}$ when $t \rightarrow \infty$. For a point $\rho = \rho_0 + (x, \lambda(\infty, x))$ we note $y_t(x) \in B(\rho_0, f)$ the point defined by $\rho_0 + (x, \lambda(t, x)) = \exp(tH_p)(\rho_0 + (y_t(x), 0))$. $\left(\frac{\partial}{\partial x}\right)\lambda(t, x) = \varphi_t(\rho_0 + (y_t(x), 0))$, $H_p(\rho) = m(t, \rho_0 + (y_t(x), 0))v_t$, where $v_t = H_p(\rho_0 + (y_t(x), 0))$, so $w_t = m(t, \rho_0 + (y_t(x), 0))((v_t)_x, 0) = ((w_t)_x, \varphi_t(\rho_0 + (y_t(x), 0))(w_t)_x) \in T_\rho E^t(\rho_0, t)$, the evolution of $(v_t)_y$ by m_t is an $\mathcal{O}(\exp(-C^{-1}t))$, therefore the distance from $H_p(\rho)$ to the space $\left\{ (\delta x, \left(\frac{\partial}{\partial x}\right)\lambda(t, x)\delta x) \right\}$ is also an $\mathcal{O}(\exp(-C^{-1}t))$. So $H_p(\rho) \in T_\rho E(\rho_0, \infty)$.

$$\frac{d}{dt}\rho(-t, \rho) = -H_p(\rho(-t, \rho)) = -(H_p(\rho(-t, \rho))_x, \left(\frac{\partial}{\partial x}\right)\lambda(\infty, x_t)H_p(\rho(-t, \rho))_x) \quad (61)$$

where $\rho(-t, \rho) = \rho_0 + (x_t, \lambda(\infty, x_t))$. We bound the $H_p(\rho(-t, \rho))_x$ component of H_p by $C_0(|H_p(\rho(-t, \rho))_x| + |H_p(\rho(-t, \rho))_y|) \leq C'_0 |H_p(\rho(-t, \rho))_x|$, using the assumption 47 and 61. In the backward evolution, the x' directions are contractive so

$$|H_p(\rho(-t, \rho))_x| \leq C \exp(-C^{-1}t) |H_p(\rho)| \quad (62)$$

this means that $|H_p(\rho(-t, \rho))| \leq C \exp(-C^{-1}t) |H_p(\rho)|$, then $\tilde{\rho} = \lim_{t \rightarrow \infty} \rho(-t, \rho)$ exist and belongs to $N' = \{\rho; H_p(\rho) = 0\}$. On the connected component of ρ_0 of N' , $p = 0$, then $\tilde{\rho} \in N$. But $p(\rho) = p(\rho(-t, \rho)) = p(\tilde{\rho}) = 0$, hence $p|_{E(\rho_0, \infty)} = 0$.

$E(\rho_0, \infty)$ is a smooth manifold of codimension r .

We shall prove that $T_\rho(E(\rho_0, \infty))^{\perp_\sigma} \subseteq T_\rho(E(\rho_0, \infty)) \cap W_-(\rho_0)^{\perp_\sigma} = W_-(\rho_0) \oplus W_0(\rho_0)$.

$$W_0(\rho_0)^{\perp_\sigma} = \text{Im } F_p(\rho_0) = W_+(\rho_0) \oplus W_-(\rho_0), \text{ so } \sigma|_{W_0(\rho_0)} \text{ is non degenerate.} \quad (63)$$

Let $v_0 \in T_\rho(E(\rho_0, \infty))^{\perp\sigma}$, $v_t = m(t, \rho)v_0$, $v_t \in (T_{\rho_t}(E(\rho_0, \infty))^{\perp\sigma})$. Then for all $\delta x \in W^+(\rho_0)$

$$\sigma(\delta x', (v_t)_y) + \sigma(\delta z, (v_t)_z) + \sigma\left(\left(\frac{\partial}{\partial x}\right)\lambda(\infty)\delta x, (v_t)_{x'}\right) = 0 \quad (64)$$

so $|(v_t)_y| + |(v_t)_z| \leq C\varepsilon |(v_t)_{x'}|$, therefore $|v_t| = \mathcal{O}(\exp(C^{-1}t))$ when $t \rightarrow -\infty$. Therefore $v_0 \in T_\rho(E(\rho_0, \infty))$. $E(\rho_0, \infty)$ is an involutive manifold. $\S$.

We shall prove now L^2 estimates. This is done by working on the real line only since we look at C^∞ singularities, let us note by $\{f, g\}$ the usual Poisson bracket.

We can now state the main result of this section.

Theorem 2 *Let $P(x, \lambda^{-1}D_x, \lambda) = (p(x, \xi) + \lambda^{-1}p_1(x, \xi, \lambda))^{\omega_\lambda}$ be a pseudo-differential operator such that $p(x, \xi)$ satisfies the assumptions $H_1, \dots, H_4$. Let λ_j be the eigenvalues of F_p with $\text{Re}\lambda_j > 0$ at the points of $N_R = \{\rho \in \mathbb{R}^{2n}; p(\rho) = dp(\rho) = 0\}$, we assume that*

$$ip_1(x, \xi, \lambda) + \sum_j (\alpha_j + 1/2)\lambda_j \neq 0, \text{ for all } \alpha_j \in \mathbb{N}. \quad (65)$$

Let $\gamma(\rho)$ and $c(\rho) \geq 0$ be smooth functions such that $\{p, \gamma\} + c\rho \geq 0$.

If ω is a small neighborhood of $\rho_0 \in N_R$, suppose that $\gamma(\rho_0) > 0$ and $\{\gamma > 0\} \cap \omega \cap OF(Pu) = \emptyset$ and $\{\gamma > 0\} \cap \partial\omega \cap OF(u) = \emptyset$, then $\rho_0 \notin OF(u)$.

Remark 2 *It is possible to make a less technical statement in the particular case where $W'_{\pm i} = \{0\}$. In this case N' is a smooth symplectic manifold of codimension $2r'$. The involutive manifolds $E'_\pm$ have a foliation, we note by $F_-(\mu)$ the leaf of E'_- through $\mu \in N'$. Then the geometric statement of Theorem 2 is : if $F_-(\rho_0) \cap \omega \cap OF(Pu) = \emptyset$ and $F_-(\rho_0) \setminus \{\rho_0\} \cap \omega \cap OF(u) = \emptyset$ then $\rho_0 \notin OF(u)$.*

This remark will be justified below when we will construct appropriate functions γ . Moreover the presence of the function $c(\rho)$ is needed to have a statement invariant by multiplication of P by an operator with a positive symbol.

5 The energy estimate.

5.1 The basic L^2 inequality.

The basic L^2 estimate will be described in the case $\text{Imp}_1 > 0$. We shall use microlocal weighted estimates. Let $\gamma(x, \xi) \in C^\infty(\mathbb{R}^{2n})$ a bounded real

valued function, we note $e_\gamma = (\lambda^{\gamma(z, \xi)})^{w_\lambda}$, we write $\mu = \ln \lambda / \lambda$. Let $e'_{-\gamma}$ be a parametrix of e_γ . If $A = (a(x, \xi))$ is a pseudo-differential operator with Weyl symbol $a(x, \xi) \in S(1, g)$, then $A_\gamma = e'_\gamma A e_{-\gamma} = (a(x, \xi) + i\mu\{a, \gamma\} + \mathcal{O}(\mu^2))^{w_\lambda}$. We write our operator as $P = (p(x, \xi) + \lambda^{-1}p_1(x, \xi, \lambda))^{w_\lambda}$. Then

$$P_\gamma = (p(x, \xi) + i\mu\{p, \gamma\}(x, \xi) + \lambda^{-1}p_1(x, \xi, \lambda) + \mathcal{O}(\mu^2))^{w_\lambda}. \quad (66)$$

We use also a multiplier $M = (m' + i\mu m'')^{w_\lambda}$, with two real functions $m' \in S(1, g)$ and $m'' \in S(1, g)$. We get an energy estimate from the computation of $\text{Im}(P_\gamma u, M^* u)$. We have

$$\text{Im}(M P_\gamma) = (m' \text{Imp} + \mu(m'' \text{Rep} + m'\{ \text{Rep}, \gamma \}) + \lambda^{-1}(m' \text{Imp}_1 + \{ \text{Rep}, m' \}) + \mathcal{O}(\mu \lambda^{-1}))^{w_\lambda}. \quad (67)$$

We must make the symbol in 67 positive. The first term $m' \text{Imp}$ is nonnegative if $m' \geq 0$. We now concentrate on the second term $m'' \text{Rep} + m'\{ \text{Rep}, \gamma \}$. Let $m''/m'(\rho) = c(\rho)$ be a C^∞ function. $c \text{Rep} + \{ \text{Rep}, \gamma \}$ is null on N' so the best possible choice of γ is to make it transversally elliptic on N' .

$\gamma = \gamma_0 + \gamma_1$, $c = c_0 + c_1$, γ_0 and c_0 are the functions which appears in the statement of Theorem 2, γ_1 and c_1 are constructed below. If γ_1 is null at the second order on N' , the hessian of $c_1 \text{Rep} + \{ \text{Rep}, \gamma_1 \}$ at $\rho \in N'$ is given by the fundamental matrix $c_1(\rho) F_{\text{Rep}}(\rho) + [F_{\text{Rep}}, F_{\gamma_1}](\rho)$.

We shall localize at points of N' , let $\rho \in N'$ we note

$$G(\rho) = c_1(\rho) F_{\text{Rep}}(\rho) + [F_{\text{Rep}}, F_{\gamma_1}](\rho). \quad (68)$$

The assumption (H_4) implies that at each point in N' there is a symplectic basis such that the hessian of F_{Rep} is a sum of terms

- (i) $Q(x, \xi) = ax, \xi$, with $\text{Spec}(a) \subseteq \{z \in \mathbb{C}; \text{Re} z > 0\}$
- (ii) $Q(x, \xi) = \alpha(x^2 + \xi^2)$, with $\alpha > 0$.
- (iii) $Q(x, \xi) = 0$.

We shall find appropriate quadratic form γ_1 and constant c_1 at ρ so that G in 68 is positive and piece them together. If we are in case (i), we chose $\gamma_1(x, \xi) = (\alpha x, x) - (\beta \xi, \xi)$ α and β are two positive matrices so that $\sigma((x, \xi), [F_{\text{Rep}}, F_{\gamma_1}](x, \xi)) \geq 1/C(x^2 + \xi^2)$; any $c_1 > 0$ will fit.

If we are in case (ii), we take $\gamma_1(x, \xi) = -k(x^2 + \xi^2)$ with $k > 0$ and small with respect to c_1 .

In case (iii) $\gamma_1 = 0$. Therefore we can construct functions $\gamma(\rho)$ and $c(\rho)$, such that

$$(c \text{Rep} + \{ \text{Rep}, \gamma \})(\rho) \geq C^{-1} d(\rho, N')^2. \quad (69)$$

Moreover if γ_1 is small with respect to γ_0 we shall have $\{\gamma > 0\} \cap \partial\omega \cap OF(u) \subseteq \{\gamma_0 > -\varepsilon\} \cap \partial\omega \cap OF(u) = \emptyset$.

We choose $m'(\rho) = \varphi(\rho)^2$, where φ is a C^∞ function supported by ω . $Imp_1 > 0$ is positive, while $\{Rep, m'\}$ is supported near $\partial\omega$. We derive the estimate

$$Im(P_\gamma u, M^* u) \geq c\mu \left(\sum_j |v_j(\varphi u)|^2 \right) + c\lambda^{-1} |\varphi u|^2 + \mathcal{O}(\lambda^{-1}) |\psi u|^2 + \mathcal{O}(\lambda^{-1}\mu) |u|^2 \quad (70)$$

the v_j form a set of equations of N' , ψ is supported near $\partial\omega$. We replace u by $e'_\gamma u$ and we note $M_\gamma = e'_\gamma M e'_\gamma$. In the following the third term in 70 could be neglected since $OF(u) \cap \partial\omega \cap \{\gamma > 0\} = \emptyset$.

We introduce the additionnal notation : let m be an order function and g a metric a symbol $a(x, \xi) \in \tilde{S}(m, g)$ if it is the sum of a symbol in $S(m, g)$ supported by a neighborhood of the support of φ and a symbol of order $-\infty$. In the following m will have the form $\lambda^m (\ln \lambda)^p$ and $g = g_0$ or $m = \lambda^\gamma (\ln \lambda)^p$ and $g' = (\ln \lambda)^2 g_0$. Then we have

$$Im(Pu, M_\gamma^* u) \geq c\mu \sum_j |v_j \varphi e'_\gamma u|^2 + \lambda^{-1} |\varphi e'_\gamma u|^2 + (R_{2\gamma-2,2} u, u). \quad (71)$$

with $R \in \tilde{S}(\lambda^{2\gamma-2} (\ln \lambda)^2, g')$. We shall use the notation $|u|_\gamma = |e'_\gamma u|$.

5.2 Concatenations.

We move the subprincipal symbol using multiplication by non elliptic operators, this is named concatenations.

$$Im(JU_N Pu, JM_\gamma^* U_N u) = Im([JU_N, P] u, JM_\gamma^* U_N u) + Im(PJU_N, M_\gamma^* JU_N u) \quad (72)$$

where $U_N u = (U_\alpha u)_{|\alpha|=N}$ and $U_\alpha = (u_1^{w_\lambda})^{\alpha_1} \dots (u_r^{w_\lambda})^{\alpha_r}$, J is a linear operator in the space $C^{N'}$ of multi-indices of length N . We can apply inequality 71 to the second term of 72.

We compute the commutator $[P, U_N]$. Let P_0 be the principal part of P , i.e. $P_0 = (p)^{w_\lambda} = \sum_{1 \leq j \leq r} (p_j u_j)^{w_\lambda}$ and $P = P_0 + i\lambda^{-1} (p_1)^{w_\lambda}$, p_1 is the sub principal symbol.

$$[U_\alpha, P_0] = \sum_{1 \leq j \leq r, p+q=\alpha_j-1} (u_1^{w_\lambda})^{\alpha_1} \dots (u_j^{w_\lambda})^p [u_j^{w_\lambda}, P_0] (u_j^{w_\lambda})^q \dots (u_r^{w_\lambda})^{\alpha_r}. \quad (73)$$

But $[u_j^{w_\lambda}, P_0] = \sum_{1 \leq k \leq r} [u_j^{w_\lambda}, (p_k u_k)^{w_\lambda}]$, $(p_k u_k)^{w_\lambda} = p_k^{w_\lambda} u_k^{w_\lambda} - \frac{1}{2i\lambda} \{p_k, u_k\}^{w_\lambda} + \mathcal{O}(\lambda^{-2})$, so $[u_j^{w_\lambda}, P_0] = \sum_{1 \leq k \leq r} [u_j^{w_\lambda}, p_k^{w_\lambda}] u_k^{w_\lambda} + \mathcal{O}(\lambda^{-2})$.

We deduce then

$$\begin{aligned} [U_\alpha, P_0] = & \sum_{1 \leq k \leq r, 1 \leq j \leq r} \frac{1}{i\lambda} \{u_j, p_k\}^{w_\lambda} \alpha_j (u_1^{w_\lambda})^{\alpha_1} \dots (u_k^{w_\lambda})^{\alpha_{k+1}} \dots (u_j^{w_\lambda})^{\alpha_{j-1}} \dots (u_r^{w_\lambda})^{\alpha_r} \\ & + \sum_{\beta < \alpha} (c_{\alpha, \beta})^{w_\lambda} U_\beta \end{aligned} \quad (74)$$

where $c_{\alpha, \beta}$ are symbols of degree $-1 - |\alpha| + |\beta|$.

We know that $\text{Spec}(\{p_k, u_j\}) \subset \{z \in \mathbb{C}, \text{Re } z > c\}$.

The operator $(z_\alpha) \rightarrow (\alpha_j z_{\alpha-(j)+(k)})$ is algebraically the operator $z_k \frac{\partial}{\partial z_j}$.

With the notation $a_{j,k} = \{u_j, p_k\}$, and

$$(\mathcal{A}_N U)_\alpha = \sum_{\alpha = \beta - (j) + (k), 1 \leq j, k \leq r} a_{k,j} \beta_j U_\beta,$$

we have

$$[U_N, P_0] = i\lambda^{-1} (\mathcal{A}_N U_N) + \left(\sum_{\beta < \alpha} (c_{\alpha, \beta})^{w_\lambda} U_\beta \right)_\alpha. \quad (75)$$

The same result will hold for P since P_1 will contribute to the second term in 75.

We construct the linear operator J , such that $J\mathcal{A}_N(\rho_0)J^{-1} = ((\sum_j \alpha_j \lambda_j) \delta_{\alpha, \beta}) + o(N)$, the contribution $o(N)$ is due to that eventually $a(\rho_0)$ cannot be made diagonal. We see that the self-adjoint part of operator $M_\gamma J\mathcal{A}_N J^{-1}$ is positive elliptic. Using the Gårding inequality for systems we have

$$\text{Im}(J\mathcal{A}_N U_N u, JM_\gamma^* U_N u) \geq cN |\varphi^{w_\lambda} e'_\gamma J U_N u|^2 - C_N (R_{2\gamma-1,2} U_N u, U_N u) \quad (76)$$

where $R_{2\gamma-1,2} \in \tilde{S}(\lambda^{2\gamma-1} (\ln \lambda)^2, g')$.

We estimate

$$\begin{aligned} \text{Im}(J[U_N, P] u, JM_\gamma^* U_N u) \geq & cN \lambda^{-1} |\varphi^{w_\lambda} e'_\gamma J U_N u|^2 + (R_{2\gamma-2,2}^{(N)} U_N u, U_N u) + \sum_{l < N} \text{Re}(J C_{N,l} U_l u, M_\gamma^* J N u) \end{aligned} \quad (77)$$

where $R^{(N)} \in \tilde{S}(\lambda^{2\gamma-2}(\ln \lambda)^2, g')$, the $C_{N,l}$ are operators of order $-1-N+l$. We estimate the third term in 77. The operator $M_\gamma = e'_\gamma(\varphi^{w_\lambda})^2 e'_\gamma + R_{2\gamma-1,2}$, then

$$\begin{aligned} & \operatorname{Re}(M_\gamma J C_{N,l} U_l u, J U_N u) \leq \\ & \varepsilon \lambda^{-1} |\varphi^{w_\lambda} e'_\gamma J U_N u|^2 + C_{N,\varepsilon} \lambda^{-1-2N+2l} |U_l u|_\gamma^2 + C_N \lambda^{-3} (\ln \lambda)^4 |U_N u|_\gamma^2 \end{aligned} \quad (78)$$

So we get

$$\begin{aligned} & \operatorname{Im}(J [U_N, P] u, J M_\gamma^* u) \geq \\ & c N \lambda^{-1} |\varphi e'_\gamma J U_N u|^2 + (R_{2\gamma-2,2}^{(N)} U_N u, U_N u) + \sum_{l < N} R_{2\gamma-2N+2l}^{(N,l)} U_l u, U_l u \end{aligned} \quad (79)$$

If we chose N such that $\operatorname{Im} p_1 + cN > 0$, using 71, 72 and 79 we obtain

$$\begin{aligned} & \operatorname{Im}(J U_N P u, J M_\gamma^* U_N u) \geq \\ & c N \lambda^{-1} |\varphi^{w_\lambda} e'_\gamma J U_N u|^2 - C \lambda^{-2} (\ln \lambda)^4 |\varphi_1^{w_\lambda} e'_\gamma U_N u|^2 \\ & - C_N \sum_{0 \leq k \leq N-1} \lambda^{-1-2N+2k} |\varphi_1^{w_\lambda} e'_\gamma U_k u|^2 + \mathcal{O}(\lambda^{-\infty}) \end{aligned} \quad (80)$$

where φ_1 is a function supported by a neighborhood of $\operatorname{supp} \varphi$.

As in [5] the proof is based on a recurrence on the H^s regularity of the $U_k u$ in the domain $\{\gamma(x, \xi) > 0\} \cap \omega$.

We must modify the Proposition 1 of [5] to take care of the last terms in 80. We shall estimate $c_k = \sum_{0 \leq l \leq k} \lambda^l |e'_\gamma \varphi^{w_\lambda} U_k u|$ by c_N , N is now fixed. To do that we use the equation

$$P u = f = \sum_{1 \leq j \leq r} p_j^{w_\lambda} u_j^{w_\lambda}(u) + \left(\frac{1}{2i\lambda} \{u_j, p_j\} + \lambda^{-1} p_1 + \mathcal{O}(\lambda^{-2}) \right) u \quad (81)$$

We note

$$p'_1 = \frac{1}{2i} \sum_{1 \leq j \leq r} \{u_j, p_j\} + p_1 \quad (82)$$

We use as above commutators with the operators U_k ; from formula 75 we obtain

$$\begin{aligned} U_n f = U_n P u &= i \lambda^{-1} A_n U_n(u) + \sum_{0 \leq j \leq n-1} Q_{n,j} U_j(u) \\ &+ \sum_{1 \leq j \leq r} p_j^{w_\lambda} u_j U_n(u) + \lambda^{-1} p'_1 U_n(u) + \mathcal{O}(\lambda^{-2})(u) \end{aligned} \quad (83)$$

The $Q_{n,j}$ are operators of order $-1 - n + j$.

It is a consequence of the assumptions that the matrix $A'_n = A_n + Id_{\mathbb{C}^n} p'_1$ is invertible for any n in a neighborhood of ρ_0 . Let B_n an operator of order γ such that $B_n A'_n = \varphi_0 e'_\gamma Id + \mathcal{O}(\lambda^{-\infty})$. We apply B_n on the members of equation 83

$$B_n(f) = i\lambda\varphi_0^{w_\lambda} e'_\gamma U_n + \sum_{1 \leq j \leq r} B_n P_j(U_n u) + \sum_{l < n} B_n Q_{n,j} U_j u \quad (84)$$

We multiply both members of 84 by φ^{w_λ} , using $\varphi^{w_\lambda} \varphi_0^{w_\lambda} = \varphi^{w_\lambda} + \mathcal{O}(\lambda^{-\infty})$, $[B_n P_j, \varphi^{w_\lambda}] \in \tilde{S}(\lambda^{\gamma-1}(\ln \lambda), g')$ and $[B_n Q_{n,j}, \varphi^{w_\lambda}] \in \tilde{S}(\lambda^{\gamma-2-n+j}(\ln \lambda), g')$ we obtain

$$\begin{aligned} |\varphi^{w_\lambda} e'_\gamma U_n u| &\leq \\ C(o(1) \sum_{1 \leq j \leq r} |\varphi^{w_\lambda} u_j U_n u|_{\gamma+1} &+ \sum_{l < n} |\varphi^{w_\lambda} U_l u|_{\gamma-n+l} + \sum_{l \leq n+1} |\varphi_1^{w_\lambda} U_l u|_{\gamma-n+l,1}) \\ &+ \mathcal{O}(\lambda^{-\infty}) \end{aligned} \quad (85)$$

This justify the notations $c_n = \sum_{j \leq n} \lambda^j |e'_\gamma \varphi^{w_\lambda} u|$ and $d_n = (\ln \lambda) \sum_{j \leq n} \lambda^j |e'_\gamma \varphi_1^{w_\lambda} u|$. We have proved

$$c_n \leq o(1)c_{n+1} + k_0 \sum_{j < n} c_j + k_1 d_{n+1} + \mathcal{O}(\lambda^{-\infty}) \quad (86)$$

k_0, k_1 are some constant.

The basic idea of Proposition 1 of [5] is to derive from 86 an upper bound of the c_j for $0 \leq j \leq N-1$ by c_N , where N is an integer choosen large enough with respect to the imaginary part of p_1 . We shall need eventually to shrink ω accordingly. The d_j are controlled by using the steps of this recurrence. So we obtain by recurrence the smoothness of u in the domain $\{(x, \xi); \gamma(x, \xi) > 0\} \cap \omega$.

Now we can finish the proof as in [5].

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