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**On regression models with non-square  
integrable martingale-like errors**

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The paper is concerned with two types of estimators of an unknown parameter  $\theta$  of the drift of an observed semimartingale  $X$ . A martingale part  $M$  of the semimartingale  $X$  is not a local square integrable martingale in general. As a rule we suppose only that  $M$  has a  $r$ -th moment,  $r \in [1, 2]$ .

The first part of the paper is devoted to an investigation of strong consistency of the least-square estimators (LS-estimators). Our approach is based on a multidimensional large numbers law for local martingales (see [1], where the results were announced particularly, see also [2] - [3]).

In the second part of the paper another type estimators of  $\theta$  are studied. They are so-called sequential estimators (SQ-estimators), and were systematically investigated in [4] for regression models with local square integrable martingales and quasi-left-continuous local martingales as errors. It was proved there that these estimators have a very important property-a guaranteed accuracy. Here we get rid of from these assumptions proved a generalisation of Novikov's [2] inequality and Metivier-Pellaumail's one [5] for general local martingales and using the approach of the paper [4].

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$  be a standard stochastic basis on which we consider all stochastic processes whose paths are regular.

Let us denote (see, for references [2])  $\mathcal{M}_{loc}(R^d)$  the set of local martingales, which values in  $R^d$   $d \geq 1$  ;

$\mathcal{A}^+_{loc}(R^d)$  the set of predicatble processes, whose values are positive definite operators (matrix) from  $R^d$  into  $R^d$  such that  $A_t - A_s \geq 0, t \geq s$ .

Let  $\lambda_1(A), \lambda_2(A)$  and  $tr(A)$  be the minimal, maximal egenvalues and the trace of the operator (matrix)  $A$ . Let us denote  $A^*$  a transpose matrix of  $A$ .

For a random process  $X$  with values in  $R^d, d \geq 1$ , let  $\{w : X_t \rightarrow\}$  be the set of  $\omega \in \Omega$  such that  $\lim_{t \rightarrow \infty} X_t(\omega) = X_\infty(\omega)$  exists for the norm  $\|\cdot\|$  of the space  $R^d$ .

If  $A, B \in \mathcal{F}$  and  $P\{A \cap \{\Omega \setminus B\}\} = 0$ , then we write  $A \leq B$  (a.s.).

Let  $M \in \mathcal{M}_{loc}(R^d)$  and

$$M_t = M_t^c + \int_0^t \int_{R_0^d} x d(\mu - \nu), \quad (1)$$

be the canonical decomposition of  $M$ , where  $R_0^d = R^d \setminus \{0\}$ ,  $M^c$  be a continuous part of  $M$  (and  $\langle M^c \rangle$  be its (matrix) quadratic characteristic),  $\mu$  be a random measure of jumps of  $M$  and  $\nu$  be its compensator (see [2]).

Theorem 1 : Assume the following conditions : (a.s.)

$$1) \lim_{t \rightarrow \infty} \lambda_1(A_t) = \infty;$$

$$2) \limsup_{t \rightarrow \infty} \frac{\lambda_1(A_t)}{\lambda_2(A_t)} < \infty;$$

$$3) \int_0^\infty \lambda_1^{-2}(A_s) d\langle M^c \rangle_s + \int_0^{+\infty} \int_{R_0^d} \lambda_1^{-r}(A_s) \|x\|^r d\nu < +\infty$$

for some  $r \in [1, 2]$ .

Then  $A_t^{-1} M_t \rightarrow 0$  (a.s.) as  $t \uparrow \infty$ .

Particularly, if  $V$  is predictable increasing process such that (a.s.)

$$\frac{d\langle M^c \rangle_t}{dV_t} + \frac{d}{dV_t} \int_0^t \int_{R_0^d} \|x\|^r d\nu \leq \xi < \infty$$

and (a.s.)

$$3') \int_0^\infty \lambda_1^{-r}(A_s) dV_s < \infty,$$

then 1), 2), 3')  $\Rightarrow A_t^{-1} M_t \rightarrow o$  (a.s.) as  $t \uparrow \infty$ .

Proof : Denote  $\mathcal{B}$  a compensator of an increasing process  $B$ . Then as in one-dimensional case (see [2]) it is proved that (a.s.).

$$\{\omega : t r < M^c >_{\infty} + \sum_s \frac{\|\Delta M_s\|^2}{1 + \|\Delta M_s\|} < \infty\} \leq \{\omega : M_t \rightarrow\}.$$

Particularly, for some  $r \in [1, 2]$  (a.s.)

$$\{\omega : t r < M^c >_{\infty} + \sum_s \|A M_s\|^r < \infty\} \leq \{\omega : M_t \rightarrow\} \quad (2)$$

The last statement follows from

$$\frac{\|x\|^2}{1 + \|x\|} \leq \|x\|^r \text{ for all } x \in R^d, r \in [1, 2].$$

Now define as in [2] - [3] the process

$$Y_t = \int_0^t A_s^{-1} d M_s.$$

Using the same arguments we have that (a.s.)

$$\{\lambda_1(A_t) \rightarrow \infty\} \cap \left\{ \lim_{t \rightarrow \infty} \frac{\lambda_2(A_t)}{\lambda_1(A_t)} < \infty \right\} \subset \{A_t^{-1} M_t \rightarrow o\}.$$

To complete the proof note that the condition 3) 3') implies (a.s.)

$$t r < Y^c >_{\infty} + \sum_s \|\Delta Y_s\|^r < \infty$$

(in the case of 3'))

and in view of (2) we get the statement of the theorem 1.

This theorem gives us a possibility to prove the strong consistency of the LS-estimators in regression models with non-square integrable martingale errors.

Consider the following regression model

$$X_t = \int_0^t f s d V_s \theta + m_t, \quad (3)$$

where  $m$  is a pure discontinuous (for simplicity) local martingale from  $\mathcal{M}_{loc}(R^d)$ , a predictable process  $V \in \mathcal{A}_{loc}^+(R^1)$ ,  $f$  is a predictable  $(d \times k)$ -matrix,  $\theta \in R^k$ ,  $k \geq 1$ , is an unknown parameter.

$$\text{Let } F_t = \int_0^t f_s^* f_s dV_s, F_t > 0, t \geq t_0.$$

In this case we can define the estimator of  $\theta$ :

$$\theta_t = F_t^{-1} \int_0^t f_s^* dX_s = \theta + F_t^{-1} \int_0^t f_s^* dm_s.$$

Theorem 2 : Suppose for the model (3) the following conditions hold (a.s.)

$$1) \lim_{t \rightarrow \infty} \lambda_1(F_t) = \infty;$$

$$2) \limsup_{t \rightarrow \infty} \frac{\lambda_2(F_t)}{\lambda_1(F_t)} < \infty;$$

$$3) \int_0^\infty \int_{R_0^d} \lambda_1^{-r}(F_s) \|f_s\|^r \|x\|^r d\nu < \infty$$

where  $r \in [1, 2]$ ,  $\nu$  is a compensator of a measure  $\mu$  of jumps of  $M$ .

Then  $\theta_t \rightarrow \theta$  (a.s.) as  $t \uparrow \infty$ .

It is possible to unify the conditions of the theorem 2, if we suppose that (a.s.)

$$\frac{d}{dV_t} \int_0^t \int_{R_0^d} \|x\|^r d\nu \leq \xi < \infty,$$

and (a.s.)

$$3) \int_0^\infty \lambda_1^{-r}(F_s) \|f_s\|^r dV_s < \infty.$$

Then 1) - 2) - 3')  $\Rightarrow \theta_t \rightarrow \theta$  (a.s.) as  $t \uparrow \infty$ .

Proof : It is sufficient to note that

$$\theta_t - \theta = A_t^{-1} M_t,$$

$$\text{where } A_t = F_t, M_t = \int_0^t \int_{R_0^d} f^* x d(\mu - \nu).$$

Using the theorem 1 we get immediatly the statement of the theorem 2.

Remark : Note that the consistency of LS-estimators for the model (3) with non-random regressors was proved by Novikov [6]. The strong consistency of the LS-estimators for this model with non-random regressors was studied also in [7]-[8].

Now consider another type of estimators of  $\theta$  in the one-dimensional model (3). These are SQ-estimators, which systematically were studied in [4]. But the case of non-square integrable errors was handed there for the quasi-left continuous martingale errors  $m$  only. Here we prove an estimate for pure-discontinuous martingales and apply it to give an upper estimate for the  $r$ -th moment of the difference between the SQ-estimator and  $\theta$ . This result gives us (in some sense) a guaranted accuracy of these estimators.

Denote  $\mathfrak{B}(R)$ - Borel  $\sigma$ -algebra of the space  $R$ . Let

$$M_t = \int_0^t \int_{R_0} x d(\mu - \nu)$$

be a purely discontinuous local martingale of the classe  $\mathcal{M}_{loc}(R^1)$  (see decomposition (1)).

Let  $U$  be a  $\mathfrak{B}(R_+^1) \otimes \mathcal{F} \otimes \mathfrak{B}(R_0^1)$ -measurable function such that for some

$$r \in [1, 2]$$

$$\int_0^t \int_{R_0} |U|^r d\nu \in \mathcal{A}_{loc}^+(R^1)$$

and

$$\int_{R_0} U(t, x, \omega) v(\{t\}, dx) = 0 \quad (4)$$

$$\text{Denote } Y_t(U) = \int_0^t \int_{R_0} U d(\mu - \nu) \text{ and } Y_t^*(U) = \sup_{s \leq t} |Y_s(U)|.$$

Theorem 3 : Suppose the function  $U$  satisfies to the condition (4) and  $\tau$  is a predictable stopping time (s.t.). Then

$$E |Y_{\tau-}^*(U)|^r \leq A_r E \int_0^{\tau-} \int_{R_0} |U|^r d\nu, \quad (5)$$

where  $A_r \leq 3 \left( \frac{r}{r-1} \right)^r$ ,  $r \in (1, 2]$ ,  $A_1 = Z$  and  $Y_{t-}$  is left limit of  $Y_t$ .

Proof. We shall use Novikov's method [5]. Let us involve the s.t. ( $a > 0$ )

$$\tau_a = \inf \left( t \leq \tau : \int_0^t \int_{R_0} |U|^r d\nu \geq a \right),$$

$$\inf(\emptyset) = \tau.$$

Of course,  $\tau_a$  is a predictable s.t.

Therefore there is a sequence of s.t.'s  $(\tau_a^n)_{n \geq 1}$  such that

$$\tau_a^n \uparrow \tau_a \text{ (a.s.) as } n \uparrow \infty,$$

$$\tau_a^n < \tau_a \text{ on the set } (\omega : \tau_a < \infty).$$

It follows from here that

$$\int_0^{\tau_a^n} \int_{R_0} |U|^r d\nu < a.$$

Let us show that  $E Y_{\tau_a^n}^*(U) < \infty$ , we have (as usually,  $I_c$  is an indicator of  $c$ )

$$\begin{aligned}
E Y_{\tau_a^n}^*(U) &\leq E \sup_{t \leq \tau_a^n} \left| \int_0^t \int_{R_0} U \cdot I_{|U|>1} d(\mu - \nu) \right| + \\
&+ E \sup_{t \leq \tau_a^n} \left| \int_0^t \int_{R_0} U \cdot I_{|U|\leq 1} d(\mu - \nu) \right| \leq \\
&\leq \text{const} \cdot E \int_0^{\tau_a^n} \int_{R_0} |U| \cdot I_{|U|>1} d\nu + \text{const} \cdot E \left[ \int_0^{\tau_a^n} \int_{R_0} |U|^2 \cdot I_{|U|\leq 1} d\nu \right]^{1/2} \leq \\
&\leq \text{const} \cdot E \int_0^{\tau_a^n} \int_{R_0} |U|^r \cdot I_{|U|>1} d\nu + \text{const} \cdot E \left[ \int_0^{\tau_a^n} \int_{R_0} |U|^r d\nu \right]^{1/2} \leq \\
&\leq \text{const} (a + a^{1/2}) < \infty.
\end{aligned}$$

Using this fact and the elementary inequality

$$\left| |x+y|^r - |x|^r \right| \leq C_r (|x|^{r-1} |y| + |y|^r)$$

we get that

$$\begin{aligned}
E \int_0^{\tau_a^n} \int_{R_0} I_{|U|>1} \left| |Y_{s-} + U|^r - |Y_{s-}|^r \right| d\nu < \infty, \\
E \left[ \int_0^{\tau_a^n} \int_{R_0} I_{|U|\leq 1} \left| |Y_{s-} + U|^r - |Y_{s-}|^r \right|^2 d\nu \right]^{1/2} < \infty \quad (6)
\end{aligned}$$

This first inequality of (6) follows from

$$E \int_0^{\tau_a^n} \int_{R_0} I_{|U|>1} \left| |Y_{s-} + U|^r - |Y_{s-}|^r \right| d\nu \leq$$



$$\leq \text{const}(r) E(1 + |Y_{\tau_a^n}|^{r-1}) \int_0^{\tau_a^n} \int_{R_0} |U|^r d\nu \leq$$

$$\leq a \cdot \text{const}(r) \cdot E(1 + |Y_{\tau_a^n}^*|^{r-1}) < \infty.$$

The second one follows from

$$\begin{aligned} E \left[ \int_0^{\tau_a^n} \int_{R_0} I_{|U| \leq 1} | |Y_{s-} + U|^r - |Y_{s-}|^r | d\nu \right]^{1/2} &\leq \\ \leq \text{const}(r) E \left[ \int_0^{\tau_a^n} \int_{R_0} I_{|U| \leq 1} (|Y_{s-}|^{r-1} |U| + |U|^r)^2 d\nu \right]^{1/2} &\leq \\ \leq \text{const}(r) E(1 + (Y_{\tau_a^n}^*)^{2(r-1)})^{1/2} \left( \int_0^{\tau_a^n} \int_{R_0} I_{|U| \leq 1} |U|^r d\nu \right)^{1/2} \end{aligned}$$

and

$$E(Y_{\tau_a^n}^*)^{r-1} \leq (E Y_{\tau_a^n}^*(U I_{|U| > 1}))^{r-1} + (E Y_{\tau_a^n}^*(U \cdot I_{|U| \leq 1}))^{r-1}$$

Now using the Ito's formula (see [2] ; p. 150-151)

we get

$$\begin{aligned} |Y_{\tau_a^n}|^r &= \int_0^{\tau_a^n} \int_{R_0} (|Y_{s-} + U|^r - |Y_{s-}|^r) d(\mu - \nu) + \\ &+ \int_0^{\tau_a^n} \int_{R_0} \{ |Y_{s-} + U|^r - |Y_{s-}|^r - r |Y_{s-}|^{r-2} Y_{s-} U \} d\nu \end{aligned} \tag{7}$$

It follows from (6) that

$$E(\text{martingale part of (7)}) = 0$$

Applying the elementary inequality

$$|x + y|^r - |x|^r - rxy|x|^{r-2} \leq B_r |y|^r,$$

where  $B_r \leq 3$ ,  $r \in [1, 2]$  and  $B_1 = 2$ ,

to the second part of (7), we have

$$E |Y_{\tau_a^n}^*|^r \leq B_r E \int_0^{\tau_a^n} \int_{R_0} |U|^r d\nu. \quad (8)$$

Using the Doob's inequality [2], we get

$$E (Y_{\tau_a^n}^*)^r \leq 3 \left( \frac{r}{r-1} \right)^r E \int_0^{\tau_a^n} \int_{R_0} |U|^r d\nu.$$

To tend  $n \uparrow \infty$  and  $a \uparrow \infty$  we complete the proof.

We note that the inequality (8) for  $r = 1$  is true with  $B_1 = 2$  and therefore  $A_1 = 2$ .

Now consider the one-dimensional regression model (3) and suppose that

$$\frac{d}{dV_t} \int_0^t \int_{R_0} |x|^r d\nu \leq \gamma_t, \quad (9)$$

where  $r \in [1, 2]$ ,  $\gamma$  is a predictable process such that

$$K_t = \int_0^t \gamma_s^{1-r} |f_s|^r dV_s \in \mathcal{A}_{loc}^+(R^1).$$

we define the following SQ-estimator

$$\theta_H = H^{-1} \int_0^{\tau_H^-} \gamma_s^{-1} f_s dX_s + H^{-1} \beta_H \gamma_{\tau_H}^{-1} f_{\tau_H} \Delta X_{\tau_H},$$

where  $H > 0$ ,  $\tau_H = \inf(t : K_t \geq H)$ ,

$\beta_H - \mathcal{F}_{\tau_H^-}$ -measurable random variable such that

$$\beta_H \in [0, 1],$$

$$\int_0^{\tau_H^-} \gamma_s^{-1} |f_s|^r dV_s + \beta_H \gamma_{\tau_H}^{-1} |f_{\tau_H}|^r \Delta V_{\tau_H} = H. \quad (10)$$

Theorem 4: Let the conditions (9) - (10) are fulfilled.

Then  $\int_0^\infty \gamma_s^{1-r} |f_s|^r dV_s = \infty (a.s.) \Rightarrow E \theta_H = \theta$  and  $E |\theta_H - \theta|^r \leq \text{const}(r) \cdot H^{-r} (H + \Delta)$ ,  
where  $\Delta = E \sup_t \Delta K_t$ ,  $r \in [1, 2]$ .

Proof: The first statement is the direct consequence of (10). Now we have, using the theorem 3, that

$$\begin{aligned} E |\theta_H - \theta|^r &= E |H^{-1} \int_0^{\tau_H^-} \int_{R_0} \gamma_s^{-1} f_s x d(\mu - \nu) + \\ &+ \beta_H H^{-1} \gamma_{\tau_H}^{-1} f_{\tau_H} \Delta M_{\tau_H}|^r \leq A_r 2^{r-1} E H^{-r} \int_0^{\tau_H^-} \int_{R_0} \gamma_s^{-r} |f_s|^r |x|^r d\nu + \\ &+ 2^{r-1} H^{-r} E \beta_H \gamma_{\tau_H}^{-r} |f_{\tau_H}|^r \cdot \int_{R_0} |x|^r \nu(\{\tau_H\}, dx) \leq \\ &\leq H^{-r} 2^{r-1} \left[ A_r E \int_0^{\tau_H^-} \gamma_s^{1-r} |f_s|^r dV_s + E \beta_H \gamma_{\tau_H}^{1-r} |f_{\tau_H}|^r \Delta V_{\tau_H} \right] \leq \\ &\leq 2^{r-1} H^{-r} [A_r H + \Delta] \leq \text{const}(r) \cdot H^{-r} (H + \Delta) \end{aligned}$$

The theorem is proved.

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