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**Convergence of Finite Element Galerkin Approximations
on Galerkin Problems**

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J.A. NITSCHÉ0. Introduction

Let the model problem

$$\begin{aligned} \dot{u} - \Delta u &= f & \text{in } \Omega \times (0, T] &, \\ u &= 0 & \text{on } \partial\Omega \times (0, T] &, \\ u_{t=0} &= u_0 & \text{in } \Omega & \end{aligned}$$

be given. Here $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with $\partial\Omega$ sufficiently smooth. With the help of a finite-element-approximation-space $S_h \subseteq \dot{H}_1^0$ the standard Galerkin approximation $u_h = u_h(t) \in S_h$ is defined by

$$\begin{aligned} (\dot{u}_h, \chi) + D(u_h, \chi) &= (f, \chi) & \text{for } \chi \in S_h &, \\ u_h(0) &= Q_h u_0 & . & \end{aligned}$$

Here $(\dots)$ is the $L_2(\Omega)$ -scalar product, $D(\dots)$ the Dirichlet integral and Q_h an appropriate projection.

The aim of this paper is to derive L_∞ -estimates for the error $e = e_h = u - u_h$. For the corresponding elliptic case this problem was solved by SCOTT, NITSCHÉ about three years ago. There is a certain feeling that the proofs for the parabolic case would (resp. should) be more or less a direct consequence.

Besides the case of one space-dimension seemingly only BRAMBLE-SCHATZ-THOMÉE-WAHLBIN have attacked this problem. Their approach is to rewrite the Galerkin-equations in the form

$$e + T_h \dot{e} = (I - R_h) u \quad .$$

Here to any f the element $U_h = T_h f = R_h(-\Delta^{-1}f)$ is the Ritz-approximation on $-\Delta^{-1}f$ defined by $U_h \in S_h$ and

$$D(U_h, x) = (f, x) \quad \text{for } x \in S_h.$$

We may also write

$$e = -\Delta^{-1}\dot{e} + (I-R_h)\Delta^{-1}\dot{e} + (I-R_h)u.$$

In this way L_∞ -estimates for e are reduced to L_∞ -estimates of the Ritz-method applied to u on the one hand and to $\Delta^{-1}\dot{e}$ on the other and L_∞ -estimates of $\Delta^{-1}\dot{e}$. The last term is bounded in L_∞ if $\dot{e}$ belongs to L_p with $p > n/2$. In this way

$$\|e\|_{L_\infty} \leq c\|\dot{e}\|_{L_p} + \dots$$

is shown. Repeating this argument an estimate of the type

$$\|e\|_{L_\infty} \leq c\|\partial_t^y e\|_{L_2} + \dots$$

can be derived. This finally leads to an optimal order of convergence - with respect to the subspaces - , but depending on the dimension the norms of the solution u entering the right hand side are stringent.

Independent of the space-dimension the validity of the estimates can be shown:

$$\|e\|_{L_\infty(L_\infty)} \leq c h^m \left\{ \|u\|_{L_\infty(W_\infty^m)} + \|\dot{u}\|_{L_\infty(W_\infty^m)} + \|\ddot{u}\|_{L_2(W_\infty^m)} \right\}.$$

The proofs are given in an article to appear in R.A.I.R.O., anal. numer. Here we discuss the case of $N = 3$ space dimensions in some detail. The general case is only sketched.

1. Notations, L_2 -Projection

In the following we will use the standard notations of the theory of partial differential equations. In addition we will work with weighted norms resp. semi-norms introduced three years ago here in Rennes.

For $x_0 \in \Omega$ and $\rho > 0$ let

$$\mu = |x - x_0|^2 + \rho^2.$$

The weighted semi-norms are defined by

$$\|\nabla^k u\|_\alpha^2 = \sum_{|\xi|=k} \iint_\Omega \mu^{-\alpha} |D^\xi u|^2 dx.$$

The first result of the mentioned paper was related to the L_2 -projection:

Theorem 1: The L_2 -projection onto a finite-element-space is bounded in the α -norm for any $\alpha \in \mathbb{R}$:

$$\|P_h u\|_\alpha \leq c_\alpha \|u\|_\alpha.$$

Generalizing this result in the same way it can be shown.

Theorem 2: Let S_h be a finite-element-space and P_h be the L_2 -projection onto S_h . Then

$$\|u - P_h u\|_\alpha + h \|\nabla(u - P_h u)\|_\alpha \leq \inf_{x \in S_h} \left\{ \|u - x\|_\alpha + h \|\nabla(u - x)\|_\alpha \right\} .$$

This theorem guarantees the simultaneous approximation property of the L_2 -projection with respect to the L_2 - and H_1 -norm.

2. A priori Estimates in Weighted Norms

We will use the identity

$$D(u, \mu^{-\alpha} u) = \|\nabla u\|_\alpha^2 + \iint u \nabla u \cdot \nabla \mu^{-\alpha} = \|\nabla u\|_\alpha^2 - \frac{1}{2} \iint u^2 \Delta \mu^{-\alpha} .$$

Now by direct differentiations we have

$$\Delta \mu^{-\alpha} = -2\alpha \mu^{-\alpha-2} (\rho^2 + (N-2\alpha-2) r^2) .$$

For $0 < \alpha < N/2 - 1$ therefore $\Delta \mu^{-\alpha}$ is negative and

$$\mu^{\alpha+1} |\Delta \mu^{-\alpha}|$$

is bounded and bounded away from zero:

Lemma 1: Let $0 < \alpha < N/2 - 1$. Then for any $u \in \overset{\circ}{H}_1$

$$\|u\|_{\alpha+1}^2 + \|\nabla u\|_\alpha^2 \leq c D(u, \mu^{-\alpha} u) .$$

The case $\alpha = N/2 - 1$ is of special interest. Then

$$\Delta \mu^{-\alpha} = -N(N-2) \rho^2 \mu^{-\alpha-2} .$$

Lemma 2: Let $\alpha = N/2 - 1$. Then for any $u \in \overset{\circ}{H}_1$

$$c \rho^2 \|u\|_{\alpha+2}^2 + \|\nabla u\|_\alpha^2 \leq D(u, \mu^{-\alpha} u) .$$

For finite elements the L_∞ -norm is bounded by the weighted norms if x_0 and ρ are chosen properly. Especially we have for $\rho = \gamma h$ with γ fixed: Let $\varphi \in S_h$ and $\alpha = N/2 - 1$. Then

$$\|\varphi\|_{L_\infty} \leq \sup_{x_0 \in \Omega} \rho \|\varphi\|_{\alpha+2}.$$

3. Error Estimates for the Galerkin Method in Case of $N = 3$ Space Dimensions.

The defining relation of the error $e = u - u_h$ is

$$(\dot{e}, x) + D(e, x) = 0 \quad \text{for } x \in S_h.$$

Now we introduce the Ritz-approximation $U_h = R_h u$ and use the splitting

$$\begin{aligned} e &= (u - U_h) - (u_h - U_h) \\ &= \epsilon - \xi \end{aligned}$$

with $\xi \in S_h$. Further we assume the initial condition

$$u_h(0) = R_h u_0.$$

Therefore we have

$$\xi(0) = 0.$$

The defining relation for ξ is

$$(\dot{\xi}, x) + D(\xi, x) = (\dot{\epsilon}, x) \quad \text{for } x \in S_h.$$

Using the estimates of Section 2 we get for $\alpha = N/2 - 1 = 1/2$

$$c^{-1} \left\{ \rho^2 \|\Phi\|_{\alpha+2}^2 + \|\nabla \Phi\|_{\alpha}^2 \right\} \leq D(\Phi, \mu^{-\alpha} \Phi)$$

and

$$D(\Phi, \mu^{-\alpha} \Phi) = D(\Phi, \mu^{-\alpha} \Phi - \chi) - (\dot{\epsilon} - \dot{\Phi}, \mu^{-\alpha} \Phi - \chi) + (\dot{\epsilon} - \dot{\Phi}, \mu^{-\alpha} \Phi) .$$

If we choose $\chi = P_h(\mu^{-\alpha} \Phi)$ the L_2 -projection then the middle term vanishes mostly. By pure approximation arguments we get

Lemma

$$\inf_{\chi \in S_h} D(\Phi, \mu^{-\alpha} \Phi - \chi) \leq c \left(\frac{h}{\rho} \right)^2 \left\{ \rho^2 \|\Phi\|_{\alpha+2}^2 + \|\nabla \Phi\|_{\alpha}^2 \right\} .$$

For $\rho = \gamma h$ with a proper γ this gives

$$\rho^2 \|\Phi\|_{\alpha+2}^2 + \|\nabla \Phi\|_{\alpha}^2 \leq c(\dot{\epsilon} - \dot{\Phi}, \mu^{-\alpha} \Phi) .$$

The left hand side is a bound of $\|\Phi\|_{L_{\infty}}^2$ if x_0 is chosen properly. The right hand side can be estimated by

$$(\dot{\epsilon} - \dot{\Phi}, \mu^{-\alpha} \Phi) \leq \|\Phi\|_{L_{\infty}} \iint \mu^{-\alpha} |\dot{\epsilon} - \dot{\Phi}| \leq \|\Phi\|_{L_{\infty}} \|\dot{\epsilon} - \dot{\Phi}\| \left\{ \iint \mu^{-2\alpha} \right\}^{1/2}$$

In the case of $N = 3$ we have $2\alpha = 1$ and the last integral is bounded. This gives because of $e = \epsilon - \Phi$

Theorem 3: Let $N = 3$. Then for any fixed time t

$$\|e\|_{L_{\infty}} \leq c \left\{ \|\epsilon\|_{L_{\infty}} + \|\dot{\epsilon}\|_{L_2} + \|\ddot{\epsilon}\|_{L_2} \right\} .$$

The L_2 -error estimates

$$\|e\|_{L_\infty(L_2)} \leq \|\varepsilon\|_{L_\infty(L_2)} + c\|\dot{\varepsilon}\|_{L_2(L_2)}$$

are well-known. We apply this with e replaced by $\dot{e}$ and ε replaced by $\dot{\varepsilon}$. In this way we come to

Theorem 4: Let $N = 3$. Then

$$\|e\|_{L_\infty(L_\infty)} \leq c\left\{\|\varepsilon\|_{L_\infty(L_\infty)} + \|\dot{\varepsilon}\|_{L_\infty(L_2)} + \|\dot{\varepsilon}\|_{L_2(L_2)}\right\}.$$

For u sufficiently regular this leads to optimal error estimates of the Galerkin approximation.

4. Two Types of Error Estimates in Arbitrary Dimensions.

Using the splitting $e = \varepsilon - \phi$ of above we get for arbitrary N of space dimensions with $\alpha = N/2 - 1$ similar to above with $\chi = P_h(\mu^{-\alpha}\phi)$

$$\begin{aligned} (\dot{\phi}, \phi)_\alpha + \|\nabla \phi\|_\alpha^2 + k\rho^2 \|\phi\|_{\alpha+2}^2 &= \\ &= D(\phi, \mu^{-\alpha}\phi - \chi) - (\dot{\varepsilon}, \mu^{-\alpha}\phi - \chi) + (\dot{\varepsilon}, \phi)_\alpha. \end{aligned}$$

By approximation arguments we come to

$$(\dot{\phi}, \phi)_\alpha \leq c\left\{\|\phi\|_\alpha^2 + \|\dot{\varepsilon}\|_\alpha^2\right\}$$

if $\rho \geq \gamma h$ with γ properly chosen. This leads to the interesting a priori inequality

$$\|\phi(t)\|_\alpha^2 \leq \|\phi(0)\|_\alpha^2 + c \int_0^t \|\dot{\varepsilon}(\tau)\|_\alpha^2 d\tau.$$

A first - but non-optimal - L_∞ -result is the consequence.

If x_0 is chosen properly then $\|\dot{\Phi}\|_\alpha$ is a bound of $h\|\Phi\|_{L_\infty}$. Therefore we get

$$\|\Phi\|_{L_\infty(L_\infty)}^2 \leq c h^{-2} \|\dot{\epsilon}\|_{L_2(\alpha)}^2 = c h^{-2} \int_0^T \|\dot{\epsilon}\|_\alpha^2 d\tau.$$

The α -norm is bounded by the L_p -norm for $p > N$. Therefore we have

Theorem 5: Let $N > 3$ be arbitrary and $p > N$. Then

$$\|e\|_{L_\infty(L_\infty)} \leq \|\epsilon\|_{L_\infty(L_\infty)} + c h^{-1} \|\dot{\epsilon}\|_{L_2(L_p)}$$

Similarly we can get

$$\|\dot{\Phi}(t)\|_\alpha^2 \leq \|\dot{\Phi}(0)\|_\alpha^2 + c \int_0^t \|\ddot{\epsilon}(\tau)\|_\alpha^2 d\tau.$$

For $\Phi(0) = 0$ we find

$$\|\dot{\Phi}(0)\|_\alpha \leq c \|\dot{\epsilon}(0)\|_\alpha$$

and in this way

$$\sup \|\dot{\Phi}(t)\|_\alpha \leq \sup \|\dot{\epsilon}(t)\|_\alpha + c \left\{ \int_0^T \|\ddot{\epsilon}\|_\alpha^2 d\tau \right\}^{1/2}.$$

The link to L_∞ -estimates of Φ and therefore of e is

Theorem 6: Let $N > 3$ and $\alpha = N/2 - 1$. For $\rho \geq \gamma h$ with γ chosen properly

$$\|\dot{\Phi}\|_{\alpha+2} + \|\nabla \Phi\|_{\alpha+1} \leq c \rho^{-1} \|\dot{\varepsilon} - \dot{\Phi}\|_{\alpha}.$$

The proof of this theorem is quite lengthy, the lines of it still being the same as in the paper three years ago.

Since now $\alpha + 2 = N/2 + 1$ for $x_0 \in \Omega$ appropriate and $\rho = \gamma h$ we get

$$\|\dot{\Phi}\|_{L_{\infty}} \leq c \rho \|\dot{\Phi}\|_{\alpha+2}.$$

This finally gives the error estimate stated at the end of the introduction.

In the mentioned paper to appear in R.A.I.R.O. a detailed bibliography is given. We suppress this here.