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A MIXED FINITE ELEMENT METHOD FOR THE NAVIER-STOKES EQUATIONS.

Claes JOHNSON

1. Introduction

We shall consider the stationary Navier-Stokes equations for an incompressible fluid:

$$\begin{cases} u\cdot\nabla u - \mu\Delta u + \nabla p = f & \text{in } \Omega, \qquad (1.1a)\\ \operatorname{div} u = 0 & \text{in } \Omega, \qquad (1.1b)\\ u = 0 & \text{on } \Gamma, \qquad (1.1c)\end{cases}$$

where Ω is a bounded domain in R^2 with Lipschitz boundary Γ, $f = (f_1,f_2)$ is a given force, $u = (u_1,u_2)$ is the velocity, p the pressure and $\mu > 0$ is the viscosity of the fluid. For simplicity we shall consider the particular boundary condition (1.1c). However, the mixed finite element method to be introduced can be applied with no additional complications also in the case of other boundary conditions (cf. Remark 2 below).

The classical variational characterization of the velocity u is the following (see [6]): Find $u \in \mathring{\mathcal{V}}$ such that

$$b(u,u,v) + \mu a(u,v) = (f,v), \qquad v \in \mathring{\mathcal{V}}, \qquad (1.2)$$

where

$$\mathring{\mathcal{V}} = \left\{ v \in [H_0^1(\Omega)]^2 : \operatorname{div} v = 0 \text{ in } \Omega \right\} ,$$

$$a(u,v) = \int_\Omega \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i}\, dx,$$

$$b(u,w,v) = \int_\Omega u_i \frac{\partial w_j}{\partial x_i} v_j \, dx,$$

$$(v,w) = \int_\Omega v\cdot w \, dx,$$

and $H_0^1(\Omega) = \{v \in H^1(\Omega):\ v = 0 \text{ on } \Gamma\}$, where $H^1(\Omega)$ is the usual Sobolev space. Here and below we use the summation convention: repeated indices indicate summation from 1 to 2. If $f \in [L^2(\Omega)]^2$ then there exists $u \in \overset{\circ}{\mathcal{V}}$ satisfying (1.1). Moreover if f is sufficiently small or ν sufficiently large then u is uniquely determined (see [6]).

In this note we shall consider a mixed finite element method for the stationary Navier-Stokes equations (1.1) where we seek an approximation u_h of the velocity u in a space $\overset{\circ}{V}_h$ of functions v satisfying the imcompressibility condition $\operatorname{div} v = 0$ exactly but where the conformity condition $\overset{\circ}{V}_h \subset [H^1(\Omega)]^2$ is relaxed; if $v \in \overset{\circ}{V}_h$ then the tangential velocity $v \cdot t$ may be discontinuous across an inter-element boundary S, t being a tangent to S. The continuity of the tangential velocity $u_h \cdot t$ will then be imposed in an approximate way by using a space H_h of Lagrange multipliers having the interpretation of stress deviatorics in mechanics. To construct the space H_h we shall use the equilibrium stress element introduced in [4].

Methods of this type, with a different choice of the space H_h, was first proposed by Fortin [3] to handle the case of a very small viscosity corresponding to a very large Reynold's number. The proof of convergence of the method was left open. Further, Raviart and Girault [5] have proposed and analyzed a somewhat similar method using as Lagrange multiplier the vorticity. That method can in fact be viewed as a finite element method of Navier-Stokes equations in the vorticity - streamfunction formulation.

An outline of the note is as follows:
In Section 2 we introduce the mixed finite element method. In Section 3 we prove existence of a finite element solution and finally in Section 4 we prove that the method will converge. The problem of estimating the rate of convergence is left open.

We shall use the following notation: By $H^s(\Sigma)$, where Σ is a bounded domain in R^2, $s \geq 0$, we will denote the usual Sobolev space with norm $\|\cdot\|_{s,\Sigma}$. When $\Sigma = \Omega$ this index will be dropped.

2. The mixed finite element method

Let us first recall the formulation of Navier-Stokes equations (1.1) used in mechanics: Find the velocity $u = (u_1, u_2)$, the pressure p and the stress deviatoric $\sigma = \{\sigma_{ij}\}$, $i,j = 1,2$, with $\sigma_{ij} = \sigma_{ji}$, such that

$$
\begin{array}{llr}
(2.1a) & \sigma = \nu\varepsilon(u) & \text{in } \Omega, \\
(2.1b) & \operatorname{div} u = 0 & \text{in } \Omega, \\
(2.1c) & -u\cdot\nabla u + \operatorname{div}\sigma - \nabla p + f = 0 & \text{in } \Omega, \\
(2.1d) & u = 0 & \text{on } \Gamma,
\end{array}
$$

where $\nu = 2\mu$

$$\varepsilon(u) = \{\varepsilon_{ij}(u)\} \quad , \quad \varepsilon_{ij}(u) = \frac{1}{2}\left\{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right\},$$

$$\operatorname{div}\sigma = \left(\frac{\partial\sigma_{1j}}{\partial x_j} \; , \; \frac{\partial\sigma_{2j}}{\partial x_j}\right) ;$$

if we eliminate σ in (2.1) we obtain (1.1).

Remark 1.

We observe that by (2.1a,b) one has

$$(2.2) \qquad \operatorname{tr}(\sigma) \equiv \sigma_{11} + \sigma_{22} = 0.$$

In continuum mechanics the (total) stress

$$\hat{\sigma} = \{\hat{\sigma}_{ij}\} \; ,$$

is decomposed according to

$$\hat{\sigma} = \sigma - p\delta \quad ,$$

where

$$\delta = \{\delta_{ij}\}, \quad \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}, \quad i,j=1,2,$$

into a deviatoric part σ satisfying (2.2) and a uniform pressure p.

Remark 2.

In general one can have the following (homogenous) boundary conditions on different parts of the boundary:

(i) $u \cdot n = 0$, $u \cdot t = 0$,

(ii) $u \cdot n = 0$, $\sigma_{nt} = 0$,

(iii) $\hat{\sigma}_{nn} = 0$, $u \cdot t = 0$,

(iv) $\hat{\sigma}_{nn} = 0$, $\sigma_{nt} = 0$,

where $n = (n_1, n_2)$ is a unit normal and $t = (t_1, t_2) = (n_2, -n_1)$ is a tangent to Γ, and $\sigma_{nn} = \sigma_{ij} n_i n_j$ and $\sigma_{nt} = \sigma_{ij} n_i t_j$ are the normal and tangential components of σ, respectively.

In order to motivate the formulation of the mixed method we shall first consider a variational formulation of the Stokes problem corresponding to (2.1):

(2.3a) $\sigma = \nu \varepsilon(u)$ in Ω,

(2.3b) $\operatorname{div} u = 0$ in Ω,

(2.3c) $\operatorname{div} \sigma - \nabla p + f = 0$ in Ω,

(2.3d) $u = 0$ on Γ,

obtained by omitting the non linear term $u \cdot \nabla u$ in (2.1). We shall seek p, u and σ in the spaces Y, V and H defined by

$$Y = L^2(\Omega) ,$$

$$V = \left\{ v \in Y^2 : \ \operatorname{div} v \in Y , \ v \cdot n = 0 \text{ on } \Gamma \right\} ,$$

$$H = \left\{ \tau \in \hat{H} : \ \operatorname{tr}(\tau) = 0 \text{ in } \Omega \right\} ,$$

$$\hat{H} \equiv \hat{H}(\operatorname{div}; \Omega) = \left\{ \tau : \tau = \tau_{ij} , \ \tau_{ij} = \tau_{ji} \in Y , \quad i,j = 1,2 , \ \operatorname{div} \tau \in Y^2 \right\} .$$

We also recall the following Green's formulas:

$$(2.4) \qquad (\tau,\varepsilon(v)) = \int_\Gamma v\cdot\tau\cdot n \, ds - (\mathrm{div}\,\tau, v) ,$$

$$(2.5) \qquad (v,\nabla q) = \int_\Gamma q \, v\cdot n \, ds - (\mathrm{div}\, v,q) ,$$

where $(\cdot,\cdot)$ denotes the scalar product in $[L^2(\Omega)]^m$, $m=1,2,4$, so that in particular

$$(\sigma,\tau) = \int_\Omega \sigma_{ij} \tau_{ij} \, dx .$$

Further,

$$\tau\cdot n = (\tau_{1j}n_j \, , \, \tau_{2j}n_j) ,$$

and $n = (n_1,n_2)$ is an outward unit normal to Γ. If $(u,\sigma,p)\in V\times H\times Y$ satisfies (2.3), then using (2.4) and (2.5) we find that

$$\begin{array}{lll} (2.6a) & (\sigma,\tau) + \nu(u, \mathrm{div}\,\tau) = 0 \quad , & \tau\in H, \\ (2.6b) & (\mathrm{div}\, u,q) = 0 \quad , & q\in Y, \\ (2.6c) & (\mathrm{div}\,\sigma,v) + (p,\mathrm{div}\, v) + (f,v) = 0, & v\in V. \end{array}$$

We note that by introducing the space

$$\overset{\circ}{V} = \left\{v\in V: \ \mathrm{div}\, v = 0 \text{ in } \Omega\right\},$$

we obtain from (2.6) the following variational characterization of u and σ not involving the pressure p : Find $(u,\sigma)\in \overset{\circ}{V}\times H$, such that

$$\begin{array}{ll} (2.7a) \\ (2.7b) \end{array} \quad \left\{ \begin{array}{ll} (\sigma,\tau) + \nu(u,\mathrm{div}\,\tau) = 0 \quad , & \tau\in H, \\ (\mathrm{div}\,\sigma,v) + (f,v) = 0 \quad , & v\in \overset{\circ}{V}. \end{array} \right.$$

Note also that the functions v in V or $\overset{\circ}{V}$ do not have to satisfy the boundary condition $v\cdot t = 0$. This condition is implicitely contained in (2.7a); if we formally integrate by parts and vary τ in (2.7a) we obtain (2.3a) and $u\cdot t = 0$.

We shall now introduce finite dimensional spaces approximating the spaces V, Y and H. For simplicity we shall assume that Ω is polygonal. Let $\{\mathcal{C}_h\}$ be a regular family of triangulations $\mathcal{C}_h$ of Ω,

$$\Omega = \bigcup_{K\in\mathcal{C}_h} K ,$$

indexed by the parameter h representing the maximum of the diameters of the triangles K. We define

$$V_h = \{v\in V : v|_K \text{ is linear on } K,\ \ K\in\mathcal{C}_h\} ,$$

$$Y_h = \{q\in Y : q|_K \text{ is constant on } K,\ \ K\in\mathcal{C}_h\} ,$$

$$H_h = \{\tau\in\hat{H} : \tau|_K \in H_K,\ \int_K tr(\tau)dx = 0,\ \ K\in\mathcal{C}_h\} ,$$

where for each $K\in\mathcal{C}_h$, H_K is a finite dimensional space defined as follows (see [4]): Let K be divided into three subtriangles T_i, $i=1,2,3$, by connecting the center of gravity with the vertices of K and set

$$H_K = \{\tau\in\hat{H}(div;K) : \tau|_{T_i} \text{ is linear } i=1,2,3\} .$$

In [4] it is proved that any $\tau\in H_K$ is uniquely determined by the following 15 degrees of freedom:

(i) the value of $\tau\cdot n$ at two points on each side S of K, n being a normal to S,

(ii) $\int_K \tau_{ij}\,dx$, $i,j=1,2$.

Note that the requirement $H_K \subset \hat{H}(div;K)$, i.e., $div\,\tau \in [L^2(K)]^2$ if $\tau\in H_K$, implies that $\tau\cdot n$ is continuous across the subtriangle boundaries, i.e., if S is a side common to the subtriangles T_i and T_j, then

$$\tau|_{T_i}\cdot n = \tau|_{T_j}\cdot n \qquad \text{on } S ,$$

where n is a normal to S. Likewise, the requirement $H_h \subset \hat{H}$ will require $\tau \cdot n$ to be continuous across interelement boundaries. As degrees of freedom for $\tau \in H_h$ one can choose

(i') the value of $\tau \cdot n$ at two points on each side S of $\mathcal{C}_h$, n being a normal to S,

(ii') $\int_K \tau_{11}\,dx = -\int_K \tau_{22}\,dx$, $\int_K \tau_{12}\,dx$ for $K \in \mathcal{C}_h$,

where the first relation in (ii') comes from the requirement

$$\int_K tr(\tau)\,dx = 0 \ , \quad K \in \mathcal{C}_h \ ,$$

for $\tau \in H_h$. Note that if $\tau \in H_h$ then it is not true in general that $tr(\tau) = 0$ in Ω and thus $H_h \not\subset H$. Further, the inclusion $V_h \subset V$ will require the velocity in the normal direction $v \cdot n$ to be continuous across interelement boundaries. As degrees of freedom for $v \in V_h$ we choose the value of $v \cdot n$ at two points on each side S of $\mathcal{C}_h$. Note however that the tangential velocity $v \cdot t$ may be discontinuous across the interelement boundary S, for $v \in V_h$, t being a tangent to S.

We now formulate the following finite element method for the Stokes equations (2.6): Find $(u_h, \sigma_h, p_h) \in V_h \times H_h \times Y_h$ such that

$$\begin{cases} (\sigma_h, \tau) + \nu(u_h, \operatorname{div} \tau) = 0 \ , & \tau \in H_h, \quad (2.8a) \\ (\operatorname{div} u_h, q) = 0 \ , & q \in Y_h, \quad (2.8b) \\ (\operatorname{div} \sigma_h, v) + (p_h, \operatorname{div} v) + (f, v) = 0, & v \in V_h \ . \quad (2.8c) \end{cases}$$

In analogy with (2.7) introducing the space

$$\overset{\circ}{V}_h = \left\{ v \in V_h : (\operatorname{div} v, q) = 0 \ , \quad q \in Y_h \right\} \ ,$$

we see that if $(u_h, \sigma_h) \in V_h \times H_h$ satisfies (2.8), then $(u_h, \sigma_h) \in \overset{\circ}{V}_h \times H_h$ and

$$
\begin{cases}
(\sigma_h,\tau) + \nu(u_h,\operatorname{div}\tau) = 0\,, & \tau\in H_h, \quad (2.9a)\\
(\operatorname{div}\sigma_h,v) + (f,v) = 0\,, & v\in \overset{\circ}{V}_h. \quad (2.9b)
\end{cases}
$$

Since div v is constant on each triangle K if $v\in V_h$, the relation $(\operatorname{div} v,q) = 0$ for $q\in Y_h$ will imply that $\operatorname{div} v = 0$ in Ω so that

$$\overset{\circ}{V}_h \subset \overset{\circ}{V}\,,$$

i.e. we will work with approximations of the velocity satisfying the incompressibility condition exactly. For simplicity, we shall below consider the formulation (2.9) and its analogy for Navier-Stokes equations. In practice we would have to work with the formulation (2.8) since we do not know of any convenient basis for $\overset{\circ}{V}_h$.

Let us now extend the formulation (2.9) to the case of Navier-Stokes equations. Since the functions in $\overset{\circ}{V}_h$ may be discontinuous we have to handle the nonlinear term $u\cdot\nabla u$ in a particular way; we shall use a method introduced by Fortin [3] producing an "upwind" dissipative scheme. This method is an extension of a method for linear hyperbolic equations using discontinuous functions introduced by Lesaint. For a given $w\in V_h$ we will for each $K\in\mathcal{C}_h$ distinguish between the part ∂K_-^w of the boundary ∂K of K where the flow is entering,

$$\partial K_-^w = \left\{ x\in\partial K :\ \ w\cdot n(x) \le 0 \right\}\,,$$

and the part where the flow is sorting,

$$\partial K_+^w = \left\{ x\in\partial K :\ \ w\cdot n(x) > 0 \right\}\,,$$

n being an ourward normal to ∂K. We note that if $w\in V_h$, then $w\cdot n$ is continuous across interelement boundaries so that for two triangles K and $\bar{K}$ with the common side S,

$$\partial K_-^w \cap S = \partial \bar{K}_+^w \cap S\,. \tag{2.10}$$

We can now formulate the mixed method for the stationary Navier-Stokes equations: <u>Find</u> $(u_h,\sigma_h) \in \overset{\circ}{V}_h \times H_h$ <u>such that</u>

$$
\begin{cases}
(\sigma_h,\tau) + \nu(u_h,\operatorname{div}\tau) = 0\ , & \tau\in H_h, \qquad (2.11a)\\
-b^*(u_h,u_h,v) + (\operatorname{div}\sigma_h,v) + (f,v) = 0\ , & v\in \overset{\circ}{V}_h. \qquad (2.11b)
\end{cases}
$$

Here

$$
b^*(w,w,v) = \sum_{K\in\mathscr{C}_h} \left\{ -\int_K w_i w_j \frac{\partial v_i}{\partial x_j}\,dx + \int_{\partial K} w\cdot n\ \tilde{w}_i v_i\ ds \right\}, \tag{2.12}
$$

where

$$
\tilde{w}|_{\partial K} = \begin{Bmatrix} \text{trace of } w|_K \text{ on } \partial K_+^w\ , \\ w^e = \text{trace of } w|_{\bar{K}} \text{ on } \partial K_-^w \cap S\ , \end{Bmatrix} \tag{2.13}
$$

where $\bar{K}$ is a triangle with the side S in common with K, $\bar{K} \neq K$. To motivate the expression corresponding to the nonlinear term $u\cdot\nabla u$, we note that by multiplying this term by v and integrating we obtain

$$
b(u,u,v) = \sum_{K\in\mathscr{C}_h} \int_K u_i \frac{\partial u_j}{\partial x_i} v_j\ dx\ .
$$

Using Green's formula on each $K\in\mathscr{C}_h$ and the fact that $\operatorname{div} u = 0$, we see that

$$
b(u,u,v) = \sum_K \left\{ -\int_K u_i u_j \frac{\partial v_i}{\partial x_j}\,dx + \int_{\partial K} u\cdot n\ u_j v_j\ ds \right\}\ .
$$

Thus, the term $b^*(w,w,v)$ is obtained from $b(w,w,v)$ by replacing the "interior trace" of w on ∂K_-^w by the "exterior trace" w^e.

3. Existence of a finite element solution.

In the proof of existence of a solution of (2.11) we shall refer to the following lemma which will also be used in the convergence proof. Here $\|\cdot\|$ denotes the norm in $[L^2(\Omega)]^m$, m=1,2,4.

Lemma 1. For $0 \le \alpha < \frac{1}{2}$ there is a constant C independent of h such that if $(w,\chi) \in \overset{\circ}{V}_h \times H_h$ satisfies

(3.1) $$(\chi,\tau) + \nu(w,\operatorname{div} \tau) = 0 , \qquad \tau \in H_h,$$

then

(3.2) $$\| w \|_{\alpha} \le C \| \chi \| .$$

Proof. The dual of $Y_\alpha = [H^\alpha(\Omega)]^2$ can be characterized (see [1]) as the closure of $C^\infty(\bar{\Omega})$ in the norm

$$\| v \|_{-\alpha} = \sup_{u \in Y_\alpha} \frac{|(u,v)|}{\| u \|_\alpha} .$$

Thus, to prove (3.2) it is sufficient to prove that

(3.3) $$|(w,v)| \le C \| \chi \| \; \| v \|_{-\alpha} , \qquad v \in C^\infty(\bar{\Omega}).$$

To prove this inequality let for a given $v \in C^\infty(\bar{\Omega})$, (φ,q) be the solution of the Stoke's problem

$$\left\{ \begin{array}{lll} \text{(3.4a)} & \nu \operatorname{div}(\varepsilon(\varphi)) + \nabla q = v & \text{in } \Omega, \\ \text{(3.4b)} & \operatorname{div} \varphi = 0 & \text{in } \tilde{\Omega}, \\ \text{(3.4c)} & \varphi = 0 & \text{on } \partial\tilde{\Omega}, \end{array} \right.$$

where $\tilde{\Omega}$ is a region with smooth boundary containing Ω and v has been extended by zero outside Ω. By well known regularity results for the Stoke's problem (see [6]) and interpolation it follows that

(3.5) $$\| \varepsilon(\varphi) \|_{1-\alpha} \le C_\alpha \| v \|_{-\alpha} .$$

Let us now introduce the interpolation operator $\Pi_h : H \to H_h$ defined for $\tau \in [H^{1-\alpha}(\Omega)]^4$, $\alpha < \frac{1}{2}$, as follows: $\Pi_h\tau$ is the unique element in H_h satisfying

(3.6) $$\int_S v\cdot(\tau - \Pi_h\tau)\cdot n \, ds = 0 \qquad \text{for } v \text{ linear},$$

for any side S of $\mathscr{C}_h$, n being a normal to S, and

(3.7) $$\int_K (\tau - \Pi_h\tau) \, dx = 0 , \qquad K\in\mathscr{C}_h .$$

We note that if $tr(\tau) = 0$ then by (3.7), we will have

$$\int_K tr(\Pi_h\tau) \, dx = 0 , \qquad K\in\mathscr{C}_h ,$$

so that $\Pi_h\tau \in H_h$ if $\tau\in H$. Further, by using a trace theorem on a reference element $\hat{K}$ and a linear mapping of $\hat{K}$ on to K, it follows that

$$\| \Pi_h\tau \|_K \leq C \| \tau \|_{1-\alpha,K} , \qquad K\in\mathscr{C}_h .$$

By using the definition of the $\| \cdot \|_{1-\alpha}$ - norm via the K-method of interpolation it follows that we can sum in this relation to obtain

$$\|\Pi_h\tau \| \leq C \| \tau \|_{1-\alpha} , \qquad \tau\in H^{1-\alpha}(\Omega) .$$

In particular, for the solution φ of the Stokes problem (3.4), we have

(3.9) $$\| \Pi_h\varepsilon(\varphi) \| \leq C \| \varepsilon(\varphi) \|_{1-\alpha} .$$

Furthermore, using (3.6), (3.7) and Green's formula on each $K\in\mathscr{C}_h$, we see that

$$(v,\mathrm{div}\ \tau) = (v,\mathrm{div}\ \Pi_h\tau) , \qquad v\in V_h .$$

Therefore, recalling (3.4a) and using the fact that $w \cdot n$ on Γ and $\operatorname{div} w = 0$ in Ω, we find that

$$(w,v) = \nu(w, \operatorname{div} \varepsilon(\varphi)) + (\nabla p, w)$$

$$= \nu(w, \operatorname{div} \Pi_h \varepsilon(\varphi)) = -(\chi, \Pi_h \varepsilon(\varphi)) ,$$

where the last equality follows from (3.1). Thus, by (3.5) and (3.9), we have

$$|(w,v)| \leq \| \chi \| \ \| \Pi_h \varepsilon(\varphi) \| \leq C_\alpha \| \chi \| \ \| v \|_{-\alpha} ,$$

which proves (3.3). This completes the proof of the lemma. ∎

We shall also use the following result.

Lemma 2. If $w \in \overset{\circ}{V}_h$, then

$$b^*(w,w,w) = \sum_S \int_S |w \cdot n| \ |[w]|^2 \, ds \geq 0 ,$$

where we sum over all sides S of $\mathcal{C}_h$, n is a unit normal to S , and $[w]$ denotes the jump of w across S .

Proof. By Green's formula we find using the fact that $\operatorname{div} w = 0$,

$$\int_K w_i w_j \frac{\partial w_i}{\partial x_j} \, dx = -\int_K \frac{\partial}{\partial x_j} (w_i w_j) \, w_i \, dx + \int_{\partial K} w_i w_j w_i n_j \, ds$$

$$= -\int_K \frac{\partial w_i}{\partial x_j} w_j w_i \, dx + \int_{\partial K} w \cdot n \, |w|^2 \, ds ,$$

where n is an outward unit normal to ∂K , so that

$$\sum_K \int_K w_i w_j \frac{\partial w_i}{\partial x_j} \, dx = \frac{1}{2} \sum_K \int_{\partial K} w \cdot n \, |w|^2 \, dx .$$

Thus, recalling the definitions (2.12) and (2.13), writing $\partial K_{\pm} = \partial K_{\pm}^{w}$,

$$b^*(w,w,w) = \sum_K \int_{\partial K} w\cdot n(\tilde{w}\cdot w - \frac{1}{2}|w|^2)\, ds$$

$$= \sum_K \left\{ \int_{\partial K_+} w\cdot n(|w|^2 - \frac{1}{2}|w|^2) ds + \int_{\partial K_-} w\cdot n w^e\cdot w ds - \frac{1}{2}\int_{\partial K_-} w\cdot n|w|^2\, ds \right\}$$

$$= \sum_K \left\{ \frac{1}{2}\int_{\partial K_+} w\cdot n|w|^2 - \int_{\partial K_+} w\cdot n w\cdot w^e + \frac{1}{2}\int_{\partial K_+} w\cdot n|w^e|^2\, ds \right\}$$

$$= \frac{1}{2}\sum_K \int_{\partial K_+} w\cdot n|w - w^e|^2\, ds \quad ,$$

since by (2.10) and the fact that $w\cdot n$ on Γ ,

$$\sum_K \int_{\partial K_-} w\cdot n w^e\cdot w ds = -\sum_K \int_{\partial K_+} w\cdot n w\cdot w^e ds \quad ,$$

$$\sum_K \int_{\partial K_-} w\cdot n|w|^2\, ds = -\sum_K \int_{\partial K_+} w\cdot n|w^e|^2\, ds \ .$$

This clearly proves the lemma, since $w\cdot n \geq 0$ on ∂K_+ . ■

We can now prove :

Theorem 1. There exists $(u_h,\sigma_h) \in \overset{\circ}{V}_h \times H_h$ satisfying (2.11).

Proof. For a given $w \in \overset{\circ}{V}_h$ let $\chi(w) \in H_h$ be defined by the relation

$$(3.10) \qquad (\chi(w),\tau) = -\nu(w,\operatorname{div}\tau) \quad , \qquad \tau \in H_h \ .$$

By Lemma 1 with $\alpha = 0$, we then have

$$(3.11) \qquad \| w \| \leq C \| \chi(w) \| \quad , \qquad w \in \overset{\circ}{V}_h \ .$$

Next, we define the mapping $P_h : \overset{\circ}{V}_h \to \overset{\circ}{V}_h$ by the relation

$$(P_h w,v) = \nu[b^*(w,w,v) - (\operatorname{div}\chi(w),v) - (f,v)] \quad , \quad v \in \overset{\circ}{V}_h \ .$$

Since $\overset{\circ}{V}_h$ is finite dimensional, P_h is clearly continuous. By Lemma 2 and (3.10) with $\tau = \chi(w)$, we then have using also (3.11)

$$(P_h w,w) \geq -\nu(\operatorname{div}\chi(w),w) - \nu(f,w) = \|\chi(w)\|^2 - \nu(f,w)$$

$$\geq C\|w\|^2 - \nu\|f\|\,\|w\| \geq \|w\|(C\|w\| - \nu\|f\|)$$

Thus,

$$(P_h w,w) \geq 0 \quad ,$$

if $\|w\|$ is sufficiently large, $w \in \overset{\circ}{V}_h$. But then it follows by a classical lemma (see e.g. [6]) that there exists $u_h \in \overset{\circ}{V}_h$ such that $P_h u_h = 0$, i.e.

$$-b^*(u_h,u_h,v) + (\operatorname{div}\chi(u_h),v) + (f,v) = 0 \quad , \qquad v \in \overset{\circ}{V}_h \quad .$$

Thus, setting $\sigma_h = \chi(u_h)$ we see that $(u_h,\sigma_h) \in \overset{\circ}{V}_h \times H_h$ satisfies (2.11) and the desired result follows. ■

4. Convergence.

We shall prove the following result:

Theorem 2. There exists a subsequence of $\{(u_h,\sigma_h)\}$, $h > 0$, again denoted by $\{(u_h,\sigma_h)\}$, where (u_h,σ_h) is the solution of (2.11), such that

$$u_h \to u \qquad \text{in } Y^2 = [L^2(\Omega)]^2 ,$$

$$\sigma_h \to \sigma \qquad \text{weakly in } Y^4 ,$$

as h tends to zero, where $u \in \mathring{\mathcal{V}}$ satisfies (1.2) and $\sigma = \nu\varepsilon(u)$. If u is uniquely determined then the whole sequence $\{(u_h,\sigma_h)\}$ will converge.

Proof. Let us first establish some a priori estimates for the finite element solution (u_h,σ_h). Taking $\tau = \sigma_h$ in (2.11a) and $v = u_h$ in (2.11b) and subtracting we obtain

$$\frac{1}{\nu} \| \sigma_h \|^2 + b^*(u_h,u_h,u_h) = (f,u_h) ,$$

so that using Lemmas 1 and 2,

$$\| \sigma_h \|^2 \le \nu(f,u_h) \le \nu \| f \| \; \| u_h \| \le C \| f \| \; \| \sigma_h \| .$$

Thus,

$$\| \sigma_h \| \le C \| f \| . \tag{4.1}$$

and hence by Lemma 1 for some $\alpha \in (0,1/2)$,

$$\| u_h \|_\alpha \le C_\alpha \| f \| . \tag{4.2}$$

By (4.1) and (4.2) it follows, since $H^\alpha(\Omega)$ is compactly inbedded in $Y = L^2(\Omega)$ for $\alpha > 0$, that there exists $(u,\sigma) \in Y^2 \times Y^4$ such that

(4.3) $\quad u_h \to u \quad$ in Y^2 ,

(4.4) $\quad \sigma_h \to \sigma \quad$ weakly in Y^4 .

Using the fact that $\operatorname{div} u_h = 0$ in Ω and $u_h \cdot n = 0$ on Γ , it follows that

$$(\nabla q, u_h) = 0 , \qquad q \in H^1(\Omega) ,$$

and thus by (4.3)

$$(\nabla q, u) = 0 , \qquad q \in H^1(\Omega) ,$$

which implies that (see [3])

(4.5) $\quad \operatorname{div} u = 0 \quad$ in Ω ,

(4.6) $\quad u \cdot n = 0 \quad$ on Γ .

Furthermore, passing to the limit in (2.11a) and using the approximability properties of H_h , we find that

$$(\sigma, \tau) + \nu(u, \operatorname{div} \tau) = 0 ,$$

for all smooth $\tau \in H$. Together with (4.5) and (4.6) this relation implies that

(4.7) $\quad \sigma = \nu\varepsilon(u) \quad$ in Ω ,

$\quad u = 0 \quad$ on Γ .

Thus, by Korn's inequality (see [2]),

$$\| v \|_1 \le C \| \varepsilon(v) \| , \qquad v \in [H_0^1(\Omega)]^2 ,$$

it follows that $u \in [H_0^1(\Omega)]^2$ so that finally $u \in \mathring{\mathcal{V}}$.

It remains to pass to the limit in the relation (2.11b), i.e., in the relation

$$(4.8) \qquad -b^*(u_h,u_h,w) + (\operatorname{div} \sigma_h,w) + (f,w) = 0 \quad w\in\overset{\circ}{V}_h ,$$

where

$$b^*(v,v,w) = b_1(v,v,w) + b_2(v,v,w) ,$$

with

$$b_1(v,v,w) = -\sum_K \int_K v_i v_j \frac{\partial w_i}{\partial x_j}\, dx ,$$

$$b_2(v,v,w) = \sum_K \int_{\partial K} v\cdot n\, \tilde{v}_i w_j\, ds .$$

Let now $v\in\overset{\circ}{\mathcal{V}}$ be a given smooth function. Then choosing $v^h\in\overset{\circ}{V}_h$ by requiring that

$$\int_S (v\cdot n - v^h\cdot n)\, q\, ds = 0 , \qquad q \text{ linear} ,$$

for all sides S of $\mathcal{C}_h$, it follows that

$$(4.9) \qquad \| v - v^h \|_{L^\infty(\Omega)} \le C h^2 ,$$

$$(4.10) \qquad \left\| \frac{\partial v_i}{\partial x_j} - \frac{\partial v_i^h}{\partial x_j} \right\|_{L^\infty(\Omega)} \le C h .$$

Let us now first consider the term $(\operatorname{div} \sigma_h,w)$. By Green's formula (2.4) we have since $v = 0$ on Γ,

$$(\operatorname{div} \sigma_h,v^h) = (\operatorname{div} \sigma_h, v^h - v) - (\sigma_h, \varepsilon(v)) ,$$

so that using (4.4), (4.9) and the inverse estimate

$$\| \operatorname{div} \sigma_h \| \le C h^{-1} \| \sigma_h \| \le C h^{-1} ,$$

we have

$$(4.11) \qquad (\operatorname{div} \sigma_h, v^h) \to -(\sigma, \varepsilon(v)) .$$

Next, by (4.3) and (4.10) we get

$$(4.12) \qquad b_1(u_h,u_h,v^h) = b_1(u_h,u_h,v) + b_1(u_h,u_h,v^h-v) \to b_1(u,u,v) .$$

To handle the term $b_2(u_h,u_h,v)$ we first note that by (2.10), (2.13) and the fact that v is continuous, we have

$$b_2(u_h,u_h,v) = 0 .$$

Further, by using the inverse estimate

$$\| w \|_{L^\infty(K)} \leq C\, h^{-1} \| w \|_{0,K} , \ w \in V_h , \quad K \in \mathcal{C}_h ,$$

we see that

$$| b_2(u_h,u_h,v^h-v) | \leq \sum_K \int_{\partial K} \| u_h \|^2_{L^\infty(K)} \| v^h - v \|_{L^\infty(K)} \, ds$$

$$\leq C \sum_K h^{-2} \| u_h \|^2_{0,K} h^2 \, h \leq C\, h \| u_h \|^2 ,$$

and therefore

$$(4.13) \qquad b_2\,(u_h,u_h,v^h) = b_2(u_h,u_h,v^h - v) \to 0 \qquad \text{as} \quad h \to 0 .$$

Now, taking $w = v^h$ in (4.8) letting h tend to zero, we conclude using (4.11)-(4.13) that

$$-b_1(u,u,v) - (\sigma,\varepsilon(v)) + (f,v) = 0 ,$$

for all smooth $v \in \mathring{\mathcal{V}}$. But integrating by parts using the fact that $\operatorname{div} u = 0$, we have

$$b_1(u,u,v) = b(u,u,v) ,$$

and thus recalling (4.7), we find that

$$b(u,u,v) + \nu(\varepsilon(u),\varepsilon(v)) = (f,v) ,$$

for all smooth $v \in \mathring{\mathcal{V}}$. Finally, it is easy to see that

$$2(\varepsilon(u),\varepsilon(v)) = a(u,v) \quad , \qquad u,v \in \mathring{\mathcal{V}} \, ,$$

and hence

$$b(u,u,v) + \mu \, a(u,v) = (f,v) \quad ,$$

for all smooth $v \in \mathring{\mathcal{V}}$ and thus for all $v \in \mathring{\mathcal{V}}$ by a density argument. This shows that $u \in \mathring{\mathcal{V}}$ satisfies (1.2) and the proof is complete. ■

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