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ORTHOGONAL MEASURES: AN EXAMPLE

by

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A family $\mathfrak{M}$ of measures, defined on a Borel field $\mathfrak{B}$ of subsets of a space X, is said to be *pairwise orthogonal* if, given $\lambda, \mu \in \mathfrak{M}$ with $\lambda \neq \mu$, there exists $H_{\lambda\mu} \in \mathfrak{B}$ such that $\lambda(H_{\lambda\mu}) = 0 = \mu(X - H_{\lambda\mu})$. $\mathfrak{M}$ will be called *uniformly orthogonal* provided there is, for each $\lambda \in \mathfrak{M}$, a set $H_\lambda \in \mathfrak{B}$ such that, for each $\mu \in \mathfrak{M} - \{\lambda\}$, $\lambda(H_\mu) = 0 = \lambda(X - H_\lambda)$. Clearly every uniformly orthogonal family is pairwise orthogonal, and every *countable* pairwise orthogonal family is uniformly orthogonal. One simple example of an uncountable pairwise orthogonal family $\mathfrak{M}$ that is not uniformly orthogonal is provided by taking X to be the unit interval I, $\mathfrak{B}$ the Borel sets of X, and $\mathfrak{M}$ to consist of Lebesgue measure, together with all 1-point measures. Here, however, the family does have an uncountable subfamily consisting of uniformly orthogonal measures; we have only to omit Lebesgue measure. The following example shows that in general we cannot obtain an uncountable uniformly orthogonal family from a pairwise orthogonal family by discarding measures -- provided the continuum hypothesis is assumed.

Theorem (CH) *There exists an uncountable family $\mathfrak{M}$ of pairwise orthogonal Borel probability measures on the unit square I^2, such that no uncountable subset of $\mathfrak{M}$ is uniformly orthogonal.*

We need a well-known lemma (see for example [1, p. 76]).

Lemma (CH) There exists a partition of the unit interval I into a family $\mathcal{N}$ of c pairwise disjoint non-empty Borel null sets such that each null set in I is covered by a countable sub-family of $\mathcal{N}$.

Proof: Well-order the null G_δ sets as $\{G_\alpha : \alpha < \omega_1\}$, define $N_\alpha = G_\alpha - \bigcup\{G_\beta : \beta < \alpha\}$, and omit empty N_α 's.

Construction Let $\mathcal{N} = \{N_\alpha : \alpha < \omega_1\}$ be a partition as in the Lemma, and let $\{y_\alpha : \alpha < \omega_1\}$ well-order I without repetition. For each $\alpha < \omega_1$, let μ_α denote the (linear) Lebesgue measure on $I \times \{y_\alpha\} \subset I^2$. For each $\alpha > 0$, take a sequence $\{u_{\alpha\beta} : \beta < \alpha\}$ of positive real numbers such that $\sum\{u_{\alpha\beta} : \beta < \alpha\} = 1/2$. Take a Borel measure $m_{\alpha\beta}$ on $N_\alpha \times \{y_\beta\}$ ($\beta < \alpha < \omega_1$) such that $m_{\alpha\beta}(N_\alpha \times \{y_\beta\}) = u_{\alpha\beta}$. Now, for each Borel set $H \subset I^2$ and $\alpha < \omega_1$, define

$$m_\alpha(H) = \frac{1}{2}\mu_\alpha(H \cap (I \times \{y_\alpha\})) + \sum\{m_{\alpha\beta}(H \cap (N_\alpha \times \{y_\beta\})) : \beta < \alpha\}$$

if $\alpha \geq 1$, and define $m_0(H) = \mu_0(H \cap (I \times \{y_0\}))$. Then put $\mathcal{M} = \{m_\alpha : \alpha < \omega_1\}$, an uncountable family of Borel probability measures on I^2. It is easy to see that they are pairwise orthogonal. On the other hand, fixing $\gamma < \omega_1$, suppose H_γ is a Borel subset of I^2 such that $m_\gamma(H_\gamma) = 1$; then also $\mu_\gamma(H_\gamma \cap (I \times \{y_\gamma\})) = 1$. That is, $\mu(H^\gamma) = 1$ where μ is Lebesgue measure and $H^\gamma = \{x \in I : (x, y_\gamma) \in H_\gamma\}$. By construction of the sets N_α, H^γ must contain all but a countable subfamily of the sets N_α, and hence H_γ can be null with repect to only countably many measures m_β with $\beta > \gamma$. It follows at once that every uniformly orthogonal subfamily of $\mathcal{M}$ is countable, as required.

<u>Remarks</u> 1. By taking a little more trouble, we could ensure that the measures m_α were all non-atomic (in addition to their other properties).

2. The continuum hypothesis is essential for the theorem. It is relatively consistent (with usual set theory) that the union of fewer than c null sets in I (with respect to any finite Borel measure) is always null. (See, for example, [2] for the case of Lebesgue measure; the same argument works for the more general measures considered here.) From this assumption it follows easily that, if $\aleph_1 < c$, each family of $\aleph_1$ pairwise orthogonal finite Borel measures on I (or, what comes to the same thing, on I^2) is uniformly orthogonal.

REFERENCES

1. J. C. Oxtoby, <u>Measure and category</u>, New York 1970.

2. J. R. Shoenfield, <u>Martin's axiom</u>, Amer. Math. Monthly 82 (1975) 610-617.

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