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Infinite interval exchange transformation with positive entropy

by

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ABSTRACT

We construct an interval exchange transformation with infinitely many intervals on $[0, 1]$ which has a positive metrical entropy with respect to the Lebesgue measure.

A transformation φ on $[0, 1]$ is called an interval exchange transformation if there exist families I_i and J_i of countably many disjoint intervals in $[0, 1]$ such that

$$(1) \quad \nu\left(\bigcup_i I_i\right) = \nu\left(\bigcup_i J_i\right) = 1,$$

$$(2) \quad \nu(I_i) = \nu(J_i) \text{ for any } i, \text{ and}$$

$$(3) \quad \varphi(\inf I_i + x) = \inf J_i + x \text{ for any } i \text{ and } x \text{ with } 0 < x < \nu(I_i),$$

where ν is the Lebesgue measure on $[0, 1]$. It is well known that if $\sum_i -\nu(I_i) \log \nu(I_i) < \infty$, then the metrical entropy $h_\nu(\varphi) = 0$. It was asked by Prof. M. Keane (Rennes) whether there exists an interval exchange transformation with positive entropy or not. Here is an answer to this question to construct such a transformation.

A real number x whose expansion to base 2 is $0.x_0x_1x_2\dots$ shall be identified with an element $((x_0, x_2, \dots), (x_1, x_3, \dots))$ of $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$. Let $\lambda = \mu \times \mu$ be the measure on $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ induced from the Lebesgue measure by this identification.

We select $F_n \subset \{0, 1\}^{2^n}$ and $F = \bigcup_{n=1}^{\infty} F_n$ so that

(4) $\{\Gamma_{\xi}; \xi \in F\}$ is a family of disjoint subsets of $\{0, 1\}^{\mathbb{N}}$ such that $\mu\left(\bigcup_{\xi \in F} \Gamma_{\xi}\right) = 1$, where for $\xi = (\xi_0, \xi_1, \dots, \xi_n)$, $\Gamma_{\xi} = \{x \in \{0, 1\}^{\mathbb{N}}; x(i) = \xi_i \text{ for } i=0, 1, \dots, n\}$.

(5) For $\xi = \xi' \xi'' \in F_n$, where ξ' is the concatenation of ξ' and ξ'' in $\{0, 1\}^{2^{n-1}}$, define $\bar{\xi} = \xi' \xi''$. Then $\{\Gamma_{\bar{\xi}}; \xi \in F\}$ is also a family of disjoint sets such that $\mu\left(\bigcup_{\xi \in F} \Gamma_{\bar{\xi}}\right) = 1$, and

(6) If $\xi' \xi'' \in F_n$, where $\xi, \xi'' \in \{0, 1\}^{2^{n-1}}$, then $\xi \xi'' \in F_n$ for any $\xi \in \{0, 1\}^{2^{n-1}}$.

Such F does exist as we will see later. For $\alpha \in \{0,1\}^{\mathbb{N}}$, let $\tau(\alpha) = n$ if $\alpha \in \overline{F_n}$ with $\bar{\epsilon} \in F_n$. Let S_k be a transformation on $\{0,1\}^{\mathbb{N}}$ such that for any $\alpha \in \{0,1\}^{\mathbb{N}}$,

$$(S_k \alpha)(n) = \begin{cases} \alpha(n+2^{k-1}) & 0 \leq n < 2^{k-1} \\ \alpha(n-2^{k-1}) & 2^{k-1} \leq n < 2^k \\ \alpha(n) & n \geq 2^k \end{cases}$$

Finally, let T be the transformation on $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ such that for any $(\alpha, \beta) \in \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$,

$$T(\alpha, \beta) = (S_{\tau(\alpha)} \alpha, S_{\tau(\alpha)} \beta).$$

This T can be considered as a transformation on $[0, 1]$ through the above correspondence between $[0, 1]$ and $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$. In this sense, it is easy to see that T is an interval exchange transformation.

THEOREM $h_\lambda(T) > 0$.

(proof) Let $A = \{A_0, A_1\}$ be the partition on $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ such that $A_i = \{(\alpha, \beta); \beta(0) = i\}$ ($i=0,1$). Then, it is sufficient to prove that $h_\lambda(T; A) = \log 2$. For simplicity, we denote $S\alpha = S_{\tau(\alpha)} \alpha$. By D_k , we denote the set of sequences $(n_0, n_1, \dots, n_{k-1})$ of positive integers such that $n_i \neq \max\{n_{i+1}, \dots, n_{k-1}\}$ for any $i=0,1, \dots, k-1$. Then, it is easy to see that $\mu\{\alpha; (\tau(\alpha), \tau(S\alpha), \dots, \tau(S^{k-1}\alpha)) \in D_k\} = 1$. For $k=1,2, \dots$, define functions $\sigma_k: \{1,2, \dots\}^k \rightarrow \{1,2, \dots\}$ inductively by

$$\sigma_1(n_0) = 2^{n_0-1}$$

$$\sigma_k(n_0, n_1, \dots, n_{k-1}) = \begin{cases} 2^{n_0-1} + \sigma_{k-1}(n_1, \dots, n_{k-1}) & n_0 > \max\{n_1, \dots, n_{k-1}\} \\ \sigma_{k-1}(n_1, \dots, n_{k-1}) & \text{else.} \end{cases}$$

It follows that if $(n_0, n_1, \dots, n_{m-1}) \in D_m$, then $\sigma_1(n_0), \sigma_2(n_0, n_1), \dots, \sigma_m(n_0, n_1, \dots, n_{m-1})$ are different from each other. Therefore, we have

$$\begin{aligned}
 & \lambda(A_{i_0} \cap T^{-1} A_{i_1} \cap \dots \cap T^{-m} A_{i_m}) \\
 &= \sum_{(n_0, n_1, \dots, n_{m-1}) \in D_m} \lambda(A_{i_0} \cap T^{-1} A_{i_1} \cap \dots \cap T^{-m} A_{i_m} \mid \tau(\alpha)=n_0, \tau(S\alpha)=n_1, \dots, \\
 & \quad \tau(S^{m-1}\alpha)=n_{m-1}) \cdot \lambda(\tau(\alpha)=n_0, \tau(S\alpha)=n_1, \dots, \tau(S^{m-1}\alpha)=n_{m-1}) \\
 &= \sum_{(n_0, n_1, \dots, n_{m-1}) \in D_m} \lambda(\beta(0)=i_0, \beta(\sigma_1(n_0))=i_1, \dots, \beta(\sigma_m(n_0, n_1, \dots, n_{m-1})) \\
 & \quad =i_m \mid \tau(\alpha)=n_0, \tau(S\alpha)=n_1, \dots, \tau(S^{m-1}\alpha)=n_{m-1}) \\
 & \quad \times \lambda(\tau(\alpha)=n_0, \tau(S\alpha)=n_1, \dots, \tau(S^{m-1}\alpha)=n_{m-1}) \\
 &= \sum_{(n_0, n_1, \dots, n_{m-1}) \in D_m} \lambda(\beta(0)=i_0, \beta(\sigma_1(n_0))=i_1, \dots, \beta(\sigma_m(n_0, n_1, \dots, n_{m-1})) \\
 & \quad =i_m) \cdot \lambda(\tau(\alpha)=n_0, \tau(S\alpha)=n_1, \dots, \tau(S^{m-1}\alpha)=n_{m-1}) \\
 &= \sum_{(n_0, n_1, \dots, n_{m-1}) \in D_m} 2^{-m-1} \cdot \lambda(\tau(\alpha)=n_0, \tau(S\alpha)=n_1, \dots, \tau(S^{m-1}\alpha)=n_{m-1}) \\
 &= 2^{-m-1}.
 \end{aligned}$$

Thus $h_\lambda(T; A) = \log 2$.

(Q.E.D.)

Now, we show how to construct F satisfying (4), (5) and (6).

Let $F_1 = \{(0, 1)\}$. Suppose that $F_1, F_2, \dots, F_n$ have been selected so that

$$(7) \quad \overline{F_i} \cap \overline{F_j} = \emptyset \text{ for any } i \neq j \text{ in } \bigcup_{i=1}^n F_i,$$

$$(8) \quad \overline{F_i} \cap \overline{F_j} = \emptyset \text{ for any } i \neq j \text{ in } \bigcup_{i=1}^n F_i, \text{ and}$$

$$(9) \quad \text{if } \xi, \eta \in F_i \quad (\xi, \eta \in \{0, 1\}^{2^{i-1}}, i=1, 2, \dots, n), \text{ then } \xi \neq \eta \text{ for any } \xi \in \{0, 1\}^{2^{i-1}}.$$

Let

$$\mu\left(\bigcup_{i=1}^{\infty} \bigcup_{\xi \in F_i} \Gamma_{\xi}\right) = \sum_{i=1}^{\infty} |F_i| 2^{-2^i} = 1 - \delta_n.$$

Then, we can select subsets G_n and H_n of $\{0,1\}^{2^n}$ satisfying that

$$(10) \quad |G_n| = |H_n| \quad \text{and} \quad |G_n| 2^{-2^n} \geq \frac{\delta_n}{3},$$

$$(11) \quad G_n \cap H_n = \emptyset,$$

$$(12) \quad \Gamma_{\xi} \cap \Gamma_{\eta} = \emptyset \text{ for any } \xi \in G_n \text{ and } \eta \in \bigcup_{i=1}^n F_i, \text{ and}$$

$$(13) \quad \Gamma_{\xi} \cap \Gamma_{\eta} = \emptyset \text{ for any } \xi \in H_n \text{ and } \eta \in \bigcup_{i=1}^n F_i.$$

Let $F_{n+1} = G_n \cup H_n = \{\xi \eta; \xi \in G_n, \eta \in H_n\}$. Then, it is easy to check that $F_1, F_2, \dots, F_{n+1}$ satisfy the conditions (7), (8) and (9) with $n+1$ for n .

Moreover, since

$$\delta_{n+1} \leq \delta_n - \left(\frac{\delta_n}{3}\right)^2,$$

$\delta_n \rightarrow 0$ as $n \rightarrow \infty$, so that

$$\mu\left(\bigcup_{\xi \in F} \Gamma_{\xi}\right) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{i=1}^n \bigcup_{\xi \in F_i} \Gamma_{\xi}\right) = 1.$$

Thus, finally we get F satisfying (4), (5) and (6).