JON AARONSON

Ergodic Theory for Inner Functions of the Upper Half Plane

Publications des séminaires de mathématiques et informatique de Rennes, 1977, fascicule 3
« Séminaire de probabilités II », , p. 1-26
<http://www.numdam.org/item?id=PSMIR_1977___3_A2_0>

ERGODIC THEORY FOR

INNER FUNCTIONS OF THE UPPER HALF PLANE

Jon Aaronson

Abstract :

The real restriction of an inner function of the upper half plane leaves Lebesgue measure quasi-invariant. It may have a finite or infinite invariant measure. We give conditions for the rational ergodicity and exactness of such restrictions.

Abstrait :

La restriction à la droite réelle d'une fonction intérieure du demi-plan supérieur laisse la mesure de Lebesgue quasi-invariante, et peut avoir une mesure invariante finie ou infinie. Nous donnons les conditions pour l'ergodicité rationnelle et l'exactitude de telles transformations.

Rennes,

December 1977

ERGODIC THEORY FOR INNER FUNCTIONS OF THE UPPER HALF PLANE

Jon Aaronson

§0 - Introduction

In this paper, we consider the ergodic properties of the real restrictions of inner functions on the open upper half plane : $\mathbb{R}^{2+} = \{x+y : x,y \in \mathbb{R},\ y > 0\}$.

Let $f:\mathbb{R}^{2+} \to \mathbb{R}^{2+}$ be an analytic function. We say that f is an <u>inner function on $\mathbb{R}^{2+}$</u> if for λ-a.e. $x \in \mathbb{R}$ the limit $\lim_{y\downarrow 0} f(x+iy)$ exists, and is real. (Here, and throughout the paper, λ denotes Lebesgue measure on $\mathbb{R}$). Consider the limit $\lim_{y\downarrow 0} f(x+iy) = Tx$. This is defined λ-a.e. on $\mathbb{R}$. We call this limit the (real) <u>restriction of f</u> , and will sometimes write this as $T = T(f)$. We will denote the class of inner functions on $\mathbb{R}^{2+}$ by $I(\mathbb{R}^{2+}) = I$, and their real restrictions by $M(\mathbb{R})$. We note that $f \in I(\mathbb{R}^{2+})$ iff $\emptyset^{-1} f\, \emptyset(z)$ is an inner function of the unit disc, according to the definition on p. 370 of [8] (where $\emptyset(z) = i(\frac{1+z}{1-z}))$.

The following characterisation of $I(\mathbb{R}^{2+})$ appears in [6] and [17].

$f \in I(\mathbb{R}^{2+})$ iff

(0-1) $$f(\omega) = \alpha\omega + \beta + \int_{-\infty}^{\infty} \frac{1+t\omega}{t-\omega}\, d\mu(t)$$ where $\alpha \geq 0$, $\beta \in \mathbb{R}$ and μ is a bounded, positive Borel measure, singular w.r.t. λ . Since we shall be refering to (0-1) rather a lot, we shall denote the class of bounded, positive, singular measures on $\mathbb{R}$ by $S(\mathbb{R})$.

G. Letac ([6]) has shown that a measurable transformation T of $\mathbb{R}$ preserves the class of Cauchy distributions iff either $T \in M(\mathbb{R})$ or $-T \in M(\mathbb{R})$. In particular, if $dP_{a+ib}(x) = \frac{b}{\pi} \frac{dx}{(x-a)^2+b^2}$ for $a+ib \in \mathbb{R}^{2+}$ and $T = T(f) \in M(\mathbb{R})$, then :

$$(0.2) \qquad P_{\omega} \circ T^{-1} = P_{f(\omega)} \quad \text{for} \quad \omega \in \mathbb{R}^{2+}$$

This equation shows that $M(\mathbb{R})$ is a class of non-singular transformations of the measure space $(\mathbb{R},\mathcal{B},\lambda)$, and is therefore an object of ergodic theory.

Let $f \in I(\mathbb{R}^{2+})$ have a fixed point $\omega_0 \in \mathbb{R}^{2+}$. By (0.2), $T(f)$ preserves the Cauchy distribution P_{ω_0}. It was shown in [16], that if f is $1-1$, then $T(f)$ is conjugate to a rotation of the circle, and shown in [15] that otherwise, $T(f)$ is mixing. We show in §1 that if f is not $1-1$ then $T(f)$ is exact.

In §2 we recall some well known facts about inner functions of $\mathbb{R}^{2+}$. The Denjoy-Wolff theorem (see [13],[14] and [18]) adapted to $\mathbb{R}^{2+}$ shows that when studying the ergodic properties of $T(f)$, for $f \in I(\mathbb{R}^{2+})$ with no fixed points in $\mathbb{R}^{2+}$, we may assume that $\alpha(f) \geq 1$. In case $\alpha(f) > 1$, $T(f)$ is dissipative, and when $\alpha(f) = 1$, $T(f)$ préserves Lebesgue measure.

In §3, we consider the case $\alpha(f) = 1$. Here, the conservativity of a restriction $T(f)$ is sufficient for its rational ergodicity ([1]) (ergodicity was established in [15]). We also give sufficient conditions for exactness, and discuss the similarity classes ([1]) of restricitions.

The ergodic theory of certain restrictions has been considered in [2], [5], [7], [10], [11], [15] and [16]

The author would like to thank B. Weiss for helpful conversations, and G. Letac, J. Neuwirth and F. Schweiger for making preprints of their works available.

§ 1 - Mixing restrictions preserving finite measures

Theorem 1.1

Let $f \in I(\mathbb{R}^{2+})$ and assume that f is not 1 - 1 . If f has a fixed point $\omega_0 \in \mathbb{R}^{2+}$, then $(\mathbb{R},\mathcal{B},P_{\omega_0},T(f))$ is an exact measure preserving transformation.

i.e. $\bigcap_{n\geq 1} T^{-1}\mathcal{B} = \{\emptyset,\mathbb{R}\}$ mod λ.

Before proving theorem 1.1., we shall need some auxiliary results. The first of these is Lin's criterion for exactness of Markov operators (theorem 4.4. in [7]) as applied to our case. To state this, we shall need some extra notation :

Let $T \in M(\mathbb{R})$, then $(\mathbb{R},\mathcal{B},\lambda,T)$ is a non-singular transformation, and so $g \in L^\infty(\mathbb{R},\mathcal{B},\lambda)$ iff $g\circ T \in L^\infty(\mathbb{R},\mathcal{B},\lambda)$. We define the dual operator of T, $\hat{T} : L^1(\mathbb{R},\mathcal{B},\lambda) \to L^1(\mathbb{R},\mathcal{B},\lambda)$ by

$$\int_{\mathbb{R}} \hat{T}h.g\,d\lambda = \int_{\mathbb{R}} h.g\circ T\,d\lambda \qquad \text{for} \quad h \in L^1 \quad \text{and} \quad g \in L^\infty$$

If we write, for $\omega = a+ib \in \mathbb{R}^{2+}$

$$\frac{dP_\omega}{d\lambda}(x) = \phi_\omega(x) = \frac{b}{\pi} \cdot \frac{1}{(x-a)^2+b^2}$$

then equation (0.2) translates to :

(1.1) $\quad \hat{T}\phi_\omega = \phi_{f(\omega)}$ for $T = T(f) \in M(\mathbb{R})$

Clearly, $\hat{T}$ is a positive linear operator, $\int_{\mathbb{R}} \hat{T}h\,d\lambda = \int_{\mathbb{R}} h\,d\lambda$ for $h \in L^1$.

<u>Lin's Criterion (for restrictions)</u> Let $T = T(f) \in M(\mathbb{R})$.

T is exact iff

(1.2) $\qquad ||\hat{T}^n u||_1 \to 0$ for every $u \in L^1$, $\int_{\mathbb{R}} u\,d\lambda = 0$. (Here, and throughout, $||u||_1 = \int_{\mathbb{R}} |u|\,d\lambda$).

We will need the following elementary lemma:

Lemma 1.2 If $\omega_n \in \mathbb{R}^{2+}$ and $\omega_n \to \omega \in R^{2+}$ then :

$$||\phi_{\omega_n} - \phi_\omega||_1 \to 0$$

Proof of the theorem 1.1

We first show that $f^n(\omega) \to \omega_0$ $\forall \omega \in \mathbb{R}^{2+}$, where $f^1(\omega) = f(\omega)$ and $f^{n+1}(\omega) = f(f^n(\omega))$.

Let $\emptyset : U = [|Z|<1] \to R^{2+}$ be a conformal map. Then $g = \emptyset^{-1} f \emptyset : U \to U$ is analytic, and $g(\emptyset(\omega_0)) = \emptyset(\omega_0)$. By the Schwartz lemma ([9]) : $|g'(\emptyset(\omega_0))| < 1$ as g is not 1 - 1 . It is now not hard to see that $g^n(Z) \to \emptyset(\omega_0)$ $\forall z \in U$, and hence that $f^n(\omega) \to \omega_0$ $\forall \omega \in \mathbb{R}^{2+}$.

Hence, by lemma 1.2

$$||\hat{T}^n \phi_\omega - \phi_{\omega_0}||_1 = ||\phi_{f^n(\omega)} - \phi_{\omega_0}||_1 \to 0 \quad \text{for} \quad \omega \in \mathbb{R}^{2+} .$$

We will now establish that

$$||\hat{T}^n u||_1 \to 0 \quad \text{for} \quad u \in L^1 \quad \text{with} \quad \int_{\mathbb{R}} u d\lambda = 0$$

which, by Lin's criterion, will ensure the exactness of T .

Let $u \in L^1$ with $\int_{\mathbb{R}} u d\lambda = 0$ and let $\varepsilon > 0$. By Wiener's Tauberian theorem (see [12] p.357), there exist $\alpha_1 \ldots \alpha_N$, $a_1 \ldots a_N \in \mathbb{Q}$ such that

$$||u - \sum_{j=1}^{N} \alpha_j \phi_{a_j+i}||_1 < \varepsilon/2$$

Clearly, this implies that $|\sum_{j=1}^{N} \alpha_j| < \varepsilon/2$ and so :

$$||\hat{T}^{n_\cdot} u||_1 \leq$$

$$\leq ||\hat{T}^{n} (u - \sum_{j=1}^{N} \alpha_j \phi_{a_j+i})||_1 + ||\hat{T}^{n_\cdot} (\sum_{j=1}^{N} \alpha_j (\phi_{a_j+i} - \phi_{\omega_0}))||_1 + ||\sum_{j=1}^{N} \alpha_j \phi_{\omega_0}||_1 \leq$$

$$\leq ||u - \sum_{j=1}^{N} \alpha_j \phi_{a_j+i}||_1 + \sum_{j=1}^{N} |\alpha_j| \; ||\hat{T}^{n_\cdot} \phi_{a_j+i} - \phi_{\omega_0}||_1 + |\sum_{j=1}^{N} \alpha_j| <$$

$< \varepsilon + o(1)$ as $k \to \infty$ □

Since $\varepsilon > 0$ was arbitrary : $||\hat{T}^{n_\cdot} u||_1 \to 0$. □

§2 - Basic Classification

Proposition 2.1 ([17] p.151-2)

Let $f \in I(\mathbb{R}^{2+})$

Then $$\frac{f(ib)}{ib} \to \begin{cases} \alpha(f) = \alpha \in [0,\infty) & \text{as } b \to \infty \quad (\alpha \text{ as in } 0.1) \\ \gamma(f) \in [\alpha,\infty] & \text{as } b \downarrow 0 \end{cases}$$

Moreover $\alpha = \gamma$ iff $f(\omega) = \alpha\omega$

Proof. From the representation 0.1 , we immediatly calculate that :

$$\frac{f(ib)}{ib} = \alpha + \frac{\beta}{ib} + \frac{1-b^2}{ib}\int_{-\infty}^{\infty}\frac{t\,d\mu(t)}{t^2+b^2} + \int_{-\infty}^{\infty}\frac{1+t^2}{t^2+b^2}\,d\mu(t) \qquad (2.1)$$

It follows from elementary integration theory that

$$\frac{f(ib)}{ib} \to \alpha = \alpha(f) \quad \text{as} \quad b \to \infty .$$

To check the limit as $b \to 0$, we "flip" f to get :

$$\tilde{f}(\omega) = -1/f(-1/\omega)$$

Since $\tilde{f} \in I(\mathbb{R}^{2+})$, we have that

$$\frac{\tilde{f}(ib)}{ib} \to \alpha(\tilde{f}) \in [0,\infty) \quad \text{as} \quad b \to \infty$$

but this decodes to :

$$\frac{f(ib)}{ib} \to \gamma(f) = \frac{1}{\alpha(\tilde{f})} \in (0,\infty] \quad \text{as} \quad b \downarrow 0 .$$

Now, if $\gamma(f) < \infty$ then, by 2.1 :

$$\gamma(f) = \alpha + \int_{-\infty}^{\infty}\frac{1+t^2}{t^2}\,d\mu(t)$$

Hence $\gamma(f) \geq \alpha(f)$ with equality iff $\mu \equiv 0$. □

Proposition 2.2

Let $f \in I(\mathbb{R}^{2+})$ and $T = T(f)$.

If $\alpha(f) > 1$ then T is dissipative .

Proof. Write $f^n(\omega) = u_n(\omega) + iv_n(\omega)$.
From the representation (0.1), we have :

$$v_{n+1}(\omega) = \alpha v_n(\omega) + v_n(\omega) \int_{-\infty}^{\infty} \frac{1+t^2}{(t-u_n)^2+v_n^2} \geq \alpha v_n$$

Hence $v_n(i) \geq \alpha^n$ for $n \geq 1$, and

$$\hat{T}^n \phi_i(t) = \frac{v_n(i)}{\pi((t-u_n)^2+v_n^2)} \leq \frac{1}{\pi\alpha^n}$$

Clearly $\sum_{n=1}^{\infty} \hat{T}^n \phi_i(t) \leq \frac{1}{(\alpha-1)}$ $\forall t \in \mathbb{R}$

and so $\sum_{n=1}^{\infty} 1_A \circ T^n < \infty$ a.e. $\forall A \in \mathbb{B}$; $\lambda(A) < \infty$ □

Proposition 2.3 (Letac [6])

Let $f \in I(\mathbb{R}^{2+})$, $T = T(f)$.

If $\alpha(f) = 1$ then $\lambda \circ T^{-1} = \lambda$.

Proof. Let $f(ib) = u(b) + iv(b)$
we have : $\frac{u(b)}{b} \to 0$ and $\frac{v(b)}{b} \to 1$ as $b \to \infty$.

Hence, for $A \in B$:

$\pi b P_{ib}(A) \to \lambda(A)$

and $\pi b P_{f(ib)}(A) \to \lambda(A)$ as $b \to \infty$.

Since $P_{ib}(T^{-1}A) = P_{f(ib)}(A)$, we have that
$\lambda(T^{-1}A) = \lambda(A)$ for $A \in \mathbb{B}$ □

The next result is the Denjoy-Wolff theorem stated on $\mathbb{R}^{2+}$, which shows shows that if $f \in I(\mathbb{R}^{2+})$ has no fixed point in $\mathbb{R}^{2+}$, then $\exists\ \tilde{f} \in I(\mathbb{R}^{2+})$ with $\alpha(\tilde{f}) = 1$, and such that $(\mathbb{R}, \mathcal{B}, \lambda, T(f))$ and $(R, B, \lambda, T(\tilde{f}))$ are conjugate, (and therefore have the same ergodic properties).

Theorem 2.4

Let $f \in I(\mathbb{R}^{2+})$ have no fixed points in $\mathbb{R}^{2+}$, and assume that $\alpha(f) < 1$; then

$\exists!\ t \in R$ such that $\alpha(\phi_t f \phi_t^{-1}) \geq 1$ where $\phi_t(\omega) = \frac{1+t\omega}{t-\omega}$. (Note that $\alpha(\phi_0^{-1} f \phi_0) = 1/\gamma(f)$).

Proof.

Let $\phi(z) = i(\frac{1+Z}{1-Z})$. Then $g = \phi^{-1} f \phi : U \to U$ is analytic, and has no fixed points in U. The Denjoy-Wolff theorem on U (see [13] or [14]) shows that $\exists!\ \rho \in T$ such that

$$(*) \qquad \mathrm{Re}\left(\frac{\rho+g(Z)}{\rho-g(Z)}\right) \geq \mathrm{Re}\left(\frac{\rho+Z}{\rho-Z}\right) \qquad \forall Z \in U$$

Now let $t = \phi(\rho)$, $\psi = i(\frac{\rho+Z}{\rho-Z})$ and $\tilde{f} = \psi g \psi^{-1} \in I(\mathbb{R}^{2+})$. It follows that $\phi\psi^{-1} = \phi_t^{-1}$ and hence that $\tilde{f} = \phi_t f \phi_t^{-1}$. Also, $(*)$ means that $\mathrm{Im}\ \psi g(Z) \geq \mathrm{Im}\ \psi(Z)$ for $Z \in U$, and hence $\mathrm{Im}\ \tilde{f}(\omega) \geq \mathrm{Im}\,\omega$ for $\omega \in \mathbb{R}^{2+}$, which implies $\alpha(\tilde{f}) \geq 1$. □

If $\alpha(\phi_t f \phi_t^{-1}) > 1$ for some t, then by proposition 2.2, $T(f)$ is dissipative. If $\alpha(\phi_t f \phi_t^{-1}) = 1$, then, by proposition 2.3, $T(\phi_t f \phi_t^{-1}) = \phi_t T(f) \phi_t^{-1}$ preserves Lebesgue measure. Hence $T(f)$ preserves the measure ν_t, where $d\nu_t(x) = dx/(x-t)^2$.

The rest of this section is devoted to odd restrictions.

(We say that a restriciton T is <u>odd</u> if $T(-x) = -T(x)$).

<u>Lemma 2.5</u>

Let $f \varepsilon I(\mathbb{R}^{2+})$ and let $T = T(f)$. The following are equivalent :

(i) T is odd (ii) $\mathrm{Re}\, f(ib) = 0$ for $b > 0$

(iii) $f(-\overline{\omega}) = -\overline{f(\omega)}$ for $\omega \varepsilon \mathbb{R}^{2+}$

(iv) $f(\omega) = \alpha\omega + \int_{-\infty}^{\infty} \frac{1+t\omega}{t-\omega}\, d\mu(t)$ where $\mu \varepsilon S(\mathbb{R})$ is symetric.

<u>Proof.</u> The implications (iv) => (iii) => (i) and (iii) => (ii) are elementary. That (ii) => (iii) is because of the Schartz reflection principle (see [9]). The fact that for $t \geq 0$:

$$e^{itf(\omega)} = \int_{-\infty}^{\infty} e^{itT(x)}\, \phi_{\omega}(t)\, dt$$

gives the implication (i) => (iii) .

We show that (iii) => (iv). Assume (iii). It is evident that

$\beta = 0$ in the representation 0.1 , so we have

$$f(\omega) = \alpha\omega + \int_{-\infty}^{\infty} \frac{1+t\omega}{t-\omega}\, d\mu(t) \quad \text{where} \quad \alpha \geq 0 \quad \text{and} \quad \mu \in S(\mathbb{R}) .$$

We must show that μ is symetric. To see this, we first rewrite the equation $v(-a+ib) = v(a+ib)$ (implied by (iii)) as :

$$(2.2) \qquad \int_{\infty}^{\infty} \phi_b(t-a)(1+t^2)\, d\mu(t) = \int_{\infty}^{\infty} \phi_b(t+a)(1+t^2)\, d\mu(t)$$

Next, we take $g(t)$ a continuous function of compact support and let $g_b(t) = \phi_{ib} * g$. for $b > 0$. It follows from (2.2) that

$$\int_{\infty}^{\infty} g_b(-t)(1+t^2)\, d\mu(t) = \int_{\infty}^{\infty} g_b(t)(1+t^2)\, d\mu(t) .$$

The symetry of μ is established by the (elementary) facts that

$$g_b(t) \to g(t) \quad \text{as} \quad b \to 0$$

$$\sup_{\substack{t\in\mathbb{R} \\ b>0}} (1+t^2)|g_b(t)| < \infty \qquad \square$$

We denote the collection of those inner functions on $\mathbb{R}^{2+}$ satisfying the conditions of the above lemma by $I_0(\mathbb{R}^{2+})$, and remark that $f \in I_0(\mathbb{R}^{2+})$ iff $\emptyset^{-1} f \emptyset$ is an essentially real inner function of U . (Here $\emptyset(z) = i(\frac{1+z}{1-z})$).

<u>Theorem 2.6</u>

Let $f \in I_0(\mathbb{R}^{2+})$ and $T = T(f)$.

If $\alpha(f) < 1 < \gamma(f)$ then T preserves a Cauchy distribution. Moreover, if $\omega f(\omega)$ is not constant, then T is exact.

<u>Proof</u>. If $f \in I_0(\mathbb{R}^{2+})$ then it follows from the lemma

$$\gamma(f) = \alpha(f) + \int_{\infty}^{\infty} \frac{1+t^2}{t^2}\, d\mu(t) .$$

Now since $\alpha(f) < 1 < \gamma(f)$, we have that

$$\int_{\infty}^{\infty} \frac{1+t^2}{t^2} d\mu(t) > 1 - \alpha > 0 .$$

But $\int_{\infty}^{\infty} \frac{1+t^2}{t^2+b^2} d\mu(t) \downarrow 0$ as $b \to \infty$ so there is a $b_0 > 0$ such that $\int_{\infty}^{\infty} \frac{1+t^2}{t^2+b_0^2} d\mu(t) = 1 - \alpha$, i.e. $f(ib_0) = ib_0$, hence

$$P_{ib_0} \circ T^{-1} = P_{ib_0} .$$

The result now follows from theorem 1.1 ▨

To illustrate the results of this section, we consider $T_x = \alpha x + \beta \tan x$ where $\alpha, \beta > 0$.

If either $\alpha > 1$, or $\alpha + \beta < 1$, T is dissipative.

If $\alpha < 1 < \alpha+\beta$, then T preserves a Cauchy distribution and is exact. (This was established in [5] for $\alpha = 0$, $\beta > 1$) .

The remaining cases ($\alpha = 1$ and $\alpha+\beta = 1$) are contained in the discussion of :

§3 - Restrictions Preserving Infinite Measures

In this section, we consider those restrictions preserving infinite measures with $\alpha = 1$, and $\gamma = 1$.

We will see that for these transformations, conservativity is sufficient for ergodicity and rational ergodicity ([1]) - a stronger property (example 1-2 in [1]). We then give sufficient conditions for exactness.

Firstly, we recall the definition of rational ergodicity. Let $(X,\mathcal{B},m,\tau)$ be a conservative, ergodic, measure preserving transformation of a non-atomic, σ-finite measure space. We say that τ is rationally ergodic if there is a set A, of positive finite measure and $K < \infty$ such that

$$\text{(B)} \qquad \int_A \left(\sum_{k=0}^{n-1} 1_A \circ \tau^k\right)^2 dm \leq K \left(\sum_{k=0}^{n-1} m(A \cap \tau^{-k}A)\right)^2 \quad \text{for} \quad n \geq 1$$

For a rationally ergodic transformation τ, we let $B(\tau)$ denote the collection of sets with the property (B) . It was shown in [1] that there is a sequence $\{a_n(\tau)\}$ such that

$$\frac{1}{a_n(\tau)} \sum_{k=0}^{n-1} m(A \cap T^{-k}A) \to m(A)^2 \quad \text{for every} \quad A \in B(\tau)$$

The sequence $\{a_n(\tau)\}_n$ is known as a <u>return sequence for τ</u> and the collection of all sequences asymptotically proportional to $a_n(\tau)$ (i.e. $\frac{a_n}{a_n(\tau)} \to c \in (0,\infty)$) is known as the <u>asymptotic type of τ</u> and denoted by $\mathcal{Q}(\tau)$. It was shown in [1] (theorem 2.4) that if τ_1 and τ_2 are rationally ergodic transformations which are both factors of the same measure preserving transformation, then

$$\mathcal{Q}(\tau_1) = \mathcal{Q}(\tau_2) \quad \left(\text{i.e. } \exists \lim_{n\to\infty} \frac{a_n(\tau_1)}{a_n(\tau_2)} \in (0,\infty)\right) .$$

We commence with the case $\alpha(f) = 1$.

<u>Lemma 3.1</u>

Let $f \in I(\mathbb{R}^{2+})$ be non-linear and let $T = T(f)$,
$f^n(\omega) = u_n(\omega) + iv_n(\omega)$ for $n \geq 1$ $\omega \in \mathbb{R}^{2+}$.

If $\alpha = 1$ then T is conservative

$$\text{iff} \quad \sum_{n=1}^{\infty} \frac{V_n(\omega)}{|f^n(\omega)|^2} = \infty \qquad \forall\ \omega \in \mathbb{R}^{2+} .$$

Proof. It will be more comfortable to work on the unit disc U. Accordingly, we let $M(z) = \emptyset^{-1} f \emptyset(z)$*. Then M is an inner function on U. Let $M(re^{i\theta}) \to \tau e^{i\theta}$ as $r \to 1$ a.e. . Denoting $\mathrm{Im}(\frac{e^{i\theta}+z}{e^{i\theta}+z})$ by $q_z(\theta)$ and $q_z(\theta)\, d\theta$ by $d\pi_z(\theta)$, we see that $\pi_z \circ \emptyset^{-1} = \pi_\emptyset P_{\emptyset(z)}$ and this combined with the fact that $\emptyset^{-1} T \emptyset = \tau$ gives us that :

$$\pi_z \circ \tau^{-1} = \pi_{M(z)} \quad .$$

So τ is a non-singular transformation of $(\mathbb{T},\lambda)$, and is conservative iff T is conservative.

Let $\hat{\tau}$ be the operator dual to τ, acting on L'. Then $\hat{\tau}\, q_z(t) = q_{M(z)}(t)$ and τ is conservative iff

$$(3.1) \qquad \sum_{n=1}^{\infty} q_{M^n(z)}(t) = \infty \qquad \text{a.e.} \quad \forall z \in U \quad .$$

We next show that $M^n(z) \to 1$ as $n \to \infty$ $\forall z \in U$. This will follow from the fact that $f^n(\omega) \to \infty$ as $n \to \infty$ $\forall \omega \in R^{2+}$ which we now demonstrate. From 0.1 :

$$v_{n+1}(\omega) = v_n(\omega) + v_n(\omega) \int_{\infty}^{\infty} \frac{(1+t^2)\ d\mu(t)}{(t-U_n)^2+v_n^2} \geq v_n(\omega) \quad .$$

Hence $v_n \uparrow v_\infty$. It is not hard to see that if $v_\infty < \infty$, we must have $|U_n| \to \infty$. Hence $M^n(z) \to 1$.

Now choose $z \in U$ and let $M^n(z) = r_n e^{i\theta_n}$. We have $r_n \to 1$ and $\theta_n \to 0$. Also :

$$q_{M^n(z)}(t) = \frac{1-r_n^2}{1-2r_n\cos(\theta_n - t)+r_n^2} \sim \frac{1-r_n}{1-\cos t} \quad \text{as } n \to \infty \text{ . For } t \neq 0 \text{ .}$$

Thus :

$$(3.2) \qquad T \text{ is conservative iff } \sum_{n=1}^{\infty} 1-|M^n(z)| = \infty \quad \forall z \in U \text{ .}$$

Since $M^n(z) \to 1$, the second condition is the same as

* where $\emptyset(z) = i\left(\frac{1+z}{1-z}\right)$

$$\sum_{n=1}^{\infty} 1 - |M^n(z)|^2 = \infty \ \forall \ z \in U .$$

Now if $\omega = a + ib \in \mathbb{R}^{2+}$, then

$$1 - \left|\frac{\omega-i}{\omega+i}\right|^2 = \frac{4b}{a^2+(b+1)^2}$$

From the definition of M , we have

$$1 - |M^n(\tfrac{\omega-i}{\omega+i})|^2 = \frac{4v_n(\omega)}{U_n(\omega)+(v_n+1)^2} \sim \frac{4v_n(\omega)}{|f^n(\omega)|^2} \quad \text{as} \quad n \to \infty$$

□

Theorem 3.2

Let $f \in I(\mathbb{R}^{2+})$ be non-linear, $T = T(f)$ and $\alpha(f) = 1$. If T is conservative then T is rationally ergodic, and $\mathcal{a}(T) = \{\sum_{k=1}^{n} \frac{v_k(\omega)}{|f^k(\omega)|^2}\}$ for every $\omega \in \mathbb{R}^{2+}$.

Proof. We first prove ergodicity, and here again, it is more comfortable to work on U . We prove the ergodicity of τ . If T is conservative then by (3.2) $\sum_{n=1}^{\infty} 1-|M^n(z)| = \infty$ $\forall z \in U$. Since $M^n(z) \to 1$, we must have that the points $\{M^n(z)\}_{n\geq 1}$ are distinct. Now, let $h \in N(U)$ (defined on p. 303 of [9]) . If $h(\bar{M}(z)) = h(z)$ for some $z \in U$ then by theorem 15-23 of [9], h must be constant. The ergodicity of τ is deduced from this as follows :

Let $A \subseteq \mathbb{T}$ be an τ-invariant measurable set.

The function $u(z) = \int_T q_z(\theta)\, 1_A(\theta)\, d\theta$ is a bounded harmonic function on U , and $u(g(z)) = u(z)$ on U . By theorem 17-26 of [8], u is the imaginary part of an analytic function $F(z) \in H^p(u)$ for $1 \leq p < \infty$ $(H^p \subseteq N)$.

Clearly $F(g(z)) = F(z) + c$ where $c \in \mathbb{R}$.

Let $F^*(e^{i\theta}) = \lim_{r\uparrow 1} F(re^{i\theta})$, then $F^*(\tau e^{i\theta}) = F^*(e^{i\theta})+c$. The conservativity of τ yields that $c = 0$ (since the set $[|F^*| \leq M]$ has positive measure for some M, and so every point of this set returns infinitely often to it under iterations of τ — an impossibility if $c \neq 0$). Thus, by step 3, F is constant and hence u is constant, hence $1_A(\theta)$.

We now turn to rational ergodicity.

Let $b_n(\omega) = \dfrac{|f^n(\omega)|^2}{v_n(\omega)}$

Since $f^n(\omega) \to \infty$, it is clear that :

(3.3) $\pi b_n(\omega)\, \hat{T}^n \phi_\omega(t) \to 1$ uniformly on compact subsets of $\mathbb{R}$.

Let $a_n(\omega) = \sum_{k=1}^{n} \dfrac{1}{\pi b_k(\omega)}$. From (3.3) we have that

(3.4) $\dfrac{1}{a_n(\omega)} \sum_{k=0}^{n-1} \hat{T}^k \phi_\omega \to 1$ uniformly on compact subset of $\mathbb{R}$.

Now, since T is a conservative ergodic transformation, it follows that $\hat{T}$ is a conservative ergodic Markov operator, and we have from (3.4), by the Chacon-Ornstein theorem (see [3]) that :

(3.5) $\dfrac{1}{a_n(\omega)} \sum_{k=0}^{n-1} \hat{T}^k f \to \int_{\mathbb{R}} f d\lambda$ a.e. $\forall f \in L'$.

Hence $\exists a_n \to \infty$ s.t. $\dfrac{a_n(\omega)}{a_n} \to 1$ for every $\omega \in \mathbb{R}^{2+}$.

We will prove rational ergodicity of T by showing that bounded intervals are in $B(T)$

Let $A = [a,b]$ where $-\infty < a < b < \infty$

Then $1_A \leq c\phi_i$

Hence, by (3.4), there is a $C_1 < \infty$ s.t.

(3.6) $\dfrac{1}{a_n} \sum_{k=0}^{n-1} \hat{T}^k 1_A(x) \leq C_1$ for $n \geq 1$, $x \in A$.

This, combined with (3.5), gives (by dominated convergence)

$$(3.7) \qquad \frac{1}{a_n}\sum_{k=0}^{n-1} \lambda(A \cap T^{-k}A) \to \lambda(A)^2$$

To complete the proof that T is rationally ergodic, we show that :

$$(3.8) \qquad \int_A \Big(\sum_{k=0}^{n-1} 1_A \circ T^k\Big)^2 d\mu \le 2\, C_1\, a_n^2 \quad \text{for} \quad n \ge 1 .$$

$$\int_A \Big(\sum_{k=0}^{n-1} 1_A \circ T^k\Big)^2 d\mu \le 2 \sum_{k=0}^{n-1}\sum_{\ell=0}^{n-1} \lambda(A \cap T^{-k}(A \cap T^{-\ell}A))$$

$$= 2 \sum_{\ell=0}^{n-1} \int_{A \cap T^{-\ell}A} \sum_{k=0}^{n-1} \hat{T}^k 1_A \, d\lambda$$

$$\le 2\, C_1\, a_n^2 \qquad \square$$

Remark : If, in addtion, we assume that $f \in I_0(\mathbb{R}^{2+})$, we have that $b_n(i) = v_n(i)$, and that (3.6) holds for every $x \in \mathbb{R}$. In this situation, we have that

$$\frac{1}{a_n}\sum_{k=0}^{n-1} p(\hat{T}^{-k}A) \to \lambda(A)$$ for p a λ-absolutely continuous probability measure, and A a bounded measurable set. (see [4] §4).

We now turn to exactness. The following elementary lemma plays a similar role to that of lemma 1.2.

Lemma 3.3

If $b_n \to \infty$, $B_n \sim b_n$ and $\frac{a_n}{b_n} \to 0$ as $n \to \infty$ then

$$||\phi_{a_n + ib_n} - \phi_{iB_n}||_1 \to 0 \quad \text{as} \quad n \to \infty .$$

Theorem 3.4

Let $f \in I(\mathbb{R}^{2+})$, $T = T(f)$ and assume

$$f(\omega) = \omega + \int_{-K}^{K} \frac{d\nu(t)}{t-\omega}$$

then : T is exact, rationally ergodic and $\mathfrak{a}(T) = \{\sqrt{n}\}$.

Proof. Let $L = \max\{\nu(\mathbb{R}), \nu(\mathbb{R})^2\}$ and assume that $K \geq \frac{1}{4}$. We write $f^n(\omega) = u_n(\omega) + iv_n(\omega)$. The assumption of the theorem means that

$$(3.9) \qquad u_{n+1} = u_n + \int_{-K}^{K} \frac{t-u_n}{(t-u_n)^2+v_n^2} \, d\nu(t)$$

$$v_{n+1} = v_n + v_n \int_{-K}^{k} \frac{d\nu(t)}{(t-u_n)^2+v_n^2}$$

The first part of the proof of this result consists of deducing the asymptotic behaviour of u_n and v_n . For this, we assume that $\omega = a + iL$ where $a \in \mathbb{R}$. The recurrence relations (3.9) show us that

$$v_n(\omega) \geq L \quad \text{for every} \quad n \geq 1 .$$

and this enables us to deduce the boundless of $|u_n(\omega)|$ as follows :

Not-ing that :

$$\left| \int_{-K}^{K} \frac{t-u_n}{(t-u_n)^2+v_n^2} \, d\nu(t) \right| \leq \frac{\nu(\mathbb{R})}{2\, v_n} \leq \frac{1}{2}$$

we see that :

If $\quad u_n \geq K$ then $-K \leq K - \frac{1}{2} \leq u_{n+1} \leq u_n$.

If $\quad u_n \leq -K$ then $u_n \leq u_{n+1} \leq -K + \frac{1}{2} \leq K$

If $\quad u_n \leq K$ then

$$u_{n+1} \leq u_n + (K-u_n) \int_{-K}^{K} \frac{d\nu}{(t-u_n)^2+v_n^2} \leq u_n + \frac{(K-u_n)}{v_n^2} \nu(R) \leq K$$

If $u_n \geq -K$ then $u_{n+1} \geq -K$.

Hence $|u_n(a+iL)| \leq |a|\gamma K$ for $n \geq 1$

The recurrence relations (3.9) now imply that $v_n \to \infty$ as $n \to \infty$ and hence

$$v_{n+1}^2 - v_n^2 = 2 v_n^2 \int_{-K}^{K} \frac{d\nu(t)}{(t-u_n)^2+v_n^2} + v_n^2\left(\int_{-K}^{K} \frac{d\nu(t)}{(t-u_n)^2+v_n^2}\right)^2$$

$$\to 2\nu(\mathbb{R}) \quad \text{as} \quad n \to \infty$$

Hence $v_n(a+iL) \sim \sqrt{2\nu n}$ as $n \to \infty$.

Lemma 3.3 now shows us that for every $a \in \mathbb{R}$:

(3.10) $$||\hat{T}^n \phi_{a+iL} - \phi_{i\sqrt{2\nu n}}|| \to 0 \quad \text{as} \quad n \to \infty \quad .$$

We now obtain exactness by Lin's criterion by an argument similar to that of theorem 1.1. (The rational ergodicity of T has already been established, and its asymptotic type characterised, by theorem 3.2).

Let $u \in L^1$, $\int_R u d\lambda = 0$, and $\varepsilon > 0$:

By Wiener's Tauberian theorem, there are $\alpha_1 \dots \alpha_N$, $a_1 \dots a_N \in R$ such that

$$||u - \sum_{k=1}^{N} \alpha_k \phi_{a_k+iL}||_1 < \varepsilon/2$$

Whence :

$$||\hat{T}^n u||_1 \leq ||\hat{T}^n(u-\sum_{k=1}^{N}\alpha_k\phi_{a_k+iL})||_1 + ||\hat{T}^n \sum_{k=1}^{N}\alpha_k\phi_{a_k+iL} - \sum_{k=1}^{N}\alpha_k\phi_{i\sqrt{2\nu n}}||_1$$

$$+ ||\sum_{k=1}^{N}\alpha_k\phi_{i\sqrt{2\nu n}}||_1$$

$$||\hat{T}^n u||_1 \le ||u-\sum_{k=1}^{N}\alpha_k\phi_{a_k+iL}||_1 + \sum_{k=1}^{N}\alpha_k\ ||\hat{T}^n\ \phi_{a_k+iL} - \phi_{i\sqrt{2vn}}||_1 + |\sum_{k=1}^{N}\alpha_k|$$

$$< \varepsilon + o(1) \qquad \square$$

We note that the "generalised Boole transformation" (proven ergodic in [7]) falls within the scope of this last theorem.

If we added $\beta \neq 0$ to f in theorem 3.4, we would obtain that for $\mathrm{Im}\omega$ large enough $|u_n(\omega)| \ge c_1 n$ and $v_n(\omega) \le c_2 \log n$ (where $f^n(\omega) = u_n(\omega) + iv_n(\omega)$). The methods of lemma 3.1 would yield that $T(f)$ is dissipative.

The following corollary follows immediately from lemma 3.1 and theorem 3.2 .

Corollary 3.5.

Let $f \in I_0(\mathbb{R}^{2+})$ and let $T = T(f)$, $f^n(i) = iv_n(i)$. If $\alpha(f) = 1$ then :

T is conservative iff $\sum_{n=1}^{\infty} \frac{1}{v_n(i)} = \infty$

and in this case, T is rationally ergodic with

$$\mathfrak{a}(T) = \{\sum_{k=1}^{n} \frac{1}{\pi v_k(i)}\} \quad .$$

Moreover, in case $f \in I_0$ and $\alpha(f) = 1$: we have that $v_n \to \infty$ and so :

$$v_{n+1}^2 - v_n^2 = 2v_n^2 \int_{-\infty}^{\infty} \frac{1+t^2}{t^2+v_n^2}\, d\mu(t) + v_n^2 \left(\int_{-\infty}^{\infty} \frac{1+t^2}{t^2+v_n^2}\, d\mu(t)\right)^2$$

$$\to 2 \int_{-\infty}^{\infty} (1+t^2)\, d\mu(t) \le \infty$$

Hence :

$$\frac{v_n(i)}{\sqrt{n}} \to \sqrt{2\int_{-\infty}^{\infty}(1+t^2)\, d\mu(t)} \le \infty$$

which means :

(a) $T \times T \times T$ is dissipative

(b) $\frac{a_n(T)}{\sqrt{n}} \to c \in [0,\infty)$ as $n \to \infty$. (in case T is r.e.) .

These last two properties are held in common with the restrictions of theorem 3.4, and with the Markov shifts of random walks on $\mathbb{Z}$.

The following example does not fall within the scope of theorem 3.4, (though theorem 3.2 does apply).

<u>Example 3.6</u> $Tx = x + \alpha \tan x$ is exact, rationally ergodic with $a_n(T) \sim \frac{\operatorname{Log} n}{\alpha}$ for $\alpha > 0$.

<u>Proof</u>. Let $f(\omega) = \omega + \alpha \tan \omega$ and $f^n(\omega) = u_n(\omega) + iv_n(\omega)$

Then :

$$u_{n+1} = u_n + \frac{2\alpha \sin 2u_n e^{2v_n}}{e^{4v_n} - 2\cos 2u_n e^{2v_n} + 1}$$

and

$$v_{n+1} = v_n + \alpha \frac{e^{4v_n}}{e^{4v_n} - 2\cos 2u_n\, e^{v_n} + 1}$$

Whence : $v_{n+1} - v_n \geq \alpha \tanh v_n \geq \alpha \tanh v_0 > 0$

so $v_n \sim \alpha n$ as $n \to \infty$.

On the other hand :

$$|u_{n+1} - u_n| \leq \frac{2\alpha e^{2v_n}}{(e^{2v_n} - 1)^2} \leq 4\alpha e^{-2v_n} \leq 4\alpha e^{-\alpha n} \quad \text{for } n \text{ large} .$$

Hence $u_n \to u_\infty$, and the argument that T is exact now proceeds identically to the last argument of theorem 3.4.

□

The following lemma will give examples of $f \in I_0(R^{2+})$ with $\alpha(f) = 1$ and $T = T(f)$ dissipative, and also uncountably many dissimilar (see [1]) rationally ergodic restrictions $T(f)$ with $f \in I_0(\mathbb{R}^{2+})$, $\alpha(f) = 1$.

Lemma 3.7

Let $\mu \in S(\mathbb{R})$ be symetric with $c(x) = \mu(|t| \geq x) \sim \frac{1}{x^\alpha}$ where $0 < \alpha < 2$.

Let $f_\alpha(\omega) = \omega + \int_{-\infty}^{\infty} \frac{1+t^2}{t-\omega} d\mu(t)$ and $f^n(i) = iv_n$.

Then : $v_n \sim cn^{1/\alpha}$ where c depends only on α .

Proof. We have

$$v_{n+1} = v_n(1+F(v_n))$$

where $F(b) = \int_{-\infty}^{\infty} \frac{1+t^2}{t^2+b^2} d\mu(t)$.

It is not difficult to see that

$$F(b) = \frac{\mu(\mathbb{R})}{b^2} + 2(b^2-1) \int_0^\infty \frac{xc(x)}{(x^2+b^2)^2} dx$$

We first show that $F(b) \sim \frac{c_1}{b^\alpha}$ as $\alpha \to \infty$

Let $\varepsilon > 0$, and M be such that

$$\frac{1-\varepsilon}{x^\alpha} \leq c(x) \leq \frac{1+\varepsilon}{x^\alpha} \quad \forall x \geq M$$

Writing

$$L_M(b) = \int_M^\infty \frac{x^{1-\alpha}}{(x^2+b^2)^2} dx$$

we have that :

$$(1-\varepsilon)\ L_M(b) = \int_M^\infty \frac{xc(x)dx}{(x^2+b^2)^2} \leq (1+\varepsilon)\ L_M(b) .$$

Now $L_M(b) = \int_M^\infty \frac{x^{1-\alpha}}{(x^2+b^2)^2} dx = \frac{1}{b^{2+\alpha}} \int_{M/b}^\infty \frac{x^{1-\alpha}dx}{(x^2+1)^2} \sim \frac{c}{b^{2+\alpha}}$ as $b \to \infty$

where $c = \int_0^\infty \frac{x^{1-\alpha}dx}{(x^2+1)^2}$

Since $\varepsilon > 0$ was arbitrary and $\alpha < 2$, we have that

$$F(b) \sim \frac{c}{b^\alpha} \quad \text{as} \quad b \to \infty \quad .$$

Clearly, $v_n \to \infty$, hence :

$$\begin{aligned} v_{n+1}^\alpha - v_n^\alpha &= v_n^\alpha[(1+F(v_n))^\alpha - 1] \\ &\sim \alpha v_n^\alpha F(v_n) \quad \text{as} \quad n \to \infty \\ &\to \alpha c \quad \text{as} \quad n \to \infty \end{aligned}$$

Thus $v_n \sim (\alpha c n)^{1/\alpha}$ as $n \to \infty$ □

We now let $T_\alpha = T(f_\alpha)$.

By corollary 3.5 :

If $0 < \alpha < 1$ then T_α is dissipative .

If $1 \leq \alpha < 2$ then T_α is rationally ergodic and

$$\mathfrak{a}(T_\alpha) = \begin{cases} \{\log n\} & \text{if} \quad \alpha = 1 \\ \{n^{1-1/\alpha}\} & \text{if} \quad 1 < \alpha < 2 \end{cases} \quad .$$

If follows from theorem 2.4. of [1] that if $1 \leq \alpha_1 < \alpha_2 < 2$ then T_{α_1} and T_{α_2} are not factors of the same measure preserving transformation.

Theorem 3.8.

Let $f \in I(\mathbb{R}^{2+})$ and $T = T(f)$

Suppose $x_0 \in \mathbb{R}$ and f is analytic in a neighbourhood around x_0.

If $Tx_0 = x_0$, $T'(x_0) = 1$ and $T''(x_0) = 0$ then T preserves the measure ν_{x_0} where $d\nu_{x_0}(x) = \frac{dx}{(x-x_0)^2}$, and is exact, rationally ergodic with asymptotic type $\{\sqrt{n}\}$

Remarks : The conditions $Tx_0 = x_0$ and $T'(x_0) = 1$ correspond to : $\alpha(\emptyset_{x_0} f \emptyset_{x_0}^{-1}) = 1$. If, in this situation, $T''(x_0) \neq 0$: then T is dissipative. By possibly considering $g(\omega) = f(\omega+x_0)-x_0$, we may (and do) assume $x_0 = 0$.

Proof. Let $f(\omega) = \omega + \sum_{n=3}^{\infty} a_n\omega^n$ for $|\omega|$ small.

Then $\frac{1}{f(\omega)} - \frac{1}{\omega} = \frac{\omega-f(\omega)}{f(\omega)} = \frac{\omega}{f(\omega)} \sum_{n=3}^{\infty} a_n\omega^n$

$\to 0$ as $\omega \to 0$.

Hence $\frac{1}{f(\omega)} = \frac{1}{\omega} + \sum_{n=1}^{\infty} b_n\omega^n$ for $|\omega|$ small.

Let $\tilde{f}(\omega) = -1/f(-\frac{1}{\omega})$.

Then :

(3.11) $\tilde{f}(\omega) = \omega + \sum_{n=1}^{\infty} b_n\omega^{-n}$ for $|\omega|$ large, say $|\omega| \geq K$ and, since $\tilde{f} \in I(\mathbb{R}^{2+})$, $\alpha(\tilde{f}) = 1$:

(3.12.) $\tilde{f}(\omega) = \omega + \beta + \int_{-\infty}^{\infty} \frac{1+t\omega}{t-\omega} d\mu(t)$ where $\mu \in S(\mathbb{R})$, $\beta \in \mathbb{R}$

In order to prove the theorem by applying theorem 3.4, we will show that

(3.13.) $\tilde{f}(\omega) = \omega + \int_{-K}^{K} \frac{d\nu(t)}{t-\omega}$ where $\nu \in S(\mathbb{R})$.

Firstly, let $g(\omega) = \tilde{f}(\omega) - \omega$. By (3.11.) :

$$- ibg(ib) \to b_1 \quad \text{as} \quad b \to \infty$$

But by (3.12.) :

$$- ibg(ib) = -ib(\beta - b^2 \int_{-\infty}^{\infty} \frac{td\mu(t)}{t^2+b^2}) + ib \int_{-\infty}^{\infty} \frac{td\mu(t)}{t^2+b^2}$$

$$+ b^2 \int_{-\infty}^{\infty} \frac{1+t^2}{t^2+b^2} d\mu(t) .$$

Hence, we obtain, from the convergence of the real part, that

$$\int_{-\infty}^{\infty} (1+t^2)\, d\mu(t) < \infty$$

and from the convergnece of the imaginary part that :

$$b^2 \int_{-\infty}^{\infty} \frac{td\mu(t)}{t^2+b^2} \to \beta \quad \text{as} \quad b \to \infty .$$

which convergence, when combined with the previous one, gives

$$\int_{-\infty}^{\infty} td\mu(t) = \beta .$$

Now, let $d\nu(t) = (1+t^2)\, d\mu(t)$, then $\nu \in S(\mathbb{R})$ and it follows easily that

(3.14.) $\tilde{f}(\omega) = \omega + \int_{-\infty}^{\infty} \frac{d\nu(t)}{t-\omega}$

Now, let $h_b(a) = \text{Im } g(a+ib) = b\int_{-\infty}^{\infty} \frac{d\nu(t)}{(t+a)^2+b^2}$. By (3.11.) g is uniformly continuous on compact subsets of $[|\omega| \geq K]$, and so $h_b(a) \to 0$ as $b \to 0$ uniformly on compact subsets of $[|a| > K]$.

Let $dQ_b(x) = h_b(x)dx$, then $Q_b = P_{ib} * \nu$, and so $Q_b(A) \to \nu(A)$ for A a compact set. If A is a compact subset of $[|x| > K]$, then

$$\nu(A) = \lim_{b \downarrow 0} Q_b(A) = \lim_{b \downarrow 0} \int_A h_b(x)dx = 0 .$$

Thus ν is concentrated on $[-K,K]$ and (3.13.) is established.

□

The transformations $T_\alpha x = \alpha x + (1-\alpha) \tan x$ for $0 \leq \alpha < 1$ fall within the scope of theorem 3.9. (It was shown in [11] that T_0 is ergodic). It follows from asymptotic type considerations that the above transformations are dissimilar to $Tx = x + \alpha \tan x$.

References :

[1] J. Aaronson : Rational Ergodicity - Israel Journal of Mathematics <u>27</u> 2 p.93-123 (1977).

[2] R. Adler and B. Weiss : The ergodic, infinite measure preserving transformation of Boole - Israel Journal of Mathematics <u>16</u> 3 p.263-278 (1973).

[3] S. Foguel : The ergodic theory of Markov processes - New York, Van-Nostrand Reinhold (1969).

[4] S. Foguel and M. Lin : Some ratio limit theorems for Markov operators. Z. Wahrscheinlichkeitstheorie <u>231</u> p.55-66.

[5] J.H.B. Kemperman : The ergodic behaviour of a class of real transformations. Stochastic Processes and related topics. Proceedings of the summer research institute on statistical inference for stochastic processes (Editor Puri) Indiana University p.249-258 Academic Press (1975).

[6] G. Letac : Which functions preserve Cauchy laws. to appear in P.A.M.S.

[7] T. Li and F. Schweiger : The generalised Boole transformation is ergodic - preprint.

[8] M. Lin : Mixing for Markov operators - Z. Wahrscheinlichkeits--theorie 19 3 p.231-243 (1971).

[9] W. Rudin : Real and Complex analysis - Mc Graw Hill (1966) .

[10] F. Schweiger : Zahlentheoretische transformationen mit σ-endlichen mass. S-Ber. Öst. Akad. Wiss. Math. - naturw. K.l. Abt. II 185 p.95-103 (1976).

[11] F. Schweiger : tan x is ergodic - to appear in P.A.M.S.

[12] K. Yosida : Functional analysis - Springer, Berlin (1968).

[13] A. Denjoy : Fonctions contractent le cercle Z 1 ; C.R.Acad. Sci. Paris 182 (1926) pp 255-257

[14] M. Heins : On the pseudo periods of the weierstrass Zeta function : Nagaoya Math. Journal 30 (1967) pp113-119

[15] J.H. Neuwirth : Ergodicity of some mapping of the circle and the line : Preprint

[16] F. Schweiger and M. Thaler : Ergodische Eigenschaften einer Klass reellen Transformationen : preprint.

[17] M. Tsuji : Potential theory in modern function theory : Maruzen, Tokyo (1959).

[18] J. Wolff : Sur l'iteraction des fonctions holomorphes dans une region : C.R.Acad.Sci. Paris, 182 (1926) p.42-43.