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FINITE ELEMENT METHODS FOR MILDLY NONLINEAR ELLIPTIC EQUATIONS AND VARIATIONAL INEQUALITIES

by

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SUMMARY

The application of finite element methods to second order mildly nonlinear elliptic boundary value problems is described, and an inequality involving the error in the finite element solution is derived. This, together with well known results from piecewise polynomial interpolation, gives a theoretical bound on the finite element error. When the solution of the mildly nonlinear problem is subjected to an additional constraint condition, a mildly nonlinear variational inequality is produced. The finite element technique is applied to this and an inequality on the error, similar to that above, is derived.

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1. Introduction

This paper is concerned with the application of finite element methods to mildly nonlinear elliptic boundary value problems. Theoretical error bounds for the finite element solutions are derived. When the mildly nonlinear problem is subjected to an additional constraint condition, this leads to a variational inequality. A finite element approximation to the solution of this is defined, and an inequality bounding the finite element error is derived. This result is an extension to the mildly nonlinear case of the similar results of Falk [5] for linear problems.

We consider second order differential problems of the type

$$\begin{aligned} -\Delta[u(x)] &= f(x,u(x)) \quad , & x \in \Omega \; , \\ u(x) &= 0 \quad , & x \in \partial\Omega , \end{aligned} \qquad (1)$$

where $\Omega \subset \mathbb{R}^2$ is a simply connected open bounded domain with boundary $\partial\Omega$ and closure $\overline{\Omega} \equiv \Omega \cup \partial\Omega$, and f is a function both of x and the unknown u. It is assumed that f and the boundary $\partial\Omega$ satisfy smoothness conditions which ensure the existence and uniqueness of the solution u.

The choice of the Laplacian operator in (1) is made for simplicity, as our main interest lies in the nonlinear right hand side. A more general equation would contain a second order, linear, self-adjoint, coercive elliptic operator and it will be seen that the technique is suitable for this. The mildly nonlinear problems considered here have been treated in the more general setting of monotone operators by Ciarlet, Schultz and Varga [3] and Varga [22] , where they appear as examples. However, it seems useful to exploit the special features of the less general problems in the mildly nonlinear case.

Our approach to the differential problems is to consider in Sections 2 and 3 generalized formulations of these problems in the setting of larger spaces of functions than those containing the classical solutions. We thus start in a Hilbert space context with both an "energy" functional associated with the differential problem and the equivalent weak formulation. These are first formulated in a general manner, seemingly independent of the differential problem, in terms of a bilinear form, and are later specialized. The notation is now introduced.

Let H be a real Hilbert space with dual H' and denote the norms on these respectively by $\|\cdot\|$ and $\|\cdot\|'$. Further denote by $(\cdot, \cdot)$ the inner product on H and by $\langle \cdot,\cdot \rangle$ the pairing between H' and H. A bilinear form $a(u,v)$, defined on H, is

<u>coercive</u> on H if there exists a constant $\rho > 0$ such that

$$a(v,v) \geq \rho\|v\|^2 , \qquad \text{for all } v \in H , \tag{2}$$

<u>continuous</u> on H if there exists a constant $\mu > 0$ such that

$$|a(u,v)| \leq \mu\|u\| \; \|v\| , \text{ for all } u,v \in H . \tag{3}$$

An operator $T : H \to H'$ is <u>antimonotone</u> on H if

$$\langle Tu - Tv, u - v\rangle \leq 0, \text{ for all } u,v \in H , \tag{4}$$

and is <u>Lipschitz continuous</u> on H if there exists a constant $\gamma > 0$ such that

$$\|Tu - Tv\| \leq \gamma\| u - v\|, \text{ for all } u,v \in H . \tag{5}$$

2. Finite Element Methods for Mildly Nonlinear Problems

Starting with a variational problem we state the following theorem, the proof of which is given in Noor and Whiteman [15].

<u>Theorem 1</u> Let $a(u,v)$ be a continuous, coercive, symmetric bilinear form on H. If the Fréchet derivative $F'(v)$ of a nonlinear functional $F(v)$, defined on H, exists and is antimonotone, then the function u which minimizes

$$I[v] = a(v,v) - 2F(v), \quad \text{for all } v \in H, \tag{6}$$

is the function $u \in H$ such that

$$a(u,v) = \langle F'(u),v \rangle \quad , \quad \text{for all } v \in H. \tag{7}$$

The converse is also true.

Note that in theorem 1 the functional F is nonlinear. When F is a linear functional, the result of theorem 1 is exactly that of Temam [21], Proposition 2, p. 9.

The finite element method can be applied to approximate the u of theorem 1. In order to do this, we choose a finite dimensional space $S^h \subset H$ and seek $u_h \in S^h$ which approximates $u \in H$. The approximate forms of (6) and (7) are then respectively

$$\text{find } u_h \in S^h \text{ solving } \min_{v_h \in S^h} (I[v_h]) \tag{8}$$

and

find $u_h \in S^h$ such that

$$a(u_h,v_h) = \langle F'(u_h),v_h \rangle \quad \text{for all } v_h \in S^h. \tag{9}$$

Problem (8) is solved using the Ritz technique and (9) using the Galerkin technique; see e.g. Whiteman [23].

Our purpose is to bound the error $\|u - u_h\|$. In the linear case the procedure is to restrict v, in the equivalent form of (6), to be an element of S^h and to subtract the equivalent form of (9) from the resticted (6), thus obtaining an orthogonality relation. For (6) and (9) as they stand here this can no longer be done because of the nonlinearity of F. We therefore, as in [4], define a <u>norm projection</u> $\bar{u}_h \in S^h$ of $u \in H$ by the orthogonality relation

$$a(u - \bar{u}_h, w_h) = 0 \qquad \text{for all } w_h \in S^h. \qquad (10)$$

A second theorem from [15] is now stated without proof.

<u>Theorem 2</u> If the hypothesis of theorem 1 is satisfied and $F'(v)$ is Lipschitz continuous on H, so that there exists a constant $\gamma > 0$ such that

$$\|F'(v) - F'(w)\| \leqq \gamma\|v-w\| \qquad \text{for all } v,w \in H,$$

then

$$\|\bar{u}_h - u_h\| \leqq \frac{\gamma}{\rho}\|u - \bar{u}_h\|, \qquad (11)$$

where ρ is the constant in (2).

Application of the triangle inequality and (11) leads immediately to

$$\|u - u_h\| \leqq (1 + \frac{\gamma}{\rho})\|u - \bar{u}_h\|. \qquad (12)$$

When use is made of the coercivity and continuity of the bilinear form $a(u,v)$ and of the orthogonality relation (10), we find that

$$\|u - u_h\| \leqq \frac{\mu}{\rho}(1 + \frac{\gamma}{\rho})\|u - w_h\| \text{ for all } w_h \in S^h. \qquad (13)$$

In particular (13) holds when w_h is taken as $\tilde{u}_h \in S^h$, where $\tilde{u}_h$ is an interpolant to u.

The effect of the nonlinearity in F has been to introduce into the right hand side of (13) the extra factor $(1 + \frac{\gamma}{\rho})$ over that found in the similar inequality of the form

$$\|u - u_h\| \leqq \frac{\mu}{\rho} \|u - w_h\| \tag{14}$$

relevant to the case of linear F.

The above analysis is now applied to the mildly nonlinear differential problem (1). We take H as the Sobolev space $\overset{\circ}{W}{}_2^1(\Omega)$ and assume that $f(x,u(x)) \in C(\bar{\Omega})$ is Lipschitz continuous and antimonotone on Ω. The weak formulation of (1), derived by multiplying by a test function $v \in \overset{\circ}{W}{}_2^1(\Omega)$ and integrating, is that of finding $u \in \overset{\circ}{W}{}_2^1(\Omega)$ such that

$$\begin{aligned} a(u,v) &= \int_\Omega f(x,u(x))v(x)dx \\ &\equiv \langle F'(u),v \rangle \quad \text{for all} \quad v \in \overset{\circ}{W}{}_2^1(\Omega), \end{aligned} \tag{15}$$

which has the form of (7), where $F'(w)$ is the Fréchet derivative of the nonlinear functional $F(w)$ defined on $\overset{\circ}{W}{}_2^1(\Omega)$ by

$$F(w) = \int_\Omega \{ \int_0^w f(x,\xi)d\xi \}dx ,$$

and

$$a(u,v) = \int_\Omega \nabla u \, \nabla v \, dx , \quad u,v \in \overset{\circ}{W}{}_2^1(\Omega) .$$

It is shown in [15] that the right hand side of (15) is a well defined pairing and that $F'(u)$ is antimonotone and Lipschitz continuous. Thus the inequality (13) is applicable to problem (1), where of course $u \in \overset{\circ}{W}{}_2^1(\Omega)$ is the weak solution. The constants ρ and μ of (2) and (3) respectively may be taken equal to unity so that we have the inequality

$$\|u - u_h\|_{\overset{\circ}{W}{}_2^1(\Omega)} \leqq (1 + \gamma) \|u - \tilde{u}_h\|_{\overset{\circ}{W}{}_2^1(\Omega)}, \tag{16}$$

where $\tilde{u}_h \varepsilon S^h$ is an interpolant to u and γ is the Lipschitz constant associated with F.

Many bounds of the form $O(h^\beta)$ have been derived for piecewise polynomial interpolants to functions defined on partitions of triangular and rectangular elements, each having generic length h; see e.g. Ciarlet and Raviart [2] and the references contained therein. These in turn can be used to bound (16) and so provide an $O(h^\beta)$ bound on the finite element error to problem (1).

An example of a problem of type (1) occurs in reactor physics where the function $f(x,u) = e^{-u(x)}$ and the solution u is subjected to a boundeness condition in order that the Lipschitz condition may satisfied. In this case u is the fuel distribution of a homogenised reactor.

The above analysis and error bounds have all been in the W_2^1 - norm. It would clearly be desirable to have bounds in the L_∞ sense. In this respect we mention here the results for linear problems of Natterer [10], Nitsche [12]-[14] and Scott [17].

The finite dimensional spaces S^h of the finite element method are constructed using piecewise polynomial basis functions having local support. The application to (9) leads to a system of nonlinear equations of the form

$$A\underline{u}_h + \alpha(\underline{u}_h) = 0 , \qquad (17)$$

where A is the global stiffness matrix, $\underline{u}_h$ is the vector of values of the unknown u_h (together possibly with values of some derivatives of u_h) at the nodal points and α is a nonlinear function of $\underline{u}_h$. The matrix A is exactly the matrix of coefficients in the linear system derived from a problem of type (1) with linear right hand side. This special form of the nonlinear equations (17) can be exploited and facilitates their solution.

3. Mildly Nonlinear Variational Inequalities

An inequality of the type (13) which is applicable to a class of constrained mildly nonlinear problems is now derived. Such problems arise when the solution u of (1) is required to satisfy an additional constraint condition $u \geq \psi$, where ψ is a given function. The approach to the finite element method is again to consider equivalent (constrained) variational and weak problems and then to associate these with the differential problem. In this case the weak formulation is a variational inequality.

The notation remains as in the previous sections, so that H is a Hilbert space with dual H'. Let $K \subset H$ be a convex subset defined by

$$K \equiv \{ v : v \in H ; v \geq \psi \} .$$

In a manner similar to Section 2 we consider a functional as in (6) but, because of the constraint on u, the functional must be minimized over K rather than over the whole of H. Such problems have been studied in an abstract setting by Lions and Stampacchia [7], Sibony [18] and Stampacchia [19]. More practical situations are considered by Fremond [6] and Strang [20].

We consider first constrained linear problems, where the function f does not involve u, and state a well known result as a theorem.

Theorem 3 Let $a(v,w)$ be a symmetric, coercive, continuous bilinear form on H, and K be a closed convex subset of H. If $f(x) \in H'$, then the function $u \in K$ minimizes the functional

$$I[v] = a(v,v) - 2 < f,v > \tag{18}$$

if and only if $u \in K$ satisfies

$$a(u,v-u) \geq < f,v-u >, \quad \text{for all} \quad v \in K . \tag{19}$$

The expression (19) is the variational inequality and theorem 3 demonstrates its equivalence to the constrained minimization problem in the linear case. It has been shown by Lions and Stampacchia [7] that there exists a unique solution $u \in K$ of (19).

We are interested here in the use of the finite element method for the approximation of the solution u over a finite dimensional convex set $K^h \subset H$, and of bounding the resulting error. The normal approach is to obtain a numerical solution to (18) using mathematical programming and to use (19) to derive error bounds. We proceed with the error bounds.

The finite dimensional form of (19) is set up by choosing a finite dimensional subspace $S^h \subset H$, and then taking a closed convex subset $K^h \subset S^h$. The problem is thus that of finding $u^h \in K^h$ such that

$$a(u_h, v_h - u_h) \geq < f,v_h - u_h >, \text{ for all } v_h \in K^h. \tag{20}$$

We remark that it is not required that $K^h \subset K$; unlike the situations considered by Natterer [11] and Nitsche [14] where $K^h = K \cap S^h$.

Bounds for the error $\|u - u_h\|$ for the linear case have recently been derived by Falk [5] and Mosco and Strang [9].

If $a(u,v)$ is as in theorem 3, and the mapping $L : H \to H'$ is defined for $u \in H$ by $a(u,v) \equiv \langle Lu,v\rangle$, for all $v \in H$, then for u the solution of (19) and u_h the solution of (20) Falk derives an inequality of the form

$$\|u - u_h\| \leqq \{\frac{\mu^2}{\rho^2}\|u - v_h\|^2 + \frac{2}{\rho}\|f - Lu\|'(\|u - v_h\| + \|u_h - v\|)\}^{1/2} ,$$

$$\text{for all } v \in K \text{ and } v_h \in K^h , \quad (21)$$

where ρ and μ are respectively the constants of (2) and (3) and $\|\cdot\|'$ is again the norm on the dual space H'. On account of the obstacle condition Lu is unequal to f in part of H'. Removal of the obstacle condition renders $Lu = f$ throughout H' so that the second term on the right and side of (21) drops out, K^h becomes S^h, and we again have inequality (14). It has been shown for a Poisson problem in [5] and [9] that, under certain conditions of smoothness on $\partial\Omega$ and with particular choices of elements and trial functions in the finite element method, the right hand side of (21) can be bounded in an $O(h)$ manner. These proofs make use of a regularity condition on the solution u of the linear problem due to Brezis and Stampacchia [1].

It is natural to try to produce an inequality similar to (21) for variational inequalities of the form (19) in which the right hand side contains a nonlinear function $f(x,u(x))$, and then to try to derive $O(h^\beta)$ error bounds for these. We thus consider the variational inequality, corresponding to (19), in which $u \in K$ satisfies

$$a(u,v-u) \geqq \langle F'(u), v - u\rangle , \quad \text{for all } v \in K , \quad (22)$$

where $F'(u)$ is the Frêchet derivative of the nonlinear functional $F(v)$ as in (7). The finite dimensional problem approximating (22) is that of finding $u_h \in K$ such that

$$a(u_h, v_h - u_h) \geqq \langle F'(u_h), v_h - u_h\rangle , \text{ for all } v_h \in K^h. (23)$$

For u and u_h as in (22) and (23) respectively Noor and Whiteman [16] have proved the following theorem:

<u>Theorem 4</u> Let $u \in K$ and $u_h \in K^h$ be respectively the solution of (22) and (23). The mapping $L : H \to H'$ is defined for $u \in H$ by

$$a(u,v) = \langle Lu,v\rangle \quad , \qquad \text{for all} \quad v \in H \ ,$$

where $a(u,v)$ is a coercive continuous symmetric bilinear form on H. If the Fréchet derivative F' is Lipschitz continuous on H, then

$$\|u - u_h\| \leqq \{\frac{3\mu^2}{\rho^2}\|u - v_h\|^2 + \frac{3\mu^2}{\rho^2}\|v - u_h\|^2 + \frac{3\gamma^2}{\rho^2}\|v_h - v\|^2$$

$$+ \frac{2}{\rho}\|F'(u)-Lu\|'\,\|u - v_h\| + \frac{2}{\rho}\|F'(u_h)-Lu_h\|'\,\|u_h - v\|\}^{1/2}, \quad (24)$$

for all $v \in K$ and $v_h \in K^h$, where γ is the Lipschitz constant for F'.

Clearly, for the problem (1) subject to the constraint condition $u \geqq \psi$ and with $F'(u)$ defined as in (15), theorem 4 can be applied so that the inequality (24) bounds the finite element error. However, unlike the linear problem considered earlier, we have been unable to obtain an $O(h^\beta)$ bound for the right hand side of (24). This is because we know of no regularity condition on u similar to that of [1] for the nonlinear case.

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