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by

John M. Franks*

In [1] Artin and Mazur introduced the zeta function for a diffeomorphism in analogy with the Weil zeta function in algebraic geometry. It is defined to be $\zeta(t) = \exp(\sum_{m=1}^{\infty} \frac{1}{m} N_m t^m)$ where N_m is the cardinality of the fixed point set of f^m. This has proven to be an important invariant for the study of the orbit structure of a large class of diffeomorphisms of compact manifolds - those which satisfy Axiom A and the no-cycle property. This class is defined and described in §2 below, but we mention that it is open in the C^1 topology, contains a representative of every isotopy class [13], and is, in fact, dense in the C^0 topology [9].

For these diffeomorphisms the zeta function is rational and in fact the quotient of integer polynomials with constant term 1. (see [5], [8] or (3.4) below). Thus for these diffeomorphisms a finite amount of data determines all of the numbers N_m.

In this article we consider a weakening of this invariant which still contains considerable information and has the advantage of being closely related to homological invariants of f.

In the following we assume that Λ_1 is a basic of f (see

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§2 for definition)

Definition: The reduced zeta function Z_i of f on Λ_i, is defined to be the rational function $\zeta(f|_{\Lambda_i})$ with all its coefficients reduced mod 2.

One of our main results ((5.8) in the text), relates the reduced zeta functions to homological invariants of f.

Theorem. Suppose $f\colon M \to M$ satisfies Axiom A and the no-cycle property, and has basic sets $\Lambda_1,\dots,\Lambda_\ell$, then the following are equal:

a) $\prod_{i=1}^{\ell} Z_i^{(-1)^{u_i}}$ where u_i = fiber dim $E^u(\Lambda_i)$

b) The reduction mod 2 of

$$\eta(f;\ R) = \prod_{k=0}^{n} \det(I - f_{*k}t)^{(-1)^{k+1}} \text{ where}$$

$f_{*k}\colon H_k(M;\ R) \circlearrowleft$ is induced by f.

c) $\eta(f;\ Z_2) = \prod_{k=0}^{n} \det(I - f_{*k}t)^{(-1)^{k+1}}$ where

$f_{*k}\colon H_k(M;\ Z_2) \circlearrowleft$ is induced by f.

The function $\eta(f;\ R)$ in b) (before reduction) is sometimes called the false or homology zeta function since it can be obtained by replacing N_m in the definition of ζ by $L(f^m)$, the Lefschetz number of f^m.

This theorem has immediate applications to what one might call the global bifurcation problem: namely how can basic sets be changed as f is isotoped to a new Axiom A, no-cycle diffeomorphism. For example, it's clear that if two basic sets, Λ_i and Λ_j can be cancelled leaving other basic sets unchanged then $Z_i^{(-1)^{u_i}} \cdot Z_j^{(-1)^{u_j}} = 1$. Also if a basic set can be removed by an isotopy which does not alter other basic sets or introduce new ones then it must have reduced zeta function 1. Examples of these phenomena are given in §6. This general problem can be viewed as an extension of the problem of simplifying a Morse function by cancelling critical points.

The reduced zeta functions also give a necessary condition for a collection of abstract basic sets to be the basic sets of a diffeomorphism (in any homotopy class) on a manifold M. This is (5.10) in the text.

Proposition. *If* $f: M \to M$ *satisfies Axiom A and the no-cycle property, then* $\Sigma(-1)^{u_i}\deg Z_i = -\chi(M)$ *where* $\chi(M)$ *is the Euler characteristic of* M.

Here degree means the degree of the numerator minus the degree of the denominator.

It is important to emphasize that Z_i contains more information than just the parity of set of fixed points of f^m on Λ_i. For example if we consider a full two-shift (see §6 for definition) and a single orbit of period 2, then both these examples satisfy $N_m \equiv 0 \bmod 2$ for all m but their zeta functions are $1/1-2t$ and $1/1-t^2$ respectively (see (2.6) and (6.1)), so their reduced zeta functions are different. Also note in the proposition above that the Z_i and the numbers u_i determine the Euler characteristic (not just the Euler characteristic mod 2).

The theorem above (5.8) is in fact a special case of more general results (5.4-7) which are generalizations of the Morse inequalities relating the Betti numbers of a manifold M to the number of critical points of a Morse function on M. In fact our (5.7) and (5.8) are simply the mod 2 analogues of Theorems 1 and 2 of [4]. The reduction mod 2 gives somewhat weaker results, but is applicable to a much larger class of diffeomorphisms, as no assumptions about orientability need be made.

The heart of the proof of all these results is a local version of the main theorem above which has the same hypothesis and is (4.1) in the text. Here Λ_i will denote a basic set of f and M_i, M_{i-1} the elements of a filtration for f (see §3) such that $\Lambda_i \subset M_i - M_{i-1}$.

Theorem: *Suppose Λ_i is a basic set of f and $u =$ fiber dim $E^u(\Lambda_i)$, then the following are equal:*

a) $z_i^{(-1)^u}$

b) *The mod 2 reduction of*

$$\eta_i(f;\, R) = \prod_{j=0}^{n} \det(I - f_{*j}t)^{(-1)^{j+1}}$$ *where*

$f_{*j}\colon H_j(M_i, M_{i-1};\, R) \circlearrowleft$ *is induced by* f.

c) $$\eta_i(f;\, Z_2) = \prod_{j=0}^{n} \det(I - f_{*j}t)^{(-1)^{j+1}}$$ *where*

$f_{*j}\colon H_j(M_i, M_{i-1};\, Z_2) \circlearrowleft$ *is induced by* f.

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§1. Preliminaries

(1.1) *Definition:* *If* $V = \bigoplus_{i=0}^{n} V_i$ *is a graded vector space over any field and* $\tau: V \to V$ *a gradation preserving linear map* $(\tau_i: V_i \to V_i)$ then we define $\tilde{\zeta}(\tau) = \prod_{i=0}^{n} \det(I - \tau_i t)^{(-1)^{i+1}}$.

$\tilde{\zeta}(\tau)$ is a rational function of t and in fact a quotient of polynomials $p(t)/q(t)$ with both p and q having coefficients in the field of V and constant term 1.

The following is a slight generalization of the classical Lefschetz-Hopf trace formula.

(1.2) *Lemma.* *If* $C = \bigoplus_{i=0}^{n} C_i$ *and* $\partial_i: C_i \to C_{i-1}$ *is a chain complex of finite dimensional vector spaces and* $\tau: C \to C$ *is a chain map then* $\tilde{\zeta}(\tau) = \tilde{\zeta}(\tau_*)$ *where* $\tau_*: H_*(C) \to H_*(C)$ *is the map on homology induced by* τ.

Proof: Suppose we have a commutative diagram of vector spaces

$$\begin{tikzcd}
0 \arrow[r] & V_1 \arrow[r, "i"] \arrow[d, "\alpha"] & V_2 \arrow[r, "j"] \arrow[d, "\beta"] & V_3 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & V_1 \arrow[r, "i"] & V_2 \arrow[r, "j"] & V_3 \arrow[r] & 0
\end{tikzcd}$$

where the horizontal rows are exact. Then choosing a basis for V_2 which begins with a basis of $i(V_1)$ we can represent β by a matrix of the form

$$A = \begin{pmatrix} A_1 & X \\ 0 & A_2 \end{pmatrix}$$

where A_1 is a matrix which represents α (with an appropriate basis) and A_2 is a matrix which represents γ. From this it follows that

$$\begin{aligned} \det(I - \beta t) &= \det(I - At) \\ &= \det(I - A_1 t)\det(I - A_2 t) \\ &= \det(I - \alpha t)\det(I - \gamma t) \end{aligned}$$

For notational simplicity we will henceforth denote $\det(I - \tau t)$ by $P(\tau)$ for any vector space endomorphism τ. Thus we have shown for the endomorphisms α, β, and γ in the diagram above $P(\beta) = P(\alpha)P(\gamma)$. We apply this result to two short exact sequences from the chain complex C. Let $\mathscr{Z}_i = \ker(\partial_i)$ and $B_i = \operatorname{im}(\partial_{i+1})$ then

$$0 \to \mathscr{Z}_i \to C_i \xrightarrow{\partial_i} B_{i-1} \to 0$$

and $0 \to B_i \to \mathscr{Z}_i \to H_i(C) \to 0$ are exact and on each of the vector spaces in these sequences there is an endomorphism induced by the chain map τ.

Applying the result above to these two cases we obtain

$$P(\tau|C_i) = P(\tau|\mathscr{Z}_i)P(\tau|B_{i-1})$$

and $P(\tau|\mathscr{Z}_i) = P(\tau|B_i)P(\tau_*|H_i(C))$. Thus

$$P(\tau|C_i) = P(\tau_*|H_i(C))P(\tau|B_i)P(\tau|B_{i-1}).$$

Hence

$$\tilde{\zeta}(\tau) = \prod_{i=0}^{n} P(\tau|C_i)^{(-1)^{i+1}}$$

$$= \prod_{i=0}^{n} (P(\tau_*|H_i(C)))^{(-1)^{i+1}} \prod_{i=0}^{n} (P(\tau|B_i)P(\tau|B_{i-1}))^{(-1)^{i+1}},$$

but since everything cancels in the product of the $P(\tau|B_i)$, we have

$$\tilde{\zeta}(\tau) = \prod_{i=0}^{n} P(\tau_*|H_i(C))^{(-1)^{i+1}} = \tilde{\zeta}(\tau_*).$$

q.e.d.

An important special case of this result which will be used subsequently is the following.

(1.3) Corollary. *If* $0 \to V_n \xrightarrow{\partial_n} V_{n-1} \to \cdots \to V_1 \xrightarrow{\partial_1} \to 0$ *is an exact sequence of vector spaces and* $\tau_i \colon V_i \to V_i$ *are endomorphisms such that* $\partial_i \circ \tau_i = \tau_{i-1} \circ \partial_i$ *then* $\prod_{i=0}^{n} \det(I - \tau_i t)^{(-1)^{i+1}} = 1$.

Proof: $C = \bigoplus_{i=0}^{n} V_i$ together with $\{\partial_i\}$ is a chain complex with $H_*(C) = 0$ and $\tau = \oplus\tau_i$ is a chain map. Thus
$\prod_{i=0}^{n} \det(I - \tau_i t)^{(-1)^{i+1}} = \tilde{\zeta}(\tau) = \tilde{\zeta}(\tau_*) = 1$ since $H_*(C) = 0$. q.e.d.

(1.4) Lemma: Suppose τ: $C \to C$ is a chain map on a finitely generated free chain complex, and $\alpha = \tau \otimes \mathrm{id}$: $C \otimes R \circlearrowleft$ and $\beta = \tau \otimes \mathrm{id}$: $C \otimes Z_2 \circlearrowleft$, then $\tilde{\zeta}(\alpha)$ with all coefficients reduced mod 2 is equal to $\tilde{\zeta}(\beta)$.

Proof: Since C is a free chain complex we can pick a basis so that the matrix of $\alpha_i = \tau_i \otimes \mathrm{id}$: $C_i \otimes R \to C_i \otimes R$ is the same integer matrix representing τ_i: $C_i \to C_i$ and such that the matrix of β_i: $C_i \otimes Z_2 \to C_i \otimes Z_2$ is the matrix of τ_i reduced mod 2. Hence $\det(I - \alpha_i)$ reduced mod 2 is equal to $\det(I - \beta_i t)$. Now,

$$\tilde{\zeta}(\alpha) = \prod_{i=0}^{n} \det(I - \alpha_i z)^{(-1)^{i+1}} \quad \text{and} \quad \tilde{\zeta}(\beta) = \prod_{i=0}^{n} \det(I - \beta_i t)^{(-1)^{i+1}}$$

where $n = \dim C$. Thus $\tilde{\zeta}(\alpha)$ with all coefficients reduced mod 2 is equal to $\tilde{\zeta}(\beta)$. q.e.d.

We wish to consider $\tilde{\zeta}(f_*)$ where f_* is the map on the homology of a space induced by a continuous map of the space. Since it will be necessary to consider different fields of coefficients we will use the following notation.

(1.5) Definition: If f: $(X,A) \to (X,A)$ is a map of a topological pair to itself and F is a field then $\eta(f; F) = \tilde{\zeta}(f_*)$ where f_*: $H_*(X,A; F) \to H_*(X,A; F)$ is the map induced by f on the homology of (X,A) with coefficients in F.

(1.6) Corollary. If f: $(X,A) \to (X,A)$ is a continuous map of a finite simplicial pair then $\eta(f; R)$ with all coefficients reduced mod 2 is equal to $\eta(f; Z_2)$.

Proof: Let C be the free oriented simplicial chain complex of (X,A) and let $\tau: C \to C$ be a chain map arising from a simplicial approximation to f. Then $H_*(X,A;\ R)$ is the homology of the complex $C \otimes R$ and $H_*(X,A;\ Z_2)$ is the homology of $C \otimes Z_2$. The maps induced by f are induced by the chain maps

$$\alpha = \tau \otimes \mathrm{id}: C \otimes R \to C \otimes R$$

and

$$\beta = \tau \otimes \mathrm{id}: C \otimes Z_2 \to C \otimes Z_2.$$

Hence by Lemmas (1.2) and (1.4), $\eta(f;\ R)$ with all coefficients reduced mod 2 is equal to $\eta(f;\ Z_2)$. q.e.d.

For later use we cite one other well known fact and give its proof since it is quite short.

(1.7) Proposition. Suppose A is an $n \times n$ real matrix then

$$\exp\left(\sum_{m=1}^{\infty} \mathrm{tr}\, A^m t^m\right) = \frac{1}{\det(I - At)}.$$

Proof: $\sum_{m=1}^{\infty} \frac{1}{m} A^m t^m$ is the formal power series for $-\log(I - At)$, (the series will of course converge for t near 0). It is also a well known fact that for any matrix B, $\exp(\mathrm{tr}\, B) = \det \exp(B)$. Hence $\exp(\Sigma \frac{1}{m}\, \mathrm{tr}\, A^m t^m) = \exp(\mathrm{tr}\Sigma \frac{1}{m} A^m t^m) = \exp[\mathrm{tr}(-\log(I - At))] = \det[\exp(-\log(I - At))] = \frac{1}{\det(I - At)}$. q.e.d.

§2. Axiom A Diffeomorphisms with the No-Cycle Property

We wish to study the structure of diffeomorphisms which satisfy Axiom A of Smale [13] and the no-cycle property, so we now briefly describe this class of diffeomorphisms.

Let $f: M \to M$ be a C^1 diffeomorphism of a compact connected manifold M. A closed f-invariant set $\Lambda \subset M$ is called hyperbolic if the tangent bundle of M restricted to Λ is the Whitney sum of two Df-invariant bundles, $T_\Lambda M = E^u(\Lambda) \oplus E^s(\Lambda)$, and if there are constants $C > 0$ and $0 < \lambda < 1$ such that

$$|Df^n(v)| \leq C\lambda^n |v| \quad \text{for} \quad v \in E^s,\ n > 0$$

and

$$|Df^{-n}(v)| \leq C\lambda^n |v| \quad \text{for} \quad v \in E^u,\ n > 0.$$

The diffeomorphism f is said to satisfy Axiom A if a) the non-wandering set of f, $\Omega(f) = \{x \in M: U \cap \bigcup_{n>0} f^n(U) \neq \emptyset$ for every neighborhood U of $x\}$ is a hyperbolic set, and b) $\Omega(f)$ equals the closure of the set of periodic points of f. If f satisfies Axiom A, one has the spectral decomposition theorem of Smale [11] which says $\Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_\ell$ where Λ_i are pairwise disjoint, f-invariant closed sets and $f|_{\Lambda_i}$ is topologically transitive.

These Λ_i are called the basic sets of f. We consider diffeomorphisms which in addition to Axiom A satisfy the no-cycle property [12] which we now define. If Λ_i is a basic set of f then its stable and unstable manifolds ([6] or [9]) are defined by

$$W^s(\Lambda_i) = \{x \in M \mid d(f^n(x), \Lambda_i) \to 0 \text{ as } n \to \infty\}$$

and

$$W^u(\Lambda_i) = \{x \in M \mid d(f^{-n}(x), \Lambda_i) \to 0 \text{ as } n \to \infty\}.$$

One says $\Lambda_i \leq \Lambda_j$ if $W^u(\Lambda_j) \cap W^s(\Lambda_i) \neq \emptyset$. If this extends to a total ordering on the basic sets Λ_i, then f is said to satisfy the <u>no-cycle property</u> and we re-index so that $\Lambda_i \leq \Lambda_j$ when $i \leq j$.

If Λ_i is a basic set of $f\colon M \to M$ then we define the <u>index</u> u_i of Λ_i with respect to f to be the fiber dimension of $E^u(\Lambda_i)$.

We review briefly the filtrations of [12] associated with a diffeomorphism which satisfies Axiom A and the no-cycle property. In fact the purpose of imposing the no-cycle condition is to obtain this filtration. It is possible to find submanifolds (with boundary and of the same dimension as M),

$$M = M_\ell \supset \cdots \supset M_1 \supset M_0 = \emptyset \text{ such that}$$

$$M_{i-1} \cup f(M_i) \subset \operatorname{int} M_i,$$

$$\Lambda_i = \bigcap_{m \in Z} f^m(M_i - M_{i-1}), \text{ and}$$

$$W^u(\Lambda_i) \cup M_{i-1} = M_{i-1} \cup \bigcap_{m \geq 0} f^m(M_i).$$

Henceforth $f\colon M \to M$ will be a diffeomorphism of a compact manifold satisfying Axiom A and the no-cycle property and $M = M_\ell \supset M_{\ell-1} \supset \cdots \supset M_0 = \emptyset$ will be a filtration for f.

(2.1) <u>Definition:</u> <u>If</u> $\Lambda_i \subset M_i - M_{i-1}$ <u>is a basic set of</u> f <u>then we</u>

define $\eta_i(f) = \prod_{j=0}^{n} \det(I - f_{*j}t)^{(-1)^{j+1}}$ where $f_{*j}\colon H_j(M_i, M_{i-1}; R) \to H_j(M_i, M_{i-1}; R)$ is the map induced by f. Alternatively $\eta_i(f) = \tilde{\zeta}(f_*)$ where $f_* = \oplus f_{*j}$.

The function η_i is sometimes called the homology or false zeta function of f on Λ_i because (as the following proposition shows) it can be obtained by taking the definition of the zeta function and replacing the number of fixed points of f^m by the number of fixed points seen by homology, i.e., the Lefschetz number of f^m.

(2.2) Proposition: $\eta_i(f) = \exp \sum_{m=1}^{\infty} \frac{1}{m} \tilde{N}_m t^m$ where $\tilde{N}_m = L(f^m; M_i, M_{i-1}) = \Sigma(-1)^j \operatorname{tr} f^m_{*j}$ and $f_{*j}\colon H_j(M_i, M_{i-1}; R) \circlearrowleft$ is the map induced by f.

Proof: We compute

$$\begin{aligned}
\exp\Big(\sum_{m=1}^{\infty} \frac{1}{m} \tilde{N}_m t^m\Big) &= \exp\Big(\sum_{m=1}^{\infty} \frac{1}{m} \big(\Sigma(-1)^j \operatorname{tr} f^m_{*j}\big) t^m\Big) \\
&= \prod_{j=0}^{n} \exp\Big[(-1)^j \sum_{m=1}^{\infty} \frac{1}{m} \operatorname{tr} f^m_{*j} t^m\Big] \\
&= \prod_{j=0}^{n} \det(I - f_{*j}t)^{(-1)^{j+1}} \quad \text{by proposition (1.6} \\
&= \eta_i(f). \qquad\qquad \text{q.e.d.}
\end{aligned}$$

We remark that $\tilde{N}_m$ is equal to $\Sigma\, I(p; f^m)$ where the sum is taken over all fixed points of f^m in Λ_i and $I(p; f^m)$ is the Lefschetz index of the fixed point p under f^m (see [3] and Lemma 3 of [4] for this). This shows than $\tilde{N}_m$ is independent of the choice of filtration elements M_i and M_{i-1} so we have the following.

(2.3) Corollary: $\eta_1(f)$ is independent of the choice of filtration for f.

It should be noted however that η_1 is not an invariant of Λ_1 and f restricted to Λ_1, but depends on how Λ_1 is embedded in M and how f extends to M.

(2.4) Definition: If Λ is a basic set of f, we say f preserves (or reverses) a u-orientation on Λ if the bundle $E^u(\Lambda)$ is orientable and Df preserves (or reverses) this orientation.

When f preserves or reverses a u-orientation on a basic set there is a close relationship between η and the zeta function of f restricted to the basic set.

(2.5) Theorem (Smale): Suppose Λ_1 is a basic set of f and ζ_1 denotes $\zeta(f|\Lambda_1)$, then

$$\eta_1^\sigma = \begin{cases} \zeta_1(t) & \text{if } f \text{ preserves a u-orientation on } \Lambda_1 \\ \zeta_1(-t) & \text{if } f \text{ reverses a u-orientation on } \Lambda_1, \end{cases}$$

where $\sigma = (-1)^{u_1}$ and u_1 = fiber dim $E^u(\Lambda_1)$ is the index of Λ_1.

Proof: Smale [11] actually proved this result only for Anosov diffeomorphisms (i.e. when $\Lambda_1 = M$), but the proof is the same for this case. Since it is short we give it. By a result of Smale [11, p. 767], if $p \in \mathrm{Fix}(f^m) \cap \Lambda_1$ then the index of p $I(p; f^m) = \Lambda_m(-1)^{u_1} = \Lambda_m \sigma$ where Λ_m is ± 1 depending on whether or not f^m preserves or reverses u-orientation on Λ_1. Thus if N_m is the cardinality of

$Fix(f^m) \cap \Lambda_1$ and $\tilde{N}_m = \Sigma I(p; f^m)$ with the sum taken over all $p \in Fix(f^m) \cap \Lambda_1$ we have $N_m = \Lambda_m \sigma \tilde{N}_m$. Now if f preserves u-orientation on Λ_1 then $\Lambda_m = 1$ for all m so

$$\zeta_1(t) = \exp(\sum_{m=1}^{\infty} \frac{1}{m} N_m t^m)$$

$$= \exp(\sigma \sum_{m=1}^{\infty} \frac{1}{m} \tilde{N}_m t^m)$$

$$= \eta_1^{\sigma}.$$

On the other hand if f reverses u-orientation on Λ_1 then $\Lambda_m = (-1)^m$ so,

$$\zeta_1(-t) = \exp(\sum_{m=1}^{\infty} \frac{1}{m} N_m (-1)^m t^m)$$

$$= \exp(\sigma \sum_{m=1}^{\infty} \frac{1}{m} \tilde{N}_m \Lambda_m (-1)^m t^m)$$

$$= \exp(\sigma \sum_{m=1}^{\infty} \frac{1}{m} \tilde{N}_m t^m) = \eta_1^{\sigma}. \qquad \text{q.e.d.}$$

(2.6) Proposition: Let $f\colon M \to M$ be a diffeomorphism with all periodic points hyperbolic, then as formal power series,

a) $\zeta(f) = \prod_{\gamma} (1 - t^{p(\gamma)})^{-1}$ where the product is taken over all periodic orbits γ and $p(\gamma)$ denotes the least period of γ.

b) $\eta(f; R) = \prod_{\gamma} (1 - \Lambda_\gamma t^{p(\gamma)})^{(-1)^{u(\gamma)+1}}$ where Λ_γ is 1 if $Df_x^{p(\gamma)}\colon E_x^u \to E_x^u$ preserves orientation for $x \in \gamma$ and -1 otherwise, and $u(\gamma) = $ fiber dim E_γ^u.

<u>If f satisfies Axiom A the same formulas hold for ζ_i and η_i if the product is taken over all periodic orbits $\gamma \subset \Lambda_i$.</u>

<u>Proof:</u> Since every periodic point is hyperbolic and we assume M is compact it follows that $\{x | x \epsilon \gamma$ and $p(\gamma) \leq n\}$ is finite for any fixed n. If γ is a single periodic orbit of period p then it is easy to check $\zeta(f|_\gamma) = (1 - t^p)^{-1}$. We now fix an integer n and let $\{\gamma_1, \ldots, \gamma_s\}$ be the set of periodic orbits with period $p(\gamma_i) \leq n$ and $K = \bigcup_{i=1}^{s} \gamma_i$, then $\zeta(f|_K) = \prod_{i=1}^{s} (1 - t^{p(\gamma_i)})^{-1}$.

But $N_n(f)$ is equal to $N_n(f|_K)$ since any fixed point of f^n is in K. Thus the coefficient of t^n in $\zeta(f) = \exp(\Sigma \frac{1}{m} N_m(f)t^m)$ is the same as the coefficient of t^n in $\zeta(f|_K) = \exp(\Sigma \frac{1}{m} N_m(f|_K)t^m) = \prod_{i=1}^{s} (1 - t^{p(\gamma_i)})^{-1}$. However, since $(1 - t^p)^{-1} = 1 + t^p + t^{2p} + \cdots$, the coefficient of t^n in $\prod_{i=1}^{s} (1 - t^{p(\gamma_i)})^{-1}$ is the same as the coefficient of t^n in $\prod_\gamma (1 - t^{p(\gamma)})^{-1}$ where the product is taken over all periodic orbits γ. Thus we have shown the coefficient of t^n in $\zeta(f)$ and $\prod_\gamma (1 - t^{p(\gamma)})^{-1}$ are the same, so this proves a).

The proof of b) is similar; we use the result of Smale [11, p. 767] that if γ has period p and $x \epsilon \gamma$ then the Lefschetz index $I(x, f^p)$ is $(-1)^{u(\gamma)} \Delta_\gamma$ where $\Delta_\gamma = +1$ if $Df^p \colon E^u_x \circlearrowleft$ preserves orientation and $\Delta_\gamma = -1$ if orientation is reversed. Now $\eta(f; R) = \exp(\sum_{m=1}^{\infty} \frac{1}{m} L(f^m)t^m)$ and $L(f^m) = \Sigma I(x, f^m)$ where this sum is over all $x \epsilon \mathrm{Fix}(f^m)$. Let K be as above and define $\rho = \exp(\Sigma \frac{1}{m} L_m(K)t^m)$ where $L_m(K)$ is the sum of $I(x, f^m)$ for all $x \epsilon \mathrm{Fix}(f^m) \cap K$. Then for $m \leq n$ we have $L_m(K) = L(f^m)$ since all points of period $\leq n$ are in K. Thus the coefficient of t^n in ρ is the same as the coefficient of t^n in $\eta(f; R)$.

But $L_m(K) = \sum_{i=1}^{s} L_m(\gamma_i)$ where $L_m(\gamma_i) = \Sigma\, I(x,f^m)$ where the sum is over $x \in \gamma_i \cap \mathrm{Fix}(f^m)$. Hence $\rho = \prod_{i=1}^{s} \exp(\sum_{m=1}^{\infty} \frac{1}{m} L_m(\gamma_i) t^m)$. Since

$$L_m(\gamma) = \begin{cases} 0 & \text{if } m \not\equiv 0 \bmod p(\gamma) \\ p(\gamma)(-1)^{u(\gamma)}\Delta^{m/p(\gamma)} & \text{if } m \equiv 0 \bmod p(\gamma), \end{cases}$$

one checks easily that $\rho = \prod_{i=1}^{s} (1 - \Delta_{\gamma_i} t^{p(\gamma_i)})^{(-1)^{u_i+1}}$. Thus, as before, the coefficient of t^n in ρ is also the same as the coefficient of t^n in $\prod_{\gamma}(1 - \Delta_\gamma t^{p(\gamma)})^{(-1)^{u(\gamma)+1}}$, and b) is proved. The proof for f restricted to a single basic set is similar. q.e.d.

§3. The Relative Double Cover for a Basic Set

In the case when the bundle $E^u(\Lambda)$ is orientable and Df preserves or reverses this orientation the zeta function is calculable from homological information using Theorem (2.5). However for many important examples things are not so nice and one must resort to other techniques. This problem was handled first by Guckenheimer [5] in his proof of the rationality of the zeta function for Axiom A diffeomorphsims satisfying the no-cycle property which was based on previous work of Williams [14].

The idea of Guckenheimer was to try to work in a double cover which orients E^u and where f has a lift which preserves u-orientation. Such a double cover exists over a neighborhood of Λ but this neighborhood is not f invariant. Hence to define a lift of f it is necessary to add to the double cover all points in filtration levels below Λ and let them cover themselves singly. The precise result we need is the following theorem implicit in [5] and explicitly worked out in the very nice appendix of [10].

(3.1) Theorem: Suppose Λ is a basic set of a diffeomorphism f satisfying Axiom A and the no-cycle property. Then there is a relative manifold $(\bar{X},\bar{A})$ and a relative double cover Π: $(\bar{X},\bar{A}) \to (X,A)$ such that

1) There exists a filtration for f with $X = M_i$, $A = M_{i-1}$ for some i and $\Lambda \subset X - A$.

2) The bundle $E^u(\Lambda)$ extends to a bundle E^u over $X - f(A)$ and Df extends to a bundle map

$$\begin{tikzcd} E_1^u \arrow[r, "\emptyset"] \arrow[d] & E^u \arrow[d] \\ X-A \arrow[r, "f"] & X-f(A) \end{tikzcd}$$

where E_1^u is the restriction of E^u to $X - A$.

3) There is a map $\bar{f}$: $(\bar{X},\bar{A}) \to (\bar{X},\bar{A})$ covering f.

4) The bundle E^u on $X - A$ lifts to an oriented bundle $\bar{E}^u$ on $\bar{X} - \bar{A}$ and for any $x \in \Pi^{-1}(\Lambda)$ $D\bar{f}_x$: $\bar{E}_x^u \to \bar{E}^u_{\bar{f}(x)}$ preserves orientation.

5) Then there is a unique covering transformation T of the double cover Π: $\bar{X} - \bar{A} \to X - A$ which reverses the orienta-tion of $\bar{E}^u$.

We will also need the following lemma.

(3.2) Lemma: Suppose Λ is a basic set and Π: $(\bar{X},\bar{A}) \to (X,A)$ is a relative double cover as above. Then if $x \in \Lambda \cap \mathrm{Fix}(f)$, $\bar{f}$ fixes the two points of $\Pi^{-1}(x)$ if and only if Df preserves the orientation of E_x^u, otherwise it switches them.

Proof: Let $y \in \Pi^{-1}(x)$ and suppose $\bar{f}(y) = y$. Then $D\Pi$: $\bar{E}_y^u \to E_x^u$ satisfies $D\Pi \cdot D\bar{f} = Df \cdot D\Pi$ so $D\bar{f}$ and Df restricted to $\bar{E}_y^u$ and E_x^u are conjugate. Since $D\bar{f}$ preserves orientation of $\bar{E}^u$ (by (4) of Theorem (3.1)) it follows that Df_x also preserves orientation.

Conversely if Df_x preserves the orientation of E_x^u then $\bar{f}(y) = y$

because, if not we can define $\hat{f} = \bar{f}\cdot T$ where T is the non-trivial deck transformation and then $\hat{f}(y) = y$ and $D\hat{f}_y$ will reverse the orientation of $\bar{E}^u_y$. Since $\hat{f}$ also covers f the same argument used above to show Df_x preserves orientation, now shows Df_x reverses orientation which is a contradiction. q.e.d.

(3.3) <u>Proposition: Suppose Λ is a basic set for f and $\Pi: (\bar{X},\bar{A}) \to (X,A)$ is a relative double cover for Λ. Then if $f_*: H_*(X,A;\ R) \circlearrowleft$ and $\bar{f}_*: H_*(\bar{X},\bar{A};\ R) \circlearrowleft$ are the maps induced by f and $\bar{f}$, u = fiber dim $E^u(\Lambda)$, and $\bar{\Lambda} = \Pi^{-1}(\Lambda)$, the following equalities hold:</u>

$$\zeta(f|_\Lambda)\tilde{\zeta}(f_*)^{(-1)^u} = \tilde{\zeta}(\bar{f}_*)^{(-1)^u} = \zeta(\bar{f}|_{\bar{\Lambda}}).$$

<u>Proof:</u> The proof of the equality $\tilde{\zeta}(\bar{f}_*)^{(-1)^u} = \zeta(\bar{f}|_{\bar{\Lambda}})$ is exactly the same as the proof of Theorem (2.5) (recall that $\bar{f}$ preserves u-orientation on $\bar{\Lambda}$). To prove the other equality we note that if $\Lambda = \Lambda_1$ then $\tilde{\zeta}(f_*) = \eta_1(f)$ (definition (2.1)) and hence by Proposition 2.2 and the remark following $\zeta(f_*) = \exp(\sum_{m=1}^{\infty} \frac{1}{m} \tilde{N}_m t^m)$ where $\tilde{N}_m$ is sum of indexes of the fixed points of f^m restricted to Λ.

Thus if we let N_m = cardinality of $\mathrm{Fix}(f^m) \cap \Lambda$ and $\bar{N}_m$ = cardinality of $\mathrm{Fix}(\bar{f}^m) \cap \bar{\Lambda}$, we have

$$\zeta(f|_\Lambda)\tilde{\zeta}(f_*)^{(-1)^u} = \exp(\sum_{m=1}^{\infty} \frac{1}{m} N_m t^m)\cdot\exp(\sum_{m=1}^{\infty} \frac{1}{m} (-1)^u\tilde{N}_m t^m)$$

$$= \exp(\sum_{m=1}^{\infty} \frac{1}{m} (N_m + (-1)^u\tilde{N}_m) t^m)$$

$$\text{and } \zeta(\bar{f}|_{\bar{\Lambda}}) = \exp(\sum_{m=1}^{\infty} \frac{1}{m} \bar{N}_m t^m)$$

So it suffices to prove $\bar{N}_m = N_m + (-1)^u \tilde{N}_m$.

By the result of Smale [11, p. 767] the index $I(p,f^m)$ of a fixed point of $f^m = \Delta(-1)^u$ where Δ is ± 1 depending on whether or not Df^m preserves or reverses orientation. Thus if we let N_m^+ be the number of fixed points of f^m where Df preserves orientation and $N_m^- = N_m - N_m^+$ then since $\tilde{N}_m = \Sigma\, I(p,f^m)$ we have $(-1)^u \tilde{N}_m = N_m^+ - N_m^-$. Since $N_m = N_m^+ + N_m^-$ it follows that $N_m + (-1)^u \tilde{N}_m = 2N_m^+$.

On the other hand Lemma (3.2) applied to $\bar{f}^m$ says that $\bar{N}_m = 2N_m^+$, and hence we have $N_m + (-1)^u \tilde{N}_m = \bar{N}_m$ as desired. q.e.d.

As a by-product we have essentially proved the following result of Guckenheimer (indeed by a method very close to that of [5] and [14]; see also Manning [8]).

(3.4) Corollary (Guckenheimer): If $f\colon M \to M$ satisfies Axiom A and the no-cycle condition then $\zeta(f)$ is a rational function. In fact it is a quotient of polynomials with integer coefficients and constant term 1. The same is true for the zeta function of f restricted to a basic set.

Proof: One checks easily (or see [11, p. 766]) that $\zeta(f) = \prod_{i=1}^{\ell} \zeta(f|_{\Lambda_i})$ so it suffices to prove the result for $\Lambda = \Lambda_i$. But since it is clear from the definition that $\tilde{\zeta}(f_*)$ and quotients of integer polynomials with constant terms 1, it then follows from Proposition (3.3) that $\zeta(f|_\Lambda)$ also has this property. q.e.d.

§4. The Reduced Zeta Function

We can now relate the reduced zeta function of a basic set to homological invariants of f. Since this is really the heart of all our results we give two quite different proofs. As before if Λ_i is a basic set, then M_i and M_{i-1} will denote the elements of a filtration for f such that $\Lambda_i \subset M_i - M_{i-1}$.

4.1 Theorem: Suppose $f: M \to M$ satisfies Axiom A and the no-cycle property and Λ_i is a basic set of index u, then the following are equal:

a) $\zeta(f|_{\Lambda_i})^{(-1)^u}$ with all coefficients reduced mod 2, i.e. $Z_i(f)^{(-1)^u}$

b) The function obtained by reducing mod 2 all coefficients of $\eta_i(f;\ R) = \prod_{j=0}^{n} \det(I - f_{*j}t)^{(-1)^{j+1}}$ where $f_{*j}\colon H_j(M_i, M_i;\ R)\circlearrowleft$ is induced by f.

c) The function $\eta_i(f;\ Z_2) = \prod_{j=0}^{n} \det(I - f_{*j}t)^{(-1)^{j+1}}$ where $f_{*j}\colon H_j(M_i, M_{i-1};\ Z_2)\circlearrowleft$ is induced by f.

Topological proof: The fact that b) is equal to c) was proved in Proposition (1.6), hence it suffices to show that a) is equal to c)

We first choose a relative double cover for Λ_i, as in §3, $\Pi\colon (\overline{X},\overline{A}) \to (X,A)$ and then a filtration such that $M_i = X$, $M_{i-1} = A$. The pair (X,A) can be triangulated and the triangulation lifted to a triangulation of $(\overline{X},\overline{A})$ so that each simplex σ which intersects

$X - A$ is covered by two simplices $\bar{\sigma}$ and $T\bar{\sigma}$ where T is the covering transformation which reverses orientation of E^u.

Let C be the oriented simplicial chain complex for the pair (X,A) and let $\bar{C}$ the oriented simplicial chain complex for the pair $(\bar{X},\bar{A})$. The map Π induces a chain map $\Pi_*: \bar{C} \to C$. Let $D = \ker \Pi_*$ so we have the short exact sequence of chain complexes $0 \to D \to \bar{C} \xrightarrow{\Pi_*} C \to 0$. The chain maps induced on C and $\bar{C}$ by f and $\bar{f}$ will be denoted by τ and $\bar{\tau}$ respectively. We then define

$$\alpha = \bar{\tau}|_D \otimes \mathrm{id}: D \otimes R \to D \otimes R$$

$$\rho = \bar{\tau} \otimes \mathrm{id}: \bar{C} \otimes R \to \bar{C} \otimes R$$

$$\gamma = \tau \otimes \mathrm{id}: C \otimes R \to C \otimes R$$

Now $H_*(\bar{C} \otimes R) = H_*(\bar{X},\bar{A}; R)$, $H_*(C \otimes R) = H_*(X,A; R)$ and, ρ and γ represent $\bar{f}_*$ and f_* on the chain level. Since $0 \to D_j \otimes R \to \bar{C}_j \otimes R \to C_j \otimes R \to 0$ is exact an application of Corollary (1.3) shows $\det(I - \alpha_j t)(\det(I - \rho_j t))^{-1}\det(I - \gamma_j t) = 1$ so $\det(I - \rho_j t) = \det(I - \alpha_j t)\det(I - \gamma_j t)$ and it follows that

$$\tilde{\zeta}(\rho) = \tilde{\zeta}(\alpha)\tilde{\zeta}(\gamma).$$

Since $\tilde{\zeta}(\rho) = \tilde{\zeta}(\bar{f}_*)$ and $\tilde{\zeta}(\gamma) = \tilde{\zeta}(f_*)$, by (1.2) it follows from Proposition 3.3 that $\tilde{\zeta}(\alpha) = \zeta(f|_{\Lambda_1})^{(-1)^u}$ where u = fiber dim $E^u(\Lambda_1)$ is the index of Λ_1. Thus $\zeta(f|_{\Lambda_1})^{(-1)^u}$ reduced mod 2 is equal to $\tilde{\zeta}(\alpha)$ reduced mod 2 which by Lemma (1.4) is equal to $\tilde{\zeta}(\beta)$ where $\beta = \tau|_D \otimes \mathrm{id}: D \otimes Z_2 \to D \otimes Z_2$. So it will suffice to show that

$\tilde{\zeta}(\beta) = \eta_1(f;\ Z_2)$.

To prove this we note that chains in D are precisely those chains in $\overline{C}$ which satisfy the condition that the coefficient of $\overline{\sigma}_j$ equals minus the coefficient of $T(\overline{\sigma}_j)$. Hence chains in $D \otimes Z_2$ are the chains with the coefficient (in Z_2) of $\overline{\sigma}_j$ equal to the coefficient of $T(\overline{\sigma}_j)$. It is now easy to see that the map $\varphi\colon C \otimes Z_2 \to D \otimes Z_2$ defined by $\varphi(\sigma_j) = \overline{\sigma}_j + T(\overline{\sigma}_j)$ is a chain isomorphism. Also it is clear that $\beta \circ \varphi = \varphi \circ \tau'$ where $\tau' = \tau \otimes \mathrm{id}\colon C \otimes Z_2$. Thus $\tilde{\zeta}(\beta) = \tilde{\zeta}(\tau')$ but by Lemma (1.2) $\tilde{\zeta}(\tau') = \tilde{\zeta}(f_*)$ where $f_*\colon H_*(X,A;\ Z_2)$ and this is precisely $\eta_1(f;\ Z_2)$. q.e.d.

<u>Algebraic Proof</u>: We again appeal to Proposition (1.6) for the equality of b) and c) and then show directly that a) is equal to b). By Proposition (2.6) we have

$$\eta_1(f;\ R)^{(-1)^u} = \prod_{\gamma \subset \Lambda_1} (1 - \Delta_\gamma t^{p(\gamma)})^{-1} \quad \text{and} \quad \zeta_1(f|_{\Lambda_1}) = \prod_{\gamma \subset \Lambda_1} (1 - t^{p(\gamma)})^{-1}$$

where both products are taken over all periodic orbits in Λ_1. Clearly these should be the same when reduced mod 2 if we can make s[illegible] of the infinite products. We do this by considering formal power series.

Let $Z[t]$ be the ring of integer polynomials and let S be the multiplicative set $1 + tZ[t]$. Then $S^{-1}Z[t]$ will denote the ring of fractions of $Z[t]$ by S. Since for the inclusion $Z[t] \to Z[[t]]$ into formal power series the image of each element of S is invertible there is a unique extension of the inclusion to a homomorphism [illegible] $S^{-1}Z[t] \to Z[[t]]$ (see [7, p. 66-69] for this).

Similarly we have $Z_2[t]$, the localization at (t), $Z_2[t]_{(t)}$ and an extension of the inclusion $Z_2[t] \to Z_2[[t]]$ to $\beta\colon Z_2[t]_{(t)} \to Z_2[[t]]$. The homomorphism β is injective since it is injective on polynomials.

Let ω: $Z[t] \to Z_2[t] \to Z_2[t]$, ψ: $S^{-1}Z[t] \to Z_2[t]_{(t)}$ and θ: $Z[[t]] \to Z_2[[t]]$ all denote reduction of coefficients mod 2. Then we have the following commutative diagram of homomorphisms

$$\begin{tikzcd}
 & S^{-1}Z[t] \arrow[dr, "\alpha"] \arrow[ddd, "\psi"] & \\
Z[t] \arrow[ur] \arrow[rr] \arrow[d, "\omega"] & & Z[[t]] \arrow[d, "\theta"] \\
Z_2[t] \arrow[rr] \arrow[dr] & & Z_2[[t]] \\
 & Z_2[t]_{(t)} \arrow[ur, "\beta"] &
\end{tikzcd}$$

where the unlabelled arrows are the natural inclusions. The diagram is commutative because it commutes for polynomials. By (2.4) and (3.4) the rational functions ζ_1 and η_1 are in $S^{-1}Z[t]$. The assertion of our theorem is that $\psi(\eta_1) = \psi(\zeta_1^{(-1)^u})$. Considering the diagram and the fact that β is injective, it suffices to show that $\theta\circ\alpha(\eta_1^{(-1)^u}) = \theta\circ\alpha(\zeta_1)$.

To do this we show they have the same coefficient of t^n. Let $\{\gamma_1,\ldots,\gamma_s\}$ be the set of periodic orbits in Λ_1 with $p(\gamma_i) \leq n$. Then the coefficient of t^n in $\alpha(\eta_1^{(-1)^u})$ is the same as that in $\alpha(\rho)$ where $\rho = \prod_{i=1}^{s}(1 - \Delta_{\gamma_i} t^{p(\gamma_i)})^{-1}$. Likewise the coefficient of t^n in $\alpha(\zeta_1)$ is the same as that in $\alpha(\hat{\rho})$ where $\hat{\rho} = \prod_{i=1}^{s}(1 - t^{p(\gamma_i)})^{-1}$. But $\psi(\rho) = \psi(\hat{\rho})$, so $\theta\circ\alpha(\rho) = \beta\circ\psi(\rho) = \beta\circ\psi(\hat{\rho}) = \theta\circ\alpha(\hat{\rho})$ and it follows that the coefficient of t^n in $\theta\circ\alpha(\eta_1^{(-1)^u})$ is equal to the coefficient of t^n in $\theta\circ\alpha(\zeta_1)$. q.e.d.

§5. Morse Inequalities

(5.1) Definition: *We define the partial homology zeta function* $\tilde{\zeta}_q = \tilde{\zeta}_q(f_*)$ *to be* $\det(I - f_{*q}t)^{-1}$ *where* $f_{*q}\colon H_q(M;\, R) \to H_q(M;\, R)$. *Thus* $\tilde{\zeta} = \tilde{\zeta}(f_*) = \prod_{q=0}^{n} \tilde{\zeta}_q^{(-1)^q}$ *where* $n = \dim M$.

In order to prove Morse inequalities we will need a standing hypothesis on the dimension of the basic sets (or on the dimension of the global unstable manifolds). Recall that the index u_i of a basic set Λ_i = fiber dim $E^u(\Lambda_i)$.

(5.2) Definition: *If* $f\colon M \to M$ *satisfies Axiom A and the no-cycle property we will say that the basic sets of* f *satisfy the dimension requirements for an integer* q *if it is true that each basic set* Λ_i *with index* $u_i \leq q$ *satisfies* $\dim W^u(\Lambda_i) \leq q$ *and each basic set* Λ_j *with* $u_j > q$ *satisfies* $\dim W^s(\Lambda_j) < n - q$ *where* $n = \dim M$.

(5.3) Remark: It is shown in Lemma 5 of [4] that $\dim \Lambda_i + u_i \geq \dim W^u(\Lambda_i)$ and similarly $\dim \Lambda_j + (n-u_j) \geq \dim W^s(\Lambda_j)$. Hence the dimension requirements above are satisfied if $\dim \Lambda_i \leq q - u_i$ when $u_i \leq q$ and $\dim \Lambda_j < u_j - q$ when $u_j > q$. From this it is clear *the dimension requirements are satisfied for all* q *if* $\dim \Lambda_i = 0$ *for all* i.

Recall that $\eta_i(f) = \prod_{j=0}^{n} \det(I - f_{*j}t)^{(-1)^j}$ where $f_{*j}\colon H_j(M_i, M_{i-1};\, R) \circlearrowleft$ and $\Lambda_i \subset M_i - M_{i-1}$. We wish now to relate these functions to the partial homology zeta functions $\tilde{\zeta}_q$.

(5.4) Proposition: *Suppose* $f\colon M \to M$ *satisfies Axiom A and the no-cycle property and the basic sets of* f *satisfy the dimension*

<u>requirements</u> (5.4) <u>for q</u>. <u>Then</u> $P^{(-1)^q} \cdot \prod_{u_i \le q} \eta_i = \prod_{k=0}^{q} \tilde{\zeta}_k^{(-1)^k}$

<u>where</u> $P(t)$ <u>is a polynomial with integer coefficients and constant term 1</u>.

<u>Proof</u>: Suppose $M = M_\ell \supset M_{\ell-1} \supset \cdots \supset M_1 \supset M_0 = \emptyset$ is a filtration for f. Define $\eta^q(M_i,M_j) = \prod_{k=0}^{q} \det(I - f_{*k}t)^{(-1)^{k+1}}$ where $f_{*k}\colon H_k(M_i,M_j;\, R) \circlearrowleft$ is the map induced by f. Consider now the exact sequence $0 \to B \to H_q(M_j) \to H_q(M_i) \to H_q(M_i,M_j) \to H_{q-1}(M_j) \to \cdots$ where $B = \ker(i_*\colon H_q(M_j) \to H_q(M_i))$ and the remainder of the sequence is the exact sequence of the pair (M_i,M_j). Note $f_{*q}(B) \subset B$ and let $P_{ij} = \det[I - (f_{*q}|B)t]$. Then applying Corollary (1.3) to this exact sequence and the endomorphisms of its elements induced by f we obtain $P_{i,j}^{(-1)^{q+1}} \cdot \eta^q(M_j)^{-1} \cdot \eta^q(M_i) \cdot \eta^q(M_i,M_j)^{-1} = 1$. Thus, if we set $j = i-1$ and denote $P_{i,i-1}$ by P_i we have,

$$\eta^q(M_i,M_{i-1}) = P_i^{(-1)^{q+1}} \cdot \eta^q(M_i) \cdot \eta^q(M_{i-1})^{-1}.$$

Taking a product over $0 \le i \le \ell$ we get $\prod_{i=1}^{\ell} \eta^q(M_i,M_{i-1}) =$ $\eta^q(M_\ell) \cdot \eta^q(M_0)^{-1} \prod_{i=1}^{\ell} P_i^{(-1)^{q+1}} = \eta^q(M) \cdot P^{(-1)^{q+1}}$ where $P = \prod_{i=1}^{\ell} P_i$, since $M_\ell = M$ and $M_0 = \emptyset$. Notice P is a polynomial with integer coefficients and constant term 1.

By hypothesis if Λ_i is a basic set with $u_i \le q$ then $\dim W^u(\Lambda_i) \le q$ so by Lemma 6 of [4], $f_{*k}\colon H_k(M_i,M_{i-1};\, R) \circlearrowleft$ is nilpotent if $k > q$. That is, $\det(I - f_{*k}t) = 1$ if $k > q$ (the characteristic polynomial of a matrix A is $t^k h(\frac{1}{t})$ for some k where $h(t) = \det(I - At)$). It follows that $\eta^q(M_i,M_{i-1}) =$

$$\eta_i = \prod_{k=0}^{n} \det(I - f_{*k}t)^{(-1)^{k+1}}, \text{ whenever } u_i \leq q.$$

On the other hand when $u_j > q$, $\dim W^s(\Lambda_j) < n - q$ so, again by Lemma 6 of [4], we have f_{*k} is nilpotent if $k \leq q$. So a similar argument shows $\eta^q(M_j, M_{j-1}) = 1$ if $u_j > q$.

Thus

$$P^{(-1)^{q+1}} \eta^q(M) = \prod_{i=1}^{\ell} \eta^q(M_i, M_{i-1}) = \prod_{u_i \leq q} \eta_i .$$

Since by definition,

$$\eta^q(M) = \prod_{k=0}^{q} \det(I - f_{*k}t)^{(-1)^k} = \prod_{k=0}^{q} \tilde{\zeta}_k^{(-1)^k}$$

we have the desired result:

$$P^{(-1)^q} \prod_{u_i \leq q} \eta_i = \prod_{k=0}^{q} \tilde{\zeta}_k^{(-1)^k}. \qquad \text{q.e.d.}$$

(5.5) <u>Corollary</u>: <u>If</u> $f: M \to M$ <u>satisfies Axiom A and the no-cycle property and has basic sets</u> $\Lambda_1, \ldots, \Lambda_\ell$, <u>then</u>

$$\prod_{i=1}^{\ell} \eta_i = \tilde{\zeta}(f_*) = \eta(f; R)$$

<u>Proof</u>: This is easily proved directly, however as remarked in (5.1), $\tilde{\zeta}(f_*) = \prod_{k=0}^{n} \tilde{\zeta}_k^{(-1)^k}$ where $n = \dim M$, and if we now apply Proposition (5.4) with $q = n$ and $q = n + 1$ we see that there are polynomials P_1 and P_2 such that

$$P_1^{(-1)^n} \prod_{i=1}^{\ell} \eta_i = \tilde{\zeta}(f_*) \text{ and } P^{(-1)^{n+1}} \prod_{i=1}^{\ell} \eta_i = \tilde{\zeta}(f_*).$$

It follows that $P_1 = P_2 = 1$ so we have the desired result. Note the dimension requirements are always satisfied. q.e.d.

(5.6) Corollary: If $f: M \to M$ satisfies Axiom A and the no-cycle property and its basic sets satisfy the dimension requirements for q and $q-1$, then $P \cdot \prod_{u_i=q} n_i^{(-1)^q} = \tilde{\zeta}_q$ for some integer polynomial P.

Proof: Take the equality of (5.4) for q and divide by the equality of (5.4) for $q-1$. q.e.d.

We can now obtain the Morse inequalities for the reduced zeta functions Z_i. The following result is analagous to Theorem 2 of [4], but uses the reduced zeta functions and thereby obviates the necessity of the hypothesis about orientability.

(5.7) Theorem: Suppose $f: M \to M$ satisfies Axiom A, the no-cycle property and the dimension requirements (5.2) for q. Then there is a polynomial $p \in Z_2[t]$ such that $p \prod_{u_i \leq q} Z_i^{\tau_i}$ is equal to the mod 2 reduction of $\dfrac{\tilde{\zeta}_q \cdot \tilde{\zeta}_{q-2} \cdots}{\tilde{\zeta}_{q-1} \cdot \tilde{\zeta}_{q-3} \cdots}$, where $\tau_i = (-1)^{q+u_i}$ and $u_i = \text{fiber dim } E^u(\Lambda_i)$.

Before giving the proof we comment on the relation of this to the Morse inequalities for a Morse function. If f is the time-one map of the flow obtained by integrating minus the gradient of a Morse function then f satisfies Axiom A, the no-cycle property and the dimension requirements for all q. The equality above then implies that the degree of $\prod_{u_i < q} Z_i^{\tau_i}$ is less than or equal to the degree of the mod 2 reduction of $\frac{\tilde{\zeta}_q \cdot \tilde{\zeta}_{q-2} \cdots}{\tilde{\zeta}_{q-1} \cdot \tilde{\zeta}_{q3} \cdots}$. One checks that these inequalities are exactly the classical Morse inequalities relating the Betti numbers of M and the number of critical points of a Morse function.

Proof of (5.7): If we take the equality of (5.4) and raise it to the power $(-1)^q$ we obtain

$$P \prod_{u_i \leq q} \eta_i^{(-1)^q} = \prod_{k=0}^{q} \tilde{\zeta}_k^{(-1)^{k+q}} = \frac{\tilde{\zeta}_q \cdot \tilde{\zeta}_{q-2} \cdots}{\tilde{\zeta}_{q-1} \cdot \tilde{\zeta}_{q-3} \cdots} .$$

By Theorem (4.1), η_i with coefficients reduced mod 2 is equal to $Z_i^{(-1)^{u_i}}$. Hence if p is equal to P with coefficients reduced mod 2 we have $p \cdot \prod_{u_i \leq q} Z_i^{\tau_i}$ = mod 2 reduction of $\frac{\tilde{\zeta}_q \cdot \tilde{\zeta}_{q-2} \cdots}{\tilde{\zeta}_{q-1} \cdot \tilde{\zeta}_{q-3} \cdots}$. q.e.d.

Applying the same type of argument to the equality of (5.5) we obtain the second of our main theorems.

(5.8) Theorem: Suppose $f\colon M \to M$ satisfies Axiom A and the no-cycle property, and has ℓ basic sets, then the following are equal:

a) $\prod_{i=1}^{\ell} Z_i^{(-1)^{u_i}}$ where u_i = fiber dim $E^u(\Lambda_i)$.

b) The reduction mod 2 of $\eta(f; R) = \prod_{k=0}^{n} \det(I - f_{*k} t)^{(-1)^{k+1}}$ where $f_{*k}\colon H_k(M; R) \circlearrowleft$ is induced by f.

c) $\eta(f; Z_2) = \prod_{k=0}^{n} \det(I - f_{*k} t)^{(-1)^{k+1}}$ where $f_{*k}\colon H_k(M; Z_2) \circlearrowleft$ is induced by f.

Proof: The fact that b) is equal to c) is a consequence of Corollary (1.6). From (5.5) we have $\prod_{i=1}^{\ell} \eta_i = \eta(f; R)$ and from Theorem (4.2) η_i reduced mod 2 is equal to $Z_i^{(-1)^{u_i}}$. It follows that a) is equal to b). q.e.d.

(5.9) Proposition: If f satisfies the dimension requirements for q and $q - 1$ (E.g. if all basic sets of f have dimension 0) then there is a polynomial $p \in Z_2[t]$ such that $p \prod_{u_i=q} Z_i$ = mod 2 reduction of $\det(I - f_{*q}t)^{-1}$ where $f_{*q}\colon H_q(M; R) \circlearrowleft$ is induced by f.

Proof: If $u_i = q$ then by (4.1) Z_i is the mod 2 reduction of $\eta_i^{(-1)^q}$. By (5.6) there is an integer polynomial P such that $P \cdot \prod \eta_i^{(-1)^q} = \tilde{\zeta}_q = \det(I - f_{*q}t)^{-1}$. Reducing mod 2 gives the result. q.e.d.

The following proposition gives a necessary condition for a collection of "abstract" basic sets to be embedded as the basic sets of any diffeomorphism f of M (no matter what the homotopy class of f). By the degree of a rational function we mean, of course, the numerator minus the degree of the denominator. The following result was inspired by the remark of Smale [11] that the degree of the homology zeta function (our $\eta(f; R)$) is minus the Euler characteristic of M.

(5.10) Proposition: If $f\colon M \to M$ satisfies Axiom A and the no-cycle property, and has basic sets $\Lambda_1, \ldots, \Lambda_\ell$ with u_i = fiber dim $E^u(\Lambda_i)$, then $\sum_{i=1}^{\ell} (-1)^{u_i} \deg Z_i = -\chi(M)$ where $\chi(M)$

is the Euler characteristic of M.

Proof: From Theorem (5.8) we have

$$\prod_{i=1}^{\ell} Z_i^{(-1)^{u_i}} = \prod_{k=0}^{n} \det(I - f_{*k}t)^{(-1)^{k+1}},$$

where f_{*k}: $H_k(M;\, Z_2) \circlearrowleft$ is induced by f. The degree of the left hand side of this equation is $\sum_{i=1}^{\ell} (-1)^{u_i} \deg Z_i$. Now f_{*k}: $H_k(M;\, Z_2) \circlearrowleft$ is an isomorphism so the degree of $\det(I - f_{*k}t)$ is rank $H_k(M;\, Z_2)$. So the degree of $\prod_{k=0}^{n} \det(I - f_{*k}t)^{(-1)^{k+1}}$ is $-\sum_{k=0}^{n} (-1)^k \operatorname{rank} H_k(M;\, Z_2) = -\chi(M)$ and the result follows. q.e.d.

§6. The Global Bifurcation Problem: Examples and Questions

We are interested in the problem of how the basic sets can change when one Axiom A, no-cycle diffeomorphism is isotoped (or even homotoped) to another. From Theorem (5.8) we have that $\Pi Z_i^{(-1)^{u_i}} = \eta(f;\ Z_2)$ and since $\eta(f;\ Z_2)$ depends only on the homotopy type of f, it follows that if f and g are homotopic then

$$\Pi Z_i(f)^{(-1)^{u_i}} = \Pi Z_j(g)^{(-1)^{u_j}}.$$

Several special cases of this give partial answers to interesting questions:

1) When can an isotopy remove a basic set Λ_i of f while leaving all others unchanged? A necessary condition is that $Z_i(f) = 1$

2) When can an isotopy "cancel" two basic sets Λ_i and Λ_j leaving all others unchanged? A necessary condition is $Z_i^{(-1)^{u_i}} \cdot Z_j^{(-1)^{u_j}} = 1$

3) When can an isotopy of f to g change a basic set $f\colon \Lambda_i \to \Lambda_i$ to a different basic set $f\colon \Lambda_i' \to \Lambda_i'$, leaving others unaltered? A necessary condition is $Z_i(f)^{(-1)^{u_i}} = Z_i(g)^{(-1)^{u_i}}$.

In order to give several examples with zero dimensional basic sets we review briefly the structure of these basic sets.

If G is an $n \times n$ matrix of zeroes and ones we define $\Sigma_A \subset \Pi\{1,2,\ldots,n\}$ by $\Sigma_A = \{(x_i)_{i=-\infty}^{\infty} | x_i \in \{1,\ldots,n\}$ and

$A_{x_i x_{i+1}} = 1$ for all i). If $\{1,\ldots,n\}$ is given the discrete topology and Σ_A a topology as a subset of the product then Σ_A is a compact metrizable space.

The shift homomorphism $\sigma\colon \Sigma_A \to \Sigma_A$ is defined by $\sigma((x_i)) = (x_i')$ where $x_i' = x_{i+1}$ (here (x_i) denotes the bi-infinite sequence whose ith element is x_i).

A result of Bowen [2] shows that on any zero-dimensional basic set Λ, f is topologically conjugate to some shift $\sigma\colon \Sigma_A \to \Sigma_A$ (the matrix A is <u>not</u> unique however).

It is not difficult to check that $N_m(\sigma)$, the number of fixed points of $\sigma^m\colon \Sigma_A \to \Sigma_A$ is tr A^m. Hence we have

$$\zeta(\sigma) = \exp(\Sigma \frac{1}{m} N_m t^m) = \exp(\Sigma \frac{1}{m} \operatorname{tr} A^m t^m) = \frac{1}{\det(I - At)} \text{ by (1.7).}$$

(6.1) <u>The Full Shift:</u> If A is the n × n matrix with all entries 1 then $\sigma\colon \Sigma_A \to \Sigma_A$ is called the full n-shift. This can be embedded as a basic set of diffeomorphism of S^2. Figures 1 and 2 illustrate this for n = 2 and 3.

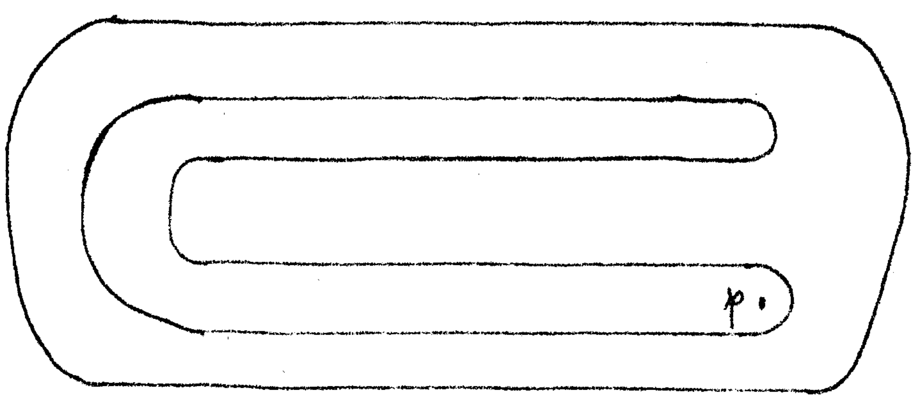


Figure 1

In both cases a disk is mapped into itself. In Figure 1 the diffeomorphism will have as basic sets a fixed point source (not shown), the fixed point sink p, and a full two shift (see [11] for an analysis of this). This diffeomorphism can be isotoped to remove the two shift without disturbing the fixed points by altering it so the disk is mapped into itself and everything tends to p.

One checks easily that if σ is the 2-shift homeomorphism $\zeta(\sigma) = \frac{1}{1-2t}$ so $Z(\sigma) = 1$ as is necessary. Exactly the same analysis works for the full n-shift if n is even.

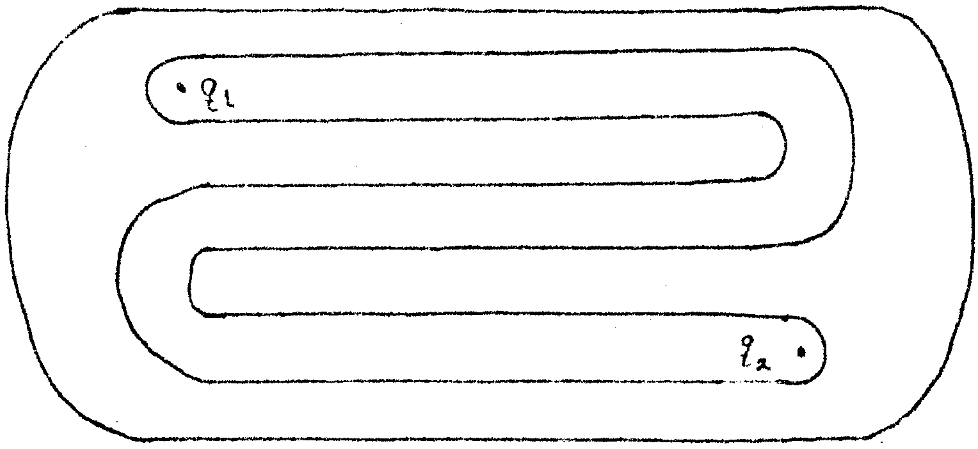


Figure 2

For the full 3-shift the basic sets are two fixed point sinks q_1, q_2, the 3-shift and a fixed point source (not shown). In this case an isotopy can replace the shift by a single hyperbolic fixed point (Figure 2) without disturbing the fixed point source and sinks.

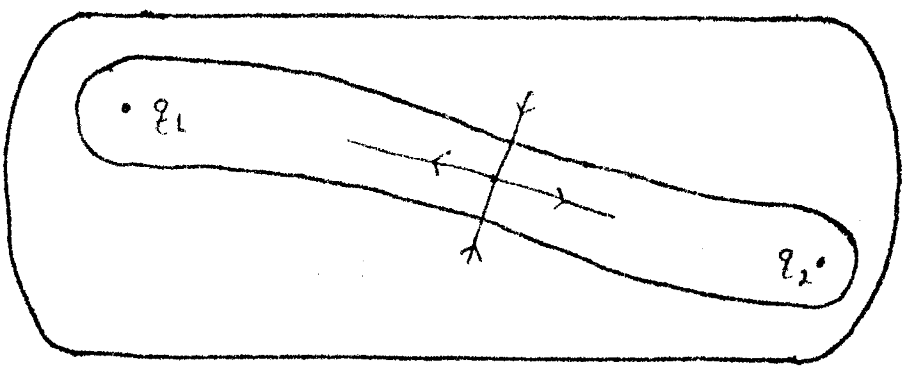


Figure 3

Also $\zeta(\sigma) = \frac{1}{1-3t}$ so $Z(\sigma) = \frac{1}{1+t}$, the same as the reduced zeta function of a single point. For any full n-shift with n odd one can do the same kind of construction and isotopy.

(6.2) Example: We give now an example of a shift which occurs a basic set of a diffeomorphism of S^2, but which cannot occur with all other basic sets as fixed points. Let σ be the shift based on the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, i.e. the square of the shift based on $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. One computes easily that $Z(\sigma) = \dfrac{1}{1 + t + t^2}$. Since for any diffeomorphism f of S^2 $\eta(f; Z_2) = \dfrac{1}{(1 + t)^2}$, it is not difficult to see that we cannot have $\eta(f; Z_2) = \Pi Z_i^{\pm 1}$ if one of the Z_i's is $Z(\sigma)$ and all others are $\frac{1}{1+t}$ (the reduced zeta function of a fixed point). The simplest way to have $\dfrac{1}{(1+t)^2} = \Pi Z_i^{(-1)^{u_i}}$ is as follows:

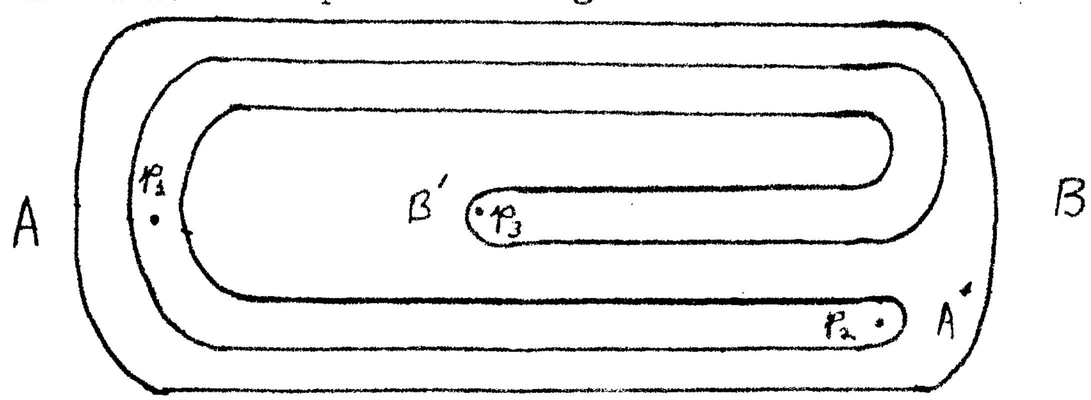
Let $Z_1 = \dfrac{1}{1+t^3}$, $u_1 = 0$ (a sink of period 3)

$Z_2 = \dfrac{1}{1+t+t^2}$, $u_2 = 1$ (the subshift for $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$)

$Z_3 = \frac{1}{1+t}$, $u_3 = 2$ (a fixed point source).

Then $\Pi Z_i^{(-1)^{u_i}} = \dfrac{1}{(1+t)^2}$, since $(1 + t)^3 = (1 + t)(1 + t + t^2)$.

In [11] Smale gives a picture of a realization of this diffeomorphism which we reproduce in Figure 4.



The disk is mapped into itself as shown. The points p_1, p_2, p_3 are an orbit of period 3 which is an attractor. The other basic sets are a point source (not shown) and the shift described (some indication of this can be found in [11]).

(6.3) Example: In [15] R. F. Williams showed that any shift $\sigma: \Sigma_A \to \Sigma_A$ which is topologically transitive can be realized as a basic set of a diffeomorphism of S^3. We give an example of a shift which cannot be realized as a basic set of a diffeomorphism of any S^n in such a way that all other basic sets are finite (i.e. periodic orbits).

If

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

then $\sigma: \Sigma_A \to \Sigma_A$ is a topologically transitive shift and $Z(\sigma) = \dfrac{1}{1 + t^2 + t^3}$. It is easy to check that in $Z_2[t]$, $(1 + t^2 + t^3)$ is an irreducible factor of $1 + t^7$, and hence in the algebraic closure of Z_2 its roots are three of the seven seventh roots of unity. On the other hand if Λ_i is a basic set which is a point of period p then $Z_i = \dfrac{1}{1 + t^p}$. In the algebraic closure of Z_2, $1 + t^p$ must have as roots, either no seventh roots of unity or all seven of them. Hence it is impossible to have $\Pi Z_i^{(-1)^{u_i}} = \eta(f; Z_2)$ if all basic sets are periodic except the one conjugate to $\sigma: \Sigma_A \to \Sigma_A$, because $\eta(f; Z_2) = \dfrac{1}{(1 + t)^2}$ or 1 for any diffeomorphism $f: S^n \to S^n$.

(6.4) Shoes (after Zeeman [16])

We have emphasized the reduced zeta function because it is an invariant of an abstract basic set, that is, the topological conjugacy type of f restricted to the basic set and does not depend on the embedding of the basic set or the extension of f to M. However, if one knows extra data it may be possible to compute the functions $\eta_i(f; R)$ which are stronger invariants and (5.7) and (5.8) can then be replaced by (5.4), (5.5) and (5.6). For example from (5.5) we have $\eta(f; R) = \prod_{i=1}^{\ell} \eta_i$ which shows that a necessary condition for an isotopy to cancel basic sets Λ_i and Λ_j is that $\eta_i\eta_j = 1$, or if a basic set Λ_i can be removed, then $\eta_i = 1$, etc.

In [16] Zeeman describes a framework for studying diffeomorphisms with zero dimensional basic sets, and a simple way of describing what amounts to the germ of an extension of f on the basic sets. What he calls a shoe is determined by two positive integer matrices A^+ (the positive intersection matrix) and A^- (the negative intersection matrix) and the index u of the basic set. The diffeomorphism on the basic set is topologically conjugate to the shift $\sigma: \Sigma_A \to \Sigma_A$ where $A = A^+ + A^-$ (see [16] for more detail). Also from (2.2), $\eta_i = \exp(\sum_{m=1}^{\infty} \frac{1}{m} \tilde{N}_m t^m)$ where $\tilde{N}_m$ is $\Sigma I(p,f^m)$ and the sum is over all $p \in Fix(f^m) \cap \Lambda_i$. It is not difficult to show that $\tilde{N}_m = (-1)^u \operatorname{tr} \tilde{A}^m$ where $\tilde{A} = A^+ - A^-$, and hence $\eta_i(f; R) = \det(I - \tilde{A}t)^{(-1)^{u+1}}$ by (1.7). Thus in this framework, where one knows both A^+ and A^-, the functions η_i are easily computable and it is more appropriate to use them than the reduced zeta functions.

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