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ALGEBRAIC K-FLOWS

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The algebraic approach to the study of Statistical Mechanics suggests an expansion of classical ergodic theory to a noncommutative ergodic theory. We indicate here how one can proceed in this spirit to generalise the classical concept of a Kolmogorov-Sinai flow.

Firstly we recall that a classical K-flow is constituted by: a probability space (Ω, Σ, μ), a measurable group $\{T(t) \mid t \in R\}$ of measure preserving transformations of (Ω, Σ, μ), and a partition $\xi \subset \Sigma$ such that: (i) $\xi \subseteq T(t)[\xi]$ $\forall t \geq 0$; (ii) $\bigcap_{t\in R} T(t)[\xi] = \hat{0}$; (iii) $\bigvee_{t\in R} T(t)[\xi] = \hat{1}$.

Secondly we construct from these elements: a separable Hilbert space $\mathfrak{H} = L^2(\Omega, \mu)$, a maximal abelian von Neumann algebra $\mathcal{R} = L^\infty(\Omega, \Sigma)$ acting on $\mathfrak{H}$, a cyclic and separating vector $\Phi(\omega) = 1$ $\forall \omega \in \Omega$ for $\mathcal{R}$ in $\mathfrak{H}$, a faithful normal state on $\mathcal{R}$ $\phi : f \in \mathcal{R} \mapsto \langle \phi; f \rangle = (f\Phi, \Phi) = \int f(\omega) d\mu(\omega)$, a continuous group of automorphisms of $\mathcal{R}$ $\alpha(t) : f \in \mathcal{R} \mapsto \alpha(t)[f] = f \circ T(t)$, and a von Neumann subalgebra $\mathcal{A}$ of $\mathcal{R}$ $\mathcal{A} = \{\chi_\Delta \mid \Delta \in \xi\}''$ with the properties: (i) $\mathcal{A} \subseteq \alpha(t)[\mathcal{A}]$ $\forall t \geq 0$; (ii) $\bigcap_{t\in R} \alpha(t)[\mathcal{A}] = CI$; (iii) $\bigvee_{t\in R} \alpha(t)[\mathcal{A}] = \mathcal{R}$.

Thirdly we notice that a standard representation theorem allows to get back to the original definition from $(\mathcal{R}, \phi, \alpha, \mathcal{A})$ where: $\mathcal{R}$ is an abelian von Neumann algebra acting on a separable Hilbert space $\mathfrak{H}$, ϕ is a faithful normal state on $\mathcal{R}$, α: $R \longrightarrow \mathrm{Aut}(\mathcal{R}, \phi)$, and $\mathcal{A}$ is a completely selfrefining, generating von Neumann subalgebra of $\mathcal{R}$.

Fourthly the generalisation to the noncommutative domain now consists exactly in taking the above as an alternative definition of a classical K-flow, and in dropping from this definition the condition that $\mathcal{R}$ be abelian. For reasons which would be too long to make explicit here we also impose that $\mathcal{A}$ be stable under the modular automorphism group $\{\sigma_\phi(t) \mid t \in R\}$ canonically associated to ϕ, and that every maximal abelian subalgebra $\mathfrak{X}$ of the centralizer $\mathcal{R}_\phi$ of $\mathcal{R}$ be already maximal abelian in $\mathcal{R}$. (Notice that both of the last two conditions are redundant when $\mathcal{R}$ is abelian since $\sigma_\phi(t) = id$ $\forall$ $t \in R$ in this case.)

We now can emphasize [1] that algebraic proofs can be given to several theorems which are well-known in the classical case, and which thus do generalize to the new situation just defined. For instance the system $(\mathcal{N}, \Phi, \alpha, \alpha)$ is ergodic (i.e. $N \in \mathcal{N}$ and $\alpha(t)[N] = N \quad \forall t \in R \Rightarrow N = \lambda I$ with $\lambda \in C$); it is mixing (i.e. $\langle \Phi; N\alpha(t)[M] \rangle \to \langle \Phi; N \rangle \langle \Phi; M \rangle$ as $t \to \pm\infty$ for all $N, M \in \mathcal{N}$); it has Lebesgue spectrum (i.e. $\alpha(t)$ is spatial and the generator H of the corresponding unitary group on $\mathfrak{H}$ has the property $Sp(H) = Sp_d(H) \cup Sp_{ac}(H)$ with $Sp_d(H) = \{0\}$ simple and $Sp_{ac}(H) = R$ has countable multiplicity). Furthermore a noncommutative entropy can be defined [2] which is strictly positive for all such systems.

We next remark that the generalisation is genuine in the sense that, in addition to the classical case where $\mathcal{N}$ is abelian, there exist [1,3,4] K-flows where $\mathcal{N}$ is of type II_1 , III_λ $(0 < \lambda < 1)$, or III_1.

Finally a link has been established [3] between certain quantum transport phenomena, governed by an evolution equation of the diffusion type, and the generalised K-flows presented here.

References

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[4] in preparation

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