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ERGODIC AUTOMORPHISMS OF COMPACT METRIC GROUPS ARE ISOMORPHIC TO BERNOULLI SHIFTS

NOBUO AOKI

I wish to discuss its title. Let X be a compact metric group and μ be its normalized Haar measure. Then (X, μ) is a Lebesgue space. Let σ be an automorphism of X, then σ is an invertible measure preserving transformation of X onto itself. Our problem is concerned with measure theoretic properties of σ.

Throughout this given a transformation of any group, a restriction on a subgroup and an induced transformation on a factor space will be denoted by the same symbol as that of the original transformation, if there is no danger of confusion.

Today we will outline a proof of the following

Theorem 1. An ergodic automorphism of a compact metric abelian group is a Bernoulli shift.

The result has recieved the most attension in the literature. In a two-dimensional torus, Adler and Weiss [1] proved the result using Ornstein's Theorems. In recently, Katznelson [4] showed the result in an n-dimensional torus. Lind [5] gave a proof for the case of an infinite-dimensional torus. The proof which Totoki and the author [2] proved was done independently of Lind's work. The techniques I use are due to Katznelson [4] and Totoki and the author [2].

In order to outline Theorem 1, we prepare the following

Proposition 1. Let X be a compact metric abelian group and σ be an ergodic automorphism of X. Then there exist subgroups X_D, X_A and X_B such that X_D is a σ-invariant totally disconnected subgroup, X_A and X_B are σ-invariant connected subgroups of X and dynamical systems (X_D, σ), (X_A, σ), (X_B, σ) are ergodic, and further (X, σ) is an algebraic factor of $(X_D \otimes X_A \otimes X_B, \sigma \otimes \sigma \otimes \sigma)$.

The proof uses the results of Entropy Theory together with the results of Group Theory.

Proposition 2. The dynamical systems (X_B, σ) and (X_D, σ) have the Bernoulli properties.

We can prove that X_B is locally connected and so by [2] we have (X_B, σ) has the Bernoulli properties. The Bernoulli properties of (X_D, σ) is an indirect application of the results of Yuzvinskii [12].

To show the dynamical system (X_A, σ) has the Bernoulli properties, let G_A be the character group of X_A and U be the dual automorphism of G_A induced by $(Ug)(x) = g(\sigma^{-1}x)$ for $g \in G_A$. Each $g \in G_A$ satisfies the following condition

(A) There exist integers $k > 0$, $n_o, n_1, \dots, n_k$ such that $(n_o, n_1, \dots, n_k) \neq (0, 0, \dots, 0)$ and

$$g^{n_o} Ug^{n_1} \dots U^k g^{n_k} = 1 .$$

We denote by $\bar{G}_A$ the minimal divisible extension of G_A and by $\bar{U}$ the automorphism of $\bar{G}_A$ extended by U. If $\bar{X}_A$ is the dual group of $\bar{G}_A$, then $\bar{X}_A$ is a compact connected metric abelian group. If $\bar{\sigma}$ is the dual automorphism of $\bar{X}_A$ induced by $\bar{U}^{-1}$, then as (X_A, σ) has the ergodic properties, it is not hard to see that $(\bar{X}_A, \bar{\sigma})$ has the ergodic properties. Since $G_A \subset \bar{G}_A$, let us define $X_1' = \mathrm{ann}(G_A, \bar{X}_A)$, then the dynamical system $(\bar{X}_A/X_1', \bar{\sigma})$ and (X_A, σ) are isomorphic. And so if $(\bar{X}_A, \bar{\sigma})$ has the Bernoulli properties, then Ornstein's theorems imply that (X_A, σ) has the Bernoulli properties. Therefore, using Propositions 1 and 2 it follows that (X, σ) has the Bernoulli properties.

We resolve this difficult with some lemmas.

Let $\bar{G}_A = \{f_1, f_2, \dots\}$. We denote by G_n the subgroup of $\bar{G}_A$ generated by $\{\bar{U}^j f_k : -\infty < j < \infty, \ k = 1,2, \dots, n\}$ for $n \geq 1$. Then we have $\mathrm{rank}(G_n) < \infty$ for $n \geq 1$. Let $X_n = \mathrm{ann}(G_n, \bar{X}_A)$ for $n \geq 1$, then we have $\bar{\sigma} X_n = X_n$, $n \geq 1$ and $X_1 \supset X_2 \supset \dots \supset \bigcap_{n=1}^{\infty} X_n = \{e\}$. Since each f_k satisfies the condition (A), we have for each $k \geq 1$

$$f_k^{\ell_k} = \bar{U} f_k^{m_1(k)} \dots \bar{U}^{p_k} f_k^{m_{p_k}(k)}$$

for some $p_k > 0$, $\ell_k > 0$ and some $(m_1(k), \dots, m_{p_k}(k)) \neq (0, \dots, 0)$.

From now on, we fix $\ell_1, \dots, \ell_n$ and put

(1) $n_o = \ell_1 \ldots \ell_n$, $k_o = \max_{1 \leqslant k \leqslant n} p_k$.

H denotes the subgroup of G_n generated by $\{f_k, \ldots, \bar{U}^{k_o} f_k : 1 \leqslant k \leqslant n\}$. Then H is finitely generated, torsionfree, $\bar{H} = \bar{G}_n$ and $\prod_{j=-\infty}^{\infty} \bar{U}^j H = G_n$. Let $X(H) = \mathrm{ann}(H, \bar{X}_A)$, then the character group of $\bar{X}_A/X(H)$ is H and so $\bar{X}_A/X(H)$ is a finite-dimensional torus.

Lemma 1. If $n_o = 1$, then $(\bar{X}_A/X_n, \bar{\sigma})$ has the Bernoulli properties.

The proof is a direct application of the result of Katznelson [4].

Lemma 2. If $n_o > 1$, then $\eta(x) = x^{n_o}$, $x \in \bar{X}_A$, is an automorphism of $\bar{X}_A$ such that $\eta\bar{\sigma} X(H) \subset X(H)$, and the induced factor $\eta\bar{\sigma}$ on $\bar{X}_A/X_n$ is ergodic.

The details of the proof are found in Chapter 1 of [13].

This lemma is essentially utilized in the proof of Lemma 13.

To obtain $(\bar{X}_A/X_n, \bar{\sigma})$ has the Bernoulli properties whenever $n_o > 1$, we construct a sequence $\{\mathcal{P}_n\}$ of weak Bernoulli partitions for the dynamical system $(\bar{X}_A/X_n, \bar{\sigma})$ such that $\mathcal{P}_n < \mathcal{P}_{n+1}$ for $n \geqslant 1$ and $\bigvee_n \mathcal{P}_n$ is the partition of $\bar{X}_A/X_n$ into single points.

Let M be a positive integer and let $\mathcal{P}$ be a partition of the interval $[0, 2\pi)$ into subintervals of the same lengths $2\pi/M$. The elements of $\mathcal{P}$ will denote successively from the left by $p_j = [a_j, a_{j+1})$, $j = 1, \ldots, M$. Let K be an arbitrarily fixed positive integer and set $N_K = n_o^{K^2+K}$ (n_o is the integer satisfying the (1)) For $k > 0$ $K_k(t)$ will denote the Fejér kernel defined on $[0, 2\pi)$.

Lemma 3. Let M be a fixed positive integer. Then there exists a positive integer $\ell = \ell(M)$ such that for each $m \geqslant 2$ there is a positive number $\delta_m = \delta(m,M)$ satisfying the following :

$$(1 + m^{-2})(1 - \frac{8}{(m^{\ell} - 1)\,\delta_m^2}) \geqslant 1 ,$$

$$3^{M+1} M^2 \delta_m < 1/m^2 .$$

The proof is elementary.

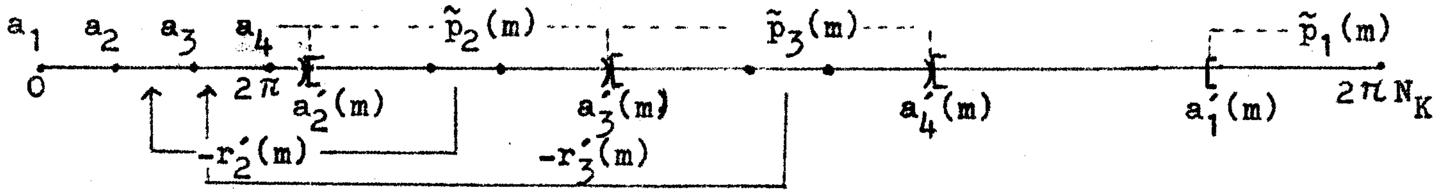
Lemma 4. Let M and δ_m be as in Lemma 3. Then

$$2\pi + 2M(N_k - 1)\delta_m < 2\pi N_K \quad (2 \leqslant m \leqslant K).$$

The proof is clear from Lemma 3.

From Lemma 4 we can manipulate as follows. For an arbitrary m such that $2 \leqslant m \leqslant K$, set $p_j(m) = [b_j, c_j)$, mod $2\pi N_K$, where $b_j = a_j - (N_K - 1)\delta_m$, $c_j = a_{j+1} + (N_K - 1)\delta_m$ $(1 \leqslant j \leqslant M)$. We translate each $p_j(m)$ by $r'_j(m)$ $(r'_1(m) = 0)$ to the right so that $c_j + r'_j(m) = b_{j+1} + r'_{j+1}(m)$ and denote the translated $p_j(m)$ by $\tilde{p}_j(m)$. Then $\tilde{p}_1(m), \dots, \tilde{p}_M(m)$ are disjoint and each $\tilde{p}_{j+1}(m)$ borders on $\tilde{p}_j(m)$ from the right and hence we may set $\tilde{p}_j(m) = [a'_j(m), a'_{j+1}(m))$ for $j = 1, 2, \dots, M$.

In the case $M = 3$ we have for instance following figure.



Noting $N_K > 2$, we have clearly

(2) $$2\delta_m < r'_j(m) < 2(j - 1)(N_K - 1)\delta_m + 2\pi j/M$$
$$(2 \leqslant m \leqslant K, \quad 2 \leqslant j \leqslant M).$$

<u>Lemma 5</u>. Let M and δ_m be as in Lemma 3. Then there exists a positive integer $K_o = K_o(M)$ such that for $K > K_o$

$$2\pi + 2M(N_K - 1)\delta_m + 3^{M+1}Mr'_M(m) < 2\pi N_K \quad (2 \leqslant m \leqslant K).$$

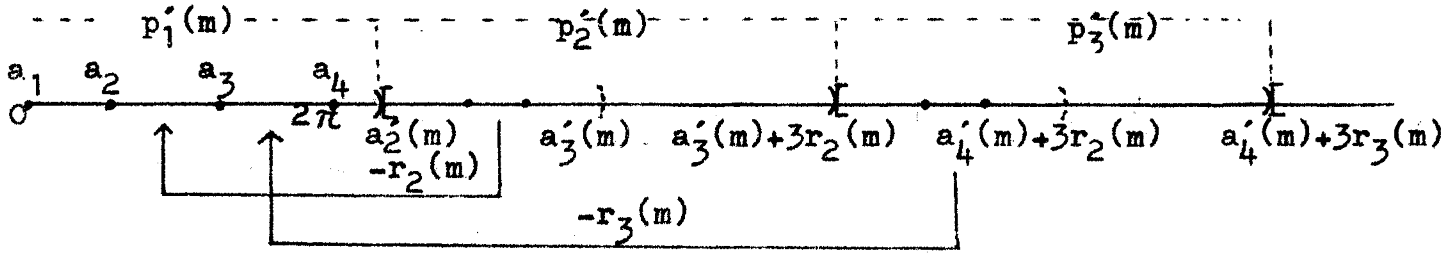
The proof is elementary.

We set for m such that $2 \leqslant m \leqslant K$ for $K > K_o$

(3) $$\begin{cases} r_1(m) = r'_1(m) = 0, \\ r_j(m) = r'_j(m) + 3r_{j-1}(m) \quad (2 \leqslant j \leqslant M) \\ p'_1(m) = \tilde{p}_1(m), \\ p'_j(m) = [a'_j(m) + 3r_{j-1}(m), a'_{j+1}(m) + 3r_j(m)) \; (2 \leqslant j \leqslant M) \end{cases}$$

Then $p'_1(m), \dots, p'_M(m)$ are disjoint and each $p'_{j+1}(m)$ borders on $p'_j(m)$ from the right.

In the case $M = 3$ we have for instance the following figure.



We have for m with $2 \leq m \leq K$ for $K > K_0$

$$3(r_2(m) + \dots + r_M(m)) < 3^{M+1} M r'_M(m).$$

Hence by Lemma 5 we have for $K > K_0$

$$\bigcup_{j=1}^{M} p'_j(m) \subset [0, 2\pi N_K) \quad (2 \leq m \leq K).$$

Lemma 6. Let M and δ_m be as in Lemma 3. Then there exists a positive integer $K_1 = K_1(M) > K_0$ such that for $K > K_1$

$$\delta_m + r_j(m)/N_K < 2\pi \quad (2 \leq m \leq K, \quad 1 \leq j \leq M).$$

The proof uses (2), (3) and Lemma 3.

Lemma 7. Let δ_m be as in Lemma 3. Then for m satisfying $m > \max(2, \sqrt{M/\pi})$ and $p_j = [a_j, a_{j+1}) \in \wp$, $j = 1, 2, \dots, M$,

$$[a_j + \delta_m, a_{j+1} - \delta_m) \neq \phi.$$

The proof is elementary.

Lemma 8. We consider the characteristic function $\chi_{p'_j(m)}$ of $p'_j(m)$ as a $2\pi N_K$-cyclic function on R^1. Then for $K > \max(2, K_1, \sqrt{M/\pi})$ and $N_K t \in [a_j + \delta_m, a_{j+1} - \delta_m)$ for $t \in [0, 2\pi)$,

$$\chi_{p'_j(m)}(N_K t + r_j(m) - s) = 1 \quad (\max(2, \sqrt{M/\pi}) < m \leq K, \quad 1 \leq j \leq M)$$

if $0 < s < N_K \delta_m$ or $2\pi N_K - N_K \delta_m - 2r_j(m) < s < 2\pi N_K$.

The lemma follows from (2), (3) and Lemma 7.

Lemma 9. Let ℓ and δ_m be as in Lemma 3 and let K_1 be as in Lemma 6. Then if $K > \max(2, K_1, \sqrt{M/\pi})$, for each m $(\max(2, \sqrt{M/\pi}) < m \leq K)$ and each $p_j \in \wp$ there exists a non-negative function $\tilde{f}_{mp_j}(t)$ on $[0, 2\pi)$ satisfying the following :

$$\tilde{f}_{mp_j}(t) \geq 1 \quad t \in [a_j + \delta_m, a_{j+1} - \delta_m),$$

for some constants c_0, c_k, c'_k $(k = 1, 2, \dots, m^{\ell})$

$$\tilde{f}_{mp_j}(t) = c_0 + \sum_{k=1}^{m^{\ell}} c_k \tilde{e}^{i(k/N_K, t)} + \sum_{k=1}^{m^{\ell}} c'_k \tilde{e}^{-i(k/N_K, t)},$$

where $\tilde{e}^{i(k/N_K, t)} = e^{ik(t/N_K)}$ and $\tilde{e}^{-i(k/N_K, t)} = e^{-ik(t/N_K)}$ for k,

$$\sum_{j=1}^{M} \tilde{f}_{mp_j}(t) \leq 1 + m^{-2} \quad t \in [0, 2\pi).$$

The proof uses the results of Lemmas 3 ~ 8 .

We denote by $e^{i(m,t)}$ an exponential function e^{mti} for m .

Lemma 10. Let ℓ and δ_m be as in Lemma 3. For each $p_j \in \mathfrak{p}$ and $K > \max(2,\sqrt{M/\pi})$ we define for $t \in [0, 2\pi)$

$$f_{mp_j}(t) = (1 + m^{-2})\hat{f}_{mp_j}(t) \quad (\max(2,\sqrt{M/\pi}) < m \leqslant K)$$

where

$$\hat{f}_{mp_j}(t) = 1/\pi \int_0^{2\pi} \chi_{p_j}(t - s) K_{m^\ell}(s)\, ds .$$

Then $f_{mp_j}(t)$ is a non-negative function on $[0, 2\pi)$ and we have the following :

$$f_{mp_j}(t) \geqslant 1 \quad t \in [a_j + \delta_m, a_{j+1} - \delta_m) ,$$

for some constants d_o, d_k, d_k' $(k = 1,2, \ldots , m)$

$$f_{mp_j}(t) = d_o + \sum_{k=1}^{m^\ell} d_k\, e^{i(k,\, t)} + \sum_{k=1}^{m^\ell} d_k'\, e^{-i(k,\, t)} ,$$

$$\sum_{j=1}^{M} f_{mp_j}(t) \leq 1 + m^{-2} \qquad t \in [0, 2\pi) .$$

The proof is direct from Katznelson [4].

We can generalize easily Lemmas 9 and 10 on a finite-dimensional torus

We assume that $\overline{X}_A/X(H)$ is r-dimensional. $\overline{X}_A/X(H)$ is algebraically isomorphic to $T^r = [0, 2\pi)^r$ whose character group is the discrete group

$$H_r = \{ e^{i(m,\, \cdot)} : m \in Z^r\}^{1)}$$

which is algebraically isomorphic to H. $\overline{H}_r$ denotes a multiplicative group $\{ e^{i(q,\, \cdot)} : q \in Q^r\}^{2)}$ which is a minimal divisible extension of H_r . Now let Y^r denote the dual group of $\overline{H}_r$, and we denote by $(e^{i(q,\, \cdot)})(y)$, $y \in Y^r$ each character $e^{i(q,\, \cdot)}$ of Y^r. We note that for $m \in Z^r$

$$(e^{i(m,\, \cdot)})(Py) = e^{i(m,\, Py)}$$

where P is the projection from Y^r onto T^r.

From the definitions of G_n and H, we have $\overline{G}_n = \overline{H}$. Thus as H_r and H are algebraically isomorphic, we have the diagram

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1) Z^r is the set of all r-dimensional integer vectors.

2) Q^r is the set of all r-dimensional rational vectors.

$$\begin{array}{ccc} \bar{X}_A/\bar{X}_n{}^{3)} & \cong & Y^r \\ & \searrow \quad \swarrow_{P} & \\ & T^r & \end{array}$$

Let τ and ξ be automorphisms of Y^r isomorphic to $\bar{\sigma}$ and η of $\bar{X}_A/\bar{X}_n$ respectively. Then $\tau\xi$ induces the endomorphism of T^r (wrtten by the same symbol $\tau\xi$) because $\eta\bar{\sigma}$ induces the endomorphism of $\bar{X}_A/X(H)$ and $\tau\xi$ is given by an $r \times r$ matrix with intger entries.

U_P denotes the linear operator from $\mathcal{C}(T^r)$ into $\mathcal{C}(Y^r)$ defined by $(U_P g)(y) = g(Py)$ for $g \in \mathcal{C}(T^r)$. The adjoint operator $\tau\xi$ on Z^r of the endomorphism $\tau\xi$ on T^r is defined by

$$e^{i(m,\ \tau\xi Py)} = e^{i(\tau\xi m,\ Py)}, \quad m \in Z^r .$$

We have for $\lambda \in Z^r$

$$(4) \qquad U_\tau^{-1}(U_P\, \tilde{e}^{i(\lambda/N_K,\cdot)})(y) = (U_P\, \tilde{e}^{i(\tau\lambda/N_K,\cdot)})(y), \ y \in Y^r.$$

Now the eigenvalue of ξ is n_o with multiplicity r . Let the eigenvalues of τ be $\lambda_1, \ldots, \lambda_k$ $(k \leqslant r)$, then the eigenvalues of $\tau\xi$ are $n_o\lambda_1, \ldots, n_o\lambda_k$.

We may consider the matrix τ as operating on R^r and so we decompose

$$R^r = V_{-k} \oplus \ldots \oplus V_o \oplus \ldots \oplus V_q$$

such that each V_j is the τ-invariant subspace of R^r corresponding to the eigenvalues of τ of modulus ρ_j' where $\rho_{-k}' < \ldots < 1 = \rho_o' < \ldots < \rho_q'$. Let $\rho_j = n_o\rho_j'$ and let $\tilde{V}_o = V_{-k} \oplus \ldots \oplus V_{-k'}$ $(k' \geqslant 0)$ be the direct sum of V_j's corresponding to ρ_j' such that $\rho_j \leqslant 1$. Then as V_j is $\tau\xi$-invariant and $\tau\xi$ is ergodic on T^r, we have

$$\tilde{V}_o \cap Z^r = \{0\}.$$

Let M be an arbitrarily fixed positive integer. Now let $\mathcal{P}$ be a partition of T^r $(= [0, 2\pi)^r)$ such that $\mathcal{P} = \otimes_{k=1}^{r} \mathcal{P}^{(k)}$, each $\mathcal{P}^{(k)}$ being a partition of $[0, 2\pi)$ into subintervals of the same lengths $2\pi/M$

For arbitrarily fixed $K > 0$ and $N > 0$ we set

$$\mathfrak{A}(K) = \bigvee_{m=1}^{K^2} \tau^{-m}P^{-1}(\mathcal{P}), \quad \mathcal{B}(K,N) = \bigvee_{m=K^2+K}^{K^2+K+N} \tau^{-m}P^{-1}(\mathcal{P}).$$

<u>Lemma 11.</u> For a sufficiently large $\ell = \ell(\mathcal{P})$ there exists

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3) $\bar{X}_n$ is the annihilator of $\bar{G}_n$ in $\bar{X}_A$.

a measurable set E_m with measure $< 1/m^2$ for each $m > \max(2, \sqrt{M/\pi})$ such that for each $p \in \wp$ there are non-negative functions $\tilde{f}_{mp}$, f_{mp} on T^r satisfying :

(a) there is a positive integer $K_1 = K_1(\wp)$ such that if $K > K_1$ and $K \geqslant m$, then

$$\tilde{f}_{mp}(t) \geqslant 1 \quad \text{on} \quad p - E_m ,$$

$$\tilde{f}_{mp}(t) = C_o + \sum_{k\,:\,\substack{1 \leqslant k_j \leqslant m^\ell \\ 1 \leqslant j \leqslant r}} [C_k \tilde{e}^{i(k/N_K,\, t)} + C_k' \tilde{e}^{-i(k/N_K,\, t)}], \quad t \in T^r,$$

where $k = (k_1, \ldots , k_r)$ are vectors of positive integers, C_o, C_k, C_k' are some constants and $\tilde{e}^{\pm i(k/N_K,\, t)} = e^{\pm ik(t/N_K)}$,

$$\sum_{p \in \wp} \tilde{f}_{mp}(t) \leqslant 1 + m^{-2} \qquad t \in T^r,$$

(b)

$$f_{mp}(t) \geqslant 1 \quad \text{on} \quad p - E_m ,$$

$$f_{mp}(t) = D_o + \sum_{k\,:\,\substack{1 \leqslant k_j \leqslant m^\ell \\ 1 \leqslant j \leqslant r}} [D_k e^{i(k,\, t)} + D_k' e^{-i(k,\, t)}] , \quad t \in T^r,$$

where k are as in (a) and D_o, D_k, D_k' are constants,

$$\sum_{p \in \wp} f_{mp}(t) \leqslant 1 + m^{-2} \qquad t \in T^r .$$

This lemma is a generalization of Lemmas 9 and 10.

Define the following functions ϕ_A and ϕ_B on Y^r for $A \in \mathfrak{A}(K)$ and $B \in \mathfrak{B}(K, N)$ where $K > K_1$,

$$\phi_A = \prod_{m=1}^{K^2} U_\tau^{-m} U_P \tilde{f}_{Kp_m}(A) , \quad \phi_B = \prod_{m=K^2+K}^{K^2+K+N} U_\tau^{-m} U_P f_{mp_m}(B) .$$

We denote by μ the normalized Haar measure on Y^r. Then we have the following

Lemma 12. Let $\varepsilon > 0$. Then there exists a positive integer $K_2 = K_2(\wp, \varepsilon) > \max(2, K_1, \sqrt{M/\pi})$ and a measurable set E in Y^r such that $\mu(E) < \varepsilon^2$ and for every $K > K_2$ and arbitrary $N > 0$

$$\phi_A \geqslant 1 \quad \text{on} \quad A - E , \quad \sum_{A \in \mathfrak{A}(K)} \int \phi_A \, d\mu \leqslant 1 + \varepsilon^2$$

and

$\Phi_B \geqslant 1$ on $B - E$, $\sum_{B \in \mathcal{B}(K,N)} \int \Phi_B \, d\mu \leqslant 1 + \varepsilon^2$.

The lemma follows from Lemma 11.

Let ℓ be as in Lemma 11. For arbitrary fixed $K > K_3$ and $N > 0$ we denote by $\tilde{Z}^r$ and $\tilde{Q}^r$ sets of all r-dimensional vectors consisting of $\{1, 2, \ldots, (K^2+K+N)^\ell\}$ and $\{1/N_K, 2/N_K, \ldots, K^\ell/N_K\}$ respectively.

Now we define an automorphism κ of Y^r by

$$\kappa y = n_0^{K^2+K+N} N_K y, \quad y \in Y^r.$$

Since for $\lambda \in Z^r$ (4) holds, we have that

$$\{(U_p \tilde{e}^{i(\sum_{m=1}^{K^2} \tau^m \lambda_m, \cdot)})(\kappa y) : \lambda_m \in \tilde{Q}^r \cup [-\tilde{Q}^r]\}$$

$$\cup \{(U_p e^{i(\sum_{m=K^2+K}^{K^2+K+N} \tau^m \lambda_m, \cdot)})(\kappa y) : \lambda_m \in \tilde{Z}^r \cup [-\tilde{Z}^r]\}$$

is a set of characters of Y^r. Further we can prove that the frequency which is common to Φ_A and Φ_B is zero for sufficiently large K. From those facts we have

Lemma 13. There exists a positive integer $K_3 > K_1$ such that for $K > K_3$ and $N > 0$

$$\int \Phi_A \Phi_B \, d\mu = \int \Phi_A \, d\mu \int \Phi_B \, d\mu.$$

Using results of Katznelson [4] and Lemmas 12 and 13, we have

Lemma 14. For $\varepsilon > 0$ there exists $\tilde{K} > \max(K_3, K_2)$ such that $\mathfrak{A}(K)$ and $\mathcal{B}(K,N)$ are 11ε-independent for $K > \tilde{K}$ and $N > 0$.

Consequently $P^{-1}(\mathfrak{p})$ is an weak Bernoulli partition on Y^r for τ. Let $\mathfrak{p}'$ be the partition of $\bar{X}_A/X(H)$ corresponding to the partition $\mathfrak{p}$ of T^r and P' be the projection of $\bar{X}_A$ onto $\bar{X}_A/X(H)$, then $P'^{-1}(\mathfrak{p}')$ is an weak Bernoulli partition on $\bar{X}_A$ for $\bar{\sigma}$. Because we have $\bigcap_{j \in Z} \bar{\sigma}^j X(H) = X_n$, $\bigvee_{\mathfrak{p}} \bigvee_{j \in Z} \bar{\sigma}^j P'^{-1}(\mathfrak{p}')$ is the partition of $\bar{X}_A$ into cosets of X_n. By Ornstein 's theorem $(\bar{X}_A/X_n, \bar{\sigma})$ has the Bernoulli partitions.

Therefore, for $n \geqslant 1$ we have showed that $(\bar{X}_A/X_n, \bar{\sigma})$ has the Bernoulli properties. Since we have

$$X_1 \supset X_2 \supset \ldots \supset \bigcap_{n=1}^{\infty} X_n = \{e\},$$

Ornstein 's theorem implies that σ on $\bar{X}_A$ is Bernoullian. (X_A, σ) has the Bernoulli properties.

We can conclude that (X, σ) has the Bernoulli properties.

Using Theorem 1, I can prove the result for the case of non-abelian. Today I do not discusse it here, but the proof of it is found in [13].

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