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A COMMON GENERALIZATION OF TOPOLOGICAL AND
MEASURE-THEORETIC ENTROPY

Günther PALM

Nowadays ergodic theory is split into two branches: measure-theoretic and topological, according to the methods used. In both branches there are similar results, proved using similar **ideas**. Therefore it is natural to look for a common generalization.

For theorems connecting spectral and mixing properties of dynamical systems Nagel [2], [3], [4] has found an appropriate generalization in terms of Banach lattices: an abstract dynamical system is a triple (E, u, T) , where E is a Banach lattice with quasi-interior point $u \in E_+$ and $T: E \rightarrow E$ is a lattice homomorphism satisfying $Tu = u$ (this definition is slightly different from that given in [2]).

For theorems concerning entropy and related questions, other mathematical structures are used: If one looks into the entropy sections of Walters' book [8], for example, the measure-theoretical and topological proofs of many analogous theorems look very similar and these proofs are based on lattice methods. Therefore I have defined entropy for a dynamical lattice (see definition 1.1.).

This definition has two advantages:

- 1) Given an abstract dynamical system (E, u, T) , the lattice of all closed ideals in E yields a dynamical lattice (see 1.3.), whose entropy reduces to the usual entropy in both the measure-theoretic and the topological case (see 1.4.).

2) In this definition of entropy it is necessary to define the entropy for not necessarily disjoint covers, even in the measure-theoretic case. But this fact allows an easy proof of Goodwyn's theorem [1] by means of a generalized version of the Kolmogoroff-Sinai theorem (see 3.4.).

In the following I want to give the basic definitions and theorems for the entropy of dynamical lattices and to sketch the proof of Goodwyn's theorem.

1. dynamical lattices

1.1. Definition: A dynamical lattice is a triple (V, m, f) , where

V is a distributive lattice with 0 and 1,

$m: V \rightarrow \mathbb{R}_+$ satisfies $m(0) = 0$ and:

$m(a) = 0 \Rightarrow m(a \vee b) = m(b)$ for every $a, b \in V$,

$f: V \rightarrow V$ satisfies $f(0) = 0, f(1) = 1$ and:

$m(a) = 0 \Rightarrow m(f(a)) = 0$ for every $a \in V$.

1.2. Definition: Two dynamical lattices (V, m, f) and (V', m', f') are

called isomorphic, if there is a lattice isomorphism

$\phi: V \rightarrow V'$ satisfying $\phi \circ f = f' \circ \phi$ and $m' \circ \phi = m$.

1.3. Definition: Let (E, u, T) be an abstract dynamical system.

Let V be the lattice of all closed (lattice-)ideals in E (see [6])

$$m: \begin{cases} V \rightarrow \mathbb{R}_+ \\ I \mapsto \sup\{\#x\# : x \in I \wedge [0, u]\} \end{cases}, \quad f: \begin{cases} V \rightarrow V \\ I \mapsto \langle T(I) \rangle; \end{cases}$$

where $\langle A \rangle$ denotes the closed ideal generated by A .

Then (V, m, f) is called the dynamical lattice of closed ideals

associated to (E, u, T) .

By the entropy of (E, u, T) we mean the entropy of the associated dynamical lattice of closed ideals.

1.4.: In the topological case we have a topological dynamical system (X, φ) , i.e. a compact Hausdorff space X and a continuous mapping $\varphi: X \rightarrow X$. Here we set $E := C(X)$, $u = 1$ and $T(f) := f \circ \varphi$. For this abstract dynamical system we get (using 1.3.) $V = \{\text{open sets in } X\}$, $m(a) = m_1(a) := \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{if } a \neq 0 \end{cases}$ and $f = \varphi^{-1}$. In the measure-theoretic case we have a dynamical system $(X, \Sigma, \mu, \varphi)$, i.e. a probability space (X, Σ, μ) and a measurable measure-preserving mapping $\varphi: X \rightarrow X$. Here we set $E := L^1(X, \Sigma, \mu)$ and again $u = 1$, $T(f) := f \circ \varphi$. For this abstract dynamical system we get V isomorphic to the measure-algebra Σ/\mathcal{N} (\mathcal{N} denoting the μ -nullsets), $m = \mu$ and $f = \varphi^{-1}$.

2. entropy

2.1. Definition: Let (V, m, f) be a dynamical lattice.

- 1) A finite subset α of V is called a cover, if $\sup \alpha = 1$.
- 2) The set \tilde{V} of all covers is ordered by:

$$\alpha \leq \beta \text{ (}\beta \text{ is a } \underline{\text{refinement}} \text{ of } \alpha \text{) if and only if for every } b \in \beta$$

there is an $a \in \alpha$ such that $b \leq a$.
- 3) $\alpha \vee \beta := \{a \wedge b : a \in \alpha, b \in \beta\}$ and $\alpha^n := \bigvee_{i=0}^{n-1} f^i(\alpha)$.
- 4) Let α be a cover and $k := \sum_{a \in \alpha} m(a)$, then we set

$$h^*(\alpha) := - \sum_{a \in \alpha} \frac{m(a)}{k} \log \frac{m(a)}{k}.$$
- 5) $\hat{h}(\alpha) := \sup\{h^*(\beta) : \beta \triangleright \alpha, N(\beta) \leq N(\alpha)\}$, $N(\alpha)$ denoting the number of elements $a \in \alpha$ such that $m(a) \neq 0$.
- 6) $h(\alpha) := \inf\{\sum_{i=1}^n \hat{h}(\beta_i) : \bigvee_{i=1}^n \beta_i \triangleright \alpha, n \in \mathbb{N}\}$.
- 7) $h(f, \alpha) := \underline{\lim} h(\alpha^n)/n$, $H(f, \alpha) := \overline{\lim} h(\alpha^n)/n$.
- 8) $h(V, m, f) := \sup\{h(f, \alpha) : \alpha \in \tilde{V}\}$, $H(V, m, f) := \sup\{H(f, \alpha) : \alpha \in \tilde{V}\}$. $h(V, m, f)$ is called the entropy of (V, m, f) .

2.2. Remarks: a) It can be proved, that in many cases $h(f, \alpha) = H(f, \alpha)$ holds for every cover α [5].

b) Step 5 of the definition should be explained:

In the measure-theoretic case we want to get the measure entropy, therefore it should be sufficient to consider disjoint covers. Now if V is a Boolean algebra and α any cover, there is a disjoint refinement β of α with $N(\beta) \leq N(\alpha)$, but if α is already disjoint, then α is the only such refinement. Therefore in step 6 we have $h(\alpha) = \inf \{ \sum_{i=1}^n \hat{h}(\beta_i) : \prod_{i=1}^n \beta_i \leq \alpha, \beta_i \text{ disjoint}, n \in \mathbb{N} \}$ and $\hat{h}(\beta) = h^*(\beta)$ for disjoint β .

c) In this general context the entropy still has many of the well-known properties of the usual entropies:

2.3. Theorem [5]: a) If (V, m, f) and (V', m', f') are isomorphic, they have the same entropy.

b) Let (V, m, f) be a dynamical lattice, where f is a lattice isomorphism such that $m \circ f = m$, then $h(V, m, f) = H(V, m, f)$ and $h(V, m, f^k) = |k| \cdot h(V, m, f)$ for $k \in \mathbb{Z}$

c) In the topological case (see 1.4.) $h(V, m, f)$ is equal to the topological entropy.

d) In the measure-theoretic case $h(V, m, f)$ is equal to the measure entropy.

3. generators

Let me define pseudometrics on V and \tilde{V} :

3.1. Definition: a) Given $a, b \in V$ let: $\delta(a, b) := \inf \{ m(d) : d \vee a = d \vee b \}$.

b) Given $\alpha, \beta \in \tilde{V}$ with $|\alpha| \leq |\beta|$ (say: let $d(\alpha, \beta) = d(\beta, \alpha) := \inf \{ \sum_{a \in \alpha} \delta(a, \pi(a)) + \sum_{b \in \beta} m(b) : \pi: \alpha \rightarrow \beta \text{ injective} \}$).

3.2. Definition: Given two covers α, β I shall write $\alpha \leq \beta$, if there is a cover α' satisfying $d(\alpha, \alpha') < \epsilon$ and $\alpha' \leq \beta$.

3.3. Definition: A cover β is called a generator, if for every cover α and every $\epsilon > 0$ there is $n \in \mathbb{N}$ such that $\alpha \leq \beta^n$.

A subset W of V is called generating, if for every cover α and every $\epsilon > 0$ there is a cover $\beta \subseteq W$ such that $\alpha \leq \beta$.

With these notions we can prove a generalized version of the well-known Kolmogoroff-Sinai theorem (along the lines of [7], see especially Lemma 5.8) [5].

3.4. Theorem: Let (V, m, f) be a dynamical lattice, V a Boolean algebra, m monotone ($a \leq b \Rightarrow m(a) \leq m(b)$) and subadditive ($m(a \vee b) \leq m(a) + m(b)$) and $m \circ f = m$, then

a) $h(f, \beta) = h(V, m, f)$ for every generator β .

b) $h(V, m, f) = \sup\{h(f, \beta) : \beta \in \tilde{V}, \beta \subseteq W\}$ for every generating $W \subseteq V$.

4. Goodwyns theorem

4.1.: Finally I will sketch a new proof of Goodwyn's theorem [1]:

Given a topological dynamical system (X, φ) and a φ -invariant regular Borel-measure μ on X , the topological entropy h_t of φ is \geq the measure entropy h_μ of φ with respect to μ .

According to 2.3. the topological entropy h_t is $h(V, m_1, f)$, where $V = \{\text{open sets in } X\}$ and $f = \varphi^{-1}$, and the measure entropy is $h(\Sigma, \mu, f)$ where Σ denotes the φ -algebra of Borel-sets.

Since μ is regular, V is a generating subset of Σ . Therefore we have (3.4.b):

$$(*) \quad h(\Sigma, \mu, f) = \sup\{h(f, \alpha) : \alpha \in \tilde{\Sigma}, \alpha \subseteq V\} = \sup\{h(f, \alpha) : \alpha \text{ open cover of } X\}$$

If α is an open cover of X , clearly $h^*(\alpha)$ computed for (V, m_1, f) is $\log N(\alpha)$, which is $\geq h^*(\alpha)$ computed for (Σ, μ, f) .

Therefore $h(f, \alpha)$ computed for (V, m_1, f) is $\geq h(f, \alpha)$ computed for (Σ, μ, f) (according to definition 2.1.).

So we can continue (*):

$$h(\Sigma, \mu, f) = \sup\{h(f, \alpha) : \alpha \in \tilde{\Sigma}, \alpha \leq V\} \leq \sup\{h(f, \alpha) : \alpha \in \tilde{V}\} = h(V, m_1, f).$$

With the same ideas the following generalization of Goodwyn's theorem can be proved [5]:

4.2. Theorem: Let X be a compact Hausdorff space and (E, u, T) an abstract dynamical system satisfying:

- a) $C(X)$ is a dense T -invariant sublattice of E .
- b) The norm of E is order-continuous.
- c) u is the function $1 \in C(X)$.
- d) T is an isometry.

Then $T|_{C(X)}$ corresponds to a homeomorphism $\varphi: X \rightarrow X$ by means of $Tf = f \circ \varphi$, and the topological entropy of φ is \geq the entropy of (E, u, T) .

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