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MINIMAL SETS GENERATED BY A
SUBSTITUTION OF NON CONSTANT LENGTH

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Let P denote the alphabet $\{0,1\}$ and let

$$\Omega = P^{\mathbb{Z}} = \{\omega = \dots \omega_{-1} \omega_0 \omega_1 \omega_2 \dots \mid \omega_i \in P\}$$

A substitution θ is a map from P to $\bigcup_{n \geq 2} P^n$. If $\text{card } \theta(0) = \text{card } \theta(1)$

the substitution is said of constant length. If not, it is said of non constant length.

For suitable i and j we can generate (1) a bisequence by the following:

$$\omega[-\text{card } \theta^{2n}(i), \text{card } \theta^{2n}(i) + \text{card } \theta^{2n}(j)] = \theta^{2n}(i) \theta^{2n}(j), \forall n \in \mathbb{N}$$

where $\omega(a,k)$ is the k -block of ω which begin at ω_a .

Under a condition of non-degeneracy, the dynamical system (O_ω, T) where O_ω is the orbit closure of ω , and T the shift, is minimal (1) and strictly ergodic (2).

We can prove that the dynamical system associated with the substitution θ_1 defined by

$$\begin{cases} \theta_1(0) = 0^{n+1-p} 1 0^p \\ \theta_1(1) = 1 0^n \end{cases} \quad \text{for } 0 \leq p \leq n,$$

has a purely discrete spectrum. (i^k , $i \in P$, $k \in \mathbb{N}$ means $\underbrace{i \ i \dots \ i}_n$ times).

(1) GOTTSCHALK. Substitution minimal sets. Trans. Amer. Math. Soc. 109 (1963)

(2) MICHEL. Stricte ergodicité d'ensembles minimaux de substitution. CRAS Paris, 278, (1974).

For this, we introduce the notion of "coincidence-value of the substitution θ " by the following :

$$\begin{aligned} \theta^\infty(0) &= \lim_{n \rightarrow \infty} \theta^{2n}(0) = \omega_1^0 \omega_2^0 \dots \omega_n^0 \dots \\ \theta^\infty(1) &= \lim_{n \rightarrow \infty} \theta^{2n}(1) = \omega_1^1 \omega_2^1 \dots \omega_n^1 \dots \end{aligned}$$

we say that n is a "coincidence value of θ " if and only if

$$\omega_n^0 = \omega_n^1$$

If we denote the density of the coincidence values of θ in \mathbb{N} by $\delta_c(\theta)$, we can prove that $\delta_c(\theta_1)$ is equal to one.

Conversely if we consider the substitution θ_2 :

$$\begin{cases} \theta_2(0) = 01 \\ \theta_2(1) = 1100 \end{cases}$$

we have $\delta_c(\theta_2) = \frac{1}{3}$ and we can prove that there is a continuous part in the spectrum of θ_2 .

Moreover, if we consider the two closely related substitutions θ_2' and θ_2'' defined by

$$\theta_2'(0) = \theta_2''(0) = 01$$

and

$$\begin{cases} \theta_2'(1) = 1010 \\ \theta_2''(1) = 1001 \end{cases}$$

we can prove that θ_2' and θ_2'' have purely discrete spectra and that

$$\delta_c(\theta_2') = \delta_c(\theta_2'') = 1.$$

More generally, we conjecture that the substitution θ_n :

$$\begin{aligned} \theta_n(0) &= 0^n 1^n = \underbrace{0\dots 0}_{n \text{ times}} \underbrace{11\dots 1}_{n \text{ times}} \\ \theta_n(1) &= 1^{2n} 0^{2n} = \underbrace{11\dots 1}_{2n \text{ times}} \underbrace{00\dots 0}_{2n \text{ times}} \end{aligned}$$

has :

- 1) a purely discrete spectrum if n is even (then $\delta_c = 1$)
- 2) a mixed spectrum (partly continuous and partly discrete) if n is odd (then $\delta_c = \frac{1}{3}$).

At last, if we consider the constant-length case (3), we know that the so-called

- regular Toeplitz substitution generates a dynamical system with discrete spectrum. In this case $\delta_c = 1$.
- generalized Morse substitution generates a dynamical system with mixed spectrum and in this case $\delta_c = 0$.

Thus we can draw the following table, and we state the question

Is the following true :

$$\delta_c = 1 \implies \text{discrete spectrum}$$

$$\delta_c < 1 \implies \text{mixed spectrum ?}$$

(see the following page)

(3) COVEN and KEANE. The structure of substitution minimal sets. (1971)

SUBSTITUTION		δ_c	SPECTRUM
constant length	Regular Toeplitz substitution $(\theta(0) \neq \widetilde{\theta(1)})$ (1)	1	Discrete
	Generalized Morse substitution $(\theta(0) = \widetilde{\theta(1)})$ (1)	0	Mixed
non constant length	$\begin{cases} \theta(0) = 0^{n+1-p} 1 0^p \\ \theta(1) = 1 0^n \end{cases}$ $0 \leq p \leq n$	1	Discrete
	$\begin{cases} \theta(0) = 01 \\ \theta(1) = 1010 \end{cases}$ and $\begin{cases} \theta(0) = 00 \\ \theta(1) = 1001 \end{cases}$	1	Discrete
	$\begin{cases} \theta(0) = 01 \\ \theta(1) = 1100 \end{cases}$	$\frac{1}{3}$	Mixed
	$\begin{cases} \theta(0) = 0^n 1^n \\ \theta(1) = 1^{2n} 0^{2n} \end{cases}$	n even 1	Discrete
		n odd $\frac{1}{3}$	Mixed
other cases		?	?

(1) with $\widetilde{\omega_k} = 1 - \omega_k$