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Topological Entropy and Homology

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TOPOLOGICAL ENTROPY AND HOMOLOGY

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Let X be a compact metric space and $f:X \to X$ a continuous map.

Definition.- (Bowen)

A (k,δ) spanning set for f is a finite set $Y \subset X$ s.t. $\forall x \in X\ \exists y \in Y$ s.t. $d(f^j x, f^j y) < \delta$ for $0 \leqslant j < k$.

$$h(f,\delta) = \limsup_{k\to\infty} \frac{1}{k}\log \text{card (minimal } (k,\delta) \text{ spanning set)}$$

The topological entropy is defined as :

$$h(f) = \sup_{\delta>0} h(f,\delta)$$

$h(f)$ thus describes the exponential growth rate of the number of types of f-orbits, or roughly speaking the amount f mixes up the points of X. $h(f) = \sup_\mu h_\mu(f)$ where μ ranges over all normalized Borel invariant measures (Dinaburg).

Conjecture.- (Shub)

Let $f:M \to M$ be a diffeomorphism of a smooth compact manifold M. Then $h(f) \geqslant \log \operatorname{sp} f_*$ where $\operatorname{sp} f_*$ is the largest modulus of any eigenvalue of the induced map of homology $f_* : H_*(M;R) \to H_*(M;R)$.

$H_i(M;R)$ is a finite dimensional real vector space describing the i dimensional structure of M, f describes the action of f on this structure and so the conjecture says that a diffeomorphism that mixes the i dimensional structure of M for some i must also have at least the corresponding amount of entropy I.e. the homology of f detects some of the dynamics of f.

The conjecture has been proved for diffeomorphisms satisfying Axiom A and the no cycle property (Shub and Williams). Hence it holds for a C^0 open and dense set of diffeomorphisms of M. For C^0 maps the inequality fails even on the manifold S^2. However, any C^0 map of M satisfies $h(f) \geq \log |\lambda|$ for each eigenvalue λ of f_* in $H_1(M;R)$ the first homology group (Manning).

For eigenvalues in higher groups differentiable hypotheses would usually be required. In this direction Misiurewicz and Przytycki showed recently that any C^1 map $f:S^2 \to S^2$ satisfies $\exp h(f) \geq |\deg(f)|$.

REFERENCE

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