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L_∞-CONVERGENCE OF FINITE ELEMENT APPROXIMATION

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0. Introduction

Projections such as L_2 - or Ritz-approximations are formulated in a Hilbert-space-setting, therefore error estimates are primarily available in Sobolev norms. The derivation of L_∞-estimates is a famous question. We mention the case in which the approximation space consists of trigonometric polynomials of degree n and the result of Faber (see Lorentz [1], p.96): The L_∞-norm of any such projection is (at least) of order $\ln n$.

The situation is better if splines or finite elements are used instead of trigonometric functions, as was shown in Nitsche [1] for linear splines in one dimension and generalized by Douglas-Dupont-Wheeler [1], Wahlbin [1], and Wheeler [1]. In higher dimensions some results are known for uniform resp. rectangular meshes, see Bramble-Nitsche-Schatz, Bramble-Schatz [1], [2], Douglas-Dupont-Wheeler [1] and Strang-Fix [1].

A first L_∞-error estimate for Ritz-approximations and linear finite elements on a general mesh was given by Nitsche [2], but with a loss of convergence-rate depending on the dimension. This was improved by Ciarlet-Raviart [2], still with a loss but independent of the dimension. Natterer [1] gets in two dimensions convergence with a power $2-\varepsilon$ of the mesh-size h. Also in two dimensions but for general finite elements Scott [1] derives optimal L_∞-error estimates. His method consists in a careful analysis of the approximability of Green's function in the L_1-norm.

In this paper we show in §2 the uniform boundedness of L_2-projections in the L_∞-norm. The idea is - similar to Natterer - first to work with weighted Sobolev-norms. In order to illustrate the power of this method we derive in §3 for Ritz-approximations a corresponding boundedness result from which optimal L_∞-estimates follow. The case of linear finite elements (for Ritz-approximations to second order problems) is excluded here, in this case logarithmic factors appear.

1. Notations, finite elements

In the following $\Omega \subseteq R^N$ denotes a bounded domain with boundary $\partial\Omega$ sufficiently smooth. For any $\Omega' \subseteq \Omega$ in the Sobolev spaces $H_k(\Omega') = W_2^k(\Omega')$ we will consider besides the usual norms also weighted semi-norms - with a positive weight-factor p -

$$(1) \qquad \|\nabla^1 v\|_{p.\Omega'} = \Big\{ \sum_{|\alpha|=1} \iint_{\Omega'} p \, |D^\alpha v|^2 \, dx \Big\}^{1/2} .$$

We omit the subscript Ω' in case of $\Omega' = \Omega$. By Γ_h a subdivision of Ω into generalized simplices Δ_i is meant, i.e. Δ_i is a simplex in case Δ_i and $\partial\Omega$ have in common at most a finite number of points and otherwise one of the faces may be curved. Γ_h is called $\varkappa$-regular if for any $\Delta_i \in \Gamma_h$ there are two spheres with radii $\varkappa^{-1}h$ and $\varkappa h$ such that Δ_i contains the one and is contained in the other.

By $H_k' = H_k'(\Gamma_h)$ we denote the space of functions $v \in L_2(\Omega)$ such that the restriction of v to any $\Delta_i \in \Gamma_h$ is in $H_k(\Delta_i)$. Parallel to (1) we introduce the 'broken' semi-norms

$$(2) \qquad \|\nabla^1 v\|_p' = \Big\{ \sum_{\Delta_i \in \Gamma_h} \|\nabla^1 v\|_{p.\Delta_i}^2 \Big\}^{1/2} .$$

We will consider finite element spaces $S_h = S_h(\Gamma_h)$ of order m : Any $\chi \in S_h$ is in $C^0(\Omega)$ and the restriction to $\Delta_i \in \Gamma_h$ is a polynomial of degree $\leq m-1$. In case of essential boundary conditions as discussed in section 3 we think of isoparametric elements as discussed by Ciarlet-Raviart [1] and Zlamal [1] .

A family of weight-factors $\{p_\rho(x)\}$ is said to be in class $\mathfrak{C}_\gamma = \mathfrak{C}_{\gamma \cdot m}$ if

(3) $$\sup_{x \in \Omega} p_\rho^{-1}(x) \mid D^\alpha p_\rho(x) \mid \leq \gamma \rho^{-|\alpha|} \quad \text{for} \quad |\alpha| \leq m \quad .$$

The standard approximation and inverse properties of finite elements may be transferred to estimates in weighted norms (see Natterer [1]).

<u>Lemma 1:</u> Let Γ_h be a $\varkappa$-regular subdivision and $\{p_\rho\}$ belong to class $\mathfrak{C}_\gamma$. There are constants $C_\nu = C_\nu(\gamma, \varkappa, m)$ $(\nu = 1,2,3)$ such that whenever $h \leq C_1 \rho$ the statements are true:

(i) To any $v \in H_1'$ with $1 \leq m$ there is a $\chi \in S_h$ according to

(4) $$\|\nabla^k (v-\chi)\|_p' \leq C_2 \, h^{1-k} \, \|\nabla^1 v\|_p' \qquad (0 \leq k < 1) \quad .$$

(ii) For any $\chi \in S_h$ Bernstein-type inequalities hold:

(5) $$\|\nabla^l \chi\|'_p \le C_3\, h^{k-l}\, \|\nabla^k l\|'_p \qquad (0 \le k < l < m) \quad .$$

With a proper approximation resp. interpolation χ it is well-known

$$\|\nabla^k (v-\chi)\|^2_{L_2(\Delta_i)} \le c_1\, h^{2(l-k)}\, \|\nabla^l v\|^2_{L_2(\Delta_i)}$$

with a constant $c_1 = c_1(\varkappa, m)$. From this we get with

$$\underline{p}_i = \inf \{p(x) | x \in \Delta_i\} \quad ,$$

$$\overline{p}_i = \sup \{p(x) | x \in \Delta_i\}$$

immediately

(6) $$\|\nabla^k (v-\chi)\|^2_p \le (\overline{p}_i/\underline{p}_i)^2\, c_1\, h^{2(l-k)}\, \|\nabla^l v\|^2_p \quad .$$

Now there are $\underline{x}$, $\overline{x} \in \Delta_i$ with $\underline{p}_i = p(\underline{x})$, $\overline{p}_i = p(\overline{x})$. Since $|\overline{x}-\underline{x}| \le \varkappa h$ and $|Dp| \le \gamma \rho^{-1} \overline{p}_i$ in Δ_i we get

$$\overline{p}_i \le \underline{p}_i + \varkappa\, \gamma\, h \rho^{-1}\, \overline{p}_i \quad .$$

The choice $C_1 = (2\varkappa\gamma)^{-1}$ guarantees $\overline{p}_i/\underline{p}_i \le 2$ if $h \le C_1 \rho$. Summation over all $\Delta_i \in \Gamma_h$ in (6) gives (4). The proof of (5) follows the same lines.

<u>Remark 1:</u> In the proof condition (3) was used only with $|\alpha| = 1$.

<u>Remark 2:</u> If $p > 0$ fulfills (3) with $|\alpha| = 1$ then also p^{-1} does.

2. L_2-projections

Let P_h be the L_2-projection onto the finite element space S_h defined by

$$u_h = P_h u \in S_h \quad ,$$

$$(u_h, \chi) = (u, \chi) \qquad \text{for} \quad \chi \in S_h \quad .$$

We first show the boundedness of P_h with respect to weighted norms. With any $\chi \in S_h$ we have

$$\|u_h\|_p^2 = \iint p u_h^2$$

$$= \iint u_h (pu_h - \chi) - \iint u(pu_h - \chi) + \iint pu\, u_h \quad .$$

With the help of

$$\iint pu\, u_h \le \frac{1}{2} \iint pu^2 + \frac{1}{2} \iint pu_h^2$$

and

$$|\iint v\, w| \le \{\iint p\, v^2\}^{1/2} \, \{\iint p^{-1} w^2\}^{1/2}$$

$$\le \delta \iint pv^2 + \frac{1}{4\delta} \iint p^{-1} w^2$$

for any $\delta > 0$ we get $(\delta = \frac{1}{4})$

$$\|u_h\|_p^2 \le \tfrac{3}{4}(\|u\|_p^2 + \|u_h\|_p^2) + 2\ \|pu_h - \chi\|^2_{p^{-1}}$$

or

$$\|u_h\|_p^2 \le 3\ \|u\|_p^2 + 8\ \|pu_h - \chi\|^2_{p^{-1}} \quad .$$

Now let χ be an approximation to pu_h according to lemma 1 with the weight-factor p^{-1} . Then

$$\|pu_h - \chi\|_{p^{-1}} \le c_2\ h^m\ \|\nabla^m(pu_h)\|'_{p^{-1}} \quad .$$

In $\Delta_1 \in \Gamma_h$ the function u_h is a polynomial of degree $\le m-1$ *). By Leibniz' rule we get

$$D^\alpha(pu_h) = \sum_{\substack{\beta\le\alpha \\ |\beta|\le m-1}} c_\beta\ (D^{\alpha-\beta}p)\ D^\beta\ u_h$$

and because of (3)

$$\text{(7)} \qquad \|\nabla^m(pu_h)\|'_{p^{-1}} \le c_2 \sum_{k=0}^{m-1} \rho^{-m+k}\ \|\nabla^k u_h\|'_p \quad .$$

Now we make use of the inverse properties (5) and come to

$$\|pu_h - \chi\|_{p^{-1}} \le c_3\ h\ \rho^{-1}\ \|u_h\|_p$$

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*) In case of isoparametric elements the m-th derivatives are linear-combinations of the lower ones, therefore (7) is valid.

with $c_3 = c_3(\gamma,\varkappa,m)$. If we choose $h \le C_4\,\rho$ with $C_4 = \mathrm{Min}\,(C_1, \frac{1}{6}\,c_3^{-1})$ we get therefore

(8)
$$\|u_h\|_p \le 2\,\|u\|_p \quad .$$

<u>Theorem 1:</u> Let Γ_h be a $\varkappa$-regular subdivision and the weight-factor be of class π_γ . For $h \le C_4(\gamma,\varkappa,m)\,\rho$ the L_2-projection $P_h : L_2 \to S_h$ is uniformly bounded.

In order to derive L_∞-estimates for P_h we specify the functions p . Let $x_o \in \bar{\Omega}$ be such that

(9)
$$u_h(x_o) = \pm\,\|u_h\|_{L_\infty(\Omega)}$$

(without loss of generality we can take the positive sign). Then we define

(10)
$$p_\rho(x) = (|x-x_o|^2 + \rho^2)^{-a}$$

with any $a > N/2$, f.i. we take $a = N$. Obviously condition (3) is met with $\gamma = \gamma(m)$. Because of

$$p_\rho(x) \le (|x-x_o|^N + \rho^N)^{-2}$$

we can estimate

$$\|u\|_p^2 \le c_4\|u\|_{L_\infty(\Omega)}^2 \int_0^\infty \frac{d\tau}{(\tau+\rho^N)^2}$$

or

(11) $$\|u\|_p \le c_5\ \rho^{-N/2}\ \|u\|_{L_\infty(\Omega)} \quad .$$

Because of the standard inverse inequality

$$\|\nabla \chi\|_{L_\infty(\Omega)} \le c_6 h^{-1} \|\chi\|_{L_\infty(\Omega)}$$

($c_6 = c_6(\varkappa, m)$) we find using (9)

(12) $$u_h(x) \ge \|u_h\|_{L_\infty(\Omega)} \left\{1 - c_6\ h^{-1} |x - x_o|\right\} \quad .$$

The volumen of the intersection between Ω and the sphere with center in x_o and radius $c_6^{-1}\ h$ is bounded from below by $c_7\ h^N$. With the help of (12) we get therefore with $c_8 = c_8(\gamma, \varkappa, m) > 0$

$$c_8\ \rho^{-2N}\ h^N\ \|u_h\|^2_{L_\infty(\Omega)} \le \|u_h\|^2_p \quad .$$

Comparing this and (11) with (8) we get finally

$$\|u_h\|_{L_\infty(\Omega)} \le c_9(\varkappa, m)\ \left(\frac{\rho}{h}\right)^{N/2}\ \|u\|_{L_\infty(\Omega)} \quad .$$

For any given h we take now $\rho = C_4^{-1} h$. Then the factor ρ/h depends only on $\varkappa, m$:

<u>Theorem 2:</u> Let Γ_h be a $\varkappa$-regular subdivision. Then the L_2-projection $P_h : L_2 \to S_h$ has uniformly bounded norm in $L_\infty(\Omega)$.

The inequality - see Alexitis [1]

$$\|u-P_h u\| \leq (1 + \|P\|) \inf_{\chi \in S_h} \|u-\chi\|$$

in connection with the approximation properties of finite elements gives

Corollary 1: Let $u \in W_\infty^n(\Omega)$ with $n \leq m$. Then

(13) $$\|u-P_h u\|_{L_\infty(\Omega)} \leq C_5 \, h^n \, \|u\|_{W_\infty^n(\Omega)}$$

with $C_5 = C_5(\varkappa, m)$.

3. Ritz-approximations

Now we consider the Dirichlet problem

$$(14)\qquad -\Delta u = f \quad \text{in } \Omega \;, \qquad u = 0 \quad \text{on } \partial\Omega \;.$$

Let S_h fulfill the boundary condition. Then the Ritz-approximation R_h is defined by

$$(15)\qquad \Phi = R_h u \in S_h \;, \qquad D(\Phi,\chi) = (f,\chi) \quad \text{for } \chi \in S_h$$

with the Dirichlet-form

$$D(\Phi,\chi) = \iint_\Omega \left\{ \sum_{i=1}^{N} \Phi_{|i}\, \chi_{|i} \right\} dx \;.$$

We may also insert u and write

$$(16)\qquad D(\Phi,\chi) = D(u,\chi) \quad \text{for } \chi \in S_h \;.$$

Besides the weight-factor p_ρ (10) with $x_o \in \Omega$ to be fixed later we introduce

$$(17)\qquad q_\rho(x) = (|x-x_o|^2 + \rho^2)^{-a-1} \;.$$

We will also write only p and q .

In analogy to theorem 1 we have in the present situation

<u>Theorem 3:</u> Assume

(i) Γ_h is a $\varkappa$-regular subdivision of Ω ,

(ii) S_h is of order $m \geq 3$ and contained in $\overset{\circ}{H}_1(\Omega)$,

(iii) p,q are defined by (10), (17) with $N/2 < a < N/2+1$

(iv) h and ρ are connected by $h \leq C_6\, \rho$ with $C_6 = C_6(\varkappa, m)$.

Then the Ritz-approximation $\Phi = R_h u$ is bounded in the sense

$$\|\Phi\|_q + \|\nabla\Phi\|_p \leq C_7(\|u\|_q + \|\nabla u\|_p) \tag{18}$$

with $C_7 = C_7(\varkappa, m)$.

The proof is divided in three steps. First in estimating the gradient of Φ we use the identity

$$\|\nabla\Phi\|_p^2 = D(\Phi, p\Phi) + \frac{1}{2} \iint \Phi^2 \Delta p \quad .$$

The second term on the right hand side is bounded by $c_{10}\|\Phi\|_q^2$ because of $\Delta p \leq c_{10} q$. The first term is handled similar to the L_2-case. So we find

$$(19) \qquad \|\nabla\Phi\|_p^2 \le c_{11}(\varkappa,m) \left\{\|\Phi\|_q^2 + \|\nabla u\|_p^2\right\} \quad .$$

Next we introduce an auxiliary function w defined by

$$(20) \qquad \begin{aligned} -\Delta w &= q\Phi \qquad && \text{in} \quad \Omega \\ w &= 0 && \text{on} \quad \partial\Omega \end{aligned}$$

by means of which we get

$$\|\Phi\|_q^2 = (\Phi, -\Delta w) = D(\Phi, w) \quad .$$

Because of (16) the equation

$$\|\Phi\|_q^2 = D(\Phi, w-\chi) - D(u, w-\chi) + D(u,w)$$

holds with $\chi \in S_h$ arbitrarily chosen. The last term can be replaced by $\iint qu\Phi$. Applying Schwarz' inequality in an appropriate way we come to

$$\|\Phi\|_q^2 \le \|u\|_q^2 + \|\nabla u\|_p^2 + \delta\|\nabla\Phi\|_p^2 + (1+\delta^{-1}) \|\nabla(w-\chi)\|_{p^{-1}}^2 \quad .$$

Now we choose $\delta < c_{11}^{-1}$ and combine the last inequality with (19):

$$(21) \quad \|\Phi\|_q^2 + \|\nabla\Phi\|_p^2 \le c_{12} \left\{\|u\|_q^2 + \|\nabla u\|_p^2 + \|\nabla(w-\chi)\|_{p^{-1}}^2\right\} \quad .$$

Since $m \geq 3$ we get with lemma 1

(22) $$\|\nabla(w-\chi)\|_{p^{-1}} \leq c_{13}\, h^2 \|\nabla^3 w\|_{p^{-1}} \quad .$$

<u>Remark 3:</u> Since $\Phi \in \overset{\circ}{H}_1(\Omega)$ and $q \in C^\infty(\Omega)$ we have $w \in H_3(\Omega)$.

The final step from (21) to (18) is done by the

<u>Lemma 2:</u> Assume $N/2 < a < N/2+1$. Let w be defined by (20). Then

(23) $$\|\nabla^3 w\|_{p^{-1}} \leq c_{14}\, \rho^{-2}(\|\Phi\|_q + \|\nabla\Phi\|_p)$$

with a numerical constant c_{14} .

The proof of the lemma is highly technical and is not given here. It only remains to couple ρ and h in order to have - see (21) - (23) - $c_{12}\, c_{13}^2\, c_{14}^2\, h^4\, \rho^{-4} < 1$.

Parallel to the L_2-case we derive now from (18)

$$\|\Phi\|_{L_\infty(\Omega)} + h\|\nabla\Phi\|_{L_\infty(\Omega)} \leq C_8(\varkappa,m)\ \left\{\|u\|_{L_\infty(\Omega)} + h\|\nabla u\|_{L_\infty(\Omega)}\right\} \quad .$$

The final result is

<u>Corollary 2:</u> Assume the order m of the finite elements used is at least 3 . Let $u \in W^n_\infty(\Omega)$ with $1 \leq n \leq m$.

Then

$$\|u-R_h u\|_{L_\infty(\Omega)} \leq c_8\, h^n \|u\|_{W^n_\infty(\Omega)}$$

$$\|u-R_h u\|_{W^1_\infty(\Omega)} \leq c_8\, h^{n-1} \|u\|_{W^n_\infty(\Omega)} \quad .$$

<u>Remark 4:</u> In case $m = 2$ the best choice is $a = N/2$.

Inequality (22) is to be replaced by

$$\text{(22')} \qquad \|\nabla(w-\chi)\|_{p^{-1}} \leq c_{13}\, h \|\nabla^2 w\|_{p^{-1}}$$

and similarily (23) by

$$\text{(23')} \qquad \|\nabla^2 w\|_{p^{-1}} \leq c'_{14}\, \rho^{-1} / \ln \rho /^{1/2}\, \|\Phi\|_q \quad .$$

In this way logarithmic terms "come in" .

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