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Reflexivity Properties of Rings

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REFLEXIVITY PROPERTIES OF RINGS

by

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(0)

Let R denote a commutative ring with unit. We ask for a property of R to ensure, that condition

Z(M) $\quad M^{+}\ (=\mathrm{Hom}_R(M,R)) = 0 = \mathrm{Ext}^1_R(M,R)$

on R-modules M implies M = 0.

As any Whitehead-group M is torsionless (i.e. $L_M : M \longrightarrow M^{++}$ is injective), Z(M) implies M = 0 in case $R = \mathbb{Z}$. We shall see that two independent properties of $\mathbb{Z}$ account for this. One of them ("parareflexive") evolves from $\mathbb{Z}$ being a PID, the other property is that $\mathbb{Z}$ is fully reflexive in the sense of the definition given below.

(1) Fully reflexive rings

Proposition 1 : The following two conditions on R are equivalent.

(i) $(R^{\mathbb{N}})^{+}$ is generated (thus freely generated) by the projections

$$\pi_n = (f \longrightarrow f(n)) \qquad (n \in \mathbb{N}).$$

(ii) $L_{R^{(\mathbb{N})}} : R^{(\mathbb{N})} \longrightarrow (R^{(\mathbb{N})})^{++}$ is bijective.

Proof : (i) implies (ii) : Consider $\varphi \in (R^{(\mathbb{N})})^{++}$, i.e. $\varphi : (R^{(\mathbb{N})})^{+} \longrightarrow R$. As $(R^{(\mathbb{N})})^{+} \cong R^{\mathbb{N}}$, and by (i), there is but a finite number of $n \in \mathbb{N}$, say $n_1,\dots,n_k$, such that $\varphi(\pi_n) \neq 0$. Now

$$\varphi = L_{R^{\mathbb{N}}} \left(\sum_{\ell=1} \varphi(\pi_{n_\ell})\, e_{n_\ell} \right)$$

here $e_n(m) = 1$ if m=n, and equals zero otherwise.

(ii) implies (i) : Consider $\alpha \in (R^{\mathbb{N}})^{+} \cong (R^{(\mathbb{N})})^{++}$. By (ii) there exists $f \in R^{(\mathbb{N})}$ such that $\alpha = L_{R^{(\mathbb{N})}}(f)$. Now $\alpha = \sum_{n \in \mathbb{N}} f(n)\, \pi_n$. q.e.d.

Définition : R is fully reflexive, if it satisfies the two conditions of proposition 1.

Here follows a listing of some known results on fully reflexive rings.

(A) As in the case of abelian groups, $\mathbb{N}$ might be replaced by any set of non-measurable cordinality.

(B) If (R,+) is torsionfree and reduced (qua abelian group), then R is fully reflexive.

(C) If R is fully reflexive (say R = $\mathbb{Z}$) or a field, then any polynomial ring $R[X_i / i \in I]$ (Corol. I non-measurable) is fully reflexive.

Fully reflexive rings were introduced in a joint paper by the author with L. Kaup and H.G. Weidner [Archiv Math. 20, 503-514 (1969)], where properties (A), (B) and (C) were proved as well.

(D) Any non-local PID is fully reflexive, and among the local PID's the complete ones are exactly those, which are not fully reflexive.

(E) If R is a Dedekind domain, R is fully reflexive if and only if $\mathrm{Ext}^1_R(Q,R) \neq 0$ (where Q denotes the field of quotients of R).

Properties (D) and (E) are due to G. Heinlein [Dissertation Erlangen, 1971].

(F) No finite ring is fully reflexive, as follows from corollory 3 of a paper by L. Kaup and M. Keane [Manuscripta Math. 1, 9-22 (1969)].

Proposition 2 : If R is a fully reflexive Dedekind domain, and M is on R-module, then Z(M) implies M = 0.

Proof : As no field is fully reflexive, R fails to be realisable in the sense of R.J. Nunke [Ill. J. Math. 3, 222-241 (1959)].
Thus, by Nunke's theorem 8.5 loc. cit. and property (E), condition Z(M) on M implies M = 0. q.e.d.

Remark : On the contrary, if the Dedekind domain R is not fully reflexive, Z(M) holds for any torsionfree, divisible R-module M. Thus within the class of Dedekind domains, the fully reflexive ones are exactly those we sought for, when posing the question on the implication M = 0 of Z(M).

Nevertheless, the problem to characterize the class of fully reflexive rings (say among noetherian rings to begin with) is not completely solved so for.

(2) Parareflexive rings

There are fully reflexive rings R (non Dededind ones, to be sure), where even finitely generated R-modules $M \neq 0$ satisfying Z(M) exist (see (H) below). So we introduce the following.

Définition : R is parareflexive, if Z(M) implies M = 0 for finitely generated R-modules M.

(G) As easily seen, any Dedekind domain is parareflexive.

Proposition 3 : If R is a reflexive (in the sense of E. Matlis) noetherian domain, then R is parareflexive.

Proof : Let M be a finitely generated R-module, such that $M^+ = 0 = \mathrm{Ext}^1_R(M,R)$. Start a finitely generated, free resolution

$$F_1 \xrightarrow{\eta} F_o \longrightarrow M \longrightarrow 0 \qquad (1)$$

of M. In ouder to prove M = 0, it will be sufficient that η^{++} is surjective. Indeed by $\eta^{++} \circ L_{F_1} \circ \eta$, and as L_{F_o} and L_{F_1} one bijective, η is surjective then.

(*) This part of the lecture is taken from a forthcoming paper of the author.

Now, it is easily seen that η^+ is injective and coker η^+ is torsionless. Thus, from sequence (1) the exact sequence

$$0 \longrightarrow F_o^+ \longrightarrow F_1^+ \xrightarrow{\eta^+} B \longrightarrow 0 \qquad (2)$$

is derived, where B is torsionless.

In notation α, A, and E instead of η^+, F_o^+, and F_1^+, respectively, sequence (2) reads

$$0 \longrightarrow A \xrightarrow{\alpha} E \xrightarrow{\pi} B \longrightarrow 0. \qquad (3)$$

We deal with sequence (3) from now on.

$\alpha^+ = \gamma \circ \delta$ is factorised via C = sin α^+ by $\delta : E^+ \longrightarrow C$ and $\gamma : C \longrightarrow A^+$.

Let $\beta : A \longrightarrow C^+$ be defined by $\delta^+ \circ \beta = L_E \circ \alpha$. As δ^+ is injective, and $\delta^+ \circ \gamma^+ \circ L_A = \alpha^{++} \circ L_A = L_E \circ \alpha = \delta^+ \circ \beta$, equation

$$\gamma^+ \circ L_A = \beta \qquad (4)$$

holds.

Next, L_{A^+} is bijective, and as $(L_A)^+ \circ L_{A^+} = 1_{A^+}$ equation

$$(L_{A^+})^{-1} = (L_A)^+ \qquad (5)$$

holds.

By (4) and (5) we get

$$\beta^+ \circ L_C = (L_A)^+ \circ \gamma^{++} \circ L_C = (L_A)^+ \circ L_{A^+} \circ \gamma = \gamma \qquad (6)$$

At lost, as L_E is bijective and L_B is injective, β is bijective. Also, L_G is bijective by assumption, as C is finitely generated and torsionless. Thus, γ is surjective (actually bijective) by (6), and α^+ is surjective as well.

q.e.d.

(H) From (C) we now obtain plenty of fully reflexive rings, which are not parareflexive. Specifiacally, $\mathbb{R}[X,Y]$ has injective dimension 2, so is not reflexive nor parareflexive.