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General Ergodic Theorems for Semigroups of Linear Operators

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Exposé de théorie ergodique

par

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GENERAL ERGODIC THEOREMS FOR SEMIGROUPS OF LINEAR OPERATORS

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0. The first operator theoretical generalization of G. D. Birkhoff's pointwise ergodic theorem was given by J. L. Doob in 1938 and then extended by S. Kakutani in 1940. However one can say that a really essential contribution to the theory was done by Eberhard Hopf in 1954. Hopf's theorem was extended into two directions. One extension is that of N. Dunford-J. T. Schwartz in 1956 and the other is that of R. V. Chacon-D. S. Ornstein in 1960. In 1961 Chacon succeeded in proving a general ergodic theorem beeing an extension of both of Dunford-Schwartz's theorem and Chacon-Ornstein's theorem. At that time he introduced a notion of admissible sequence.

Let $(X,\mathcal{F},\mu)$ be a σ-finite measure space . Consider a linear contraction T on $L_1 = L_1(X)$ and a sequence $\{p_n\}_{n\geq 0}$ of nonnegative-measurable (not necessarily integrable) functions on X . Then, $\{p_n\}_{n\geq 0}$ is called T-admissible if $f\in L_1$ and $|f|\leq p_n$ for some n imply $|Tf|\leq p_{n+1}$. Chacon's general ergodic theorem is

Theorem 1. [1] . Let T be a linear contraction on L_1 and let $\{p_n\}_{n\geq 0}$ be T-admissible. Then, for every $f\in L_1$,

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} T^k f}{\sum_{k=0}^{n} p_k}$$

exists and is finite a.e. in $\{x : \sum_{k=0}^{\infty} p_k(x) > 0\}$.

1. I will state a general ergodic theorem for a semigroup of linear contractions which i

theorem 1.

Let $\{T_t\}_{t\geq 0}$ be a sem

means that

(t. 1) each T_t is a linea

(t. 2) $T_{s+t} = T_t T_s$ for eve

(t. 3) $\lim_{t\downarrow 0} ||T_t f - f||_1 =$

Concerning a semigrou

Proposition 2. [3] .

contractions on L_1. Then th

tive linear contractions on

$|T_t f| \leq S_t |f$

Next, consider a famil

necessarily integrable) func

$\{T_t\}$-admissible if it has th

(i) $f\in L_1$ and $|f| \leq p_s$

every t ;

(ii) there exists a strictl

$\lim_{t\downarrow s} ||\, |p_t - p_s| \wedge p$

stands for min (a,b) .

We have two typical ex

Example. 1. Consider

Let $0\leq g\in L_1$ and define

sible.

Example. 2. Consider

traction on L_∞ in the sen

$f \in L_1 \cap L_\infty$. Define $p_t = 1$. Then $\{p_t\}_{t>0}$ is $\{T_t\}$-admissible.

From now on, we have to deal with integrals $\int_0^\alpha T_t f(x)\,dt$ and $\int_0^\alpha p_t(x)\,dt$. However, when x is fixed, each of $T_t f(x)$ and $p_t(x)$ may not be Lebesgue measurable as a function of a variable t . Hence we have to consider a suitable measurable version of each of them.

Proposition 3. [5] . Let $\{g_t\}_{t\geq 0}$ be a family of measurable functions on X . Suppose that there exists a strictly positive function $g \in L_1$ such that

$\lim_{t \downarrow s} \| \, |g_t - g_s| \wedge g \|_1 = 0$ for every s . Then there exists a function $g(t,x)$ on $[0,\infty) \otimes X$ such that

a) $g(t,x)$ is $\mathcal{M} \otimes \mathcal{F}$-measurable, where $\mathcal{M}$ is the σ-algebra of all Lebesgue measurable subsets of $[0,\infty)$;

b) for every t , $g(t,x) = g_t(x)$ for a.a.x.

Such a function $g(t,x)$ is uniquely determined up to on a set of $m \otimes \mu$-measure zero, where m is Lebesgue measure on $\mathcal{M}$.

$g(t,x)$ is called the good version of $g_t(x)$ and denoted by $g_t(x)$ itself.

By virtue of Proposition 3 each of $T_t f(x)$ and $p_t(x)$ has its good version denoted by itself. Then, by Fubini's theorem, for a.a.x., each of the good versions is Lebesgue measurable as a function of t , and moreover $T_t f(x)$ is Lebesgue integrable on any bounded subintervals of $[0,\infty)$.

Our general ergodic theorem is

Theorem 4. [5] . Let $\{T_t\}_{t\geq 0}$ be a semigroup of linear contractions on L_1 and let $\{p_t\}_{t\geq 0}$ be $\{T_t\}$-admissible. Then, for every $f \in L_1$,

$$\lim_{\alpha\to\infty} \frac{\int_0^\alpha T_t f(x)\,dt}{\int_0^\alpha p_t(x)\,dt}$$

exists and is finite a.e. i

The proof of the theo
1 by making use of Propositi

2. Next, I will ta
ergodic theorem for a semifl
was worked out firstly by N.
tended to the case of semigr
in 1969 and also by D. S. Or
R. V. Chacon, T. R. Terrell
neralizations of Krengel-Orn
most general local ergodic t

Theorem 5. [2] , [3]
(i) a semigroup of positiv
tractions) on L_1 ,
or
(ii) a semigroup of linear
on L_1 .
Then, for every $f \in L_1$,

$$\lim_{\alpha\downarrow 0} \frac{1}{\alpha} \int_0^\alpha$$

The proof for the cas
ergodic inequality.

Proposition 6. [2] .
tive linear operators on L_1

$$\limsup_{\alpha \downarrow 0} \frac{1}{\alpha} \int_0^\alpha T_t f(x$$

implies

$$\int_E f^- \, d\mu \leq \int$$

where f^- and f^+ are the negative and positive part of f , respectively.

On the other hand, the case (ii) of Theorem 5 is reduced to the case (i) by virtue of Proposition 2 .

Now, Kubokawa and I would like to propose an open question.

<u>Question 1.</u> Let $\{T_t\}_{t \geq 0}$ be a semigroup of linear operators on L_1 and let $f \in L_1$. Does

$$\lim_{\alpha \downarrow 0} \frac{1}{\alpha} \int_0^\alpha T_t f(x)\, dt = f(x)$$

a. e. ?

If it were possible to extend Proposition 2 to the case where each T_t is not necessarily a contraction, then the question can be reduced to the case (i) of Theorem 5. However, I think Proposition 2 is not extended to the general case, because the assumption of contraction property of each T_t in Proposition 2 is really essential. Thus it might be impossible to reduce the question to the case (i) of Theorem 5. But, I think the question will be answered affirmatively.

<u>3</u>. Now, I will talk about random ergodic theorems. A random ergodic theorem for measure preserving transformations was proved by H. R. Pitt in 1942 and also by S. M. Ulam-J. von Neumann in 1945. After that, S. Kakutani, C. Ryll-Nardzewski, S. Gladysz, A. Beck-J. T. Schwartz, E. Kin and I extended the theory. We shall state now one of the most general random ergodic theorems.

Consider two σ-finite measure spaces $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{Q}, \nu)$. Let $\{\varphi_t\}_{t \geq 0}$ be a semiflow of measure preserving transformations on Y . This means that

(φ. 1) each φ_t is a measure preserving transformation on Y ;

(φ. 2) $\varphi_{s+t} = \varphi_t \varphi_s$ for every s, t ;

(φ. 3) the mapping : $(t,y) \in$
surable.

Let $\{T_t(y)\}_{t \geq 0}$, $y \in Y$
tions on $L_1(X)$ associated
means that

(T. 1) each $T_t(y)$ is a li

(T. 2) $T_{s+t}(y) = T_t(y)\, T_s(\varphi$

(T. 3) for every t , $T_t(y)$
every t and every
$\mathcal{Q}$-measurable functi

(T. 4) for a.a.y , $T_t(y)$
a.a.y and every $f \in$

$$\lim_{t \downarrow 0} \| T_t(y)$$

Then, given $f \in L_1(X$
for a.a.y , and so we can de
in general, when x and y
Lebesgue measurable as a fun

<u>Proposition 7.</u> [4] .
$[0,\infty) \otimes X \otimes Y$ having the p

a) $g(t,x,y)$ is $\mathcal{M} \otimes$

b) for every t , th
ν-measure zero s
$g(t,x,y) = T_t(y$

Such a function $g(t,$
set of $m \otimes \mu \otimes \nu$-measure ze
$g(t,x,y)$ is called t
denoted by $T_t(y) f (x, \varphi_t y)$

Next, we consider a f

(not necessarily integrable) functions on $X \otimes Y$. $\{p_t\}_{t\geq 0}$ is called $\{T_t(y)\}$-admissible if it has the properties :

(i) $f \in L_1(X \otimes Y)$ and $|f(x,y)| \leq p_s(x,y)$ a.e. for some s imply $|T_t(y)\, f\,(x,\varphi_t y)| \leq p_{s+t}(x,y)$ a.e. for every t ;

(ii) there exists a strictly positive function $p \in L_1(X \otimes Y)$ such that

$$\lim_{t \downarrow s} \| \, |p_t - p_s| \wedge p \, \|_{L_1(X \otimes Y)} = 0 \quad \text{for every } s .$$

Note that $p_t(x,y)$ has its good version by virtue of Proposition 3. We have two typical examples of $\{T_t(y)\}$-admissible family.

Example. 1. Consider the case where each $T_t(y)$ is positive. Let $0 \leq g \in L_1(X \otimes Y)$ and define $p_t(x,y) = T_t(y)\, g(x,\varphi_t y)$. Then $\{p_t\}_{t\geq 0}$ is $\{T_t(y)\}$-admissible.

2. Consider the case where each $T_t(y)$ is also a contraction on $L_\infty(X)$. Define $p_t(x,y) = 1$. Then $\{p_t\}_{t\geq 0}$ is $\{T_t(y)\}$-admissible .

We are now in a position to state a general random ergodic theorem which is more general than that in [4] .

Theorem 8. Let $\{T_t(y)\}_{t\geq 0,\, y\in Y}$ be a quasi-semigroup of linear contractions on $L_1(X)$ associated with a semiflow $\{\varphi_t\}_{t\geq 0}$ of measure preserving transformations on Y . Let $\{p_t\}_{t\geq 0}$ be $\{T_t(y)\}$-admissible. Then, for every $f \in L_1(X \otimes Y)$,

$$\lim_{\alpha\to\infty} \frac{\int_0^\alpha T_t(y)\, f\,(x,\varphi_t y)\, dt}{\int_0^\alpha p_t(x,y)\, dt}$$

exists and is finite a.e. in $\{(x,y) : \int_0^\infty p_t(x,y)\, dt > 0\}$.

The theorem is proved in the same way as in [4] by making use of Theorem 4 .

Concerning Theorem 8
question. Our formulation of
comes from quasi-semiflow of
$\{\psi_t(y)\}_{t\geq 0,\, y\in Y}$ be a quasi-
mations on X associated wi

(ψ. 1) each $\psi_t(y)$ is a mea

(ψ. 2) $\psi_{s+t}(y) = \psi_s(\varphi_t y)\, \psi$

(ψ. 3) the mapping : (t,x,y
measurable .

Now, if we consider an

$T_t(y)\, f\, (x$

then $\{T_t(y)\}_{t\geq 0,\, y\in Y}$ is an
that (ψ. 2) induces (T. 2
However, in view point of fu
is not satisfactory for us,
pears in the first operator
quasi-semigroup, (T. 2) shou
(T. 2^*) $T_{s+t}(y) = T_t(\varphi_s y)$
Then a questions arises.

Question 2. Conside
satisfying (T. 2^*) instead

$$\lim_{\alpha\to\infty} \frac{\int_0^\alpha T_t(y \cdots}{\int_0^\alpha \cdots}$$

exist a.e. in $\{(x,y) : \int_0^\infty p$

I have still nothing t

R E F E R E N C E S

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