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THE FINITE ELEMENT METHOD--LINEAR AND NONLINEAR APPLICATIONS

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Numerical analysis is a crazy mixture of pure and applied mathematics. It asks us to do two things at once, and on the surface they do appear complementary: (i) to propose a good algorithm; and (ii) to analyze it. In principle, the analysis should reveal what makes the algorithm good, and suggest how to make it better. For some problems--computing the eigenvalues of a large matrix, for example, which used to be a hopeless mess--this combination of invention and analysis has actually succeeded. But for partial differential equations, which come to us in such terrible variety, there seems to be a long way to go.

We want to speak about an algorithm which, at least in its rapidly developing extensions to nonlinear problems, is still new and flexible enough to be improved by analysis. It is known as the finite element method, and was created to solve the equations of elasticity and plasticity. In this instance, the "numerical analysts" were all engineers. They needed a better technique than finite differences, especially for complicated systems on irregular domains; and they found one. Their method falls into the framework of the Ritz-Galerkin technique, which operates with problems in "variational form"--starting either from an extremum principle, or from the weak form of the differential equation, which is the engineer's equation of virtual work. The key idea which has made this classical approach a success is to use piecewise polynomials as trial functions in the variational problem.¹

¹The most important applications are still to structural problems, but no longer to the design of airplanes; that has now been superseded by the safety of nuclear reactors.

We plan to begin by describing the method as it applies to linear problems. Because the basic idea is mathematically sound, convergence can be proved and the error can be estimated. This theory has been developed by a great many numerical analysts, and we can summarize only a few of the most essential points--the conditions which guarantee convergence, and which govern its speed. This linear analysis has left everyone happier, and some divergent elements have been thrown out, but the method itself has not been enormously changed. For nonlinear problems the situation is entirely different. It seems to me that numerical analysts, especially those in optimization and nonlinear systems, can still make a major contribution. The time is actually a little short, because the large scale programs for plasticity, buckling, and nonlinear elasticity are already being written. But everyone is agreed that they are tremendously expensive, and that new ideas are needed.

Nonlinear problems present a new challenge also to the analyst who is concerned with error estimates. The main aim of this paper is to describe some very fragmentary results (Section III) and several open questions (Section IV). We are primarily interested in those nonlinearities which arise, in an otherwise linear problem, when the solution is required to satisfy an inequality constraint. This is typical of the problems in plasticity. The solution is still determined by a variational principle, but the class of admissible functions becomes a convex set instead of a subspace. In other words, the equation of virtual work becomes a variational inequality.

At the end we look in still a different direction, at linear programming constrained by differential equations. Here we need not only good algorithms and a proper numerical analysis, but also answers to

the more fundamental questions of existence, uniqueness, and regularity.

II. Linear Equations. The finite element method applies above all to elliptic boundary value problems, which we write in the following form:

Find u in the space of admissible functions V such that

$$(1) \quad a(u,v) = \ell(v) \quad \text{for all } v \text{ in } V.$$

Standard example: $\iint (u_x v_x + u_y v_y) dx dy = \iint f v dx dy$ for all v in $H_0^1(\Omega)$.

This is the weak form of Poisson's equation $-\Delta u = f$. Because the expression $a(u,v)$ is in this case symmetric and positive definite, the problem is equivalent to: Minimize $J(v) = a(v,v) - 2\ell(v)$ over the admissible space V . The "strain energy" $a(v,v)$ is the natural norm in which to estimate the error.

The error comes from changing to a finite-dimensional problem:

Find u_h in S_h such that

$$(2) \quad a_h(u_h, v_h) = \ell_h(v_h) \quad \text{for all } v_h \text{ in } S_h.$$

It is this problem which the computer actually solves, once it is given a basis ϕ_1, \dots, ϕ_N for the space S_h . Very briefly, it has to form the stiffness matrix $K_{ij} = a_h(\phi_i, \phi_j)$ and the load vector $F_j = \ell_h(\phi_j)$, solve the linear system $KQ = F$, and print out the approximate solution $u_h = \sum Q_j \phi_j$. That sounds straightforward, but it is nearly impossible unless the basis functions ϕ_j are extremely simple, and nearly useless unless they can provide a good approximation to the true solution u . The finite element method manages to combine both properties.²

²We shall have to refer to the book [1] and to its bibliography, both for the construction of piecewise polynomials and for the proof of their approximation properties. Perhaps the favorites, when derivatives of order m appear in the energy $a(v,v)$, are the polynomials of degree $m+1$.

Our plan in this section is to summarize four of the main points in the theory of convergence. Each of them is concerned with the change in solution when there is a change in the problem--when the admissible space V is replaced by S_h , or the given a and l are approximated by a_h and l_h . To give some kind of order to the discussion, we formulate all four as applications of the "fundamental theorem of numerical analysis":

CONSISTENCY + STABILITY \Leftrightarrow CONVERGENCE.

1. The classical Ritz-Galerkin case: The energy $a(v,v)$ is symmetric positive definite; S_h is a subspace of V ; $a_h = a$ and $l_h = l$.

Since every v_h is an admissible v , we may compare (1) and (2): $a(u, v_h) = a(u_h, v_h)$. This means that in the "energy inner product," u_h is the projection of u onto the subspace S_h . In other words, the positive definiteness of $a(v,v)$ implies two properties at once [1, p. 40]: the projection u_h is no larger than u itself,

$$(3) \quad a(u_h, u_h) \leq a(u, u),$$

and at the same time u_h is as close as possible to u :

$$(4) \quad a(u - u_h, u - u_h) \leq a(u - v_h, u - v_h) \text{ for all } v_h \text{ in } S_h.$$

Property (3) represents stability; the approximations are uniformly bounded. Given that u can be approximated by the subspace S_h --in this Ritz-Galerkin context, consistency is the same as approximability--convergence follows immediately from (4).

2. The indefinite case: u is only a stationary point of the functional $J(v)$. This corresponds to the use of Lagrange multipliers in optimization; the form $a(v,v)$ can take either sign, and v may include two different types of unknowns--both displacements and stresses, in the "mixed method" and "hybrid method."

Consistency reduces as before to approximation by polynomials. But stability is no longer automatic; even the simplest indefinite form $J(v) = v_1 v_2$ --which has a unique stationary point at the origin, if V is the plane R^2 --will collapse on the one-dimensional subspace given by $v_2 = 0$. Therefore, for each finite element space S_h and each functional $J(v)$, it has to be proved that a degeneracy of this kind does not occur.

The proper stability condition is due to Babuska and Brezzi:

$$(5) \quad \sup_{\|v\|=1} |a(v,w)| \geq c \|w\|.$$

Brezzi has succeeded in verifying this condition for several important hybrid elements. For other applications the verification is still incomplete, and the convergence of stationary points--which is critical to the whole theory of optimization--remains much harder to prove than the convergence of minima.

3. The modified Galerkin method: a and l are changed to a_h and l_h (numerical integration of the stiffness matrix and load vector), and v_h may lie outside V (non-conforming elements).

The effect on u can be estimated by combining (1) and (2):

$$(6) \quad a_h(u-u_h, u-u_h) = (a_h-a)(u, u-u_h) - (l_h-l)(u-u_h).$$

Stability, in this situation, means a lower bound for the left side:

$$(7) \quad a_h(u-u_h, u-u_h) \geq c a(u-u_h, u-u_h).$$

Consistency is translated into an upper bound for the right side, and it is checked by applying the patch test: whenever the solution is in a "state of constant strain"--the highest derivatives in $a(u,u)$ are all constant--then u_h must coincide with u .³

The patch test applies especially to nonconforming elements, for which $a(v_h, v_h) = \infty$; the derivatives of v_h introduce delta-functions, which are simply ignored in the approximate energy a_h . This is extremely illegal, but still the test is sometimes passed and the approximation is consistent. Convergence was established by the author for one such element, and Raviart, Ciarlet, Crouzeix, and Lesaint have recently made the list much more complete.

³The patch test is also an ideal way to check that a finite element program is actually working.

4. Superconvergence: Extra accuracy of the finite element approximation at certain points of the domain. It was recognized very early that in some special cases-- $u'' = f$ with linear elements, or $u'''' = f$ with cubics--the computed u_h is exactly correct at the nodes. (The Green's function lies in S_h .) And even earlier there arose the difficulty of interpreting the finite element output in a more general problem; u_h and its derivatives can be evaluated at any point in the domain, but which points do we choose? This question is as important as ever to the engineers.

In many problems the error $u - u_h$ oscillates within each element, and there must be points of exceptional accuracy. Thomee discovered superconvergence at the nodes of a regular mesh, for $u_t = u_{xx}$, and his analysis has been extended by Douglas, Dupont, Bramble, and Wendroff. It is not usually carried out in our context of consistency and stability, but perhaps it could be: consistency is checked by a patch test at the superconvergence points, to see which polynomial solutions and which derivatives are correctly reproduced, and stability needs to be established in the pointwise sense.⁴

III. Variational Inequalities. What happens when a constraint like $v \leq \psi$ is enforced on the admissible functions v , so that the functional $J(v)$ is minimized only over a convex subset K of the original space V ? This occurs naturally in plasticity theory, when v represents the stress; wherever the yield limit ψ is reached, the differential equation (Hooke's law) is replaced by plastic flow. For the minimizing u , the "free boundary" which marks out this plastic region $u = \psi$ is not known in advance.⁵ Since such a solution u lies on the edge of the convex set K , $J(u) \leq J(u + \epsilon(v - u))$ is guaranteed only for $\epsilon \geq 0$. This translates into the variational inequality which determines u :

and Scott

⁴Convergence in L_∞ will be discussed by Bramble, at this Congress; it is one of the outstanding problems in the linear theory.

⁵This boundary cannot be found by solving the original linear problem and then replacing u by $\min(u, \psi)$!

$$(8) \quad a(u, v-u) \geq \ell(v-u) \quad \text{for all } v \text{ in } K.$$

In the finite element method, we minimize an approximate functional $J_h(v) = a_h(v, v) - 2\ell_h(v)$ over a finite-dimensional convex set K_h .

For example, the piecewise polynomials may be constrained by $v_h \leq \psi$ at the nodes of the triangulation. Again the minimizing u_h is determined by a variational inequality,

$$(9) \quad a_h(u_h, v_h - u_h) \geq \ell_h(v_h - u_h) \quad \text{for all } v_h \text{ in } K_h;$$

now a polygonal free boundary is to be

The practical problem is to carry out this minimization, and compute u_h ; we are in exactly the situation described in the introduction, with many proposed algorithms and a difficult task of comparison and analysis. The theoretical problem, which assumes that u_h has somehow been found, is to estimate its distance from the true solution u . We want to report on this latter problem, and it is natural to ask the same four questions about convergence which were answered in the linear case.

The easiest way is to take the questions in reverse order:

4. Superconvergence is almost certainly destroyed by the error in determining the free boundary. Even in one dimension with $u'' = 1$, u differs from u_h by $O(h^2)$.

3. The approximation of a and ℓ by a_h and ℓ_h lead to no difficulties; the identity (6) simply becomes an inequality, if we combine (8) and (9), and the patch test is still decisive. The same is true for nonconforming elements, and the extra term Δ in the error estimates [1, p. 178] is exactly copied from the linear case.

2. It is an open problem, both for K and for the discrete K_h , to show how stability can compensate for the indefiniteness of $a(v, v)$.

1. This is the basic question in the nonlinear Ritz-Galerkin method: if the trial functions in K_h can approximate u to a certain accuracy, how close is the particular choice u_h ? It is no longer exactly optimal, because it is no longer the projection of u . But we hope to prove, in the natural norm $\|v\|^2 = a(v, v)$, that $\|u - u_h\| \leq c \min \|u - v_h\|$.

First, we ask how large this minimum is, choosing v_h to be the piecewise polynomial u_I in S_h which interpolates u at the finite

element nodes. The answer depends on the degree of the polynomial and on the regularity of u . For our obstacle problem, with $-\Delta u = f$ in the elastic part and $u = \psi$ in the plastic part, it is now known that u lies in $W^{2,\infty}$. (Brezis and Kinderlehrer may announce this long-sought result in Vancouver.) At the free boundary there is a jump in the second derivative of u --which absolutely limits the accuracy of the interpolation. Courant's linear approximation, on triangles of size h , is still of order $\|u-u_I\| = O(h)$. But for polynomials of higher degree, and a smooth free boundary, this is improved only to $O(h^{3/2})$ --and no elements can do better. There are $O(1/h)$ triangles in which the gradient is in error by $O(h)$. Therefore there is no justification for using cubic polynomials, and the question is whether quadratics are worthwhile; we believe so.

To prove that the actual error $u-u_h$ is of the same order as $u-u_I$, we depend on an a priori estimate of Falk [2]. It resembles (4), but the change in (8) and (9) from equations to inequalities produces a new term:

$$(10) \quad \|u-u_h\|^2 \leq \|u-v_h\|^2 + 2\int(f+\Delta u)(u-v_h+u_h-v).$$

We may choose any v_h in K_h and any v in K --and for simplicity we have specialized to $\ell = \ell_h = \int fv$ and $a = a_h = \int |\nabla v|^2$. The new term is automatically zero in the elastic part, where $-\Delta u = f$, but elsewhere $f + \Delta u > 0$.

To estimate (10), we take $v = \psi$ and $v_h = u_I$ --which lies in K_h because it cannot exceed ψ at the nodes, where it agrees with u . (In the case of quadratic polynomials, some members of K_h will go above the yield limit ψ within the triangles, but we have to be lenient enough to permit that; it doesn't hurt the error estimate, and anyway it is only constraints on v_h at nodal "checkpoints" which can be enforced in practice.) With this choice of v and v_h , the terms in (10) are

(i) With Courant's linear finite elements:

$$\|u-u_I\|^2 \sim h^2; \quad \int(f+\Delta u)(u-u_I) \sim h^2; \quad \int(f+\Delta u)(u_h-\psi) \leq 0.$$

(ii) With quadratic finite elements:

$$\|u-u_I\|^2 \sim h^3; \quad \int(f+\Delta u)(u-u_I) \sim h^3; \quad \int(f+\Delta u)(u_h-\psi) \sim h^3.$$

(The next-to-last integral is split into a part completely within the plastic region, where $u-u_I \sim h^3$, and a part formed from those triangles which cross the free boundary. This transition region has area $O(h)$, and the integrand $u-u_I$ is $O(h^2)$.) Substituting back into (10), the rates of convergence are h and $h^{3/2}$ in the two cases-- these rates are confirmed by experiment.

IV. Open Problems. True plasticity theory is a deeper mathematical problem than the model we have used above. The reason is that the history of the loading f has to be taken into account; a part of the domain can go from elastic to plastic and back again, as the external loads are increased. Therefore incremental theory introduces a time parameter, and a rate of loading f in the functional J --and it computes the stressrate $\dot{\sigma}$. In other words, as Maier and Capurso have shown, we have a time-dependent variational inequality;

$$(11) \quad \min_{\dot{v} \in K(t)} J(\dot{v}) = J(\dot{u}), \text{ with } K = \{\dot{v} \in H_0^1, \dot{v} \leq 0 \text{ where } u(t) = \psi\}.$$

Notice that at each instant the convex set depends on the current state u . In a practical problem the state is actually a vector of stresses and plastic multipliers, but we hope that this quasi-static obstacle problem will serve as a reasonable model. We also hope that the new results on regularity can be extended to $u(t)$. But even on this assumption, there remain three new problems in numerical analysis:

(i) Keeping time continuous, to prove convergence of the finite element approximations. The difficulty is that the convex set K , and therefore the minimizing \dot{u} , depend discontinuously on the current state u ; therefore it is not true that \dot{u}_h is close to \dot{u} whenever u_h is close to u .

(ii) To admit finite difference approximations in time, and to determine the stability limits on the interval Δt .

(iii) To find a quick way of solving, with adequate accuracy, the obstacle problem which arises at each time step.

We believe that these are among the most important questions in nonlinear finite element analysis, and that answers can be found.

A second class of problems, of an entirely different type, arises if we are interested only in the multiple λ of the load f which will induce plastic collapse. This is known as limit analysis, and no longer requires us to follow the loading history. In place of minimizing a quadratic functional, the problem falls into the framework of infinite-dimensional linear programming. Here is a typical example, with unknown stresses $\sigma_{ij}(x,y)$ and multiplier λ : Maximize λ , subject to

Equilibrium: $\sum \frac{\partial \sigma_{ij}}{\partial x_i} = \lambda f_j$ in Ω , $\sum \sigma_{ij} n_i = \lambda g_j$ on $\partial\Omega$, and

Piecewise Linear
Yield Conditions: $\sum b_{ij}^\alpha \sigma_{ij} \leq c^\alpha$ in Ω , $1 \leq \alpha \leq M$.

Suppose we make this problem finite-dimensional, by assuming that the stresses (and also the displacements, which are the unknowns in the dual program) belong to piecewise polynomial spaces S_h . The continuous linear programming problem is then approximated, in a completely natural way, by a discrete one [3]. But we know nothing about the rate of convergence.

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