Publications mathématiques et informatiques de Rennes

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Publications des séminaires de mathématiques et informatique de Rennes, 1974, fascicule S4
« Journées éléments finis », , p. 1-28
<http://www.numdam.org/item?id=PSMIR_1974___S4_A5_0>

DUAL ITERATIVE TECHNIQUES FOR SOLVING A FINITE ELEMENT APPROXIMATION OF THE BIHARMONIC EQUATION

Ph. Ciarlet, R. Glowinski

Abstract. A finite element approximation of the Dirichlet problem for the biharmonic operator is described. Its main feature is that it is equivalent to solving a sequence of discrete Dirichlet problems for the operator $-\Delta$. This method, which has already been shown to be convergent, is particularly well-suited for problems in Fluid Dynamics.

Résumé. On décrit une méthode d'approximation par éléments finis du problème de Dirichlet pour l'opérateur biharmonique. Le principal intérêt de cette méthode est d'équivaloir à la résolution d'une suite de problèmes de Dirichlet pour l'opérateur $-\Delta$. Cette méthode, dont la convergence a été démontrée par ailleurs, est particulièrement bien adaptée aux problèmes de la Dynamique des Fluides.

1. Introduction. Throughout this paper, Ω denotes the interior of a convex polygon in the plane, with boundary Γ, $\frac{\partial}{\partial\nu}$ denotes the exterior normal derivative along Γ, and f is a given function belonging to the space $L^2(\Gamma)$. We consider the Dirichlet problem for the biharmonic operator, which reads formally as :

$$\Delta^2 u = f \text{ in } \Omega, \tag{1.1}$$

$$u = \frac{\partial u}{\partial\nu} = 0 \text{ on } \Gamma. \tag{1.2}$$

We first study in §2 a variational formulation of problem (1.1)-(1.2), recently introduced in [1], whose main feature is that the associated solution lies in the product space $H^1_0(\Omega) \times L^2(\Omega)$, instead of the space $H^2_0(\Omega)$ for the "classical" variational formulation. Therefore, when this approach is discretized, it suffices to use finite element of class $\mathcal{C}^0$, whereas finite element of class $\mathcal{C}^1$ are needed for the usual conforming methods, such as the well-known 21-degree of freedom triangle of Argyris [2]. Moreover, it is not necessary to use finite element of Hermite type, as is the case for standard nonconforming methods. For general discussions about finite element methods for solving problem (1.1)-(1.2), we refer the reader to [3,4] and the various references quoted therein.

In addition, from the point of view of Fluid Mechanics, the present method seems even more interesting than any conforming method, since it not only yields a continuous approximation of the stream function u, but also of the vorticity Δu, whereas a standard approximation using finite elements of class $\mathcal{C}^1$ would result in a discontinuous approximation of the vorticity.

As was already pointed out in [1], it seems that a safe way to perform variational crimes, such as curved boundaries or numerical integration (cf. [5]), when one deals with fourth-order problems is thus to use this method or a similar one, since, in essence, the associated computations are exactly the same as in the case of second order problems.

Indeed, a variational crime was already performed on this method in [6], where convergence was proved for subspaces made up of piecewise polynomials of degree 1 (which are not included in the error analysis of [1]). As was pointed out in [6], the discrete equations associated with this type of subspaces are identical to the usual 13-point finite-difference approximation of the operator Δ^2.

As regards the convergence of the method, the following was proved in [1]: If the trial functions are piecewise polynomials of degree k, then a sequence (u_h, ϕ_h) of approximations of the solution u (of (1.1)-(1.2)) and $-\Delta u$ is obtained which satisfies

$$(1.3) \qquad \|u-u_h\|_{H^1(\Omega)} + \|\Delta u+\phi_h\|_{L^2(\Omega)} < \mathcal{K} \|u\|_{H^{k+2}(\Omega)} h^{k-1}$$

for some constant $\mathcal{K}$ independent of h. As a consequence, an $O(h)$ convergence is therefore obtained with polynomials of degree only 2. Let us add that this method falls in the general category of <u>mixed finite element methods</u>. For general results concerning these methods which however do not contain the present ones, see the works of Oden and Reddy [7,8,9].

The main object of this paper is to show (Theorems 4,5,8 and 9) that <u>solving either the continuous or the discrete problem amounts to solving a sequence of Dirichlet problems for either the operator $-\Delta$ or its variational approximation</u>.

The method presented here is thus an answer to a problem which several authors have considered, for either the continuous problem or its various possible discretizations; see for instance [10,11,12,13,14] and the references therein.

As a practical consequence, all that is needed for approximating the solution of problem (1.1)-(1.2) is therefore a finite element program for solving second order problems.

Following and generalizing a method due to the second author [6], the basic idea consists in applying Uzawa's method [15, Chapter 2] for solving the saddle-point equations of the Lagrangian associated with the present variational formulation; see §§2 and 3. As usual, this method is convergent provided a certain parameter ρ lies in some interval. In the case of the discrete problem, we show (Theorem 10) that this interval is of the form $0 < \rho < 2\sigma_h^2$, where

$$\lim_{h\to 0} \sigma_h = \inf_{v\in H^2(\Omega)\cap H_0^1(\Omega)} \frac{\| \Delta v \|_{L^2(\Omega)}}{\| \frac{\partial v}{\partial \nu} \|_{L^2(\Gamma)}} , \tag{1.4}$$

for regular families of finite element subspaces and provided an appropriate inner-product is chosen in the space of the discrete Lagrange multipliers. Thus the quantity σ_h may be estimated in practice, at least for simple geometric domains Ω. This result generalizes Theorem 6 of [11], where an analogous result was proved for a finite-difference approximation over a rectangle, using Fourier series technique.

The above results were announced in [16]; complete proofs for the discrete problem will be found in [17]. Finally, we refer to a forthcoming paper of Bourgat in the same Journal, where numerical results will be presented.

The duality pairing between a space X and its dual X' will be denoted by

$$\langle \cdot , \cdot \rangle .$$

Given a mapping $f : X \to Y$, where X and Y are normed vector spaces, its Fréchet derivative at the point $a \in X$ will be denoted (assuming its existence)

$$Df(a).$$

At several places, we shall use Green formulas in Sobolev spaces, for which we refer to Nečas [18].

2. The continuous problem.

The standard variational formulation of problem (1.1)-(1.2) consists in finding a function $u \in H_0^2(\Omega)$ which satisfies

$$(2.1) \qquad \mathcal{J}(u) = \min_{v \in H_0^2(\Omega)} \mathcal{J}(v),$$

where

$$(2.2) \qquad \mathcal{J}(v) = \frac{1}{2} \int_\Omega |\Delta v|^2 dx - \int_\Omega f v dx,$$

and it can be shown [19] that the unique solution u of the above problem belongs to the space $H^3(\Omega) \cap H_0^2(\Omega)$.

We may also consider that we minimize the functional

$$(2.3) \qquad \mathscr{J}(v,\psi) = \frac{1}{2} \int_\Omega |\psi|^2 dx - \int_\Omega f v dx,$$

when the functions $v \in H_0^2(\Omega)$ and $\psi \in L^2(\Omega)$ satisfy the equation $-\Delta v = \psi$. Since the corresponding subspace of the product space $H_0^2(\Omega) \times L^2(\Omega)$ can be alternatively described as the space

$$(2.4) \quad \mathscr{V} = \{(v,\phi) \in H_0^1(\Omega) \times L^2(\Omega);\ \forall \mu \in H^1(\Omega),\ \beta((v,\psi),\mu) = 0\},$$

with

$$(2.5) \qquad \beta((v,\psi),\mu) = \int_\Omega \text{grad} v \ \text{grad} \mu \ dx - \int_\Omega \psi \ \mu \ dx,$$

we have the following result (for the proofs, see Theorem 1 of [1]).

Theorem 1. Let u denote the solution of the minimization problem (2.1). Then we also have

$$(2.6) \qquad \mathscr{J}(u,-\Delta u) = \min_{(v,\psi) \in \mathscr{V}} \mathscr{J}(v,\psi),$$

where $\mathscr{J}$ is the functional defined in (2.3). ∎

To solve the constrained minimization problem (2.6), we will use a standard technique in duality theory: Let there be given a subspace of the space $H^1(\Omega)$ such that we may write the direct sum

$$H^1(\Omega) = H^1_0(\Omega) \oplus \mathcal{M}. \tag{2.7}$$

We next introduce the space

$$\mathcal{W} = \{(v,\psi) \in H^1_0(\Omega)\times L^2(\Omega);\ \forall\mu \in H^1_0(\Omega),\ \beta((v,\psi),\mu) = 0\}, \tag{2.8}$$

and the Lagrangian

$$\mathcal{L}((v,\psi),\mu) = \mathcal{J}(v,\psi) + \beta((v,\psi),\mu), \tag{2.9}$$

with $\mathcal{J}$ and β defined as in (2.3) and (2.5), respectively.

In view of decomposing the original problem as a sequence of Dirichlet problems for the operator $-\Delta$, the following result will play a central role.

Theorem 2. *Given a function $\lambda \in \mathcal{M}$, the problem : Find a pair $(u_\lambda,\phi_\lambda) \in \mathcal{W}$ such that*

$$\mathcal{L}((u_\lambda,\phi_\lambda),\lambda) = \min_{(v,\psi)\in\mathcal{W}} \mathcal{L}((v,\psi),\lambda) \tag{2.10}$$

has one and only one solution, which may also be obtained by solving the following Dirichlet problems for the operator $-\Delta$:

(i) *Find a function $\phi_\lambda \in H^1(\Omega)$ such that*

$$\phi_\lambda-\lambda \in H^1_0(\Omega), \tag{2.11}$$

$$\forall v \in H^1_0(\Omega),\ \int_\Omega \operatorname{grad}\phi_\lambda \operatorname{grad}v\, dx = \int_\Omega f\, v\, dx. \tag{2.12}$$

(ii) *Find a function $u_\lambda \in H^1_0(\Omega)$ such that*

$$\forall v \in H^1_0(\Omega),\ \int_\Omega \operatorname{grad}u_\lambda \operatorname{grad}v\, dx = \int_\Omega \phi_\lambda\, v\, dx. \tag{2.13}$$

Proof. First, we observe that problem (2.10) has one and only one solution since the leading term in the Lagrangian : $\frac{1}{2}\int_\Omega |\psi|^2dx$, is the square of a norm over the space $\mathcal{W}$.

By definition (see (2.8)), a pair $(v,\psi) \in H_0^1(\Omega) \times L^2(\Omega)$ belongs to the space $\mathscr{W}$ if

$$\forall\mu \in H_0^1(\Omega), \int_\Omega \text{grad}v \ \text{grad}\mu \ dx = \int_\Omega \psi \ \mu \ dx.$$

Since for any given function $\psi \in L^2(\Omega)$, there exists a unique function $v \in H_0^1(\Omega)$ which satisfies the above relations, we may write $v = A\psi$, the linear operator

$$A : L^2(\Omega) \to H_0^1(\Omega) \tag{2.14}$$

defined in this fashion being continuous.

We may thus consider that we minimize a function of two variables (λ is fixed), namely :

$$\mathscr{L}((\cdot,\cdot),\lambda) \ : \ (v,\psi) \in H_0^1(\Omega) \times L^2(\Omega) \to \mathscr{L}((v,\psi),\lambda),$$

when these two variables satisfying the constraint

$$\Phi(v,\psi) = 0,$$

where the mapping

$$\Phi : H_0^1(\Omega) \times L^2(\Omega) \to H_0^1(\Omega) \tag{2.15}$$

is defined by $\Phi(v,\psi) = A\psi - v$.

Both functions $\mathscr{L}((\cdot,\cdot),\lambda)$ and Φ being differentiable, there exists a unique Lagrange multiplier $\Xi_\lambda \in H^{-1}(\Omega)$ such that :

$$D\mathscr{L}((u_\lambda,\phi_\lambda),\lambda) = \Xi_\lambda \cdot D\Phi(u_\lambda,\phi_\lambda).$$

Taking partial derivatives with respect to the two variables, we find that

$$\forall v \in H_0^1(\Omega), < \Xi_\lambda, v > + \int_\Omega \text{grad}\lambda \ \text{grad}v \ dx = \int_\Omega f \ v \ dx, \tag{2.16}$$

$$\forall\psi \in L^2(\Omega), < \Xi_\lambda, A\psi > + \int_\Omega \psi \ \lambda \ dx = \int_\Omega \phi_\lambda \ \psi \ dx, \tag{2.17}$$

and thus, equations (2.16) and (2.17) together with the equation

(2.18) $$u_\lambda = A\phi_\lambda ,$$

allow to determine the functions u_λ and ϕ_λ. In order to put relations (2.16) and (2.17) in more explicit a form, it is convenient to introduce the unique function $\xi_\lambda \in H^1_0(\Omega)$ which satisfies

(2.19) $$\forall v \in H^1_0(\Omega),\ < \Xi_\lambda, v > = \int_\Omega \operatorname{grad}\xi_\lambda \operatorname{grad}v\, dx.$$

Then equations (2.16) become

(2.20) $$\forall v \in H^1_0(\Omega),\ \int_\Omega \operatorname{grad}(\xi_\lambda+\lambda) \operatorname{grad}v\, dx = \int_\Omega f\, v\, dx,$$

on the one hand, and in view of the definition of the mapping A of (2.14), equations (2.17) may be rewritten as

(2.21) $$\forall \psi \in L^2(\Omega),\ \int_\Omega (\xi_\lambda+\lambda-\phi_\lambda)\psi\, dx = 0,$$

on the other hand. Thus

(2.22) $$\phi_\lambda = \xi_\lambda+\lambda,$$

which achieves the proof. ∎

Notice that since the boundary of the domain is smooth enough, the function u_λ is in fact in the space $H^2(\Omega) \cap H^1_0(\Omega)$, so that equations (2.13) can be equivalently written as

(2.23) $$\phi_\lambda = -\Delta u_\lambda .$$

We now show that the solution of the original problem is also the first argument of the saddle-point of the Lagrangian $\mathcal{L}$.

<u>Theorem 3</u>. <u>Let</u> u <u>denote the solution of problem</u> (2.1) <u>and let</u> λ^* <u>be the function in the space</u> $\mathcal{M}$ <u>with the property that the function</u> $(\Delta u+\lambda^*)$ <u>belongs to the space</u> $H^1_0(\Omega)$. <u>Then the triple</u> $((u,-\Delta u),\lambda^*)$ <u>is the unique</u>

<u>saddle-point of the Lagrangian</u> $\mathcal{L}$ <u>of</u> (2.9) <u>over the space</u> $\mathcal{W} \times \mathcal{M}$, <u>i.e.</u>,

$$\mathcal{L}((u,-\Delta u),\mu) \leq \mathcal{L}((u,-\Delta u),\lambda^*) \leq \mathcal{L}((v,\psi),\lambda^*) \tag{2.24}$$

<u>for all pairs</u> $(v,\psi) \in \mathcal{W}$ <u>and all</u> $\mu \in \mathcal{M}$.

<u>Proof</u>. Since the pair $(u,-\Delta u)$ belongs to the space $\mathcal{V}$ defined in (2.4), we have

$$\forall \mu \in \mathcal{M} \ , \ \mathcal{L}((u,-\Delta u),\mu) = \mathcal{J}(u,-\Delta u),$$

and thus the first inequality of (2.24) is proved. To prove the second inequality of (2.24), it suffices to show that

$$\forall v \in H_0^1(\Omega), \int_\Omega \mathrm{grad}(-\Delta u) \ \mathrm{grad} v \ dx = \int_\Omega f \ v \ dx, \tag{2.25}$$

in view of Theorem 2; we have already noticed that the function u is in the space $H^3(\Omega) \cap H_0^2(\Omega)$. Using Green's formula, we obtain

$$\forall v \in H_0^2(\Omega), \int_\Omega \Delta u \ \Delta v \ dx = \int_\Omega \mathrm{grad}(-\Delta u) \ \mathrm{grad} v \ dx = \int f \ v \ dx ,$$

and equations (2.25) follow since the space $H_0^2(\Omega)$ is a dense subspace of $H_0^1(\Omega)$.

Let now $((u^*,\phi^*),\lambda^*)$ be any saddle-point of the Lagrangian $\mathcal{L}$. From Theorem 2, we infer that

$$\phi^* - \lambda^* \in H_0^1(\Omega), \tag{2.26}$$

$$\forall v \in H_0^1(\Omega), \int_\Omega \mathrm{grad}\phi^* \ \mathrm{grad} v \ dx = \int_\Omega f \ v \ dx, \tag{2.27}$$

$$\forall v \in H_0^1(\Omega), \int_\Omega \mathrm{grad} u^* \ \mathrm{grad} v \ dx = \int_\Omega \phi^* \ v \ dx, \tag{2.28}$$

since

$$\mathcal{L}((u^*,\phi^*),\lambda^*) = \min_{(v,\psi)\in \mathcal{W}} \mathcal{L}((v,\psi),\lambda^*).$$

Likewise, the inequalities

$$\forall \mu \in \mathcal{M} , \ \mathcal{L}((u^*,\phi^*),\mu) \leq \mathcal{L}((u^*,\phi^*),\lambda^*),$$

imply that

$$\forall \mu \in \mathcal{M}_\gamma\,,\ \int_\Omega \mathrm{grad} u^* \,\mathrm{grad}\mu \; dx = \int_\Omega \phi^* \mu \; dx. \tag{2.29}$$

Combining (2.28) and (2.29), we deduce that $(u^*,\phi^*) \in \mathcal{V}$, i.e.,

$$u^* \in H_0^2(\Omega) \text{ and } \phi^* = -\Delta u^*. \tag{2.30}$$

Using Green's formula, (2.27) and (2.30), we get

$$\forall v \in H_0^2(\Omega),\ \int_\Omega \Delta u^* \,\Delta v \; dx = \int_\Omega \mathrm{grad}(-\Delta u^*)\,\mathrm{grad} v \; dx = \int_\Omega f\, v \; dx,$$

i.e., $u^* = u$, and the conclusion follows using ((2.26) and (2.30). ∎

As a corollary of Theorem 3, we can write (see [20, Chapter VI])

$$\mathcal{L}\big((u,-\Delta u),\lambda^*\big) = \max_{\lambda \in \mathcal{M}_\gamma} g(\lambda), \tag{2.31}$$

where the function g is defined by

$$g : \lambda \in \mathcal{M}_\gamma \to g(\lambda) = \min_{(v,\psi)\in \mathcal{W}} \mathcal{L}\big((v,\psi),\lambda\big). \tag{2.32}$$

Using the same notations as in Theorem 2, we may thus write

$$g(\lambda) = \mathcal{L}\big((u_\lambda,\phi_\lambda),\lambda\big) = -\frac{1}{2}\int_\Omega |\phi_\lambda|^2 \; dx \tag{2.33}$$

for any $\lambda \in \mathcal{M}_\gamma$.

Lemma 1. *Assume that the space $\mathcal{M}_\gamma$ is equipped with a norm equivalent to the norm of the space $H^1(\Omega)$. Then the function g defined in (2.32) is differentiable and its derivative $Dg(\lambda) \in \mathcal{M}_\gamma'$ is given by*

$$< Dg(\lambda),\mu > = \beta\big((u_\lambda,\phi_\lambda),\mu\big) \tag{2.34}$$

for all $\lambda,\mu \in \mathcal{M}_\gamma$.

Proof. Write $g = f \circ h$, with the functions f and h defined by

$$f : v \in H^1(\Omega) \to -\frac{1}{2}\int_\Omega |v|^2 dx,$$
$$h : \lambda \in \mathcal{M}_\gamma \to \phi_\lambda \in H^1(\Omega).$$

Let us show that the affine mapping h is continuous : From (2.11) and (2.12), we obtain

$$\int_\Omega \operatorname{grad}(\phi_\lambda-\lambda)\ \operatorname{grad}(\phi_\lambda-\lambda)dx = \int_\Omega f(\phi_\lambda-\lambda)dx$$

$$\leq \| f \|_{L^2(\Omega)} \| \phi_\lambda-\lambda \|_{L^2(\Omega)},$$

so that, for some constant C solely dependent upon the set Ω, we have

$$\| \phi_\lambda-\lambda \|_{H^1(\Omega)} \leq C \| f \|_{L^2(\Omega)},$$

and thus the mapping h is continuous since

$$\| \phi_\lambda \|_{H^1(\Omega)} \leq \| \lambda \|_{H^1(\Omega)} + C \| f \|_{L^2(\Omega)}.$$

The mapping h being affine, it suffices to compute its derivative when $f = 0$, which yields for all $\lambda,\mu \in \mathcal{M}$,

(2.35) $$< Dh(\lambda),\mu > = \phi_\mu^o,$$

where the function $\phi_\mu^o \in H^1(\Omega)$ satisfies

(2.36) $$\phi_\mu^o - \mu \in H_o^1(\Omega),$$

(2.37) $$\forall v \in H_o^1(\Omega),\ \int_\Omega \operatorname{grad}\phi_\mu^o\ \operatorname{grad}v\ dx = 0.$$

Since, for all $v,w \in H^1(\Omega)$,

(2.38) $$< Df(v),w > = -\int_\Omega v\ w\ dx,$$

we obtain, from (2.35) and (2.38),

$$< Dg(\lambda),\mu > = -\int_\Omega \phi_\lambda\ \phi_\mu^o\ dx.$$

To transform this last expression, we observe that

(2.39) $$-\int_\Omega \phi_\lambda\ \phi_\mu^o\ dx = \int_\Omega \phi_\lambda(\mu-\phi_\mu^o)dx - \int_\Omega \phi_\lambda\ \mu\ dx$$

$$= \beta\big((u_\lambda,\phi_\lambda),\mu\big),$$

as a simple consequence of (2.13), (2.36) and (2.37). ∎

Using the differentiability of the function g which we just established, we now apply the gradient method to the maximization problem (2.31), a technique known as Uzawa's method for the original problem. Given any function λ° in the space $\mathcal{M}$ and a parameter $\rho > 0$ to be specified later (cf. Theorem 5), the method consists in defining a sequence of functions $\lambda^n \in \mathcal{M}$ such that

(2.40) $\quad \forall \mu \in \mathcal{M}, \; (\lambda^{n+1}-\lambda^n,\mu)_{\mathcal{M}} = -\rho < Dg(\lambda^n),\mu >, \; n \geq 0,$

where $(\cdot,\cdot)_{\mathcal{M}}$ is an inner-product in the space $\mathcal{M}$ whose associated norm is equivalent to the norm in the space $H^1(\Omega)$.

As a simple corollary of Theorem 2 and Lemma 1, we then have :

Theorem 4. *Each iteration of Uzawa's method consists of the following steps* :

(i) *Given a function* $\lambda^n \in \mathcal{M}$, *find the function* $\phi^n \in H^1(\Omega)$ *which satisfies* :

(2.41) $$\phi^n - \lambda^n \in H^1_0(\Omega),$$

(2.42) $$\forall v \in H^1_0(\Omega), \int_\Omega \operatorname{grad}\phi^n \operatorname{grad}v \, dx = \int_\Omega f \, v \, dx.$$

(ii) *Find the function* $u^n \in H^1_0(\Omega)$ *which satisfies* :

(2.43) $$\forall v \in H^1_0(\Omega), \int_\Omega \operatorname{grad}u^n \operatorname{grad}v \, dx = \int_\Omega \phi^n v \, dx.$$

(iii) *Find the function* $\lambda^{n+1} \in \mathcal{M}$ *which satisfies*

(2.44) $$\forall \mu \in \mathcal{M}, \; (\lambda^{n+1}-\lambda^n,\mu)_{\mathcal{M}} = \rho \, B((u^n,\phi^n),\mu).$$

The Uzawa method described in the previous theorem thus consists in solving a sequence of Dirichlet problems. That this scheme yields indeed the solution of the original problem will now be proved.

Theorem 5. The method described in Theorem 4 is convergent, in the sense that

(2.45) $$\lim_{n\to\infty} u^n = u \text{ in } H_0^1(\Omega),$$

(2.46) $$\lim_{n\to\infty} \phi^n = -\Delta u \text{ in } L^2(\Omega),$$

provided that

(2.47) $$0 < \rho < 2c^2\sigma^2,$$

where the quantity σ is defined by

(2.48) $$\sigma = \inf_{v\in H^2(\Omega)\cap H_0^1(\Omega)} \frac{\|\Delta v\|_{L^2(\Omega)}}{\|\frac{\partial v}{\partial \gamma}\|_{L^2(\Gamma)}},$$

and c is any constant such that

(2.49) $$\forall\mu \in H^1(\Omega),\ c\,\|\mu\|_{L^2(\Gamma)} < \sqrt{(\mu,\mu)_{\mathcal{M}}}.$$

Proof. We recall that the space $\mathcal{M}$ is equipped with a norm equivalent to the norm $\|\cdot\|_{H^1(\Omega)}$. To prove the convergence (cf. (2.45) and (2.46)), it suffices to prove that $\lim_{n\to\infty} u^n = 0$ in $H_0^1(\Omega)$ and $\lim_{n\to\infty} \phi^n = 0$ in $L^2(\Omega)$, in the special case where $f = 0$.

Let us define a mapping

(2.50) $$B : L^2(\Omega) \to \mathcal{M}$$

as follows : for any given $\psi \in L^2(\Omega)$, the function $B\psi \in \mathcal{M}$ is uniquely determined by the condition that

$$\forall\mu \in \mathcal{M}\ ,\ (B\psi,\mu)_{\mathcal{M}} = \beta\big((A\psi,\psi),\mu\big),$$

where the mapping $A : L^2(\Omega) \to H_0^1(\Omega)$ is the mapping of (2.14). As a consequence, equations (2.44) may be rewritten as

$$\lambda^{n+1} = \lambda^n + \rho\, B\, \phi^n,$$

so that by making use of (2.39) (recall that $f = 0$), we deduce :

$$(\lambda^{n+1},\lambda^{n+1})_{\mathcal{M}_\gamma} = (\lambda^n,\lambda^n)_{\mathcal{M}_\gamma} - 2\rho \|\phi^n\|^2_{L^2(\Omega)} + \rho^2 (B\phi^n, B\phi^n)_{\mathcal{M}_\gamma} \quad ,$$

and therefore,

(2.51) $$(\lambda^n,\lambda^n)_{\mathcal{M}_\gamma} - (\lambda^{n+1},\lambda^{n+1})_{\mathcal{M}_\gamma} \geq (2\rho - \rho^2 \|B\|^2) \|\phi^n\|^2_{L^2(\Omega)},$$

with

(2.52) $$\|B\| = \|B\|_{\mathcal{L}(L^2(\Omega);\mathcal{M}_\gamma)} = \sup_{\psi \in L^2(\Omega)} \frac{\sqrt{(B\psi,B\psi)_{\mathcal{M}_\gamma}}}{\|\psi\|_{L^2(\Omega)}} .$$

From (2.51), we may conclude that

$$\lim_{n\to\infty} \phi^n = 0 \text{ in } L^2(\Omega),$$

and also, in view of the continuity of the mapping A, that

$$\lim_{n\to\infty} u^n = 0 \text{ in } H^1_0(\Omega),$$

provided that

(2.53) $$0 < \rho < \frac{2}{\|B\|^2} .$$

It therefore remains to obtain a lower bound for the quantity $\frac{2}{\|B\|^2}$.

We have, for any function $\psi \in L^2(\Omega)$,

$$\sqrt{(B\psi,B\psi)_{\mathcal{M}_\gamma}} = \sup_{\mu \in \mathcal{M}_\gamma} \frac{|(B\psi,\mu)_{\mathcal{M}_\gamma}|}{\sqrt{(\mu,\mu)_{\mathcal{M}_\gamma}}} ,$$

and using Green's formula, we can write

$$(B\psi,\mu)_{\mathcal{M}_\gamma} = \beta\big((A\psi,\psi),\mu\big) = \int_\Gamma \frac{\partial v}{\partial \nu}\, \mu \, d\gamma,$$

where $v = A\psi$. As a consequence :

$$|(B\psi,\mu)_{\mathfrak{M}}| \le \left\|\frac{\partial v}{\partial \nu}\right\|_{L^2(\Gamma)} \|\mu\|_{L^2(\Gamma)} \le \frac{1}{c}\left\|\frac{\partial v}{\partial \nu}\right\|_{L^2(\Gamma)} \sqrt{(\mu,\mu)_{\mathfrak{M}}},$$

with the constant c defined as in (2.49). Since $-\Delta v = \psi$, we finally obtain the estimate

$$\|B\| \le \frac{1}{c} \sup_{v \in H^2(\Omega)\cap H^1_0(\Omega)} \frac{\left\|\frac{\partial v}{\partial \nu}\right\|_{L^2(\Gamma)}}{\|\Delta v\|_{L^2(\Omega)}}, \tag{2.54}$$

from which the conclusion follows. ∎

Remark 1. Assume the space $\mathfrak{M}$ is chosen to be the orthogonal complement of the space $H^1_0(\Omega)$ in the space $H^1(\Omega)$. Then it is known that the space $\mathfrak{M}$ is isomorphic to the space $H^{1/2}(\Gamma)$. That the space $\mathfrak{M}$ can be thought of as a space of traces on Γ is also reflected by the identity

$$\beta((v,\psi),\mu) = \int_\Gamma \frac{\partial u}{\partial \nu}\mu \, dx, \tag{2.55}$$

valid for any pair $(v,\psi) \in \mathcal{W}$.

Indeed it is possible to develop a treatment analogous to that considered here with the space $\mathfrak{M}$ replaced by a Hilbert space of functions defined over Γ; this is basically what has been done in [6], with the space $L^2(\Gamma)$ in lieu of $\mathfrak{M}$. Let us review briefly the main steps in this case : One looks for the saddle-point of the Lagrangian

$$L(v,\mu) = \mathcal{J}(v) + \int_\Gamma \frac{\partial v}{\partial \nu}\mu \, dx \tag{2.56}$$

($\mathcal{J}$ defined as in (2.1)) over the product space $W \times L^2(\Gamma)$ with

$$W = H^2(\Omega) \cap H^1_0(\Omega). \tag{2.57}$$

The Lagrange multiplier is the trace of the function $-\Delta u$ over Γ; Compare with Theorem 3.

As shown in [6], one iteration of Uzawa's method consists in the following steps :

(i) Given a function $\lambda^n \in L^2(\Gamma)$, find the function $u^n \in W$ which satisfies :

$$L(u^n,\lambda^n) = \min_{v\in W} L(v,\lambda^n). \tag{2.58}$$

(ii) Find a function $\lambda^{n+1} \in L^2(\Gamma)$ such that

$$\lambda^{n+1} = \lambda^n + \rho \frac{\partial u^n}{\partial \nu} . \tag{2.59}$$

Notice that the minimization problem (2.58) amounts to solving, at least formally,

$$\begin{aligned} \Delta^2 u^n &= f \text{ in } \Omega, \\ u^n &= 0 \text{ on } \Gamma, \\ -\Delta u^n &= \lambda^n \text{ on } \Gamma, \end{aligned} \tag{2.60}$$

a problem which can be obviously decomposed into two Dirichlet problems for $-\Delta$. Notice also that equations (2.44) have now been replaced by the explicit equation (2.59), since we are now working in the space $L^2(\Gamma)$. For the same reason, we could prove the convergence of the above algorithm provided the parameter ρ belongs to the interval $]0,2\sigma^2[$; compare with Theorem 5.

However there is one good reason for "prefering" the space $\mathfrak{M}$: the functional set-up being in this case exactly the same for both the continuous and discrete problems, the proofs are the same in the discrete case, as we will see in the next section.

3. The discrete problem. We are given a finite-dimensional subspace V_h of the space $H^1(\Omega)$ and we define the spaces

$$(3.1) \qquad V_{oh} = \{v_h \in V_h;\ v_h = 0 \text{ on } \Gamma\},$$

and

$$(3.2) \qquad \mathcal{V}_h = \{(v_h,\psi_h) \in V_{oh}\times V_h;\ \forall \mu_h \in V_h,\ \beta((v_h,\psi_h),\mu_h) = 0\},$$

where β is defined as in (2.5).

Remark 2. It is worth pointing out that the same space V_h approximates both spaces $L^2(\Omega)$ and $H^1(\Omega)$; compare (3.2) and the definition (2.4) of the space $\mathcal{V}$. Indeed, we could carry out a seemingly more general discretization in which another space, say Y_h, would approximate the space $L^2(\Omega)$, but since we eventually need the inclusion $Y_h \subset V_h$ to derive the error estimates (cf. [1, Theorem 7]) it is therefore appropriate to assume at the outset that $Y_h = V_h$. In addition, this assumption results in simpler statements and proofs at several places. ∎

In analogy with Theorem 1, we define the discrete problem : Find a pair $(u_h,\phi_h) \in \mathcal{V}_h$ such that

$$(3.3) \qquad \mathcal{J}(u_h,\phi_h) = \min_{(v_h,\psi_h)\in\mathcal{V}_h} (v_h,\psi_h).$$

It was proved in [1, Theorem 3] that this problem has a unique solution.

Let there be given a subspace $\mathcal{M}_h$ of the space V_h such that

$$(3.4) \qquad V_h = V_{oh} \oplus \mathcal{M}_h.$$

We define the space

$$(3.5) \qquad \mathcal{W}_h = \{(v_h,\psi_h) \in V_{oh}\times V_h; \forall \mu_h \in V_{oh},\ \beta((v_h,\psi_h),\mu_h) = 0\}.$$

The following is the discrete analog of Theorem 2.

Theorem 6. Given a function $\lambda_h \in \mathcal{M}_h$, the problem: Find a pair $(u_h, \phi_h) \in \mathcal{W}_h$ such that

(3.6) $$\mathcal{L}((u_{\lambda_h}, \phi_{\lambda_h}), \lambda_h) = \min_{(v_h, \psi_h) \in \mathcal{W}_h} \mathcal{L}((v_h, \psi_h), \lambda_h)$$

has one and only one solution, which may also be obtained by solving the following discrete Dirichlet problems :

(i) Find a function $\phi_{\lambda_h} \in V_h$ such that

(3.7) $$\phi_{\lambda_h} - \lambda_h \in V_{oh},$$

(3.8) $$\forall v_h \in V_{oh}, \int_\Omega \operatorname{grad}\phi_{\lambda_h} \operatorname{grad} v_h \, dx = \int_\Omega f \, v_h \, dx.$$

(ii) Find a function $u_{\lambda_h} \in V_{oh}$ such that

(3.9) $$\forall v_h \in V_{oh}, \int_\Omega \operatorname{grad} u_{\lambda_h} \operatorname{grad} v_h \, dx = \int_\Omega \phi_{\lambda_h} v_h \, dx.$$

Proof. The proof is basically the same as the proof of Theorem 2 and, for this reason, will be omitted. We shall just mention that an essential use is made of the fact that the same space V_h approximates both spaces $L^2(\Omega)$ and $H^1(\Omega)$; cf. Remark 2. ∎

The pair (u_h, ϕ_h) being the solution of the discrete problem (3.3), we let ϕ_{oh} be the unique function in the space V_{oh} such that the function $(\phi_h - \phi_{oh})$ belongs to the space $\mathcal{M}_h$. We then have the following theorems whose proofs follow the same lines as the proofs of Theorems 3, 4 and 5, respectively.

Theorem 7. The triple $((u_h, \phi_h), \phi_h - \phi_{oh})$ is the unique saddle-point of the Lagrangian $\mathcal{L}$ of (2.9) over the space $\mathcal{W}_h \times \mathcal{M}_h$. ∎

In the next theorem, $(\cdot,\cdot)_{\mathcal{M}_h}$ is any inner-product in the space $\mathcal{M}_h$ and ρ is a strictly positive parameter.

Theorem 8. Each iteration of Uzawa's method consists of the following steps :

(i) Given a function $\lambda_h^n \in \mathcal{M}_h$, find the function $\phi_h^n \in V_h$ which satisfies

$$\phi_h^n - \lambda_h^n \in V_{oh}, \tag{3.10}$$

$$\forall v_h \in V_{oh},\ \int_\Omega \operatorname{grad}\phi_h^n \operatorname{grad} v_h \, dx = \int_\Omega f\, v_h \, dx. \tag{3.11}$$

(ii) Find the function $u_h^n \in V_{oh}$ which satisfies

$$\forall v_h \in V_{oh},\ \int_\Omega \operatorname{grad} u_h^n \operatorname{grad} v_h \, dx = \int_\Omega \phi_h^n \, v_h \, dx. \tag{3.12}$$

(iii) Find the function $\lambda_h^{n+1} \in \mathcal{M}_h$ which satisfies

$$\forall \mu_h \in \mathcal{M}_h,\ (\lambda_h^{n+1} - \lambda_h^n, \mu_h)_{\mathcal{M}_h} = \rho\ \beta\big((u_h^n, \phi_h^n), \mu_h\big). \tag{3.13}$$

As in (2.14), we define a mapping

$$A_h : V_h \to V_{oh} \tag{3.14}$$

by the condition that $v_h = A_h \psi_h$ be equivalent to the equations

$$\forall \mu_h \in V_{oh},\ \int_\Omega \operatorname{grad} v_h \operatorname{grad} \mu_h \, dx = \int_\Omega \psi_h \, \mu_h \, dx. \tag{3.15}$$

We also define a mapping

$$B_h : V_h \to \mathcal{M}_h \tag{3.16}$$

as follows : for any function $\psi_h \in V_h$, the function $B_h \psi_h$ satisfies the equations (compare with (2.50)) :

$$\forall \mu_h \in \mathcal{M}_h,\ (B_h \psi_h, \mu_h)_{\mathcal{M}_h} = \beta\big((A_h \psi_h, \psi_h), \mu_h\big). \tag{3.17}$$

In the sequel it will be understood that $\| B_h \|$ is the norm of the linear mapping B_h when the spaces V_h and $\mathcal{M}_h$ are respectively equipped with the norms $\|\cdot\|_{L^2(\Omega)}$ and $\sqrt{(\cdot,\cdot)_{\mathcal{M}_h}}$.

Theorem 9. *The method described in Theorem 8 is convergent in the sense that*

(3.18) $$\lim_{n\to\infty} u_h^n = u_h \text{ in } V_{oh},$$

(3.19) $$\lim_{n\to\infty} \phi_h^n = \phi_h \text{ in } V_h,$$

provided that

(3.20) $$0 < \rho < 2\sigma_h^2,$$

where

(3.21) $$\sigma_h = \frac{1}{\|B_h\|}.$$

It is worth pointing out that the convergence of the present method is thus guaranteed for *any* choice of subspace $\mathcal{M}_h$ satisfying (3.4) and *any* choice of inner-product over the space $\mathcal{M}_h$. What is *not* independent upon these data however is the quantity σ_h of (3.21) and in practive, it is of course desirable to get an asymptotic estimate of this quantity. This will be achieved in the next theorem, but first, we need to develop some preliminaries.

In the sequel, triangulations $\mathcal{E}_h$, made up of finite elements K, are established over the set $\bar{\Omega}$ in the sense that $\bar{\Omega} = \bigcup_{K\in\mathcal{E}_h} K$, the finite elements satisfying the usual geometrical restrictions about their respective positions. It is assumed that for all the triangulations which we consider, all the finite elements K are the image $F_K(\hat{K})$ of a reference finite element $\hat{K}$ through an affine mapping F_K. With such a triangulation, we associate the space

(3.22) $$V_h = \{v_h \in \mathcal{C}^\circ(\bar{\Omega}),\ \forall K \in \mathcal{E}_h,\ v_h|_K \in P_K\},$$

where

(3.23) $$P_K = \{v : K \to \mathbb{R}\ ;\ v = \hat{v}\circ F_K^{-1}, \forall \hat{v} \in \hat{P}\},$$

and $\hat{P}$ is a given finite-dimensional space of functions $\hat{v} : \hat{K} \to \mathbb{R}$ which satisfies the inclusion

(3.24) $$P_1 \subset \hat{P},$$

where P_1 denotes the set of all polynomials of degree ≤ 1 in two variables.

The space V_{oh} is then defined as in (3.1).

By a <u>regular family</u> $(\mathcal{E}_h)$ of triangulations, we mean that for some constants α and β independent of h, we have

(3.25) $$\max_{K \in \mathcal{E}_h} \left(\frac{h(K)}{\rho(K)}\right) < \alpha,$$

(3.26) $$\tau \max_{K \in \mathcal{E}_h} h(K) \leq \min_{K \in \mathcal{E}_h} h(K),$$

where $h(K)$ = diameter of K, $\rho(K)$ = sup{diameter of inscribed spheres in K}. Finally we let

(3.27) $$h = \max_{K \in \mathcal{E}_h} h(K).$$

Although the space $\mathcal{M}_h$ is not uniquely determined by the sole condition that the direct sum (3.4) shall hold, there is a "canonical" choice for the space $\mathcal{M}_h$: Since we are considering subspaces made up of piecewise polynomials defined by their values at nodes, we shall henceforth assume that the space $\mathcal{M}_h$ consists of those functions in the space V_h whose values are zero at the interior nodes, i.e., in Ω. Defined in this fashion, the space $\mathcal{M}_h$ also appears as a natural discrete analog of the space $\mathcal{M}$, which essentially consists of traces over Γ; cf. Remark 1.

Theorem 10. *With the above choice for the space $\mathcal{M}_h$, assume that the inner-product $(\cdot,\cdot)_{\mathcal{M}_h}$ is the inner-product of the space $L^2(\Gamma)$. Then for subspaces V_h, V_{oh}, and $\mathcal{M}_h$, satisfying the above conditions, and associated with regular families of triangulations, we have*

$$\lim_{h\to o} \sigma_h = \sigma, \tag{3.28}$$

where σ_h and σ are defined as in (2.48) and (3.21), respectively.

Proof. Given two functions ψ and μ in the space $H^1(\Omega)$, there exist two sequences, (ψ_h) and (μ_h), of functions in the space V_h such that

$$\lim_{h\to o} \psi_h = \psi \text{ and } \lim_{h\to o} \mu_h = \mu \text{ in } H^1(\Omega), \tag{3.29}$$

in view of (3.24). Using the operators of (2.14) and (3.14), it is easily established that

$$\lim_{h\to o} A_h \psi_h = A\psi \text{ in } H^1(\Omega). \tag{3.30}$$

Given any function $\mu_h \in V_h$, we can write in a unique fashion

$$\mu_h = \mu_{oh} + \mu_{\Gamma,h} \text{ with } \mu_{oh} \in V_{oh},\ \mu_{\Gamma,h} \in \mathcal{M}_h,$$

so that, as a consequence of our present choice for the space $\mathcal{M}_h$ and its inner-product, we have

$$\int_\Gamma B_h\psi_h \mu_h \, d\gamma = (B_h\psi_h, \mu_{\Gamma,h})_{\mathcal{M}_h}$$

$$= \int_\Omega \operatorname{grad} A_h\psi_h \operatorname{grad}\mu_{\Gamma,h} dx - \int_\Omega \psi_h\mu_{\Gamma,h}\, dx,$$

in view of the definition of the operator B_h of (3.16). Since on the other hand, from (3.15),

$$\int_\Omega \operatorname{grad} A_h\psi_h \operatorname{grad}\mu_{oh}\, dx = \int_\Omega \psi_h\, \mu_{oh}\, dx,$$

we may always write

$$(3.31) \qquad \int_\Gamma B_h\psi_h\, \mu_h\, d\gamma = \int_\Omega \operatorname{grad} A_h\psi_h\, \operatorname{grad}\mu_h\, dx - \int_\Omega \psi_h\, \mu_h\, dx,$$

for all functions $\mu_h \in V_h$. As a consequence of (3.29), (3.30) and (3.31), we thus have :

$$(3.32) \qquad \lim_{h\to o} \int_\Gamma B_h\psi_h\, \mu_h\, d\gamma = \int_\Omega \operatorname{grad} A\psi\, \operatorname{grad}\mu\, dx - \int_\Omega \psi\, \mu\, dx$$

$$= \int_\Gamma \frac{\partial v}{\partial \nu}\, \mu\, d\gamma,$$

where the function $v \in H^2(\Omega) \cap H^1_o(\Omega)$ satisfies $-\Delta v = \psi$.

By definition (cf. (3.21)), we have for all h :

$$(3.33) \qquad \sigma_h < \frac{\| \psi_h \|_{L^2(\Omega)}}{\| B_h\psi_h \|_{L^2(\Gamma)}} < \frac{\| \psi_h \|_{L^2(\Omega)} \| \mu_h \|_{L^2(\Gamma)}}{\left| \int_\Gamma B_h\psi_h\, \mu_h\, d\gamma \right|},$$

so that we easily deduce from (3.29) and (3.32) that

$$(3.34) \qquad \limsup_{h\to o} \sigma_h < \sigma.$$

Let us now derive the opposite inequality. For any h, we let ψ_h and μ_h be two arbitrary functions in the spaces V_h and $\mathcal{M}_{\gamma_h}$, respectively. We let

$$(3.35) \qquad u_h = A_h\psi_h \quad \text{and } \tilde{u}_h = A\psi_h.$$

By definition of σ, we have

$$\frac{1}{\sigma} > \frac{\| \frac{\partial v}{\partial \nu} \|_{L^2(\Gamma)}}{\| \Delta v \|_{L^2(\Omega)}} > \frac{\left| \int_\Omega \operatorname{grad} A\psi\, \operatorname{grad}\mu\, dx - \int_\Omega \psi\, \mu\, dx \right|}{\| \psi \|_{L^2(\Omega)} \| \mu \|_{L^2(\Gamma)}},$$

for all functions $\psi = -\Delta v \in L^2(\Omega)$ and $\mu \in H^1(\Omega)$, and thus

$$(3.36) \qquad \frac{1}{\sigma} > \frac{\left| \int_\Omega \operatorname{grad} A\psi_h\, \operatorname{grad}\mu_h\, dx - \int_\Omega \psi_h\, \mu_h\, dx \right|}{\| \psi_h \|_{L^2(\Omega)} \| \mu_h \|_{L^2(\Gamma)}},$$

in particular. Since

$$\int_\Omega \text{grad}A\psi_h\ \text{grad}\mu_h\ dx - \int_\Omega \psi_h\mu_h\ dx = \int_\Gamma B_h\psi_h\mu_h\ d\gamma +$$
$$+ \int_\Omega \text{grad}(\tilde{u}_h - u_h)\ \text{grad}\mu_h\ dx,$$

we have

$$(3.37)\qquad \left|\int_\Omega \text{grad}A\psi_h\ \text{grad}\mu_h\ dx - \int_\Omega \psi_h\mu_h\ dx\right| \geq$$
$$\geq \left|\int_\Gamma B_h\psi_h\mu_h\ d\gamma\right| - |\tilde{u}_h - u_h|_{1,\Omega}\ |\mu_h|_{1,\Omega},$$

where, in general, $|v|_{1,\Omega} = \int_\Omega |\text{grad}v|^2 dx$. Since we are considering regular families of triangulations, it is easily established on the one hand that

$$(3.38)\qquad |\tilde{u}_h - u_h|_{1,\Omega} \leq Ch\ \|\psi_h\|_{L^2(\Omega)},$$

and on the other hand that

$$(3.39)\qquad |\mu_h|_{1,\Omega} \leq \frac{\gamma}{\sqrt{h}}\ \|\mu_h\|_{L^2(\Gamma)}$$

for some constants C and γ independent of h, the inequality of (3.39) making use of the fact that the functions of the space $\mathcal{M}_h$ vanish outside the "boundary" finite elements. From (3.36), (3.37), (3.38) and (3.39), we obtain

$$(3.40)\qquad \frac{1}{\sigma} \geq \frac{\left|\int_\Gamma B_h\psi_h\mu_h\ d\gamma\right|}{\|\psi_h\|_{L^2(\Omega)}\|\mu_h\|_{L^2(\Gamma)}} - C\gamma\sqrt{h},$$

for all $\psi_h \in V_h$ and $\mu_h \in \mathcal{M}_h$. Since

$$\frac{1}{\sigma_h} = \| B_h \| = \sup_{\psi_h \in V_h} \frac{\| B_h \psi_h \|_{L^2(\Gamma)}}{\| \psi_h \|_{L^2(\Omega)}} = \sup_{\psi_h \in V_h} \sup_{\mu_h \in \mathcal{M}_h} \frac{\left| \int_\Gamma B_h \psi_h \mu_h \, d\gamma \right|}{\| \psi_h \|_{L^2(\Omega)} \| \mu_h \|_{L^2(\Gamma)}} ,$$

we eventually obtain

$$\frac{1}{\sigma_h} < \frac{1}{\sigma} + C\gamma \sqrt{h},$$

and therefore

$$\liminf_{h \to 0} \sigma_h > \sigma. \tag{3.41}$$

The conclusion then follows from (3.34) and (3.41). ∎

Remark 3. With this choice for the inner-product in the space $\mathcal{M}_h$, solving problem (iii) (cf. Theorem 8) amounts in general to solving a linear system of roughly $\alpha\sqrt{N}$ equations, α : constant independent of h, whereas solving either problem (i) or (ii) (cf. Theorem 8) requires the solution of N linear equations.

As a consequence, the amount of work required for solving problem (iii) is negligible with respect to the total amount of work required in one iteration, at least asymptotically.

If this is still considered to be too much, there remains the possibility to use a numerical integration procedure over Γ, and this is precisely why Theorem 9 was proved with an arbitrary inner-product over the space $\mathcal{M}_h$. ∎

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