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ON THE EXISTENCE, UNIQUENESS AND APPROXIMATION OF SADDLE-POINT PROBLEMS ARISING FROM LAGRANGIAN MULTIPLIERS.

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INTRODUCTION

The present paper has been suggested by the recent development of the so called "dual analysis" and in particular of the method of Lagrangian multipliers in elasticity problems; we shall refer for now only to a few papers, and in particular to[12]-[14],[24],[25],[28],[29], ando to the references contained in such papers; many other references, however, will be given in the following. Although the equilibrium, hybrid and mixed methods contained in the mentioned works are often quite satisfactory from a numerical point of view, a complete study of the convergence of these methods and of the behaviour of the error has not been done until last years, and however, only in some particular case (see e.g.[20]and especially [18],[19] fo the "**mixed** methods" and [5] for the "assumed stresses hybrid method"; other references on this subject can be founded in[31]). The interest of these methods, and in particular of hybrid methods has been increased by papers [15],[27],[32], in which the theory of "non conforming" (or "delinquent") elements (see e.g.[30],[21], etc.) is presented as a "particular case" (in some sense) of hybrid methods. On the other hand, a careful analysis, for instance, of the work[11]on the Stoke's equations shows that the greatest difficulties in proving convergence and error bounds are connected with the use of the method of Lagrangian multipliers itself, rather than with the physical meaning of the problem. In this sense, the "general strategy" employed in[11]and in [5], in order to have convergence and error bounds for discretizations of different problems is, in fact, quite similar.

These considerations have suggested the author to develop the present "abstract theory" about saddle-point problems. More generally the problem treated here is the following.

$$(P) \qquad \begin{cases} \text{Find } (u,\psi) \text{ in } V\times W \text{ such that:} \\ a(u,v)+b(v,\psi)=\langle f,v\rangle \ \forall v\in V, \\ b(u,\varphi)=\langle g,\varphi\rangle \ \forall \varphi\in W, \end{cases}$$

where V,W are real Hilbert spaces, $a(u,v)$ and $b(v,\varphi)$ are continuous bilinear forms on VXV and VXW respectivelly and f,g are given functionals in V' and W' resp.

In paragraph 1 we give necessary and sufficient conditions on $a(u,v)$ and $b(v,\varphi)$ in order to have existence and uniqueness of the solution of problem (P) for all given (f,g) in V'XW'. In paragraph 2 we introduce the "approximate problem":

$$
(P_h) \quad \begin{cases} \text{find } (u_h,\psi_h) \text{ in } V_h\times W_h \text{ such that:} \\ a(u_h,v_h)+b(v_h,\psi_h)=\langle f,v_h\rangle \quad \forall v_h\in V_h , \\ b(u_h,\varphi_h)=\langle g,\varphi_h\rangle \quad \forall \varphi_h\in W_h , \end{cases}
$$

(where V_h and W_h are closed subspaces of V and W resp.), and we give, under suitable assumptions, an upper bound for the "error":

$$E_h=||u-u_h||+||\psi-\psi_h||$$

The third paragraph is dedicated to further considerations concerning "numerical integration" and "non conforming" approximation of W (that is $W_h\not\subseteq W$); this latest topic has been suggested by the papers[17],[32]and can be applied for instance to the "strongly diffusive" elements (equilibrium models) by F. de Veubeke.

Of course, the theoretical results given here, do not answer any question related to the mentioned methods and in general to the problems in which the method of Lagrangian multipliers is employed. In some particular cases the greatest difficulty will often be the verification of the abstract hypotheses proposed here. It is reasonable, however, to think that the knowledge of a "winning strategy" will be, in any case, useful.

Some of the results of this paper were also reported in a previous note (see [7]); I wish to thank Prof.J.L. Lions for presenting it to the C.R. Acad. Sc.. Thanks are also due to Prof. P.A. Raviart for their help in useful personel conversations.

0.- PRELIMINARIES

Let X be a real Hilbert space; we denote by X' its dual space; if $x' \in X'$ and $x \in X$ the value of x' at the point x will be indicated by $\langle x',x\rangle$. The scalar product and the norm in X will be indicated by $(\, , \,)$ and $\| \;\|$ (resp.) or by $(\, , \,)_X$ and $\| \;\|_X$ whenever confusion may rise. We denote also by J_X Riesz's "representation operator" from X' onto X, defined by

$$(J_X x',x)=\langle x',x\rangle \qquad \forall x \in X,\ x' \in X'.$$

It is well known that J_X is a norm preserving isomorphism from X' onto X. Let now Y be another real Hilbert space and let T be a continuous linear operator from $D(T)$ into Y. The domain $D(T')$ of the dual operator is defined by:

$$D(T')=\{y' \mid y' \in Y', x \to \langle y',Tx\rangle \text{ is continuous on } X\}.$$

Then the dual operator T' from $D(T')$ into X' is defined by:

$$\langle T'y',x\rangle=\langle y'Tx\rangle \qquad \forall x \in X,\ y' \in D(T').$$

We want now to prove a theorem that will be useful in the following.

<u>Theorem 0.1.-</u> <u>Let</u> X,Y <u>be real Hilbert spaces; let</u> $\mathcal{C}(x,y)$ <u>be a continuous bilinear form on</u> $X\times Y$ <u>and let</u> T <u>be the continuous linear operator from</u> X <u>into</u> Y' <u>associated to</u> $\mathcal{C}(x,y)$, <u>defined by</u>:

$$\langle Tx,y\rangle= \mathcal{C}(x,y) \qquad \forall x \in X, y \in Y.$$

<u>For all</u> $k>0$ <u>the three following statements are equivalent:</u>

i) $\displaystyle\sup_{x\in X-\{0\}} \frac{\mathcal{C}(x,y)}{\|x\|} \geqslant k\|y\| \qquad \forall y \in Y,$

ii) $\|T'y\| \geqslant k\|y\| \qquad \forall y \in Y,$

iii) $\exists\, S \in \mathcal{L}(Y',X)$ (1) <u>such that</u> $TS=I$ (<u>identity</u>) <u>on</u> Y' <u>and</u> $\|S\| \leqslant k^{-1}$.

---

(1) If H_1 and H_2 are Hilbert spaces, $\mathcal{L}(H_1,H_2)$ will be the space of all linear continuous operators from H_1 into H_2, with the norm:

$$\|S\|=\|S\|_{\mathcal{L}(H_1,H_2)} =\sup_{x\in H_1-\{0\}} \frac{\|Sx\|}{\|x\|}.$$

Proof.

i)$\Leftrightarrow$ii) follows obviously from :

$$\sup_{x\in X-\{0\}} \frac{\mathcal{B}(x,y)}{||x||} = \sup_{x\in X-\{0\}} \frac{<T'y,x>}{||x||} = ||T'y||, \qquad \forall\, y\in Y.$$

iii)$\Rightarrow$ii) follows obviously from the relations ($y\neq 0$):

$$\begin{cases} \sup\limits_{x\in X-\{0\}} \dfrac{\mathcal{B}(x,y)}{||x||} \geq \dfrac{\mathcal{B}(SJ_Y^{-1}y,y)}{||SJ_Y^{-1}y||} = \dfrac{||y||^2}{||SJ_Y^{-1}y||}, \\ ||ST_Y^{-1}y|| \leq k^{-1}||J_Y^{-1}y|| = k^{-1}||y||. \end{cases}$$

ii)$\Rightarrow$iii). Let $N=\ker(T)$ the kernel of T; setting

$N^{\perp}=\{x \mid x\in X,\ (x,\xi)=0 \ \text{ if } T\xi=0\}$,

T_1 =restriction of T to $N^{\perp}$,

from ii) and the closed range theorem (cfr. e.g. Yosida [33] pag 205), we have that T_1 is an isomorphism from $N^{\perp}$ onto Y'. From i) we easily get that, for all y in Y,

$$\sup_{x\in N^{\perp}-\{0\}} \frac{\mathcal{B}(x,y)}{||x||} \geq k||y||.$$

Then (see part i)$\Leftrightarrow$ii) of this proof) $||(T_1')^{-1}|| \leq k^{-1}$; hence $||T_1^{-1}|| \leq k^{-1}$, and setting $S=T_1^{-1}$ the proof is completed.

Corollary 0.1.- Under the hypotheses of theorem 0.1 for all k and $\bar{k}$ positive numbers the three following statements are equivalent:

I) $\sup\limits_{x\in X-\{0\}} \dfrac{\mathcal{B}(x,y)}{||x||} \geq k||y||\ \forall y\in Y$ and $\sup\limits_{y\in Y-\{0\}} \dfrac{\mathcal{B}(x,y)}{||y||} \geq \bar{k}||x|| \quad \forall x\in X$;

II) $||Tx|| \geq \bar{k}||x||\ \forall x\in X$ and $||T'y|| \geq k||y||\ \forall y\in Y$,

III) T is an isomorphism from X onto Y', with $||T^{-1}|| \leq k^{-1}$ and $||(T')^{-1}|| \leq \bar{k}^{-1}$.

Proof.- It is sufficient to apply theorem 0.1 to the form $\mathcal{B}(x,y)$ and to the form $\mathcal{B}'(y,x)=\mathcal{B}(x,y)$ (defined on $Y\times X$).

Remark 0.1.- The results contained in theorem 0.1 and in corollary 0.1 are of classical type and might not be new. For instance part I)$\Rightarrow$III) of corollary 0.1 was used by Babuska [3].

1.- EXISTENCE AND UNIQUENESS

Let now V and W be real Hilbert spaces, and let $a(u,v)$ and $b(v,\varphi)$ be continuous bilinear forms on $V\times V$ and $V\times W$ respectively. For any given pair (f,g) in $V'\times W'$ we consider the problem:

$$(1.1)\qquad \begin{cases} \underline{\text{find}}\ (u,\psi)\ \underline{\text{in}}\ V\times W\ \underline{\text{such that}}: \\ a(u,v)+b(v,\psi)=\langle f,v\rangle\ \forall v\in V, \\ b(u,\varphi)=\langle g,\varphi\rangle\ \forall\varphi\in W. \end{cases}$$

We remark that, if, for instance, $a(u,v)$ is symmetric and V-elliptic, in the sense that there exists a positive constant δ such that

$$a(v,v)\geq \delta||v||^2\ \forall v\in V,$$

then problem 1.1 is equivalent to the research of the saddle point on $V\times W$ of the functional

$$\mathcal{L}(v,\varphi)=\frac{1}{2}a(v,v)+b(v,\varphi)-\langle f,v\rangle-\langle g,\varphi\rangle.$$

We look now for (necessary and) sufficient conditions in order that for each (f,g) in $V'\times W'$ problem (1.1) has a unique solution. In other words, if $A\in\mathcal{L}(V,V')$ and $B\in\mathcal{L}(V,W')$ are the operators associated to $a(u,v)$ and $b(v,\varphi)$ resp., we search for (necessary and) sufficient conditions in order that the operator $\Lambda:V\times W\to V'\times W'$, defined by

$$(1.2)\qquad \Lambda(v,\varphi)=(Av+B'\varphi,Bv),$$

results an isomorphism.

For this, first of all we introduce the space:

$$(1.3)\qquad Z=\mathrm{Ker}(B)=\{v\,|\,v\in V,\ b(v,\varphi)=0\ \ \forall\varphi\in W\},$$

which is a closed subspace of V. Let Z' be the dual space of Z; Z' can be identified with a closed subspace of V', consisting of all $f\in V'$ such that

$$(1.4)\qquad \langle f,v\rangle=0\ \text{ if }(v,w)=0\ \ \forall w\in Z.$$

Let us denote by $\pi:V'\to Z$ the orthogonal projection from V' onto Z'. The closed subspace of V' consisting of all $f\in V'$ such that $\pi f=0$ (polar set of Z) will be indicated by Z^0.

We can now prove the following theorem.

THEOREM 1.1.- The operator Λ defined in (1.2) is an isomorphism from $V\times W$ onto $V'\times W'$ iff the two following conditions are satisfied:

(1.5) πA is an isomorphism from Z onto Z', [1]

(1.6) $\exists\, k>0$ such that $\|B'\varphi\|\geq k\|\varphi\| \quad \forall\varphi\in W$.

Proof.- Suppose that Λ is an isomorphism. Let us define, for all g in W', Sg as the first element of the pair $\Lambda^{-1}(0,g)$, that is:

(1.7) $w=Sg \Leftrightarrow \exists\,\chi\in W,\ \Lambda(w,\chi)=(0,g)$.

We have from (1.2) and (1.7) that $BS=I$; since Λ is an isomorphism, $S\in\mathcal{L}(W',V)$ and therefore, by theorem 0.1, (1.6) holds. We define now, for all $f\in Z'$, Qf as the first element of the pair $\Lambda^{-1}(f,0)$, that is:

(1.8) $w=Qf\Leftrightarrow\exists\,\chi\in W,\ \Lambda(w,\chi)=(f,0)$.

Since, by the closed range theorem and (1.6), $\pi B'\varphi=0\ \forall\varphi\in W$, we get from (1.8) and (1.2) that $\pi AQf=\pi f=f$. So $\pi AQ=I$ and then πA is surjective. Suppose now that $z\in Z$ and $\pi Az=0$; then $Az\in Z^o$ and by (1.6) and by the closed range theorem there exists a χ in W such that $B'\chi=-Az$. So $\Lambda(\chi,z)=(0,0)$ and then $z=0$. Therefore πA is also injective and, obviously, continuous; hence (1.5) holds. Suppose now, conversely, that (1.5) and (1.6) hold. From (1.6) and theorem 0.1 the problem

(1.9) $\Lambda(u,\psi)=(f,g)$

is equivalent the problem

(1.10) $\Lambda(w,\psi)=(f-A\bar{u},0)$

with $u=w+\bar{u}$ and $B\bar{u}=g$. Hence Λ is an isomorphism from $V\times W$ onto $V'\times W'$ if Λ_o, restriction of Λ to $Z\times W$, is an isomorphism from $Z\times W$ onto $V'\times\{0\}$. Let now be $f\in V'$, and let $w\in Z$ be the unique solution of $\pi Aw=\pi f$, which existence follows from (1.5). Since $\pi(f-Aw)=0$ we have $f-Aw\in Z^o$ and then from (1.6) there exists a unique ψ in W such that $B'\psi=-Aw+f$; we have proved in this way that for each $f\in V'$ there exists a unique $(w,\psi)\in Z\times W$ such that $\Lambda_o(w,\psi)=(f,0)$. Then Λ_o is a continuous one to one mapping and therefore an isomorphism.
The following proposition expresses the norm of Λ^{-1} and $(\Lambda')^{-1}$ as function of the constants related to A and B in theorem 1.1.

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(1) For sufficient conditions in order that Λ be an isomorphism, in a much more general case, see [2]

Proposition 1.1.- Suppose that A and B are such that (1.5) and (1.6) are satisfied. Let us define

$$\gamma = \sup_{u\in Z-\{0\}} \frac{||\pi Au||}{||u||}, \quad \gamma' = \sup_{u\in Z-\{0\}} \frac{||\pi A'u||}{||u||},$$

(1.11) $\qquad \alpha=||A||=||A'||, \qquad \beta=||B||=||B'||.$

Then, setting

(1.12) $\quad M(\alpha,\gamma,k)=\max\{(\gamma^{-1}+k^{-1}(1+\alpha\gamma^{-1})),(k^{-1}+\alpha k^{-2})(1+\alpha\gamma^{-1})\},$

we have :

(1.13) $\qquad ||\Lambda^{-1}||\leq M(\alpha,\gamma,k),$

(1.14) $\qquad ||(\Lambda')^{-1}||\leq M(\alpha,\gamma',k).$

Proof.- Let $(f,g)\in V'\times W'$ and let $(u,\psi)=\Lambda^{-1}(f,g)$, that is:

(1.15) $\qquad \begin{cases} Au+B'\psi=f \\ Bu=g \end{cases}$

From (1.6) and theorem 0.1 there exists a w in V such that Bw=g and

(1.16) $\qquad ||w||\leq k^{-1}||g||.$

Setting now v=u-w we get, from (1.15),

(1.17) $\qquad \pi Av=\pi f-\pi Aw,$

and from (1.11),

(1.18) $\qquad |v||\leq\gamma^{-1}(||f||+\alpha||w||);$

so we have:

(1.19) $\qquad ||u||\leq||v||+||w||\leq\gamma^{-1}||f||+k^{-1}(\gamma^{-1}\alpha+1)||g||.$

Since from (1.15) we get

(1.20) $\qquad ||B'\psi||\leq||f||+||Au||\leq||f||+\alpha||u||,$

from (1.6) we obtain

(1.21) $\qquad ||\psi||\leq k^{-1}||f||+k^{-1}\alpha||u||,$

and from (1.19) and (1.21) we have (1.13); the proof of (1.14) can be performed in a similar manner.

Remark 1.1.- It can be easily verified that Λ is the operator associated to the form

(1.22) $$\mathcal{C}((u,\psi),(v,\varphi))=a(u,v)+b(u,\varphi)+b(v,\psi).$$

So by corollary 0.1 with $X=Y=V\times W$, Λ is an isomorphism iff there exists $\tau,\bar{\tau}>0$ such that

(1.23) $$\sup_{x\in X-\{0\}}\frac{\mathcal{C}(x,y)}{||x||}\geq\tau||y|| \ \forall y\in Y \text{ and } \sup_{y\in Y-\{0\}}\frac{\mathcal{C}(x,y)}{||y||}\geq\bar{\tau}||x|| \forall x\in X.$$

On the other hand it can be shown that condition (1.23) holds iff (1.5) and (1.6) hold. Then, this can be another way, which extends and generalizes the idea of Babuska [4], in order to prove theorem 1.1.

The following corollary will be useful in the applications.

<u>Corollary</u> 1.1.- <u>If</u> $a(u,v)$ <u>is</u> Z-<u>elliptic and</u> (1.6) <u>holds</u>, <u>then</u> Λ <u>is an isomorphism</u>.

The proof is immediate.

<u>Remark</u> 1.2.- In many applications (see e.g. Raviart-Thomas [27] and Thomas [32]) we are led to the problem (1.1) by the following procedure. Let V_0 and V be real Hilbert spaces, with V_0 closed subspace of V, and let $a(u,v)$ be a continuous bilinear form on $V\times V$ which is V_0-elliptic; we want to solve the problem:

(1.24) $$\begin{cases} \text{find } u \text{ in } V_0 \text{ such that} \\ a(u,v)=\langle f,v\rangle \quad \forall v\in V_0 \end{cases}$$

where f is a given element in V'. For this we consider the space $W=V_0^{\circ}$ (polar space of V_0) which is a closed subspace of V'; problem (1.24) is now equivalent to:

(1.25) $$\begin{cases} \text{find } (u,\psi) \text{ in } V\ W \text{ such that:} \\ a(u,v)+\langle\psi,v\rangle=\langle f,v\rangle \quad \forall v\in V, \\ \langle\varphi,u\rangle=0 \quad \forall\varphi\in W, \end{cases}$$

and setting

(1.26) $$b(v,\varphi)=\langle\varphi,v\rangle \quad , \quad v\in V \ , \varphi\in W\subseteq V',$$

problem (1.25) is of the form (1.1). We note also that from (1.26) we have, in this case, $B'=I$ (identity), so (1.6) is automatically satisfied; moreover we have obviously

(1.27) $Z=\ker(B)=V_o$,

and then, since $a(u,v)$ is V_o-elliptic, corollary 1.1 is immediatelly applicable.

Exemples.- We shall report here only a few exemples, related to the applications of the hybrid methods by Pian and Tong to plate bending problem (Dirichlet problem for the biharmonic operator Δ^2). The field of application of the theory is (however) quite large; for further examples of applications and for all the details we shall refer to others papers (i.e. Pian-Tong [25], Brezzi [5], [6], F. de Veubeke [15], Raviart-Thomas [27], Thomas [32], Brezzi-Marini [8], etc.) which have suggested the abstract theory which is presented here.

Example 1.1.- Let us consider the problem:

$$(1.28)\qquad \begin{cases} \Delta^2 w=p & \text{in } \Omega, \\ w=\frac{\partial w}{\partial n}=0 & \text{on } \Gamma=\partial\Omega, \end{cases}$$

where Ω is a convex polygon, $p(x,y)$ an element of $L^2(\Omega)$ and n is the outward normal direction to $\partial\Omega$. We apply to this problem the first hybrid method ("assumed stresses hybrid method") by Pian and Tong [25]. For this let us consider, for any given decomposition of Ω into polygonal subdomains Ω_i $(i=1,\dots,N)$, the spaces

$$(1.29)\qquad F=\{v\in(L^2(\Omega))^3;\ v_{1,xx}+2v_{2,xy}+v_{3,yy}\in L^2(\Omega_i)\quad (i=1,\dots,N)\},$$

$$(1.30)\qquad V=\{v\in F;\ v_{1,xx}+2v_{2,xy}+v_{3,yy}=0 \text{ in } \Omega_i\quad (i=1,\dots,N)\},$$

$$(1.31)\qquad W=\{\varphi\in H^2_o(\Omega);\quad \Delta^2\varphi=0 \quad \text{in}\quad \Omega_i\ (i=1,\dots,N)\},$$

and let f be an element of F such that:

$$(1.32)\qquad f_{1,xx}+2f_{2,xy}+f_{3,yy}=p \quad \text{in}\quad \Omega_i\quad (i=1,\dots,N).$$

Finally we consider the bilinear form $b(v,\varphi)$ defined on $V\times W$ by:

$$(1.33)\qquad b(v,\varphi)=\sum_{i=1}^{N}\left\{\int_{\Omega_i}(v_1\varphi_{,xx}+2v_2\varphi_{,xy}+v_3\varphi_{,yy})\,dxdy-\int_{\Omega_i}(v_{1,xx}+2v_{2,xy}+v_{3,yy})\varphi\,dxdy\right.$$

Setting now, for every u,v in F,

(1.34) $$[u,v]=\int_\Omega (u_1v_1+2u_2v_2+u_3v_3)\,dxdy,$$

we define

(1.35) $$\begin{cases} a(u,v)=[u,v] \ , & u,v\in V,\\ L(v)=-[f,v] \ , & v\in V,\\ T(\varphi)=-b(f,\varphi) \ , & \varphi\in W,\end{cases}$$

and we introduce the norms:

(1.36) $$||v||^2_V=[v,v] \ , \quad v\in V,$$

(1.37) $$||\varphi||^2_W=|\varphi|^2_{2,\Omega}=||\varphi_{,xx}||^2_{L^2(\Omega)}+2||\varphi_{,xy}||^2_{L^2(\Omega)}+||\varphi_{,yy}||^2_{L^2(\Omega)}, \quad \varphi\in W.$$

Then by corollary 1.1 the problem

(1.38) $$\begin{cases} \text{find } (u,\psi) \text{ in } V\times W \text{ such that:}\\ a(u,v)+b(v,\psi)=L(v) \quad \forall v\in V,\\ b(u,\varphi)=T(\varphi) \quad \forall\varphi\in W,\end{cases}$$

has a unique solution. It can be shown (see Brezzi-Marini [8]) that the solution (u,ψ) of (1.38) is related to the solution w of (1.28) by:

(1.39) $$(w_{,xx},w_{,xy},w_{,yy})=u+f \quad \text{in } \Omega$$

(1.40) $$(w,w_{,x},w_{,y})=(\psi,\psi_{,x},\psi_{,y}) \qquad \text{on } \Sigma=\bigcup_{i=1}^{N}\partial\Omega_i.$$

Example 1.2.- We want to apply now to problem (1.28) the second hybrid method ("assumed displacements hybrid method") by Pian and Tong [25]; for this we consider, for any given decomposition of Ω into polygonal subdomains Ω_i (i=1,...,N), the spaces:

(1.41) $$V=\{v\,|\,v\ \prod_{i=1}^{N}H^2(\Omega_i), \quad v=\frac{\partial v}{\partial n}=0 \text{ on } \partial\Omega\},$$

(1.42) $$\tilde{W}=\{M\,|\,M\in(L^2(\Omega))^3, \ M_{1,xx}+2M_{2,xy}+M_{3,yy}=0 \text{ in } \Omega_i \ (i=1,\dots,N)\},$$

(1.43) $\tilde{W}_0=\{M \mid M\in\tilde{W},\ M_1=\varphi_{,xx},\ M_2=\varphi_{,xy},\ M_3=\varphi_{,yy}$ in each $\Omega_i,\ \varphi\in V\}$

We define the bilinear form $b(v,M)$ on $V\times\tilde{W}$ by

(1.44) $$b(v,M)=\sum_{i=1}^{N}\int_{\Omega_i}(v_{,xx}M_1+2v_{,xy}M_2+v_{,yy}M_3)\,dx\,dy$$

and then we define :

(1.45) $$W=\{M \mid M\in\tilde{W}_0,\ \ b(v,M)=0\ \ \forall v\in H_0^2(\Omega)\}.$$

Using Southwell stress functions U,V, defined by

(1.46) $$M_1=V_{,y}\ ,\quad M_2=-\frac{1}{2}(V_{,x}+U_{,y}),\quad M_3=U_{,x}\ ,$$

(cfr. F. de Veubeke-Zienkiewicz [16]), W can be characterized as the set of M in $\tilde{W}_0$ such that $U_n=U\nu_x+V\nu_y$ and $U_t=-V\nu_x+U\nu_y$ ($\nu_x=\cos n_i x$, $\nu_y=\cos n_i y$) are "continuous" across the interelement boundaries $\partial\Omega_i$.

In fact setting (see e.g. F. de Veubeke [15]).

(1.47) $$M_n=\frac{\partial U_t}{\partial s}\ ,\quad M_{nt}=-\omega-\frac{\partial U_n}{\partial s}\ ,\quad Q_n=\frac{\partial\omega}{\partial s}\ ,\quad \omega=\frac{1}{2}(V_{,x}-U_{,y}),$$

$b(v,M)$ can be written formally as:

(1.48) $$\begin{aligned} b(v,M)&=\sum_{i=1}^{N}\int_{\partial\Omega_i}\left(M_n\frac{\partial v}{\partial n}+M_{nt}\frac{\partial v}{\partial s}-Q_n v\right)d\sigma_i=\\ &=\sum_{i=1}^{N}\int_{\partial\Omega_i}\left(M_n\frac{\partial v}{\partial n}-\frac{\partial U_n}{\partial s}\ \frac{\partial v}{\partial s}\right)d\sigma_i\ ,\end{aligned}$$

and formula (1.48) can be justified from a mathematical point of view as a pairing between spaces of the type $H^{1/2}(\partial\Omega_i)$ (1) (for $\frac{\partial v}{\partial n}$ and $\frac{\partial v}{\partial s}$), and their duals.

Let finally $f\in(L^2(\Omega))^3$ be such that

(1.49) $$f_{1,xx}+2f_{2,xy}+f_{3,yy}=p \text{ in each } \Omega_i\ ,$$

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(1) For the definitions and the properties of Sobolew spaces $H^s(\Omega)$ and $H^s(\Gamma)$ we refer to Lions-Magenes [22].

and let, for all v in V,

$$F(v)=\sum_{i=1}^{N}\int_{\Omega_i}(v,_{xx}f_1+2v,_{xy}f_2+v,_{yy}f_3)\,dxdy \tag{1.50}$$

we define

$$a(u,v)=\sum_{i=1}^{N}\int_{\Omega_i}(u,_{xx}v,_{xx}+2u,_{xy}v,_{xy}+u,_{yy}v,_{yy})\,dxdy \qquad u,v\in V, \tag{1.51}$$

If V and W are equipped with the norms

$$||u||_V^2=\sum_{i=1}^{N}||v||^2_{2,\Omega_i}\,(^3) \quad , \quad ||M||_W^2=\sum_{i=1}^{N}||M||^2_{(L^2(\Omega_i))^3} \tag{1.52}$$

it is easily proved that the conditions of corollary 1.1 are satisfied. Then the problem

$$\begin{cases} \text{Find } (u,\overline{M})\in V\times W \text{ such that :} \\ a(u,v)+b(v,\overline{M})=F(v) \quad \forall v\in V, \\ b(u,M)=0 \quad \forall M\in W, \end{cases} \tag{1.53}$$

has a unique solution. It can also be verified that, if w is the solution of (1.28) and $(u,\overline{M})$ is the solution of (1.53), then :

$$w=u, \tag{1.54}$$

$$(w_{xx},w_{xy},w_{yy})=-\overline{M}+f. \tag{1.55}$$

(3) Such notations are classical; see e.g. Ciarlet-Raviart [9], [10], Strang-Fix [31].

2.- APPROXIMATION

Let now V_h and W_h be closed subspaces of V and W respectively. We substitute to problem (1.1) the "approximated problem":

$$(2.1)\qquad \begin{cases} \underline{\text{find}}\ (u_h,\psi_h)\ \underline{\text{in}}\ V_h\times W_h\ \underline{\text{such that}}: \\ a(u_h,v_h)+b(v_h,\psi_h)=\langle f,v_h\rangle\ \forall v_h\in V_h, \\ b(u_h,\varphi_h)=\langle g,\varphi_h\rangle\ \forall\varphi_h\in W_h. \end{cases}$$

We want now, at first, to find sufficient conditions on V_h and W_h in order that (2.1) has a unique solution, and, after that, to evaluate the distance between the "approximate solution" (u_h,ψ_h) of (2.1) and the "exact solution" (u,ψ) of (1.1).

First of all we suppose that the following hypothesis is satisfied.

H1.- <u>There exists a positive constant</u> k_h <u>such that</u>:

$$(2.2)\qquad \sup_{v_h\in V_h-\{0\}} \frac{b(v_h,\varphi_h)}{||v_h||} \geq k_h||\varphi_h||\ \forall\varphi_h\in W_h.$$

We define now:

$$(2.3)\qquad Z_h=\{v_h|v_h\in V_h,\ b(v_h,\varphi_h)=0\ \forall\varphi_h\in W_h\},$$

and we remark that, in general, $Z_h\not\subset Z$. Therefore we need also the following hypothesis.

H2.- <u>There exist two positive constants</u> γ_h <u>and</u> γ_h' <u>such that</u>:

$$(2.4)\qquad \sup_{u_h\in Z_h-\{0\}} \frac{a(u_h,v_h)}{||u_h||} \geq \gamma_h'||v_h||\ \forall v_h\in Z_h,$$

$$(2.5)\qquad \sup_{v_h\in Z_h-\{0\}} \frac{a(u_h,v_h)}{||v_h||} \geq \gamma_h||u_h||\ \forall u_h\in Z_h.$$

Let ρ_h be the projection operator from $V\times W$ onto $V_h\times W_h$; identifying $V_h'\times W_h'$ with a closed subspace of $V'\times W'$ we can define the projection operator ρ_h' from $V'\times W'$ onto $V_h'\times W_h'$. Let now $\Lambda_h:V_h\times W_h\to V_h'\times W_h'$ be defined by:

$$(2.6)\qquad \Lambda_h(v_h,\varphi_h)=\rho_h'\Lambda(v_h,\varphi_h)\quad \forall v_h\in V_h,\ \varphi_h\in W_h.$$

It is clear that the solution (u_h,ψ_h) of (2.1) (if it exists) is such that

(2.7) $$\Lambda_h(u_h,\psi_h)=\rho_h'(f,g)$$

Therefore the following proposition gives an answer to our first question about existence and uniqueness of the solution of (2.1).

Proposition 2.1.- Under the hypotheses H1 and H2 the operator $\Lambda_h=\rho_h'\Lambda$ is an isomorphism between $V_h\times W_h$ and $V_h'\times W_h'$; moreover we have:

(2.8) $$||\Lambda_h^{-1}||\leq M(\alpha,\gamma_h,k_h),$$

(2.9) $$||(\Lambda_h')^{-1}||\leq M(\alpha,\gamma_h',k_h),$$

where $M(\alpha,\gamma_h,k_h)$ is always expressed by (1.12).

The proof is immediate by theorem 1.1 and proposition 1.1.

We can now prove the following theorem.

Theorem 2.1.- Under the hypotheses H1 and H2, for every pair (f,g) in $V'\times W'$ let

(2.10) $$(u,\psi)=\Lambda^{-1}(f,g),$$

(2.11) $$(u_h,\psi_h)=\Lambda_h^{-1}\rho_h'(f,g).$$

Then we have:

(2.12) $$||u-u_h||+||\psi-\psi_h||\leq\sigma_h(\operatorname*{Inf}_{v_h\in V_h}||u-v_h||+\operatorname*{Inf}_{\varphi_h\in W_h}||\psi-\varphi_h||),$$

where

(2.13) $$\sigma_h=M(\alpha,\gamma_h',k_h)(\alpha+\beta)+1.$$

Proof.- First of all, we remark that, from (2.6), (2.10), (2.11), we have for every (w_h,χ_h) in $V_h\times W_h$:

(2.14) $$<\Lambda_h(u_h,\psi_h),(w_h,\chi_h)>=<f,w_h>+<g,\chi_h>=<\Lambda(u,\psi),(w_h,\chi_h)>.$$

So if (v_h,φ_h) is any other pair in $V_h\times W_h$ we have:

(2.15) $$<\Lambda_h(u_h-v_h,\psi_h-\varphi_h),(w_h,\chi_h)>=<\Lambda(u-v_h,\psi-\varphi_h),(w_h,\chi_h)>$$

Then by (2.9) and corollary 0.1 (part III)$\Rightarrow$I)) we have

$$(2.16)\qquad ||u_h-v_h||+||\psi_h-\varphi_h||\leq c||\Lambda||(||u-v_h||+||\psi-\varphi_h||)$$

with $c=||(\Lambda_h')^{-1}||\leq M(\alpha,\gamma_h',k_h)$. On the other hand it is immediate to verify that

$$(2.17)\qquad ||\Lambda||\leq\alpha+\beta;$$

hence from (2.16),(2.17) we get

$$(2.18)\qquad ||u-u_h||+||\psi-\psi_h||<\left[c(\alpha+\beta)+1\right](||u-v_h||+||\psi-\varphi_h||)$$

for every (v_h,φ_h) in $V_h\times W_h$, and the result follows immediatelly.

Corollary 2.1.- Suppose that H1 holds and that there exists a constant $\delta_h>0$ such that

$$(2.19)\qquad a(v_h,v_h)\geq\delta_h||v_h||^2\quad\forall v_h\in Z_h;$$

then Λ_h is an isomorphism from $V_h\times W_h$ onto $V_h'\times W_h'$. Moreover, for every pair (f,g) in $V'\times W'$, if $(u,\psi)=\Lambda^{-1}(f,g)$ and $(u_h,\psi_h)=\Lambda_h^{-1}\rho_h'(f,g)$, then:

$$(2.20)\qquad ||u-u_h||+||\psi-\psi_h||\leq\bar\sigma_h(\operatorname*{Inf}_{v_h\in V_h}||u-v_h||+\operatorname*{Inf}_{\varphi_h\in W_h}||\psi-\varphi_h||),$$

with $\bar\sigma_h=M(\alpha,\delta_h,k_h)(\alpha+\beta)+1$.

The proof is immediate.

Remark 2.1.- Suppose for instance that $a(u,v)$ is Z-elliptic and that, for simplicity, $g=0$.
Then the first element u of the solution of (1.1) can be characterized as the solution of:

$$(2.21)\qquad \begin{cases}a(u,v)=\langle f,v\rangle\ \forall v\in Z,\\ u\in Z.\end{cases}$$

Suppose now that V_h and W_h are closed subspaces of V and W, such that H1 is verified, and let again Z_h be the space

$$Z_h=\{v_h|v_h\in V_h,\ b(v_h,\varphi_h)=0\ \ \forall\varphi_h\in W_h\}.$$

If $a(u,v)$ is Z_h elliptic then the first element u_h of the solution of (2.1) can be presented as the solution of:

$$(2.22)\qquad \begin{cases}a(u_h,v_h)=\langle f,v_h\rangle\ \forall v_h\in Z_h,\\ u_h\in Z_h.\end{cases}$$

Since $Z_h \not\subset Z$, (2.22) can be regarded as an approximation of (2.21) by "non-conforming elements" (see e.g. [17], [21], [30], [31]); therefore given $Z \subseteq V$, $a(u,v)$ and $f \in V'$, and given a closed subspace $Z_h \subset V$, the existence of W, W_h, $b(v,\varphi)$ such that the hypotheses of corollary 1.1 are satisfied, constitutes some kind of "abstract patch-test" for the elements of Z_h (see also, in the case of the elasticity and plate bending problems, F. de Veubeke [15]; also important in this contest are the papers by Raviart-Thomas [27] and Thomas [32]).

Example 2.1.- We return to the situation of the example 1.1, and we suppose, for sake of simplicity, that Ω is the square $]0,\ell[\times]0,\ell[$ and that Ω_i are also squares of lenght $h=\ell N^{-1/2}$. Let $\hat{K}$ be the unit square $]0,1[\times]0,1[$ and let $\hat{P}_V$ be a finite dimensional space of smooth vectors functions $(\hat{v}_1,\hat{v}_2,\hat{v}_3)$ defined on K and self-equilibrating, in the sense that

$$\hat{v}_{1,xx}+2\hat{v}_{2,xy}+\hat{v}_{3,yy}=0 \quad \text{on } \hat{K};$$

let $\hat{P}_W$ be a finite dimensional space of smooth functions $\hat{\varphi}$ defined on $\hat{K}$ and such that $\Delta^2\hat{\varphi}=0$ in $\hat{K}$. For each Ω_i let F_i be the "affine" inversible transformation that maps $\hat{K}$ on Ω_i and let $P_{V,i}$, $P_{W,i}$ the images of $\hat{P}_V$ and $\hat{P}_W$ (resp.) through F_i. We consider now the spaces

$$V_h=\{v_h \mid v_h \in V, \quad v_{h|\Omega_i} \in P_{V,i} \quad (i=1,\dots,N)\}$$

$$W_h=\{\varphi_h \mid \varphi_h \in W, \quad \varphi_{h|\Omega_i} \in P_{W,i} \quad (i=1,\dots,N)\}$$

It can be verified (cfr. Brezzi [5], Brezzi-Marini [8]) that if $\hat{P}_V$ and $\hat{P}_W$ verify the following hypothesis:

$$\hat{H}1 \quad \begin{cases} \sup\limits_{v \in \hat{P}_V} \left(\int_{\hat{K}} \hat{v}_1\hat{\varphi}_{,xx}+2\hat{v}_2\hat{\varphi}_{,xy}+\hat{v}_3\hat{\varphi}_{,yy}\,dxdy\right) \|\hat{v}\|^{-1}_{(L^2(\hat{K}))^3} \geq \hat{\lambda}|\hat{\varphi}|_{2,\hat{K}} \\ \underline{\text{for all}}\ \hat{\varphi}\ \underline{\text{in}}\ \hat{P}_W,\ \underline{\text{with}} \quad \hat{\lambda}>0, \end{cases}$$

then V_h and W_h satisfy hypothesis H1 with constant $k_h > \lambda > 0$, λ independent of h. Since, obviously, in this case

$$\alpha=\gamma_h'=1$$

for every decomposition, then the constant σ_h which appears in (2.12), is in fact independent of h. We remark that in practice, since the value of $\hat{\varphi}$ in $\hat{K}$ is depending only on the values of $\hat{\varphi}$ and $\frac{\partial\hat{\varphi}}{\partial n}$ on $\partial\hat{K}$, $\hat{P}_W$ will be chosen as a space of biharmonic functions such that $\hat{\varphi}$ and $\frac{\partial\hat{\varphi}}{\partial n}$ are polynomials of assigned degree on $\partial\hat{K}$. Of course we ignore the value of $\hat{\varphi}$ at the interior of $\hat{K}$, but this is not a difficulty since we can use sistematically, for the computations, Green's formula

$$\int_{\hat{K}} (\hat{v}_1\hat{\varphi}_{xx}+2\hat{v}_2\hat{\varphi}_{xy}+\hat{v}_3\hat{\varphi}_{,yy})\,dxdy=\int_{\partial\hat{K}} (\hat{M}_n \frac{\partial\hat{\varphi}}{\partial n} +\hat{M}_{nt} \frac{\partial\hat{\varphi}}{\partial s} -\hat{Q}_n\hat{\varphi})\,d\sigma,$$

where

$$\begin{cases} \hat{M}_n=\hat{v}_1\nu_x^2+2\hat{v}_2\nu_x\nu_y+\hat{v}_3\nu_y^2, \\ \hat{M}_{nt}=\hat{v}_1\nu_x\nu_y+(\nu_y^2-\nu_x^2)\hat{v}_2-\hat{v}_3\nu_x\nu_y, \\ Q_n=(\hat{v}_{1,x}+\hat{v}_{2,y})\nu_x+(\hat{v}_{2,x}+\hat{v}_{3,y})\nu_y, \\ \nu_x=\cos nx, \quad \nu_y=\cos ny, \end{cases}$$

valid whenevever $\hat{v}$ is sufficiently smooth and $\hat{v}_{1,xx}+2\hat{v}_{2,xy}+\hat{v}_{3,yy}=0$ on $\hat{K}$. For further details we refer to Brezzi [5] and Brezzi-Marini [8].

Example 2.2.- We return now to the situation of the example 1.2, and we suppose again that Ω and all the Ω_i are squares, as in example 2.1. Let $\hat{K}$ be the unit square and let $\hat{P}_V$ be a finite dimensional linear space of smooth functions and $\hat{P}_W$ a finite dimensional linear space of smooth self equilibrating vectors of the type $\hat{M}=(\hat{\varphi}_{,xx},\hat{\varphi}_{,xy},\hat{\varphi}_{,yy})$. We define, for each i (i=1,...,N) $P_{V,i}$ and $P_{W,i}$ as the images of $\hat{P}_V$ and $\hat{P}_W$ through the "affine" inversible transformation F_i which maps $\hat{K}$ on Ω_i. We consider the spaces:

$$V_h=\{v\,|\,v\in V,\quad v_{|\Omega_i}\in P_{V,i}\quad (i=1,\dots,N)\},$$

$$W_h=\{M\,|\,M\in W,\quad M_{|\Omega_i}\in P_{W,i}\quad (i=1,\dots,N)\}.$$

Suppose that P_V and P_W verify hypothesis

$$\hat{H}1\begin{cases} \underline{\text{There exists a constant}}\ \hat{\lambda}>0\ \underline{\text{such that}} \\ \sup\limits_{\hat{v}\in\hat{P}_V-\{0\}} \|\hat{v}\|_{2,\hat{K}}^{-1}\int_{\hat{K}} (\hat{v}_{,xx}\hat{\varphi}_{,xx}+2\hat{v}_{,xy}\hat{\varphi}_{,xy}+\hat{v}_{,yy}\hat{\varphi}_{,yy})\,dxdy\geq\hat{\lambda}|\hat{\varphi}|_{2,\hat{K}} \\ \underline{\text{for all}}\ (\hat{\varphi}_{,xx},\hat{\varphi}_{,xy},\hat{\varphi}_{,yy})\ \underline{\text{in}}\ \hat{P}_W. \end{cases}$$

Then we can prove that H1 is satisfied with constant $\chi_h > \lambda > 0$, λ independent of h. The chiefest difficulty is now, in the particular cases, to prove that a(u,v) defined by (1.51), is Z_h-elliptic (where Z_h is always defined by (2.3)) with constant δ_h independent of h. If this is the case, then we get that the constant $\bar{\sigma}_h$ which appears in (2.20) is in fact independent of h. The verification of $\hat{H}1$ can, also in this case, be easily performed using (1.44); (1.47), (1.48), if we know the value of $\hat{U}$ and $\hat{V}$ on $\partial\hat{K}$. F. de Veubeke has shown (cfr. [15]) that in this case Z_h is in fact a space of non conforming approximations of $H^2_0(\Omega)$ and that we can find in this way all the classical non conforming elements for the biharmonic problem (see [30], [31], [21]); for further details we refer to F. de Veubeke [15]; in a forthcoming paper we shall treat this case from a mathematical point of view.

3.- FURTHER CONSIDERATIONS

In many applications, the exact computation of $a(u_h,v_h)$, $b(v_h,\varphi_h)$, $\langle f,v_h\rangle$, $\langle g,\varphi_h\rangle$ which appear in the approximated problem (2.1) is rather difficult or, in some case, impossible. Therefore some kind of numerical integration (see e.g. Strang-Fix [31], Ciarlet-Raviart [10]) is needed in order to solve numerically problem (2.1). We shall shown that the classical results about the use of numerical integration for variational problems and for variational inequalities can be easily extended to our case. For this, let $a_h(u_h,v_h)$ and $b_h(v_h,\varphi_h)$ be (continuous) bilinear forms on $V_h\times V_h$ and on $V_h\times W_h$ respectively; suppose moreover that f_h and g_h are (continuous) linear functionals on V_h and W_h resp., and consider the problem:

$$(3.1)\qquad \begin{cases} \underline{\text{find}}\ (u_h^*,\psi_h^*)\ \underline{\text{in}}\ V_h\times W_h\ \underline{\text{such that}}: \\ a_h(u_h^*,v_h)+b_h(v_h,\psi_h^*)=\langle f_h,v_h\rangle \quad \forall v_h\in V_h, \\ b_h(u_h^*,\varphi_h)=\langle g_h,\varphi_h\rangle \quad \forall \varphi_h\in W_h. \end{cases}$$

We suppose that the following hypotheses are satisfied:

H1*) <u>There exists a positive constant</u> k_h^* <u>such that</u>:

$$\sup_{v_h\in V_h-\{0\}} ||v_h||^{-1} b_h(v_h,\varphi_h)\geq k_h^*||\varphi_h|| \quad \forall\varphi_h\in W_h.$$

H2*) <u>There exist two positive constants</u> γ_h^* <u>and</u> $\bar{\gamma}_h^*$ <u>such that</u>:

$$\sup_{u_h\in Z_h-\{0\}} ||u_h||^{-1} a_h(u_h,v_h)\geq\bar{\gamma}_h^*||v_h|| \quad \forall v_h\in Z_h^*,$$

$$\sup_{v_h\in Z_h-\{0\}} ||v_h||^{-1} a_h(u_h,v_h)\geq\gamma_h^*||u_h|| \quad \forall u_h\in Z_h^*,$$

<u>where, of course,</u> Z_h^* <u>is defined by</u>:

$$(3.2)\qquad Z_h^*=\{v_h\,|\,v_h\in V_h,\ b_h(v_h,\varphi_h)=0\ \forall\varphi_h\in W_h\}.$$

Then, always from theorem 1.1 we get

<u>Proposition</u> 3.1.- <u>Under the hypotheses</u> H1* <u>and</u> H2*, <u>for all</u> (f_h,g_h) <u>in</u> $V_h'\times W_h'$, <u>problem</u> (3.1) <u>has a unique solution</u>.

We want now to evaluate the distance between (u_h^*,ψ_h^*), solution of (3.1), and

(u,ψ), solution of (1.1). For this we define at first the operator $\Lambda_h^*: V_h \times W_h \to V_h' \times W_h'$ by:

$$
(3.3) \quad \begin{cases} \langle \Lambda_h^*(v_h,\varphi_h),(w_h,\chi_h)\rangle = a_h(v_h,w_h)+b_h(w_h,\varphi_h)+b_h(v_h,\chi_h), \\ \text{for all } (v_h,\varphi_h) \text{ and } (w_h,\chi_h) \text{ in } V_h\times W_h. \end{cases}
$$

Let now (u_h,ψ_h) be the solution of (2.1) and let

$$
(3.4) \qquad \alpha^* = \sup_{u_h,v_h\in V_h-\{0\}} ||u_h||^{-1}||v_h||^{-1} a_h(u_h,v_h).
$$

From proposition 1.1 we get that there exists a $(v_h,\varphi_h)\in V_h\times W_h$ such that:

$$
(3.5) \quad \begin{aligned} &||u_h-u_h^*||+||\psi_h-\psi_h^*|| \leq \\ &\leq M(\alpha^*,\bar{\gamma}_h^*,k_h^*)\langle \Lambda_h^*(u_h-u_h^*,\psi_h-\psi_h^*),(v_h,\varphi_h)\rangle (||v_h||+||\varphi_h||)^{-1}. \end{aligned}
$$

Moreover we have

$$
(3.6) \quad \begin{aligned} &\langle \Lambda_h^*(u_h-u_h^*,\psi_h-\psi_h^*),(v_h,\varphi_h)\rangle = \langle (\Lambda_h^*-\Lambda_h)(u_h,\psi_h),(v_h,\varphi_h)\rangle + \\ &+\langle f-f_h,v_h\rangle + \langle g-g_h,\varphi_h\rangle, \end{aligned}
$$

and also :

$$
(3.7) \quad \begin{aligned} &\left|\langle (\Lambda_h^*-\Lambda_h)(u_h,\psi_h),(v_h,\varphi_h)\rangle\right| \leq \left|a_h(u_h,v_h)-a(u_h,v_h)\right| + \\ &+\left|b_h(v_h,\psi_h)-b(v_h,\psi_h)\right| + \left|b_h(u_h,\varphi_h)-b(u_h,\varphi_h)\right|. \end{aligned}
$$

Setting now :

$$
\mathcal{F} = \sup_{v_h\in V_h-\{0\}} ||v_h||^{-1}\left|\langle f-f_h,v_h\rangle\right|,
$$

$$
\mathcal{G} = \sup_{\varphi_h\in W_h-\{0\}} ||\varphi_h||^{-1}\left|\langle g-g_h,\varphi_h\rangle\right|,
$$

$$
\mathcal{A}(u_h) = \sup_{v_h\in V_h-\{0\}} ||v_h||^{-1}\left|a(u_h,v_h)-a_h(u_h,v_h)\right|,
$$

$$
\mathcal{B}(u_h) = \sup_{\varphi_h\in W_h-\{0\}} ||\varphi_h||^{-1}\left|b(u_h,\varphi_h)-b_h(u_h,\varphi_h)\right|,
$$

$$\mathcal{B}^T(\psi_h) = \sup_{v_h \in V_h - \{0\}} ||v_h||^{-1} \left| b(v_h,\psi_h) - b_h(v_h,\psi_h) \right| ,$$

we have from (3.5), (3.6), (3.7), that:

(3.8) $$||u_h - u_h^*|| + ||\psi_h - \psi_h^*|| \leq M(\alpha^*, \bar{\gamma}_h^*, k_h^*)\,(\mathcal{Y} + \mathcal{G} + \mathcal{A}(u_h) + \mathcal{B}(u_h) + \mathcal{B}^T(\psi_h)) .$$

Therefore we can conclude with the following theorem.

Theorem 3.1.- If hypotheses $H1^*$, $H2^*$ are satisfied and if (u_h^*, ψ_h^*) and (u,ψ) are the solutions of (3.1) and (1.1) respectively, we have:

$$||u - u_h^*|| + ||\psi - \psi_h^*|| \leq ||u - u_h|| + ||\psi - \psi_h|| +$$
$$+ M(\alpha^*, \bar{\gamma}_h^*, k_h) \left(\mathcal{Y} + \mathcal{G} + (\mathcal{A} + \mathcal{B} + \mathcal{B}^T)(||u_h|| + ||\psi_h||)\right) ,$$

where (u_h, ψ_h) is the solution of (2.1) and where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}^T$ are defined by:

$$\mathcal{A} = \sup_{w_h \in V_h - \{0\}} \mathcal{A}(w_h)\, ||w_h||^{-1} ,$$

$$\mathcal{B} = \sup_{w_h \in V_h - \{0\}} \mathcal{B}(w_h)\, ||w_h||^{-1} ,$$

$$\mathcal{B}^T = \sup_{\varphi_h \in W_h - \{0\}} \mathcal{B}^T(\varphi_h)\, ||\varphi_h||^{-1} .$$

The proof follows immediatelly from (3.8) and the triangular inequality.

Remark 3.1.- As in corollary 2.1, hypothesis $H2^*$ can be substituted, in the applications, by the Z_h-ellipticity, i.e. there exists a positive constant δ_h^* such that

$$a_h(v_h, v_h) \geq \delta_h^* ||v_h||^2 \qquad \forall\, v_h \in V_h .$$

Remark 3.2.- In the applications the fact that $Z_h \not\subset Z$ is sometimes a difficulty. Then, it can happens that a choice of a "greater" W_h is needed, in order to have $Z_h \subseteq Z$. This cannot, in general, be obtained unless $W_h \not\subseteq W$; therefore it is of some interest to consider the case of an "external approximation" of W. We shall give, in the following, some idea of the general case, but we refer for more precise results, in a large class of examples, to the papers by Raviart-Thomas [27] and Thomas [32] which contain the best treatement of the question from a mathematical point of view. On the other hand, from a numerical point of view, it is recommended to refer to the works by F.de Veubeke and his associateds (cfr. e.g. [12], [13], [14], [28], [29]) who have developed the theory of "strongly diffusive" elements, which is the most important case of application of the abstract situation described above.

We suppose then that another real Hilbert space, H, is given, such that

(3.9) $$W \subseteq H$$

with continuous injection. We suppose that W is dense in H and so we can identify H' with a dense subspace of W'. Let now V_h be a closed subspace of V and W_h a closed subspace of H, such that

(3.10) $$B(V_h) \subseteq H',$$

and that, for all v_h in V_h, if

$$_{H'}\langle Bv_h, \varphi_h\rangle_H = 0 \quad \forall \varphi_h \in W_h,$$

then:

$$_{H'}\langle Bv_h, \varphi\rangle_H = 0 \quad \forall \varphi \in W$$

We consider now the following approximation of problem (1.1).

(3.11) $$\begin{cases} \text{Find } (\bar{u}_h, \bar{\psi}_h) \text{ in } V_h \times W_h \text{ such that:} \\ a(\bar{u}_h, v_h) + \langle Bv_h, \bar{\psi}_h\rangle = \langle f, v_h\rangle \quad \forall v_h \in V_h, \\ \langle B\bar{u}_h, \varphi_h\rangle = \langle g, \varphi_h\rangle \quad \forall \varphi_h \in W_h, \end{cases}$$

where, of course, g is supposed to belong to H'. We always suppose that hypothesis H2 is satisfied[1] and we substitute H1 with the following condition.

$\overline{H1}$) <u>There exists a positive constant $\bar{k}_h$ such that:</u>

(3.12) $$\sup_{v_h \in V_h - \{0\}} \|v_h\|^{-1} \langle Bv_h, \varphi_h\rangle \geq \bar{k}_h \|\varphi_h\|_H \quad \forall \varphi_h \in W_h.$$

Then by $\overline{H1}$, H2 and theorem 1.1 we get immediatelly that problem (3.11) has a unique solution. In order to evaluate the distance between (u, ψ) and $(\bar{u}_h, \bar{\psi}_h)$ we define at first the space $\mathcal{V}$ in the following way:

(3.13) $$\mathcal{V} = \{v \mid v \in V, \ \ Bv \in H'\},$$

and we reamrk that $V_h \subseteq \mathcal{V}$ from (3.10), and also $u \in \mathcal{V}$, since g is supposed in H' and (u, ψ) is the solution of (1.1). We define then $\bar{\Lambda}_h : \mathcal{V} \times W \to V_h' \times W_h'$ by:

(1) In this case, of course, we will use $\bar{Z}_h = \{v_h \mid v_h \in V_h, \ Bv_h = 0\} \subseteq Z$; we remark therefore that, if $a(u,v)$ is Z-elliptic, then H2 is automatically satisfied.

$$(3.14)\quad \begin{cases} \langle \bar{\Lambda}_h(v,\varphi),(w_h,\chi_h)\rangle = a(v,w_h)+b(w_h,\varphi)+\langle Bv,\chi_h\rangle \\ \text{for all } (v,\varphi) \text{ in } \mathcal{V}\times W \text{ and } (w_h,\chi_h) \text{ in } V_h\times W_h. \end{cases}$$

Let now (v_h,φ_h) be a pair in $V_h\times W_h$; from $\overline{H1}$, H2 and the proposition 1.1 we get that there exists a (w_h,χ_h) in $V_h\times W_h$ such that:

$$(3.15)\quad \begin{aligned} &\langle \bar{\Lambda}_h(\bar{u}_h-v_h,\bar{\psi}_h-\varphi_h),(w_h,\chi_h)\rangle \geq \\ &\geq M(\alpha,\gamma_h',\bar{k}_h)^{-1}\cdot(||w_h||+||\chi_h||_H)\cdot(||\bar{u}_h-v_h||+||\bar{\psi}_h-\varphi_h||_H). \end{aligned}$$

On the other hand we have:

$$(3.16)\quad \langle \bar{\Lambda}_h(\bar{u}_h-v_h,\bar{\psi}_h-\varphi_h),(w_h,\chi_h)\rangle = \langle \bar{\Lambda}_h(u-v_h,\psi-\psi_h),(w_h,\chi_h)\rangle \quad \forall\,(v_h,\varphi_h),(w_h,\chi_h)\in V_h\times W_h.$$

Observing now that

$$(3.17)\quad |\langle \bar{\Lambda}_h(v,\varphi),(w_h,\chi_h)\rangle| \leq \alpha||v||\cdot||w_h|| + \bar{\beta}||w_h||\cdot||\varphi||_H + \bar{\beta}||v||\cdot||\chi_h||_H,$$

where

$$(3.18)\quad \bar{\beta} = \sup_{\substack{v\in\mathcal{V}-\{0\} \\ \varphi\in W\cup W_h-\{0\}}} ||v||^{-1}\,||\varphi||_H^{-1}\langle Bv,\varphi\rangle,$$

we get from (3.15),(3.16),(3.17), that:

$$||u-\bar{u}_h||+||\psi-\bar{\psi}_h||_H \leq ((\alpha+\bar{\beta})M(\alpha,\gamma_h',\bar{k}_h)+1)(\inf_{v_h\in V_h}||u-v_h|| + \inf_{\varphi_h\in W_h}||\psi-\varphi_h||_H).$$

We can conclude with the following theorem.

<u>Theorem</u> 3.2.- <u>Under the hypotheses of corollary</u> 1.1, <u>if</u> (3.9),(3.10) <u>and</u> H1 <u>are satisfied, for every</u> (f,g) <u>in</u> $V'\times W'$ <u>we have that, if</u> (u,ψ) <u>and</u> $(\bar{u}_h,\bar{\psi}_h)$ <u>are the solutions of</u> (1.1) <u>and</u> (3.11) <u>respectivelly, then</u>:

$$||u-\bar{u}_h||+||\psi-\bar{\psi}_h||_H \leq ((\alpha+\bar{\beta})M(\alpha,\gamma_h',\bar{k}_h)+1)(\inf_{v_h\in V_h}||u-v_h|| + \inf_{\varphi_h\in W_h}||\psi-\varphi_h||_H).$$

<u>where</u> $\bar{\beta}$ <u>is given by</u> (3.18).

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