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Strongly Mixing g-Measures

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by

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SUMMARY

Let T be an n - to - 1 covering transformation of the compact metric space X (e.g. (X,T) the n - shift). For suitable functions g on X an " inverse " φ_g of T is defined : φ_g is a Markov kernel. If g is strictly positive and satisfies a Lipschitz condition, then there exists a unique φ_g - invariant measure, strongly mixing under T. Conversely, we associate to any T - invariant probability measure a suitable g, and if g is " nice ", then strong mixing is present. Examples include all Bernoulli and Markov measures on the n - shift. The strong mixing criterion is useful, and applications to harmonic analysis, ergodic theory, and symbolic dynamics are given. For example : if G is any infinite subgroup of the group of roots of unity, there exist uncountably many (explicitly constructible) continuous Morse sequences whose corresponding dynamical systems are pairwise non - isomorphic and all have as eigenvalue group exactly the given group G.

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§ 1 - PRELIMINARIES. -

Let X be a compact metric space with metric $|.,.|$.

The following notation will be necessary :

$\mathbb{C}$ (X) = the continuous real - valued functions on X.

$|f| = \sup_{x \in X} | f(x) |$ for $f \in C(X)$.

$C^*(X)$ = the finite signed measures on the Borel sets of X, or, equivalently, the continuous linear forms on C (X).

$\mathcal{P}(X)$ = the probability measures in $C^*(X)$.

Suppose that T is a homomorphism of the space X. Then T maps each of C (X), $C^*(X)$, and $\mathcal{P}(X)$ into itself, and because $\mathcal{P}(X)$ is a weakly compact convex subset of C (X), the space

$$\mathcal{P}_T(X) = \{ \mu \in \mathcal{P}(X) \mid T\mu = \mu \}$$

is again a non - empty compact convex subset of $C^*(X)$. Suppose $\mu \in \mathcal{P}_T(X)$. Then

i) μ is <u>ergodic</u> iff μ is an extrem point of $\mathcal{P}_T(X)$.

ii) μ is <u>strongly mixing</u> iff for each pair f, g $\in$ C (X) we have

$\mu(f.T^k g) \longrightarrow \mu(f).\ \mu(g)$.

Our purpose is to study $\mathcal{P}_T(X)$ for special pairs (X,T). We call T a (minimal) <u>covering transformation</u> of X if there exist an integer $n \geq 2$ and $\rho > 1$ real such that

i) T is everywhere n - to - 1,

ii) T is a local homeomorphism,

iii) for sufficiently small $\delta > 0$, $|x,y| = \delta$ implies $|Tx,Ty| \geq \rho\ \delta$, and

iv) for each $x \in X$, $\bigcup_{n \geq 1} T^{-n}(x)$ is dense in X.

Let μ be any measure in $C^*(X)$, and denote the measure in $C^*(X)$ obtaibed by lifting μ locally via T^{-1} by Q μ. The total mass of Q μ is n times that of μ . If $\mu \in \mathcal{P}_T(X)$ then obviously μ is absolutely continuous with respect to Q μ, and we can form the Radon - Nikodym derivative

$$g = \frac{d\mu}{dQ\mu} \quad .$$

Moreover, $0 \leq g \leq 1$ and

$$\sum_{z \in T^{-1}(x)} g(z) = 1$$

for μ - almost every $x \in X$. Therefore, we are led to the following definitions. Set

$$G = \{g : X \longrightarrow [0,1] \mid g \text{ measurable}, \sum_{z \in T^{-1}(x)} g(z) = 1 \text{ for each } x \in X\} .$$

A probability measure μ on X is called a g - measure for a given $g \in G$ if $\frac{d\mu}{dQ\mu} = g \bmod \mu$. For any $g \in G$ and $\mu \in C^*(X)$, define the measure $\varphi_g \mu \in C^*(X)$ by

$$\frac{d\varphi_g \mu}{dQ\mu} = g.$$

Then, φ_g maps $\mathcal{P}(X)$ into $\mathcal{P}(X)$ and the following theorem is valid.

Theorem. - A probability measure μ is T - invariant if and only if μ is a g - measure for some $g \in G$. For each continuous $g \in G$ there exists at least one g - measure.

Proof : The first statement in the theorem is obvious from the preceding explanations. To prove the rest, note that if $g \in G$ is continuous, then φ_g is a weakly continuous map from $\mathcal{P}(X)$ into itself. By a fixed point theorem, there exists a $\mu \in \mathcal{P}(X)$ with $\varphi_g \mu = \mu$, i.e.   is a g - measure.

An example of a $g \in G$ with no corresponding g - measure will be given in § 4. Examples for covering transformations T are provided by taking for X the circle and for T an n - fold wrapping, or for (X,T) the one - sided shift space on n symbols. If X has a differentiable structure, we set

$$C^1(X) = \{f : X \longrightarrow \mathbb{R} \mid f \text{ continuously differentiable}\} .$$

In general, let

$$L(X) = \{f \in C(X) \mid \text{there exists } K > 0 \text{ with } |f(x) - f(y)| \leq K\, |x,y| \text{ for each } x,y \in X\} .$$

We also define the map φ_g for real - valudd functions f on X by setting

$$\varphi_g f(x) = \sum_{z \in T^{-1}(x)} g(z) f(z).$$

Then if f is μ - integrable, we have

$$\int_X \varphi_g f \, d\mu = \int_X f \, d \varphi_g \mu ,$$

and if $g \in C(X)$, then $\varphi_g : C^*(X) \longrightarrow C^*(X)$ is the dual transformation to $\varphi_g : C(X) \longrightarrow C(X)$.

§ 2 - THE STRONG MIXING CRITERION FOR THE CIRCLE. -

In this paragraph, a proof is given of the basic result for the circle group $X = \mathbb{R}/\mathbb{Z}$ under the transformation T defined by $Tx = 2x \pmod 1$. Points $x \in X$ are assumed to lie in the interval $[0,1[$. The result can be stated as follows.

Theorem : Let $g \in G \cap C^1(X)$ be strictly positive. Then there exists exactly one g - measure μ_g, and μ_g is strongly mixing.

Proof : The idea of the proof is to show that the sequence $\varphi_g^k f$ converges to a constant for any $f \in C^1(X)$, using the Arzela - Ascoli theorem. This yields the measure $\mu_g(f) = \lim \varphi_g^k f$, and it is easy to see that μ_g is unique and strongly mixing.

1. $\{\varphi_g^k f \mid k \geq 0\}$ is relatively compact in $C(X)$ if $f \in C^1(X)$:

Let $D = \frac{d}{dx}$. Then

$$D(\varphi_g f)(x) = \frac{1}{2}\varphi_g(Df)(x) + \frac{1}{2}\varphi_{Dg} f(x).$$

φ_{Dg} being defined in the obvious manner, and

$$|D\varphi_g f| \leq \frac{1}{2}|Df| + |Dg| . |f| ,$$

since φ_g is a contraction of $C(X)$. Therefore

$$|D\varphi_g^k f| \leq \frac{1}{2}|D\varphi_g^{k-1} f| + |Dg| . |f|$$

$$\leq \dots$$

$$\leq \frac{1}{2^k}|Df| + \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{k-1}}\right)|Dg| . |f|$$

$$\leq |Df| + 2|Dg| . |f| .$$

But

$$|\varphi_g^k f| \leq |f| \qquad (k = 1,2,\dots)$$

so that 1. follows from the Arzela - Ascoli theorem.

2. Choose $\{n_k\}$ such that $h := \lim_k \varphi_g^{n_k} f \in C(X)$. Then

$h = \text{const.} = \lim \varphi_g^k f$:

For $\tilde{f} \in C(X)$ set

$$\alpha(\tilde{f}) = \inf_{x \in X} \tilde{f}(x)$$

$$\beta(f) = \sup_{x \in X} \tilde{f}(x)$$

Because $g \in G$,

$$\alpha(f) \leq \alpha(\varphi_g(f)) \leq \dots \leq \alpha(h) \leq \beta(h) \leq \dots \leq \beta(\varphi_g f) \leq \beta(f),$$

and if we set

$$\alpha = \alpha(h) = \alpha(\varphi_g h) = \dots$$

$$\beta = \beta(h) = \beta(\varphi_g h) = \dots ,$$

then it suffices to show that $\alpha = \beta$. Now if $\tilde{f} \in C(X)$ and $\alpha(f) = \alpha(\varphi_g \tilde{f}) = \tilde{f}(y)$, then

$$\varphi_g \tilde{f}(y) = g\left(\frac{y}{2}\right) \tilde{f}\left(\frac{y}{2}\right) + g\left(\frac{1}{2} + \frac{y}{2}\right) \tilde{f}\left(\frac{1}{2} + \frac{y}{2}\right)$$

and g strictly positive imply $f\left(\frac{y}{2}\right) = \tilde{f}\left(\frac{1}{2} + \frac{y}{2}\right)$. Therefore if

$$A = \{x \in X \mid h(x) = \alpha\} ,$$

and if y satisfies $\varphi_g^k h(y) = \alpha$, then

$$\left\{ \frac{y + j}{2^k} \mid 0 \leq j < 2^k \right\} \subseteq A.$$

A similar argument holds for $\beta(h)$, and we have $h = \alpha = \beta$.

3. For $\tilde{f} \in C(X)$, $\varphi_g^k \tilde{f}$ converges uniformly to a constant :

Choose $\varepsilon > 0$ and $f \in C^1(X)$ with $|\tilde{f} - f| < \varepsilon$.

Let $\alpha = \lim \varphi_g^k f$. Then

$$|\varphi_g^k f - \varphi_g^k f| < \varepsilon$$

implies

$$\lim \left[\beta(\varphi_g^k f) - \alpha(\varphi_g^k f) \right] \leq 2\varepsilon ,$$

and φ_g^k converges to a constant.

4. Define $\mu_g(f) = \lim \varphi_g^k f \quad (f \in C(X))$. If μ is a g - measure, then $\mu = \mu_g$:

For any $f \in C(X)$,

$$\mu(f) = \mu(\varphi_g^k f) \longrightarrow \mu(\mu_g(f)) = \mu_g(f)$$

by the denominated convergence theorem and $\mu = \mu_g$.

5. μ_g is stoongly mixing :

Let $f, h \in C(X)$. Since $\varphi_g T f = f$,

$$\mu_g(T^k f . h) = \mu_g(f . \varphi_g^k h) \longrightarrow \mu_g(f) . \mu_g(h)$$

as k tends to infinity, and μ_g is strongly mixing.

The condition that g be strictly positive can be relaxed. For such a modification only step 2. of the proof needs to be checked, the other parts being independent of this condition.

Theorem : Let $g \in G \cap C^1(X)$ satisfy one of the following conditions :

a) g has only one zero in X.

b) g has finitely many zeroes, none of which wander into periodic orbits under T.

c) the zeroes of g lie in $[\frac{1}{4}, \frac{3}{4})$ or $(\frac{1}{4}, \frac{3}{4}]$.

Then there exists exactly one g - measure $_g$, and $_g$ is strongly mixing.

Proof : The notation of the preceding proof is used.

a) Let $g(z) = 0$ and $x \in X$ with $\varphi_g^k h(x) = \alpha$. Either $x < z$ or $h(\frac{x+2^k-1}{2^k}) = \alpha$. Since $h \in C(X)$, we have $\alpha = \beta$.

b) Let $A = \{z \mid g(z) = 0\}$ have r elements. The convexity argument of 2. can be applied to $\varphi_g^k h(y) = \alpha$ unless for some $1 \leq i \leq k$ and $0 \leq j < 2^i$,

$$\frac{y+j}{2^i} \in A.$$

Now, if $\frac{y+j}{2^i} = z \in A$ does not wander into a periodic orbit, then i is uniquely determined by z. Thus

$$h\left(\frac{y+j}{2^k}\right) = \alpha$$

for all $0 \leq j < 2^k$ except possibly those of the form $2^i p + q_i$, for at most r different values of i. As k increases, the subset on which $h = \alpha$ still becomes dense.

c) If $\varphi_g^k h(y) = \alpha$ and $y \in [\frac{1}{4}, \frac{3}{4}]$, then either $\frac{y}{2} \in (0, \frac{1}{4})$ or $\frac{1}{2} + \frac{y}{2} \in (\frac{3}{4}, 1)$ and we can apply the technique used in a).

Condition c) and the proofs of a) and c) were suggested to me by L. KAUP. In § 4, we give an example of a $g \in G \cap C^1(X)$ with two g - measures because of zeroes at the points of periodicity of T. The above theorem can certainly be sharpened.

§ 3 - THE STRONG MIXING CRITERION FOR COVERING TRANSFORMATIONS. -

In this paragraph, the results of § 2 are extended to covering transformations T of the compact metric space X.

Theorem : Let $g \in G \cap L\ (X)$ be strictly positive. Then there exists exactly one g - measure μ_g, and μ_g is strongly mixing.

Proof : Only part 1 of the proof of § 2 needs modification to :

1. If $f \in C\ (X)$, then $\{\varphi_g^k f \mid k \geq 0\}$ is relatively compact in $C\ (X)$:

For any $f \in C\ (X)$ and $\delta > 0$, let

$$\varepsilon\ (f, \delta) = \sup_{|x,y| \leq \delta} |\ f\ (x) - f\ (y)|$$

Let $f \in C\ (X)$ and suppose that K is a Lipschitz constant for g.

Then for sufficiently small $\delta > 0$, $|x,y| \leq \delta$ implies

$\max_{1 \leq i \leq n} |\ z_i\ ,\ z_i| \leq \rho^{-1}\delta$, where $T^{-1}\ x = \{\ z_1,\ldots,z_n\}$, $T^{-1}y = \{z_1,\ldots,z$

and thus

$$\varepsilon\ (\varphi_g f, \delta) = \sup_{|x,y| \leq \delta} |\sum_{i=1}^{n} (g(z_i)f(z_i) - g\ (z_i)f(z_i))|$$

$$\leq \sup_{|x,y| \leq \delta} \sum_{i=1}^{n}\ g\ (z_i)|\ f\ (z_i) - f\ (z_i)|$$

$$+ \sup_{|x,y| \leq \delta} \sum_{i=1}^{n}\ |f\ (z_i)||\ \ g\ (z_i) - g\ (z_i)|$$

$$\leq \varepsilon\ (f,\rho^{-1}\delta) + n\ .\ |\ f|\ .\ K\ .\ \rho^{-1}\delta$$

By induction we conclude that

$$\varepsilon\ (\varphi_g^k f, \delta) \leq \varepsilon(f,\rho^{-k}\delta) + n\ K\ |\ f\ |.\ \delta\ (\rho^{-1} + \rho^{-2} + \ldots + \rho^{-k+1})$$

$$\leq \text{constant} + n\ K\ \ |f|\ \ \delta \frac{\rho^{-1}}{1 - \rho} \qquad ,$$

and this implies the equi - continuity of the set $\{\varphi_g^k\ f \mid k \geq 0\}$.

In special cases, the strict positivity of g can be relaxed as in § 2, e. g. for (X,T) a one - sided shift space.

We note that $\mu_g \neq \mu_{g'}$ implies $\mu_g \perp \mu_{g'}$, because of the strong mixing property.

The idea behind the existence of measures as shown in this and the preceding paragraph is not new - similar and more general existence problems have been handled in various settings (e.g. [1] , [3] , [5] , [6]). What is original is the strong mixing of all these measures with respect to the same map T and the resulting orthogonality.

§ 4. EXAMPLES. -

1. Bernoulli schemes.

Let $X = \Omega_\eta = \prod_0^\infty \{0, 1, \ldots, n-1\}$ and let T be the shift transformation on X. Under the metric defined by

$$|w, \eta| = |\inf \{ i \mid w_i \neq \eta_i \}|^{-1}$$

X is compact and T is a covering transformation of X.

If $p = (p_0, p_1, \ldots, p_{n-1})$ satisfies $p_k > 0$ and $\Sigma p_k = 1$, then let μ^p denote the product measure on X with distribution p in each component. An easy calculation shows that μ^p is a g^p - measure with

$$g^p(w) = p_k \qquad (w \in X, \; w_0 = k)$$

Since g^p is a continuous, locally constant, and positive function on X, we have $g^p \in G \cap L(X)$ and the results of § 3 apply to Bernoulli schemes.

2. Markov measures.

Let (X,T) be as in 1., and let $P = (p_{ij})$ be a Markov kernel on $\{0, 1, \ldots, n-1\}$. Choose a probability vector $\pi = (\pi_0, \ldots, \pi_{n-1})$ with $\pi P = \pi$ and denote by m the Markov measure on X generated by the initial distribution and transition probabilities P.

If $a_0 \, a_1 \ldots a_k$ is a sequence of states, let

$$[a_0 \, a_1 \ldots a_k] = \{ w \in X \mid w_i = a_i \quad \text{for } 0 \leq i \leq k \}$$

Now, $m \in \mathcal{P}_T(X)$ because P = , so that m is a g - measure for some $g \in G$. We may calculate g as follows. Fixing the states i and j, the ratio

$$\frac{m([ija_2 \ldots a_k])}{m([ja_2 \ldots a_k])} = \frac{\pi_i p_{ij}}{\pi_j}$$

is independent of $a_2, \ldots, a_k$. Therefore if

$$g(w) = \frac{\pi_{w_0} \, p_{w_0 w_1}}{\pi_{w_1}} \qquad (w \in X),$$

m is a g - measure. If P and π are strictly positive, then m is the unique g - measure and is strongly mixing, since $g \in L(X)$.

3. Let $X = \mathbb{R}/\mathbb{Z}$ and $Tx = nx \mod 1$ for some $n \geq 2$. For some α with $|\alpha| \leq 1$, set

$$g(x) = \frac{1 + \alpha \cos 2\pi x}{n} \qquad (x \in X).$$

Then obviously $0 \leq g(x) \leq 1$ and

$$\sum_{z \in T^{-1}(x)} g(z) = 1 + \frac{\alpha}{n} \sum_{j=0}^{n-1} \cos 2\pi \frac{z+j}{n} = 1.$$

Therefore $g \in G \cap C^1(X)$ and § 2 shows that exists a unique g - measure μ_g, which is strongly mixing. In particular, the ergodicity of λ (Lebesgue measure) and μ_g implies $\lambda \perp \mu_g$, and we have a singular measure μ_g on X. It is easy to see that μ_g is continuous iff $\alpha \neq +1$, and μ_g is point mass at 0 if $\alpha = +1$.

4. Let $X = \mathbb{R}/\mathbb{Z}$ and $Tx = 2x \mod 1$. If $g \in G$ with $g(0) = g(\frac{1}{3}) = g(\frac{2}{3}) = 1$, then point mass ε_0 at zero and the measure $\frac{1}{2}(\varepsilon_{\frac{1}{3}} + \varepsilon_{\frac{2}{3}})$ are both g - measures, and our method does not produce results because of the periodicities in the orbits of the zeroes of g.

5. Let $X = \mathbb{R}/\mathbb{Z}$ and $Tx = 2x \mod 1$. We set

$$g(x) = \begin{cases} \dfrac{1 + \cos 2\pi x}{2} & (x \neq 0, \frac{1}{2}) \\ \dfrac{1}{2} & (x = 0 \text{ or } \frac{1}{2}) \end{cases}$$

Then, $g \in G$ and for $f \in C(X)$, $\mathcal{L}_g^n f$ converges to $f(0)$, but ε_0 is not a g - measure because of the discontinuity of g at 0.

§ 5. APPLICATIONS TO HARMONIC ANALYSIS. -

Let $X = \mathbb{R}/\mathbb{Z}$ and $T x = n x \bmod 1$, $|\alpha| \leq 1$, $n \geq 3$. Products of the form

$$p_k(x) = \prod_{j=1}^{k} (1 + \alpha \cos 2\pi n^j x)$$

are special cases of Riesz products (see [8]). Let g be as in § 4, example 3. Then if λ denotes Lebesgue measure, we have

$$\varphi_g^k \lambda = p_k \cdot \lambda$$

and the theorems of § 2 show that $\varphi_g^k \lambda$ converges to the continuous singular measure μ_g of the example. If $n = 2$ and $\alpha \neq \pm 1$, then the measure μ_g remains singular and continuous, but if $\alpha = + 1$ and $n = 2$, we get $\mu_g = \varepsilon_0$. In this case, the products $p_k(x)$ are just the Fejer kernels

$$K_{2^k-1}(x) = \frac{1}{2^{k-1}} \left\{ \frac{\sin 2^{k-1} \cdot 2\pi x}{2 \sin \pi x} \right\}^2$$

and form an approximate identity. Our methods yield in fact for $n = 2$ an approximate identity whenever $\frac{1}{2}$ is the only zero of g, and it is conceivable that approximate identities with desirable properties could be constructed. We note also the combinatorial connections : if $\hat{\mu}_g$ denotes the Fourier transform of the measure μ_g, a simple calculation yields

$$\hat{\mu}_g(k) = \sum_r \left(\frac{\alpha}{2}\right)^r .$$

Number of ways to write

$$k = \pm n^{j_1} \pm \dots \pm n^{j_r} ,$$

and $\hat{\mu}_g(k) = \lim_{j \to \infty} \varphi_g^j (e^{-2\pi i k x})$ gives an analytic expression for this combinatorial quantity.

§ 6. SPECTRAL CALCULATIONS FOR MORSE SEQUENCES. -

In this paragraph, we calculate the spectral measures corresponding to the continuous spectrum of a generalized Morse sequence. We assume familiarity with [4] , and begin by describing the results in [4] that we need.

Denote by

$$\Omega = \{0,1\}^{\mathbb{Z}}$$

the space of bisequences of zeroes and ones, with the left shift σ . Let each of b^0, b^1, b^2, ... be a finite block of zeroes and ones of length at least two and starting with 0. To exclude periodic cases, assume that an infinity of the b^i are different from 00 ... 0, and an infinity different from 0101 ... 010. Assume also that the sequence

$$x = b^0 \times b^1 \times b^2 \times \dots$$

is a <u>continuous Morse sequence</u> (see definitions 7,8 and theorem 9 of [4]).
This implies the following :

I - The orbit closure $\mathcal{O}$ of x in (Ω,σ) is strictly ergodic.

II - Denote by m_x the unique σ - invariant probability measure concentrated on $\mathcal{O}_x$ and set

$$\mathcal{D}_x = \{f \in L^2(\mathcal{O}_x, m_x) \mid f = \tilde{f}\}$$
$$\mathcal{E}_x = \{f \in L^2(\mathcal{O}_x, m_x) \mid f = -\tilde{f}\}$$

Then, $L^2(\mathcal{O}_x, m_x) = \mathcal{D}_x \oplus \mathcal{E}_x$ and $\mathcal{D}_x$ and $\mathcal{E}_x$ are σ - invariant.

III - σ has discrete spectrum on $\mathcal{D}_x$ with eigenvalue group

$$\mathcal{J}_x = \{ \exp (2\pi i \frac{j}{n_0 \dots n_k}) \mid j,k=0,1,2,\dots\} \quad ,$$

where n_i is the length of b^i. There is a map $\gamma \longrightarrow f_\gamma$ from $\mathcal{J}_x$ to $\mathcal{D}_x$ such that $\sigma f_\gamma = \gamma \cdot f_\gamma$, $f_{\gamma\delta} = f_\gamma f_\delta$,
$\{f_\gamma \mid \gamma \in \mathcal{J}_x\}$ is a complete orthonormal basis for $\mathcal{D}_x$ and

$$f_\gamma = \sum_{j=0}^{n_0 \dots n_k - 1} \gamma^j 1_{A_j} \qquad (\gamma = \exp(2\pi i \frac{j'}{n_0 \dots n_k}))$$

with $\sigma^j A_0 = A_j$ for $0 \leq j \leq n_0 \dots n_k - 1$, the A_j being open and closed in $\mathcal{O}_x$.

IV - σ has continuous spectrum (no eigenvalues) on $\mathcal{E}_x$. If we put

$$h(w) = (-1)^{w_0}$$

then $\{h.f_\gamma | \gamma \in \mathcal{J}_x\}$ spans $\mathcal{E}_x$.

V - Let $b^0 = b_0 b_1 \dots b_{n-1}$ and $y = b^1 \times b^2 \times \dots$
Then y is also a continuous Morse sequence. If $\gamma = \exp(2\pi i \frac{j}{n})$ then the sets $A_0, A_1, \dots, A_{n-1}$ in III can be chosen (choose them as in § 4 of [4]) such that the (strictly ergodic) systems $(\mathcal{O}_y, \sigma)$ and (A_j, σ^n) are isomorphic, and

$$\sigma^i (h\, 1_{A_{i+j}}) = (-1)^{b_i + b_{i+j}}\, h\, 1_{A_j}$$

for $0 \leq j \leq i + j < n$.

Now, we can begin our spectral calculations for σ on $\mathcal{C}_x$.
Denote by $\mu_{\gamma,\delta}$ the measure on $\mathbb{R}/\mathbb{Z}$ such that

$$< \sigma^k (hf_\gamma), hf_{\delta^{-1}} > = \hat{\mu}_{\gamma\,\delta}(k) = \int_0^1 \exp(-2\pi ikt)\, \mu_{\gamma,\delta}(dt)$$

for each $k \in \mathbb{Z}$, $< . , . >$ denoting the scalar product in $L^2(\mathcal{O}_x, m_x)$. Then
$< \sigma^k (h.f_\gamma), hf_{\delta^{-1}} > = \gamma^n < \sigma^n (h.1), hf_{\delta^{-1}\gamma^{-1}} >$
and $f_1 = 1$, so that the measures $\mu_{\gamma\,\delta}$ and $\mu_{1,\gamma\delta}$ are related by a translation of $\log \gamma = \frac{j}{n_0 \dots n_k}$ on $\mathbb{R}/\mathbb{Z}$.

Now let $y = b^1 \times b^2 \times \dots$, $b^0 = b_0 b_1 \dots b_{n-1}$,
$\gamma = \exp(2\pi i \frac{j'}{n})$, and let ν be the measure $\mu_{1,1}$ for the point y, c'est à dire

$$\hat{\nu}(k) = < \sigma^k h, h >_{L^2(\mathcal{O}_y, m_y)}$$

For fixed $m \in \mathbb{Z}$, we obtain, using III and IV,

$$\hat{\mu}_{1,\gamma}(mn) = \langle \sigma^{mn} h, hf_\gamma \rangle$$

$$= \sum_{j=0}^{m-1} \gamma^j \langle \sigma^{mn} h, h 1_{A_j} \rangle = \hat{\nu}(m) \sum_{j=0}^{n-1} \gamma^j$$

$$\hat{\mu}_{1,\gamma}(mn+1) = \langle \sigma^{mn+1} h, hf_\gamma \rangle = \sum_{j=0}^{n-1} \gamma^j \langle \sigma^{mn+1} h, h 1_{A_j} \rangle$$

$$= \sum_{j=0}^{n-2} \gamma^j \langle \sigma^{mn+1}(h1_{A_{j+1}}), h1_{A_j} \rangle + \gamma^{n-1} \langle \sigma^{(m+1)n} h, \sigma^{n-1}(h1)_{A_{n-1}} \rangle$$

$$= \hat{\nu}(m) \sum_{j=0}^{n-2} (-1)^{b_j + b_{j+1}} \gamma^j + \hat{\nu}(m+1)(-1)^{b_0 + b_{n-1}} \gamma^{n-1}$$

and in general for $0 \leq j \leq n-1$.

$$\hat{\mu}_{1,\gamma}(mn+j) = \hat{\nu}(m) \cdot \sum_{i=0}^{n-1-j} (-1)^{b_i + b_{i+j}} \gamma^i + \hat{\nu}(m+1) \sum_{i=n-j}^{n-1} (-1)^{b_i + b_{i+j-n}} \gamma^i$$

Denoting by Q the operation defined in § 1 for the transformation $Tt = nt \bmod 1$, we see by simple operations with Fourier transforms that $\mu_{1,\gamma}$ is absolutely continuous with respect to $Q\nu$ and

$$\frac{d\mu_{1,\gamma}}{dQ\nu} = \frac{1}{n} \sum_{k=0}^{n-1} \alpha(b^0,\gamma,k) \exp(2\pi ikt)$$

$$+ \frac{1}{n} \sum_{k=0}^{n-1} \beta(b^0,\gamma,k) \exp(2\pi i(k-n)t),$$

where

$$\alpha(b^0,\gamma,k) = \frac{1}{n} \sum_{i=0}^{n-1-k} \gamma^i (-1)^{b_i + b_{i+k}}$$

$$\beta(b^0,\gamma,k) = \frac{1}{n} \sum_{i=n-k}^{n-1} \gamma^i (-1)^{b_i + b_{i+k-n}}$$

In particular, for $\gamma = 1$, we have $\alpha(b^0,1,0) = 1$ and

$\alpha(b^0,1,k) = \beta(b^0,1,n-k)$, $k = 1, \dots, n-1$

Thus, if we set

$$g\,(b^{o}) = g\,(b^{o},t) = \frac{1 + \sum_{k=1}^{n-1} 2\,\alpha(b^{o},1,k)\cos 2\pi ikt}{n},$$

$g\,(b^{o}) \in G$ and

$$\mu_{1,1} = \varphi_{g\,(b^{o})}\ \nu,$$

where ν is itself the measure $\mu_{1,1}$ corresponding to the sequence y. In general, denote by λ_j the (probability) measure $\mu_{1,1}$ corresponding to the product $b^{j} \times b^{j+1} \times \ldots$, and set

$$g_j\,(t) = g\,(b^{j},t)$$

$$\varphi_j = \varphi_{gj}$$

Lemma.

For each $j \geq 1$, $\lambda_{j-1} = \varphi_{j-1}\,\lambda_j$.

Moreover, for any f, f' $\in \mathcal{C}_{\pi}$, the measure μ defined by

$$\langle \sigma^{k} f, f' \rangle = \int_0^1 \exp\,(-2\pi ikt)\mu(dt) \qquad (k \in \mathbb{Z})$$

satisfies $\mu \ll \lambda_o$.

Proof.

The first statement is the result of the preceding calculations. Since any $f \in \mathcal{C}_{\pi}$ can be expressed as an infinite linear combination of the f_{γ}, it suffices to show that $\mu_{\gamma,\delta} \ll \lambda_o$ for each pair $\gamma, \delta \in \mathcal{J}_{\pi}$. This follows from the facts that $g\,(b^{i})$ has only a finite number of zeroes, all measures λ_i are continuous and equivalent to their translations by amounts of the form $\frac{j}{n_o \ldots n_k}$.

Restating the conclusion of the lemma, the spectral measure corresponding to σ on $\mathcal{C}_{\pi}$ is absolutely continuous with respect to λ_o. Thus the class of measures equivalent to λ_o is an isomorphism invariant for σ on $\mathcal{C}_{\pi}$.

Next we show how to calculate the measure λ_o and derive some of its properties.

Theorem.

For any $f \in C(\mathbb{R}/\mathbb{Z})$,

$$\lim_{j \to \infty} \varphi_j \varphi_{j-1} \cdots \varphi_0 f = \lambda_0 (f)$$

uniformly.

Proof :

Let f be a trigonometric polynomial of degree k. Applying φ_j and using the special form of $g(b^j)$ given above, we see that φ_j f is again a trigonometric polynomial of degree

$$1 + \left[\frac{k-1}{n_j}\right] \leq 1 + \left[\frac{k-1}{2}\right] < k ,$$

provided that $k > 1$. Thus deg (f) = k implies

$$\varphi_j \varphi_{j-1} \cdots \varphi_0 f = a_0^j + a_1^j e(2\pi it) + \bar{a}_1^j e(-2\pi it)$$

for $j \geq k$. Applying φ_{j+1}, we obtain

$$| a_1^{j+1} | = \frac{|a_1^j|}{n_{j+1}} \leq \frac{| a_1^j |}{2} ,$$

in view of $|\alpha(b^{j+1}, 1, n_{j+1}-1)| = \frac{1}{n_{j+1}}$.

Thus, $\varphi_j \cdots \varphi_0$ f converges uniformly to a constant μ (f) for trigonometric polynomials and hence for each f $C(R/Z)$. Now

$$\lambda_0 (f) = (\varphi_0 \cdots \varphi_j)' \lambda_{j+1} (f) = \lambda_{j+1} (\varphi_j \cdots \varphi_0 f) \to \mu (f)$$

and

$$\mu(f) = \lambda_0 (f).$$

Theorem (Strong mixing property)

For $f, g \in C(\mathbb{R}/\mathbb{Z})$

$$\lim_{j \to \infty} \frac{\lambda_0(f . T_0 T_1 \ldots T_j g)}{\lambda_0(T_0 T_1 \ldots T_j g)} = \lambda_0 (f),$$

where $T_j t = n_j t \bmod 1$ and if the denominator remains non - zero.

Proof :

Because $\varphi_j T_j$ is the identity,

$$\lambda_o (f \cdot T_o \ldots T_j g) = \varphi_o \ldots \varphi_j T_j \ldots T_o \lambda_o (f \cdot T_o \ldots T_j g)$$

$$= \lambda_{j+1} (g \cdot \varphi_j \ldots \varphi_o f)$$

and $\varphi_j \ldots \varphi_o f$ converges uniformly to $\lambda_c (f)$. Since $\lambda_{j+1} (g) = \lambda_o (T_o \ldots T_j g)$, the theorem follows.

We can use now our information about λ_c to show that in general, Morse sequences are non - isomorphic. Let $x = b^o \times b^1 \times \ldots$ and $x' = c^o \times c^1 \times \ldots$ be continuous Morse sequences with length (b^i) = length $(c^i) = n_i$ for each i, and denote the corresponding basic measures by λ_c and λ'_c.

Lemma.

Either $\lambda_o \perp \lambda'_o$ or $\lambda_o \sim \lambda'_o$. If $\lambda_o \sim \lambda'_o$, then $||\lambda_j - \lambda'_j|| \longrightarrow 0$, where $|| \cdot ||$ is the variation norm.

Proof.

For each n write

$$\lambda_n = \lambda_n^s + f_n \cdot \lambda'_n$$

where

$\lambda_n^s \perp \lambda'_n$. Then $\lambda_n^s = \varphi_n \lambda_{n+1}^s$ for each n, so that

$$\lambda_o^s (f) = \varphi_o \ldots \varphi_n \lambda_{n+1}^s (f)$$

$$= \lambda_{n+1}^s (\varphi_n \ldots \varphi_o f) \longrightarrow c \cdot \lambda_o (f).$$

Thus, either $\lambda_o^s = 0$ or λ_o. If $\lambda_o = f_o \cdot \lambda'_o$, then

$$\lambda_{n+1} = (\varphi'_n \cdot \varphi'_{n-1} \ldots \varphi'_o f) \cdot \lambda'_{n+1}$$

Now, if f is continuous, $\varphi'_n \ldots \varphi'_o f$ converges uniformly to 1, so that by approximation, any $f \in L^1_+(\lambda'_o)$ with $\int f \, d\lambda'_c = 1$ satisfies

$$\int |\varphi'_n \cdots \varphi_c f - 1| \, d\lambda'_n \longrightarrow 0$$

Theorem.

If

1) there exists a constant K such that

$I = \{ i \geq 0 \mid n_i \leq K \text{ and } g(b^i) \neq g(c^i) \}$

is infinite, and

2) λ_i (or λ'_i) converges weakly to a continuous measure ν along some subsequence along some subsequence of $I + 1 = \{i+1 \mid i \in$

then $\lambda_c \perp \lambda'_o$.

Proof :

There are only a finite number of blocks of length not exceeding so that we can choose a sequence $i' + 1$ in $I + 1$ such that $\lambda_{i'+1} \longrightarrow \nu$ and $b^{i'} = b$, $c^{i'} = c$ with $g(b) \neq g(c)$. This implies that

$g(b) \, . \, Q \, \nu = g(c) \, . \, Q \, \nu$.

But because of the form of $g(b)$ and $g(c)$, $\{ t \mid g(b,t) = g(c,t) \}$ is finite. Since ν is continuous, we have a contradiction.

We note two simple consequences of this theorem. The first is that if $x = b \times b \times \ldots$ and $x' = c \times c \times \ldots$ are continuous Morse sequences with $g(b) \neq g(c)$, then $\lambda_o \perp \lambda'_o$ and the dynamical systems $(\mathcal{O}_x, m_x)$ and $(\mathcal{O}_{x'}, m_{x'})$ are not isomorphic. For a direct proof note that λ_o and λ'_o are $g(b)$ respectively $g(c)$ measures in the sense of § 1. Since $\lambda_o \neq \lambda'_c$ and since both are ergodic under $T\,t = n\,t \mod 1$, n being the common length of b and c, we have $\lambda_c \perp \lambda'_o$. The second consequence is the following theorem.

Theorem :

Let $\mathcal{G}$ be an infinite subgroup of the group of roots of unity. There exists an uncountable number of dynamical systems whose eigenvalue group is exactly $\mathcal{G}$, such that any two of the systems are non - isomorphic.

This theorem generalizes the result in [2] , where the case $\mathcal{G} = \{ \lambda \mid \ n : \lambda^{2^n} = 1 \}$ is dealt with.

§ 7 - Miscellaneous.

Let $G_o \subseteq X = \mathbb{R}/\mathbb{Z}$ be the subgroup of all dyadic rationals. In [7] an example of a quasi - ergodic measure class different from the Lebesgue measure class with respect to the group G_o was given. If we define $Tx = 2x \mod 1$ and let $g \in G \cap C^1(X)$ be strictly positive, then it is easy to see that the measure μ_g is quasi - invariant and quasi - ergodic with respect to G_o. Moreover, we obtain different classes for different g, and thus uncountably many such classes exist.

We note that there is one - to - one correspondence between the invariant measures on the one - sided n - shift and those on the two - sided n - shift, since these measures are uniquely determined by their values on cylinder sets. The properties of ergodicity and strong mixing are compatible with this correspondence, so that the examples for the one - sided shift are also valid for the two - sided shift.

We remark that the theorem in § 2 answers negatively a conjecture of KARLIN [3] , since our measures are singular with respect to Lebesgue measure.

There are a number of questions left answered :

1. If $g \in G \cap C(X)$ is strictly positive, is there only one g - measure ? In [3] , KARLIN states a theorem to this effect, but the proof seems to use derivatives of g. It would suffice to show that the Cesaro means of $\varphi_g^n f$ converge uniformly.

2. The entropy of the g - measure μ should be $- \int \log g \, d\mu$. Is it ?

3 - Which dynamical systems (X, μ_g, T) are isomorphic ?

4 - Let $b = b_o \ldots b_{n-1}$ be a 0 - 1 sequence, and call $c = c_o \ldots c_{n-1}$ similar to b if it is obtained from b by interchange of 0 and 1 and for order reversal. If b and c are similar, then $g(b) = g(c)$.

Does g (b) = g (c) imply that b and c are similar ?

Hopefully, the criteria in § 2 and § 3 will turn out to be effective in proving the ergodicity of dynamical systems. A note announcing the results of this paper has appeared in Comptes Rendus, March 1931.

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