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A PROBABILISTIC PROOF OF BOCHNER'S THEOREM ON
POSITIVE DEFINITE FUNCTIONS

by

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1. Introduction

The purpose of this paper is to give a proof of Bochner's theorem on positive definite functions in a locally compact second countable abelian group by a purely probabilistic method without using the theory of Banach algebras or general theory of representations. We reduce the problem to the Daniell-Kolmogorov construction of a stochastic process on the basis of finite dimensional distributions. The Lévy-Cramér continuity theorem is deduced as a corollary.

2. Preliminaries

Let X be a locally compact second countable abelian group and Y its character group. Y consists of all continuous homomorphisms from X into the multiplicative circle group K of complex numbers with modulus unity. We shall denote the Haar measure on Y by H and use dy to denote integration with respect to H . For any $x \in X$, $y \in Y$, we shall denote by $\langle x, y \rangle$ the value of the character y at x . We assume the fact that X and Y are character groups of each other (cf. Rudin [2]).

A complex valued function φ defined on Y is said to be positive definite if for all $y_1, y_2, \dots, y_k \in Y$, complex numbers $c_1, c_2, \dots, c_k$ and positive integers k , the inequality

$$(2.1) \quad \sum_{1 \leq i, j \leq k} c_i \bar{c}_j \varphi(y_i - y_j) \geq 0$$

is satisfied. With these notations we have the following theorem due to Bochner [2].

Theorem 2.1 Let φ be a complex-valued positive definite function defined on Y such that φ is continuous at e and $\varphi(e) = 1$, e being the identity element of Y . Then there exists a unique probability measure μ defined on the Borel σ -field of X such that

$$\varphi(y) = \int_X \langle x, y \rangle d\mu(x), \quad y \in Y$$

3. The case when X is a finite-dimensional torus

In this section we shall consider the case when $X = K^r$ where K is the multiplicative circle group consisting of all complex numbers of modulus unity and K^r is the r -fold cartesian product of K . Then $Y = I^r$, where I is the additive group of all integers. Any point $x \in X$ can be represented by $\underline{\theta} = (\theta_1, \theta_2, \dots, \theta_r)$ where $0 \leq \theta_j < 2\pi$, $j = 1, 2, \dots, r$. With this representation the group operation becomes coordinatewise addition modulo 2π . Any point $y \in Y$ can be represented by $\underline{n} = (n_1, n_2, \dots, n_r)$ where all the n_j 's are integers. Then $\langle \underline{\theta}, \underline{n} \rangle = \exp i \sum_{j=1}^r n_j \theta_j$. Let φ be a positive

definite function on I^r and $\varphi(\underline{0}) = 1$. φ is automatically continuous because I^r is discrete. Even though an elementary proof of Theorem 2.1 is well known in this case, we reproduce it for the sake of completeness.

Consider the measure μ_N which is absolutely continuous with respect to the uniform distribution on K^r and which has the density function

$$f_N(\underline{\theta}) = \frac{1}{N^r} \sum_{\substack{1 \leq m_j \leq N \\ 1 \leq n_j \leq N \\ j=1, \dots, r}} \varphi(\underline{m} - \underline{n}) \langle \underline{\theta}, \underline{n} - \underline{m} \rangle$$

Since $\langle \underline{\theta}, \underline{n} - \underline{m} \rangle = \langle \underline{\theta}, \underline{n} \rangle \overline{\langle \underline{\theta}, \underline{m} \rangle}$, it follows from (2.1) that $f_N(\underline{\theta}) \geq 0$ for all $\underline{\theta}$. Trivially

$$\frac{1}{(2\pi)^r} \int_0^{2\pi} \dots \int_0^{2\pi} f_N(\underline{\theta}) d\theta_1 \dots d\theta_r = 1$$

An easy computation shows that, for $\underline{j} = (j_1, j_2, \dots, j_r)$, we have

$$\begin{aligned} \frac{1}{(2\pi)^r} \int f_N(\underline{\theta}) \langle \underline{\theta}, \underline{j} \rangle d\underline{\theta} &= \varphi(\underline{j}) \prod_{k=1}^r \left(1 - \frac{|j_k|}{N} \right), \quad \text{if } |j_k| < N, \\ & \quad k=1, 2, \dots, r. \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Since any set of probability measures on K^Y is weakly conditionally compact, $\{\mu_N\}$ has a convergent subsequence. Let μ be any limit of $\{\mu_N\}$. Then

$$\int \langle \underline{Q}, \underline{j} \rangle d\mu = \lim_N \int \langle \underline{Q}, \underline{j} \rangle d\mu_N = \varphi(\underline{j})$$

for all $\underline{j}$. This establishes the existence of a μ with the required property. Uniqueness follows from the well known fact that any continuous function on K^Y is a uniform limit of linear combinations of functions of the form $\langle \underline{Q}, \underline{j} \rangle$. This completes the proof.

Before proceeding to the proof of Theorem 2.1 in the general case we shall mention a simple corollary. To this end we need some notations. For any two abstract sets A and B , we shall denote by A^B the set of all functions defined on B and taking values in A . Then K^Y is the space of all functions defined on Y and taking values in K . Any element $\underline{Q} \in K^Y$ can be represented as a function $\underline{Q}(y)$ where $0 \leq \underline{Q}(y) < 2\pi$ for all $y \in Y$. Let π_y denote the projection map $\underline{Q} \rightarrow \underline{Q}(y)$ from K^Y onto K . Let $\mathcal{G}$ denote the smallest σ -field of subsets of K^Y with respect to which all the π_y are measurable. We shall denote by $\mathcal{N}^p$ the character group of K^Y . $\mathcal{N}^p$ is the set of all integer valued functions $\underline{n}(y)$ defined on Y and vanishing outside a finite subset. We shall denote by $\underline{Q}$ the function which is identically 0. With these notations we have the following corollary.

Corollary 3.1 Let $\psi(\underline{n})$, $\underline{n} \in \mathcal{N}^p$ be any positive definite function defined on $\mathcal{N}^p$ and $\psi(\underline{Q}) = 1$. Then there exists a unique probability measure

P on $\mathcal{S}$ such that

$$\int \left[\exp \left(i \sum_{y \in Y} \underline{n}(y) \underline{\theta}(y) \right) \right] dP(\underline{\theta}) = \psi(\underline{n})$$

for all $\underline{n} \in \mathcal{N}^0$.

Proof For any finite subset $F \subseteq Y$, let

$$\mathcal{N}_F = \left\{ \underline{n} : \underline{n} \in \mathcal{N}^0, \underline{n}(y) = 0 \text{ if } y \notin F \right\}$$

When ψ is restricted to $\mathcal{N}_F$ we obtain a positive definite function on $\mathbb{I}^F$.

Hence by the validity of Theorem 2.1 for K^F , we deduce the existence of a measure P_F on the Borel σ -field of K^F , satisfying the property

$$\int_{K^F} \left[\exp \left(i \sum_{y \in F} \underline{n}(y) \underline{\theta}(y) \right) \right] dP_F(\underline{\theta}) = \psi(\underline{n}), \quad \underline{n} \in \mathcal{N}_F.$$

It follows from the uniqueness of P_F for every F that if $F_1 \supseteq F_2$ are two finite subsets of Y , the measure P_{F_1} induces P_{F_2} through the natural projection from K^{F_1} onto K^{F_2} obtained by dropping the y th coordinate whenever $y \notin F_2$.

In other words the family of measures $\{P_F : F \subseteq Y, F \text{ finite}\}$ is consistent.

Hence, by the Daniell-Kolmogorov theorem [1] there exists a unique probability measure P on $\mathcal{S}$ with the required properties.

b. The general case

Before proceeding to the proof of Theorem 2.1 we need a simple lemma.

Lemma 4.1 Let $f(y)$ be a complex-valued Haar-measurable function satisfying the following conditions:

- (1) $|f(y)| = 1$ for all $y \in Y$
- (2) $f(y) \cdot f(y') = f(y+y')$ a.e. $y, y' (H \times H)$.

Then there exists a unique $x \in X$ such that

$$f(y) = \langle x, y \rangle \quad \text{a.e. } y (H).$$

Proof Consider the set function

$$\mu(C) = \int_C f(y) dy$$

for all Haar-measurable sets contained in a compact set. If $C+y'$ denotes the translate of C by y' , we have from the invariance of Haar measure under translations,

$$\begin{aligned} \mu(C+y') &= \int_{C+y'} f(y) dy \\ &= \int_C f(y+y') dy \\ &= f(y') \mu(C) \quad \text{a.e. } y' (H). \end{aligned}$$

Choose and fix a compact set C such that $\mu(C) \neq 0$. Then

$$f(y) = \frac{\mu(C+y)}{\mu(C)} \quad \text{a.e. } y \in (H).$$

We shall now prove that $\mu(C+y)$ is continuous in y . Indeed, we have from condition (1) of the lemma

$$\begin{aligned} & |\mu(C+y_1) - \mu(C+y_2)| \\ &= \left| \int \chi_{C+y_1}(y) f(y) dy - \int \chi_{C+y_2}(y) f(y) dy \right| \\ &\leq \int |\chi_{C+y_1}(y) - \chi_{C+y_2}(y)| dy \\ &= \int |\chi_{C+y_1-y_2}(y) - \chi_C(y)| dy \end{aligned}$$

where χ_C denotes the indicator function of C . The last term on the right hand side of the above inequalities is simply the Haar measure of $(C+y_1-y_2) \Delta C$ which tends to zero as $y_1 - y_2$ approaches the identity.

Thus $f(y)$ is almost everywhere equal to a continuous function $g(y)$.

It follows from the conditions of the lemma that $|g(y)| = 1$ for all $y \in Y$ and $g(y) \cdot g(y') = g(y+y')$ for all $y, y' \in Y$. In other words g is a character on Y . Hence, by the duality theorem, there is a unique point $x \in X$ such that $g(y) = \langle x, y \rangle$ for all y . This completes the proof of the lemma.

Proof of Theorem 2.1

Adopting the notations described before the statement of corollary 3.1, we introduce the function ψ defined on $\mathcal{N}^B$ by

$$\psi(\underline{n}) = \varphi\left(\sum_{y \in Y} \underline{n}(y)y\right)$$

The positive definiteness of φ in Y implies the positive definiteness of ψ in $\mathcal{N}$. Further $\psi(\underline{0}) = \varphi(e) = 1$. Thus, by corollary 3.1, there exists a probability measure P on $\bar{\mathcal{S}}$ such that

$$\int_{K^Y} \exp\left[i \sum_{y \in Y} \underline{n}(y)\underline{\theta}(y)\right] dP(\underline{\theta}) = \psi(\underline{n}), \quad \underline{n} \in \mathcal{N}$$

Consider the probability space $(K^Y, \bar{\mathcal{S}}, P)$ where $\bar{\mathcal{S}}$ denotes the P -completion of $\mathcal{S}$. Treating the elements of Y as a time variable, define the stochastic process

$$z(y, \underline{\theta}) = \exp i \underline{\theta}(y), \quad y \in Y, \underline{\theta} \in K^Y.$$

For all $y_1, y_2, \dots, y_k \in Y$, integers $n_1, n_2, \dots, n_k$ and positive integers k , we have

$$(4.1) \quad E \prod_{j=1}^k z(y_j, \underline{\theta})^{n_j} = \varphi\left(\sum_{j=1}^k n_j y_j\right)$$

where E denotes expectation with respect to P . In particular,

$$(4.2) \quad E z(y, \underline{\theta}) z(y', \underline{\theta}) z(y+y', \underline{\theta})^{-1} = 1, \quad y, y' \in Y.$$

Since the random variable within the expectation sign above is of modulus unity, it follows that

$$(4.3) \quad z(y+y', \underline{\theta}) = z(y, \underline{\theta}) z(y', \underline{\theta}) \quad \text{a.e. } \underline{\theta} \quad (P)$$

for every pair $y, y' \in Y$. We also have

$$(4.4) \quad E z(y, \underline{\theta}) = \varphi(y), \quad y \in Y.$$

From (4.1) an easy computation gives

$$(4.5) \quad E |z(y, \underline{\theta}) - z(y', \underline{\theta})|^2 = 2 - \varphi(y-y') - \varphi(y'-y)$$

Since $\varphi(e) = 1$ and φ is continuous at the identity, the above equation shows that $z(y, \underline{\theta})$ is stochastically continuous in y . Therefore, by a theorem similar to Theorem 2.6, page 61 of Doob [1], we may assume that $z(y, \underline{\theta})$ is $H \times P$ -measurable. By applying Fubini's theorem to the function $|z(y, \underline{\theta}) z(y', \underline{\theta}) - z(y+y', \underline{\theta})|$ in the space $C \times C \times K^Y$ for every compact $C \subseteq Y$, we conclude that there exists a set $A \subseteq K^Y$ such that $P(A) = 1$, and for every $\underline{\theta} \in A$,

$$z(y, \underline{\theta}) z(y', \underline{\theta}) = z(y+y', \underline{\theta}) \quad \text{a.e. } y, y' \quad (H \times H)$$

It now follows from lemma 4.1 that there exists an element $x(\underline{\theta}) \in X$ such that

$$(4.6) \quad \langle x(\underline{\theta}), y \rangle = z(y, \underline{\theta}) \quad \text{a.e. } y \quad (H)$$

for every $\underline{\theta} \in \Lambda$. Define $x(\underline{\theta}) = e$, the identity of X , for every $\underline{\theta} \notin \Lambda$. Since $\langle x(\underline{\theta}), y \rangle$ is continuous in y for every $\underline{\theta}$, it follows that $\langle x(\underline{\theta}), y \rangle$ is P -measurable in $\underline{\theta}$ for every $y \in Y$. Since the smallest σ -field with respect to which all the characters are measurable, is the Borel σ -field in X , it follows that $x(\underline{\theta})$ is an X -valued random variable defined on $(K^Y, \overline{\mathcal{G}}, P)$. Let μ be the distribution of $x(\underline{\theta})$ in X .

Using Fubini's theorem, we deduce from equation (4.6) that

$$\int \langle x(\underline{\theta}), y \rangle dP(\underline{\theta}) = \int z(y, \underline{\theta}) dP(\underline{\theta}) \quad \text{a.e. } y \text{ (H)}$$

Since the left hand integral in the above equation is simply $\int \langle x, y \rangle d\mu(x)$, equation (4.4) gives

$$(4.7) \quad \int \langle x, y \rangle d\mu(x) = \varphi(y) \quad \text{a.e. } y \text{ (H)}.$$

Further, (4.4) and (4.5) imply the continuity of φ at all points of Y . Hence we have

$$(4.8) \quad \int \langle x, y \rangle d\mu(x) = \varphi(y), \quad y \in Y.$$

In order to prove the uniqueness of μ , choose and fix a dense subset $\{y_k\}$, $k = 1, 2, \dots$ of Y . Consider the map

$$T : x \rightarrow (\langle x, y_1 \rangle, \langle x, y_2 \rangle, \dots)$$

of X into the space K^{ω} which is a countable product of K . Since characters

are continuous and separate points, it follows that τ is a one-one and continuous map of X into K^∞ . Since X is a countable union of compact sets, it follows by considering the restriction of τ to these compact sets that τ^{-1} is measurable. In other words τ is a Borel isomorphism between X and the σ -compact set $\tau(X)$. If μ and ν are two measures satisfying (4.8), then $\mu\tau^{-1}$ and $\nu\tau^{-1}$ are measures in K^∞ with the same finite dimensional distributions. Further $\mu\tau^{-1}$ and $\nu\tau^{-1}$ are concentrated in $\tau(X)$. Therefore $\mu = \nu$. This completes the proof of Theorem 2.1.

5. The continuity theorem

For any probability measure μ defined on the Borel σ -field of X , the function φ defined on Y by the equation

$$\varphi(y) = \int \langle x, y \rangle d\mu(x)$$

is called the characteristic function of μ . It is easy to show that φ is continuous and positive definite. Bochner's theorem asserts that every continuous positive definite function arises in this way. Further the correspondence

$\mu \leftrightarrow \varphi$ is one-one. The uniform continuity of the characters implies the uniform continuity of φ . Further if a sequence of probability measures μ_n converges weakly to μ , then the corresponding characteristic functions φ_n converge uniformly over compact sets to the characteristic function φ of μ .

The following theorem asserts a much stronger version of the converse.

Theorem 5.1 Let $\{\mu_n\}$, $n = 1, 2, \dots$ be a sequence of probability measures defined on the Borel σ -field of X and $\{\varphi_n\}$, $n = 1, 2, \dots$ be the corresponding characteristic functions. If φ_n converges pointwise to a function φ which is continuous at the identity of X , then φ is the characteristic function of a probability measure μ and μ_n converges weakly to μ .

Proof First of all it is obvious that φ is a positive definite function on X and $\varphi(e) = 1$. Hence by Theorem 2.1 there exists a measure μ with φ as its characteristic function.

In order to prove the weak convergence consider the map T described in the last paragraph of section 4. T maps X into the space K^∞ . It is clear that the characteristic function of $\mu_n T^{-1}$ converges to the characteristic function of μT^{-1} . Since the set of all probability measures in K^∞ is weakly compact, it follows that $\mu_n T^{-1}$ converges weakly to μT^{-1} in K^∞ .

Choose and fix a sequence $\{C_j\}$ of pairwise disjoint Borel subsets of X whose union is X and such that the boundary of each C_j has μ -measure 0 and $\overline{C_j}$ is compact. Then for any closed set $F \subseteq X$, we have

$$\begin{aligned}
 (5.1) \quad \overline{\lim_{n \rightarrow \infty} \mu_n(F)} &= \overline{\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} \mu_n(F \cap C_j)} \\
 &\leq \sum_{j=1}^{\infty} \overline{\lim_{n \rightarrow \infty} \mu_n(F \cap \overline{C_j})}.
 \end{aligned}$$

Since T is a Borel isomorphism which maps compact subsets of X onto closed subsets of K^∞ and $\mu_n T^{-1}$ converges weakly to μT^{-1} , we have

$$\begin{aligned}
 (5.2) \quad \overline{\lim_{n \rightarrow \infty}} \mu_n(F \cap \bar{C}_j) &= \overline{\lim_{n \rightarrow \infty}} \mu_n T^{-1}[T(F \cap \bar{C}_j)] \\
 &\leq \mu T^{-1}[T(F \cap \bar{C}_j)] \\
 &= \mu(F \cap \bar{C}_j) = \mu(F \cap C_j)
 \end{aligned}$$

Combining (5.1) and (5.2) we have

$$\overline{\lim_{n \rightarrow \infty}} \mu_n(F) \leq \mu(F) .$$

Since F is an arbitrary closed set, it follows that μ_n converges weakly to μ . This completes the proof.

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