Publications mathématiques de l'I.H.É.S.

Burt Totaro

Euler characteristics for p-adic Lie groups

Publications mathématiques de l'I.H.É.S., tome 90 (1999), p. 169-225
<http://www.numdam.org/item?id=PMIHES_1999__90__169_0>

EULER CHARACTERISTICS FOR p-ADIC LIE GROUPS

by BURT TOTARO

Lazard [23] found definitive results about the cohomology of p-adic Lie groups such as $GL_n\mathbf{Z}_p$ with coefficients in vector spaces over $\mathbf{Q}_p$. These results, applied to the image of a Galois representation, have been used many times in number theory. It remains a challenge to understand the cohomology of p-adic Lie groups with integral coefficients, and especially to relate the integral cohomology of these groups to the cohomology of suitable Lie algebras over the p-adic integers $\mathbf{Z}_p$. In this paper, we do enough in this direction to compute a subtle version of the Euler characteristic, arising in the number-theoretic work of Coates and Howson ([14], [13]), for most of the interesting p-adic Lie groups.

The Euler characteristics considered in this paper have the following form. Let G be a compact p-adic Lie group with no p-torsion. Let M be a finitely generated $\mathbf{Z}_p$-module on which G acts, and suppose that the homology groups $H_i(G, M)$ are finite for all i. They are automatically 0 for i sufficiently large [27]. Then we want to compute the alternating sum of the p-adic orders of the groups $H_i(G, M)$:

$$\chi(G, M) := \sum_i (-1)^i \operatorname{ord}_p |H_i(G, M)|,$$

where $\operatorname{ord}_p(p^a) := a$. These Euler characteristics determine the analogous Euler characteristics for the cohomology of G with coefficients in a discrete "cofinitely generated" $\mathbf{Z}_p$-module such as $(\mathbf{Q}_p/\mathbf{Z}_p)^n$; see section 1 for details. If the module M is finite, Serre gave a complete calculation of these Euler characteristics in [29].

The first result on these Euler characteristics with M infinite is Serre's theorem that $\chi(G, M) = 0$ for any open subgroup G of $GL_2\mathbf{Z}_p$ with $p \geqslant 5$, where $M = (\mathbf{Z}_p)^2$ with the standard action of G [29]. This is the fact that Coates and Howson need for their formula on the Iwasawa theory of elliptic curves ([14], [13]). In fact, Serre's paper [29] and the later paper by Coates and Sujatha [15] prove the vanishing of similar Euler characteristics for many p-adic Lie groups other than open subgroups of $GL_2\mathbf{Z}_p$, but only for groups which are like $GL_2\mathbf{Z}_p$ in having an abelian quotient group of positive dimension. For example, it was not clear what to expect for open subgroups of $SL_2\mathbf{Z}_p$.

We find that the above Euler characteristic, for sufficiently small open subgroups of $SL_2\mathbf{Z}_p$, is equal to 0 for all $p \neq 3$ and to -2 for $p = 3$. We also compute the Euler characteristic of these groups with coefficients in a representation of SL_2 other than the standard one: it is again 0 except for finitely many primes p. The phenomenon simplifies curiously for larger groups (say, reductive groups of rank at least 2), as the following main theorem asserts: the Euler characteristic is 0 for all primes p and all representations of the group for which it makes sense. The proof of this theorem is completed at the end of section 9.

Theorem **0.1.** — *Let p be any prime number. Let* G *be a compact p-adic Lie group of dimension at least 2, and let* M *be a finitely generated $\mathbf{Z}_p$-module with* G*-action. Suppose that the homology of the Lie algebra $\mathfrak{g}_{\mathbf{Q}_p}$ of* G *acting on* $M \otimes \mathbf{Q}_p$ *is* 0*; this is equivalent to assuming that the homology of any sufficiently small open subgroup* G_0 *acting on* M *is finite, so that the Euler characteristic* $\chi(G_0, M)$ *is defined. (For $\mathfrak{g}_{\mathbf{Q}_p}$ reductive, this assumption is equivalent to the vanishing of the coinvariants of $\mathfrak{g}_{\mathbf{Q}_p}$ on* $M \otimes \mathbf{Q}_p$*.) Then the Euler characteristics* $\chi(G_0, M)$ *are the same for all sufficiently small open subgroups* G_0 *of* G *(that is, all open subgroups contained in a certain neighborhood of 1).*

The common value of these Euler characteristics is 0 *if every element of the Lie algebra $\mathfrak{g}_{\mathbf{Q}_p}$ has centralizer of dimension at least 2 (example: $\mathfrak{g}_{\mathbf{Q}_p}$ reductive of rank at least 2). Otherwise, there is an element of $\mathfrak{g}_{\mathbf{Q}_p}$ whose centralizer has dimension 1 (example: $\mathfrak{g}_{\mathbf{Q}_p} = \mathfrak{sl}_2\mathbf{Q}_p$), and then we give an explicit formula for the common value of the above Euler characteristics; in particular, this common value is not* 0 *for some choice of the module* M.

Remarks. — 1) The dimensions of centralizers play a similar role in the case of finite coefficient modules: if G is a compact p-adic Lie group with no p-torsion, then $\chi(G, M) = 0$ for all finite p-torsion G-modules M if and only if every element of G has centralizer of dimension at least 1, by Serre [29], Corollary to Theorem C.

2) There are simple sufficient conditions for G to be "sufficiently small" that $\chi(G, M)$ is equal to the value which we compute. For example, if M is a faithful representation of G, it suffices that G should act trivially on M/p if p is odd and on $M/4$ if $p = 2$. In fact, for the most natural p-adic Lie groups, we can avoid this assumption completely: Corollary 11.6 shows that $\chi(G, M) = 0$ for all compact open subgroups G of a reductive algebraic group of rank at least 2 when p is big enough. In particular, such groups G (including $SL_n\mathbf{Z}_p$, for example) need not be pro-p groups.

3) It is somewhat surprising that Euler characteristics of this type are the same for all sufficiently small open subgroups, given that G has dimension at least 2. Other types of Euler characteristics tend instead to be multiplied by r when passing from G to a subgroup H of finite index r. Of course, these two properties are the same when the Euler characteristics of G and H are both 0.

4) The theorem is false for G of dimension 1. In this case, for a given module M as above, G has an open subgroup isomorphic to $\mathbf{Z}_p$ such that

$$\chi(p^n\mathbf{Z}_p, M) = \chi(\mathbf{Z}_p, M) + n \dim(M \otimes \mathbf{Q}_p)$$

for all $n \geqslant 0$. That is, the Euler characteristics for open subgroups need not attain a common value when G has dimension 1.

The key to the proof of Theorem 0.1 is to relate the homology of p-adic Lie groups to the homology of Lie algebras. Lazard did so for homology with $\mathbf{Q}_p$

coefficients. There is more to be discovered about the relation between group homology and Lie algebra homology without tensoring with $\mathbf{Q}_p$, but at least we succeed in showing that the Euler characteristic of a p-adic Lie group (of dimension at least 2) is equal to the analogous Euler characteristic of some Lie algebra over $\mathbf{Z}_p$. The proof in sections 8 and 9 sharpens Lazard's proof that the group and the Lie algebra have the same rational cohomology, giving an explicit upper bound for the difference between the integral cohomology of the two objects. We compute these Euler characteristics for Lie algebras in sections 3 to 7, using in particular Kostant's theorem on the homology of the "upper-triangular" Lie subalgebra of a semisimple Lie algebra over a field of characteristic zero [22]. Sections 1 and 2 give some preliminary definitions and results.

The rest of the paper goes beyond Theorem 0.1 in several directions. First, using the general results we have developed on the integral homology of p-adic Lie groups, Theorem 10.1 computes the whole homology with nontrivial coefficients of congruence subgroups, not just the Euler characteristic. Section 11 extends the earlier arguments to prove the vanishing of Euler characteristics for many p-adic Lie groups which are not pro-p groups, namely open subgroups of a reductive group of rank at least 2. The proof uses that for sufficiently large prime numbers p, all pro-p subgroups of a reductive algebraic group are valued in the sense defined by Lazard. A sharper estimate of the primes p with this property is given in section 12, using the Bruhat-Tits structure theory of p-adic groups. Finally, section 13 shows that the results of section 11 on vanishing of Euler characteristics do not extend to open subgroups of $SL_2\mathbf{Z}_p$.

I am grateful for John Coates's questions about these Euler characteristics. Jean-Pierre Serre, Ottmar Venjakob, and the referee made useful suggestions on the exposition.

CONTENTS

1. Homology vs. cohomology

The main results of this paper are about the homology of groups or Lie algebras with coefficients in finitely generated $\mathbf{Z}_p$-modules. We will explain here how to deduce analogous results for cohomology, or for coefficients in a module of the form $(\mathbf{Q}_p/\mathbf{Z}_p)^n$.

Throughout the paper, a Lie algebra $\mathfrak{g}$ over a commutative ring R is always assumed to be a finitely generated free R-module. We use Cartan-Eilenberg [11], Chapter XIII, as a reference for the homology and cohomology of Lie algebras. The homology of a Lie algebra $\mathfrak{g}$ depends on the base ring R as well as on $\mathfrak{g}$, but (as is usual) we will not indicate that in the notation. One relation between homology and cohomology for Lie algebras is the naive duality:

Lemma **1.1.** — *For any* $\mathfrak{g}$*-module* M *and any injective* R*-module* I, *there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{R}}(\mathrm{H}_i(\mathfrak{g}, \mathrm{M}), \mathrm{I}) = \mathrm{H}^i(\mathfrak{g}, \mathrm{Hom}_{\mathrm{R}}(\mathrm{M}, \mathrm{I})).$$

Typical cases are $\mathrm{I} = \mathrm{R}$, when R is a field, and $\mathrm{I} = \mathbf{Q}_p/\mathbf{Z}_p$, when $\mathrm{R} = \mathbf{Z}_p$. Also, there is a canonical Poincaré duality isomorphism for any $\mathfrak{g}$-module M ([11], p. 288):

Lemma **1.2.**

$$\mathrm{H}_i(\mathfrak{g}, \mathrm{M}) \cong \mathrm{H}^{n-i}(\mathfrak{g}, \wedge^n \mathfrak{g} \otimes_{\mathrm{R}} \mathrm{M}).$$

Either of these lemmas can be used to translate the results of this paper about Lie algebras from homology to cohomology.

A reference for the homology of profinite groups G is Brumer [9]. For a prime number p, let $\mathbf{Z}_p\mathrm{G}$ denote the completed group ring of G over the p-adic integers,

$$\mathbf{Z}_p\mathrm{G} := \varprojlim \mathbf{Z}_p[\mathrm{G}/\mathrm{U}],$$

where U runs over the open normal subgroups of G. Define a pseudocompact $\mathbf{Z}_p\mathrm{G}$-module to be a topological G-module which is an inverse limit of discrete finite p-torsion G-modules. The category of pseudocompact $\mathbf{Z}_p\mathrm{G}$-modules is an abelian category with exact inverse limits and enough projectives. So we can define the homology groups $\mathrm{H}_*(\mathrm{G}, \mathrm{M})$ of a profinite group G with coefficients in a pseudocompact $\mathbf{Z}_p\mathrm{G}$-module M as the left derived functors of the functor $\mathrm{H}_0(\mathrm{G}, \mathrm{M}) = \mathrm{M}_{\mathrm{G}} := \mathrm{M}/\mathrm{I}(\mathrm{G})\mathrm{M}$, where $\mathrm{I}(\mathrm{G}) = \ker(\mathbf{Z}_p\mathrm{G} \to \mathbf{Z}_p)$. We have

$$\mathrm{H}_i(\mathrm{G}, \mathrm{M}) = \varprojlim \mathrm{H}_i(\mathrm{G}/\mathrm{U}, \mathrm{M}/\mathrm{I}(\mathrm{U})\mathrm{M}),$$

where U runs over the open normal subgroups of G ([9], Remark 1, p. 455). Furthermore, the category of pseudocompact $\mathbf{Z}_p$G-modules is dual, via Pontrjagin duality

$$M^* := \mathrm{Hom}_{\mathrm{cont}}(M, \mathbf{Q}_p/\mathbf{Z}_p),$$

to the category of discrete p-torsion G-modules ([9], Proposition 2.3, p. 448). The category of discrete p-torsion G-modules has enough injectives, and the cohomology of a profinite group G with coefficients in such a module can be defined as a right derived functor [28]. As a result, the homology theory of profinite groups G with coefficients in pseudocompact $\mathbf{Z}_p$G-modules is equivalent to the better-known cohomology theory with coefficients in discrete p-torsion G-modules, via Pontrjagin duality:

Lemma **1.3.**

$$H_i(G, M)^* = H^i(G, M^*).$$

So the main results of this paper, about Euler characteristics associated to the homology of a p-adic Lie group with coefficients in a finitely generated $\mathbf{Z}_p$-module, are equivalent to statements about the cohomology of such a group with coefficients in a discrete "cofinitely generated" $\mathbf{Z}_p$-module such as $(\mathbf{Q}_p/\mathbf{Z}_p)^n$.

2. Euler characteristics for Lie algebras

This section discusses some simpler situations where Euler characteristics can be shown to vanish. Most of the results and definitions here will be needed for the later results on Euler characteristics for p-adic Lie groups.

A simple fact in this direction is that a compact connected real Lie group G, viewed as a real manifold, has Euler characteristic 0 unless the group is trivial, in which case the Euler characteristic is 1. This fact can be reformulated as a statement about Euler characteristics in Lie algebra homology, by E. Cartan's theorem that

$$H_*(G, \mathbf{R}) = H_*(\mathfrak{g}, \mathbf{R})$$

for a compact connected Lie group G with Lie algebra $\mathfrak{g}$ over the real numbers [10]. There is a much more general vanishing statement about Euler characteristics of Lie algebras, as follows.

Proposition **2.1.** — *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k, and let* M *be a finite-dimensional representation of $\mathfrak{g}$. Then the Euler characteristic*

$$\chi(\mathfrak{g}, M) := \sum_i \dim_k H_i(\mathfrak{g}, M)$$

is equal to 0 *if $\mathfrak{g} \neq 0$, and to the dimension of* M *if $\mathfrak{g} = 0$.*

Proof. — For $\mathfrak{g}=0$, $H_0(\mathfrak{g}, M)=M$ and the higher homology is 0. For $\mathfrak{g}\neq 0$, we consider the standard complex which computes the Lie algebra homology $H_*(\mathfrak{g}, M)$ ([11], p. 282):

$$\to \wedge^2\mathfrak{g}\otimes_k M \to \mathfrak{g}\otimes_k M \to M \to 0,$$

where

$$d((x_1\wedge \ldots \wedge x_r)\otimes m) = \sum_i (-1)^i (x_1 \wedge \ldots \wedge \widehat{x_i} \wedge \ldots \wedge x_r)\otimes x_i m$$
$$+\sum_{i<j}(-1)^{i+j}([x_i, x_j]\wedge x_1 \wedge \ldots \wedge \widehat{x_i}\wedge \ldots \wedge \widehat{x_j}\wedge \ldots \wedge x_r)\otimes m$$

for $x_1, \ldots, x_r \in \mathfrak{g}$ and $m \in M$. Since $\mathfrak{g}$ and M are finite-dimensional, this is a bounded complex of finite-dimensional vector spaces. The basic fact about Euler characteristics is that, in this situation, the Euler characteristic of the homology of this complex (that is, of $H_*(\mathfrak{g}, M)$) is equal to the alternating sum of the dimensions of the vector spaces in the complex. Thus, if we let n be the dimension of $\mathfrak{g}$, the Euler characteristic is

$$\sum_{i=0}^{n}(-1)^i \binom{n}{i} \dim M,$$

which is 0 for $n > 0$. □

The same argument applies to the homology of Lie algebras over a discrete valuation ring Γ with coefficients in a Γ-module of finite length, as the following proposition says. We have in mind the case of a Lie algebra over the p-adic integers $\mathbf{Z}_p$ acting on a finite $\mathbf{Z}_p$-module. A $\mathbf{Z}_p$-module A is finite if and only if it has finite length, and in that case

$$\mathrm{ord}_p|A| = \mathrm{length}_{\mathbf{Z}_p} A.$$

Proposition **2.2.** — *Let* $\mathfrak{g}$ *be a Lie algebra over a discrete valuation ring* Γ, *and let* M *be a* Γ*-module of finite length on which* $\mathfrak{g}$ *acts. Then the Euler characteristic*

$$\chi(\mathfrak{g}, M) := \sum_i (-1)^i\, \mathrm{length}_\Gamma H_i(\mathfrak{g}, M)$$

is equal to 0 *if* $\mathfrak{g}\neq 0$, *and to* $\mathrm{length}_\Gamma M$ *if* $\mathfrak{g}=0$.

Proof. — For any $\mathfrak{g}$-module M, the homology groups $H_*(\mathfrak{g}, M)$ are the homology of the standard complex

$$\to \wedge^2\mathfrak{g}\otimes_\Gamma M \to \mathfrak{g}\otimes_\Gamma M \to M \to 0.$$

For M of finite length, as we assume, this is a bounded complex of Γ-modules of finite length. So the basic fact about Euler characteristics says that the Euler characteristic

$\chi(\mathfrak{g}, M)$ is equal to the alternating sum of the lengths of the Γ-modules in the complex, which is 0 for $\mathfrak{g} \neq 0$ by the same calculation as in the proof of Proposition 2.1. □

We now consider a more subtle situation, which is essentially the main topic of this paper. Let $\mathfrak{g}$ be a Lie algebra over a discrete valuation ring Γ, the case of interest being $\Gamma = \mathbf{Z}_p$. Let M be a finitely generated Γ-module on which $\mathfrak{g}$ acts, and suppose that $H_*(\mathfrak{g}, M) \otimes F = 0$, where F is the quotient field of Γ. Then the homology groups $H_i(\mathfrak{g}, M)$ are Γ-modules of finite length, and we can try to compute the Euler characteristic

$$\chi(\mathfrak{g}, M) := \sum_i (-1)^i \operatorname{length}_\Gamma H_i(\mathfrak{g}, M).$$

In this situation, the standard complex which computes $H_*(\mathfrak{g}, M)$,

$$\to \wedge^2 \mathfrak{g} \otimes_\Gamma M \to \mathfrak{g} \otimes_\Gamma M \to M \to 0,$$

does not consist of Γ-modules of finite length, and so the basic fact about Euler characteristics is not enough to determine $\chi(\mathfrak{g}, M)$.

We do, however, have the following results on "independence of M" and "independence of $\mathfrak{g}$." For part (2), we need to define relative Lie algebra homology for Lie algebras $\mathfrak{h} \subset \mathfrak{g}$ and a $\mathfrak{g}$-module M. Namely, let $H_*(\mathfrak{g}, \mathfrak{h}; M)$ be the homology of the mapping cone of the map of chain complexes $C_*(\mathfrak{h}, M) \to C_*(\mathfrak{g}, M)$ which compute the homology of $\mathfrak{h}$ and $\mathfrak{g}$. Then there is a long exact sequence

$$H_j(\mathfrak{h}, M) \to H_j(\mathfrak{g}, M) \to H_j(\mathfrak{g}, \mathfrak{h}; M) \to H_{j-1}(\mathfrak{h}, M).$$

Proposition **2.3.** — *Let $\mathfrak{g}$ be a Lie algebra over a discrete valuation ring Γ. Let* F *be the quotient field of* Γ.

1) *Suppose $\mathfrak{g} \neq 0$. Let M_1 and M_2 be $\mathfrak{g}$-modules, finitely generated over Γ, which become isomorphic after tensoring with* F. *Then $H_*(\mathfrak{g}, M_1) \otimes F = 0$ if and only if $H_*(\mathfrak{g}, M_2) \otimes F = 0$, and if either condition holds then*

$$\chi(\mathfrak{g}, M_1) = \chi(\mathfrak{g}, M_2).$$

2) *Suppose that $\mathfrak{g}$ has rank at least 2 as a free Γ-module. Let* M *be a $\mathfrak{g}$-module which is finitely generated over Γ. Let $\mathfrak{h} \subset \mathfrak{g}$ be an open Lie subalgebra, meaning a Lie subalgebra such that the Γ-module $\mathfrak{g}/\mathfrak{h}$ has finite length. Then the relative Lie algebra homology groups $H_*(\mathfrak{g}, \mathfrak{h}; M)$ have finite length as Γ-modules, and the corresponding Euler characteristic $\chi(\mathfrak{g}, \mathfrak{h}; M)$ is* 0. *It follows that $H_*(\mathfrak{g}, M) \otimes F = 0$ if and only if $H_*(\mathfrak{h}, M) \otimes F = 0$, and if either condition holds then*

$$\chi(\mathfrak{g}, M) = \chi(\mathfrak{h}, M).$$

The assumption that $\mathfrak{g}$ has rank at least 2 as a free Γ-module is essential in statement (2). Indeed, for the Lie algebra $\mathfrak{g} = \mathbf{Z}_p$ acting on a finitely generated

$\mathbf{Z}_p$-module M such that the space of coinvariants of $\mathfrak{g}$ on $M \otimes \mathbf{Q}_p$ is 0 (so that these Euler characteristics are defined), $\chi(p^n\mathfrak{g}, M)$ is equal to $\chi(\mathfrak{g}, M) + n \dim(M \otimes \mathbf{Q}_p)$, not to $\chi(\mathfrak{g}, M)$. A general calculation of $\chi(\mathfrak{g}, M)$ for $\mathfrak{g}$ isomorphic to $\mathbf{Z}_p$ can be found in Proposition 6.1.

Proof. — Since $H_*(\mathfrak{g}, M) \otimes F = H_*(\mathfrak{g}, M \otimes F)$, we have the first part of statement 1). Furthermore, we can multiply a given $\mathfrak{g}$-module isomorphism $M_1 \otimes F \to M_2 \otimes F$ by a suitable power of a uniformizer π of Γ to get a $\mathfrak{g}$-module homomorphism $M_1 \to M_2$ which becomes an isomorphism after tensoring with F. That is, the kernel and cokernel have finite length. Then 1) follows from Proposition 2.2.

Since $H_*(\mathfrak{g}, M) \otimes F = H_*(\mathfrak{g} \otimes F, M \otimes F)$, the vanishing of $H_*(\mathfrak{g}, \mathfrak{h}; M) \otimes F$ follows from the isomorphism $\mathfrak{h} \otimes F \cong \mathfrak{g} \otimes F$. So the Γ-modules $H_*(\mathfrak{g}, \mathfrak{h}; M)$ have finite length, and the Euler characteristic $\chi(\mathfrak{g}, \mathfrak{h}; M)$ is defined. Furthermore, Proposition 2.2 shows that $\chi(\mathfrak{g}, M_{\mathrm{tors}}) = \chi(\mathfrak{h}, M_{\mathrm{tors}}) = 0$, and so $\chi(\mathfrak{g}, \mathfrak{h}; M_{\mathrm{tors}}) = 0$. Therefore, to show that $\chi(\mathfrak{g}, \mathfrak{h}; M) = 0$, it suffices to show that $\chi(\mathfrak{g}, \mathfrak{h}; M/M_{\mathrm{tors}}) = 0$. That is, we can assume that the finitely generated Γ-module M is free.

The map of chain complexes $C_*(\mathfrak{h}, M) \to C_*(\mathfrak{g}, M)$ associated to the inclusion $\mathfrak{h} \subset \mathfrak{g}$ has the form $(\Lambda^*\mathfrak{h}) \otimes_\Gamma M \to (\Lambda^*\mathfrak{g}) \otimes_\Gamma M$. Since M is a finitely generated free Γ-module, this map is injective. So the relative Lie algebra homology $H_*(\mathfrak{g}, \mathfrak{h}; M)$ is the homology of the cokernel complex $C_*(\mathfrak{g}, \mathfrak{h}; M)$ of this map. Here $C_j(\mathfrak{g}, \mathfrak{h}; M)$ is a Γ-module of finite length with

$$\begin{aligned}\mathrm{length}_\Gamma C_j(\mathfrak{g}, \mathfrak{h}; M) &= \mathrm{length}_\Gamma(\Lambda^j(\mathfrak{g})/\Lambda^j(\mathfrak{h}))\ \mathrm{rank}_\Gamma M\\ &= \binom{n-1}{j-1} \mathrm{length}_\Gamma(\mathfrak{g}/\mathfrak{h})\ \mathrm{rank}_\Gamma M,\end{aligned}$$

where n denotes the rank of the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ as free Γ-modules. The formula for $\mathrm{length}_\Gamma(\Lambda^j(\mathfrak{g})/\Lambda^j(\mathfrak{h}))$ which I have used here applies to any inclusion of a free Γ-module $\mathfrak{h}$ of rank n into another, $\mathfrak{g}$, of the same rank. It follows from the special case where $\mathfrak{g}/\mathfrak{h} \cong \Gamma/\pi$. That special case can be proved by writing out a basis for $\Lambda^j(\mathfrak{g})/\Lambda^j(\mathfrak{h})$.

Since $C_*(\mathfrak{g}, \mathfrak{h}; M)$ is a bounded complex of Γ-modules of finite length, we have

$$\begin{aligned}\chi(\mathfrak{g}, \mathfrak{h}; M) &= \chi(C_*(\mathfrak{g}, \mathfrak{h}; M))\\ &= \sum_j (-1)^j \binom{n-1}{j-1} \mathrm{length}_\Gamma(\mathfrak{g}/\mathfrak{h})\ \mathrm{rank}_\Gamma M\\ &= 0,\end{aligned}$$

using the assumption that the dimension n of $\mathfrak{g}$ is at least 2.

The statements in Proposition 2.3(2) about the homology of $\mathfrak{g}$ and $\mathfrak{h}$ follow from those about $H_*(\mathfrak{g}, \mathfrak{h}; M)$ by the long exact sequence before the proposition. □

The following definition makes sense thanks to Proposition 2.3.

Definition **2.4.** — *Let* $\mathfrak{g}_{\mathbf{Q}_p}$ *be a Lie algebra of dimension at least 2 over* $\mathbf{Q}_p$. *Let* $\mathrm{M}_{\mathbf{Q}_p}$ *be a finite-dimensional* $\mathfrak{g}_{\mathbf{Q}_p}$*-module such that* $\mathrm{H}_*(\mathfrak{g}_{\mathbf{Q}_p}, \mathrm{M}_{\mathbf{Q}_p}) = 0$. *Define* $\chi_{\mathrm{fin}}(\mathfrak{g}_{\mathbf{Q}_p}, \mathrm{M}_{\mathbf{Q}_p})$ *to be the Euler characteristic* $\chi(\mathfrak{g}, \mathrm{M})$ *for any integral models* $\mathfrak{g}$ *and* M *of* $\mathfrak{g}_{\mathbf{Q}_p}$ *and* $\mathrm{M}_{\mathbf{Q}_p}$. *That is,* $\mathfrak{g}$ *is a Lie algebra over* $\mathbf{Z}_p$ *and* M *is a* $\mathfrak{g}$*-module, finitely generated as a* $\mathbf{Z}_p$*-module, such that tensoring up to* $\mathbf{Q}_p$ *gives the Lie algebra* $\mathfrak{g}_{\mathbf{Q}_p}$ *and its module* $\mathrm{M}_{\mathbf{Q}_p}$.

A slight extension of this definition is sometimes useful. Let K be a finite extension of the p-adic numbers, with ring of integers o_{K}. Let $\mathfrak{g}_{o_{\mathrm{K}}}$ be a Lie algebra over o_{K}, and let $\mathrm{M}_{o_{\mathrm{K}}}$ be a finitely generated o_{K}-module on which $\mathfrak{g}_{o_{\mathrm{K}}}$ acts. Suppose that $\mathrm{H}_*(\mathfrak{g}_{o_{\mathrm{K}}}, \mathrm{M}_{o_{\mathrm{K}}}) \otimes_{o_{\mathrm{K}}} \mathrm{K} = 0$. Then we can define an Euler characteristic, extending our earlier definition for Lie algebras over the p-adic integers, by

$$\chi(\mathfrak{g}_{o_{\mathrm{K}}}, \mathrm{M}_{o_{\mathrm{K}}}) = [\mathrm{K} : \mathbf{Q}_p]^{-1} \sum_j (-1)^j \operatorname{ord}_p |\mathrm{H}_j(\mathfrak{g}_{o_{\mathrm{K}}}, \mathrm{M}_{o_{\mathrm{K}}})|.$$

This rational number does not change if we tensor the Lie algebra $\mathfrak{g}_{o_{\mathrm{K}}}$ and the module $\mathrm{M}_{o_{\mathrm{K}}}$ with o_{L} for some larger p-adic field L, as a result of the flatness of o_{L} over o_{K}. Also, for a Lie algebra of rank at least 2 as an o_{K}-module, Proposition 2.3 shows that this number only depends on the Lie algebra and its module after tensoring with K, so we have an invariant $\chi_{\mathrm{fin}}(\mathfrak{g}_{\mathrm{K}}, \mathrm{M}_{\mathrm{K}})$. Combining this with the previous observation shows that the following invariant is well defined.

Definition **2.5.** — *Let* $\mathfrak{g}_{\overline{\mathbf{Q}}_p}$ *be a Lie algebra of dimension at least 2 over the algebraic closure of* $\mathbf{Q}_p$, *and let* $\mathrm{M}_{\overline{\mathbf{Q}}_p}$ *be a finite-dimensional* $\mathfrak{g}_{\overline{\mathbf{Q}}_p}$*-module such that* $\mathrm{H}_*(\mathfrak{g}_{\overline{\mathbf{Q}}_p}, \mathrm{M}_{\overline{\mathbf{Q}}_p}) = 0$. *Define* $\chi_{\mathrm{fin}}(\mathfrak{g}_{\overline{\mathbf{Q}}_p}, \mathrm{M}_{\overline{\mathbf{Q}}_p})$ *to be the rational number* $\chi(\mathfrak{g}_{o_{\mathrm{K}}}, \mathrm{M}_{o_{\mathrm{K}}})$ *for any models of* $\mathfrak{g}_{\overline{\mathbf{Q}}_p}$ *and* $\mathrm{M}_{\overline{\mathbf{Q}}_p}$ *over the ring of integers* o_{K} *of some finite extension* K *of* $\mathbf{Q}_p$.

3. Reductive Lie algebras in characteristic zero

The following lemma is a reformulation of the basic results on the cohomology of reductive Lie algebras in characteristic zero, due to Chevalley-Eilenberg [12] and Hochschild-Serre [20]. By definition, a finite-dimensional Lie algebra $\mathfrak{g}$ over a field K of characteristic zero is called reductive if $\mathfrak{g}$, viewed as a module over itself, is a direct sum of simple modules. Equivalently, $\mathfrak{g}$ is the direct sum of a semisimple Lie algebra and an abelian Lie algebra. Beware that if $\mathfrak{g}$ is reductive but not semisimple, finite-dimensional $\mathfrak{g}$-modules are not all direct sums of simple modules, contrary to what the name "reductive" suggests.

Lemma **3.1.** — *Let* $\mathfrak{g}$ *be a reductive Lie algebra over a field* K *of characteristic zero. Then any finite-dimensional* $\mathfrak{g}$*-module* M *splits canonically as a direct sum of modules all of whose simple subquotients are isomorphic. Also, if the space* $\mathrm{M}_{\mathfrak{g}}$ *of coinvariants or the space* $\mathrm{M}^{\mathfrak{g}}$ *of invariants is* 0, *then* $\mathrm{H}^*(\mathfrak{g}, \mathrm{M})$ *and* $\mathrm{H}_*(\mathfrak{g}, \mathrm{M})$ *are* 0.

Proof. — Let $\mathfrak{g}$ be a reductive Lie algebra over a field K of characteristic zero. Hochschild and Serre ([20], Theorem 10, p. 598), extending Chevalley and Eilenberg, showed that if M is a nontrivial simple $\mathfrak{g}$-module of finite dimension, then $H^*(\mathfrak{g}, M) = 0$. I will only describe the proof for $\mathfrak{g}$ semisimple. In that case, the Casimir operator in the center of the enveloping algebra $U\mathfrak{g}$ acts by 0 on the trivial module K, and by a nonzero scalar on every nontrivial simple module M; so

$$H^*(\mathfrak{g}, M) = \mathrm{Ext}^*_{U\mathfrak{g}}(K, M) = 0.$$

In particular, for $\mathfrak{g}$ reductive and a nontrivial simple $\mathfrak{g}$-module M, $H^1(\mathfrak{g}, M) = 0$, which says that there are no nontrivial extensions between the trivial $\mathfrak{g}$-module K and a nontrivial simple $\mathfrak{g}$-module. So every finite-dimensional $\mathfrak{g}$-module M splits canonically as the direct sum of a module with all simple subquotients trivial and a module with all simple subquotients nontrivial.

Therefore, if M is a finite-dimensional $\mathfrak{g}$-module with $M_{\mathfrak{g}} = 0$ or $M^{\mathfrak{g}} = 0$, then all simple subquotients of M are nontrivial, and so $H^*(\mathfrak{g}, M) = 0$ by Hochschild and Serre's theorem. The analogous statement for homology follows from naive duality, Lemma 1.1, which says that

$$H_i(\mathfrak{g}, M)^* = H^i(\mathfrak{g}, M^*),$$

where M^* denotes the dual of the vector space M. Finally, the splitting of any finite-dimensional $\mathfrak{g}$-module as a direct sum of modules all of which have the same simple subquotient follows from the vanishing of $\mathrm{Ext}^1_{\mathfrak{g}}(S_1, S_2) = H^1(\mathfrak{g}, \mathrm{Hom}_K(S_1, S_2))$ for any two non-isomorphic simple modules S_1 and S_2. That vanishing follows from what we have proved about the vanishing of cohomology, since

$$\begin{aligned} H^0(\mathfrak{g}, \mathrm{Hom}_K(S_1, S_2)) &= \mathrm{Hom}_{\mathfrak{g}}(S_1, S_2) \\ &= 0. \qquad \square \end{aligned}$$

4. The case of abelian Lie algebras

Let K be a finite extension of the field of p-adic numbers, with ring of integers o_K. We will show that the Euler characteristics we consider are 0 for any abelian Lie algebra over o_K of rank at least 2 as an o_K-module. (They are not 0 for a Lie algebra of rank 1 as an o_K-module, as I mentioned after the statement of Proposition 2.3.) This is the first step in a sequence of generalizations, the next step being Theorem 5.1 which proves the same vanishing for all reductive Lie algebras of rank at least 2.

Proposition **4.1.** — *Let $\mathfrak{h}$ be an abelian Lie algebra of the form $(o_K)^r$ for some $r \geqslant 2$. Let* M *be a finitely generated o_K-module with $\mathfrak{h}$-action such that $M_{\mathfrak{h}} \otimes K = 0$. Then the homology groups $H_*(\mathfrak{h}, M)$ are finite and the resulting Euler characteristic $\chi(\mathfrak{h}, M)$ (defined in section 2) is* 0.

Proof. — At first let $\mathfrak{h} = (o_K)^r$ for any r. Let $\mathfrak{h}_K = \mathfrak{h} \otimes_{o_K} K$. For any $\mathfrak{h}_{o_K}$-module M, finitely generated over o_K, such that the coinvariants of $\mathfrak{h}_K$ on M_K are 0, Lemma 3.1 shows that $H_*(\mathfrak{h}_K, M_K) = 0$. It follows that $H_j(\mathfrak{h}, M)$ is a finite o_K-module for all j.

Now suppose that $\mathfrak{h}$ has rank $r \geqslant 2$ as an o_K-module; we want to show that $\chi(\mathfrak{h}, M) = 0$. By Definition 2.5, which only makes sense for a Lie algebra of dimension at least 2, it is equivalent to show that $\chi_{\text{fin}}(\mathfrak{h}_{\overline{\mathbf{Q}}_p}, M_{\overline{\mathbf{Q}}_p}) = 0$, given that $(M_{\overline{\mathbf{Q}}_p})_{\mathfrak{h}_{\overline{\mathbf{Q}}_p}} = 0$. By Lemma 3.1, the assumption $(M_{\overline{\mathbf{Q}}_p})_{\mathfrak{h}_{\overline{\mathbf{Q}}_p}} = 0$ implies that $M_{\overline{\mathbf{Q}}_p}$ is an extension of nontrivial simple $\mathfrak{h}_{\overline{\mathbf{Q}}_p}$-modules; so it suffices to show that $\chi_{\text{fin}}(\mathfrak{h}_{\overline{\mathbf{Q}}_p}, M_{\overline{\mathbf{Q}}_p}) = 0$ for a nontrivial simple $\mathfrak{h}_{\overline{\mathbf{Q}}_p}$-module $M_{\overline{\mathbf{Q}}_p}$. Since $\mathfrak{h}_{\overline{\mathbf{Q}}_p}$ is abelian, such a module has dimension 1 by Schur's lemma. Changing the definition of the original Lie algebra $\mathfrak{h}$, we know that there is some p-adic field K and some models $\mathfrak{h}$ and M over o_K for $\mathfrak{h}_{\overline{\mathbf{Q}}_p}$ and the 1-dimensional module $M_{\overline{\mathbf{Q}}_p}$, and we are done if we can show that $\chi(\mathfrak{h}, M) = 0$.

Since $\mathfrak{h}$ has rank $r \geqslant 2$ as an o_K-module, there is a Lie subalgebra $\mathfrak{l} \subset \mathfrak{h}$ of rank $r - 1$ which is a direct summand as an o_K-module and which acts nontrivially on M. Since M has rank 1 over o_K, it follows that the coinvariants of $\mathfrak{l}$ on $M \otimes K$ are 0. So the homology groups $H_*(\mathfrak{l}, M)$ are finite as shown above. Consider the Hochschild-Serre spectral sequence

$$E^2_{ij} = H_i(\mathfrak{h}/\mathfrak{l}, H_j(\mathfrak{l}, M)) \Rightarrow H_{i+j}(\mathfrak{h}, M),$$

where $\mathfrak{h}/\mathfrak{l} \cong o_K$. We have $\chi(\mathfrak{h}/\mathfrak{l}, N) = 0$ for any finite $\mathfrak{h}/\mathfrak{l}$-module N, by Proposition 2.2, and so this spectral sequence shows that $\chi(\mathfrak{h}, M) = 0$. (This concluding argument is a version for Lie algebras of the argument in Coates-Sujatha about the Euler characteristic of a p-adic Lie group which maps onto $\mathbf{Z}_p$ [15].) □

5. The case of reductive Lie algebras

The rank of a reductive Lie algebra over a field K of characteristic zero is defined to be the dimension of the centralizer of a general element; the standard definition is equivalent ([6], Ch. VII, sections 2 and 4). The rank does not change under field extensions. For example, for the Lie algebra $\mathfrak{g}_K$ of a reductive algebraic group G over K, the rank of $\mathfrak{g}_K$ is the rank of G over the algebraic closure of K.

Theorem **5.1.** — *Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{Z}_p$ such that $\mathfrak{g}_{\mathbf{Q}_p}$ is reductive of rank at least 2. Let M be a finitely generated $\mathbf{Z}_p$-module with $\mathfrak{g}$-action such that the coinvariants of $\mathfrak{g}_{\mathbf{Q}_p}$ on $M_{\mathbf{Q}_p}$ are 0. Then the homology groups $H_*(\mathfrak{g}, M)$ are finite and the resulting Euler characteristic $\chi(\mathfrak{g}, M)$ is equal to 0.*

The optimal generalization of this statement is Theorem 7.1, which proves the same vanishing for all Lie algebras over $\mathbf{Z}_p$ in which every element has centralizer of dimension at least 2. See section 6 for the calculation of Euler characteristics, which are sometimes nonzero, for reductive Lie algebras of rank 1.

Proof. — We will prove the analogous statement for Lie algebras $\mathfrak{g}$ over the ring of integers o_{K} of any finite extension K of $\mathbf{Q}_p$, not just over $\mathbf{Z}_p$. The homology groups $\mathrm{H}_*(\mathfrak{g}, \mathrm{M})$ are finitely generated o_{K}-modules. Since we assume that $(\mathrm{M}_{\mathrm{K}})_{\mathfrak{g}_{\mathrm{K}}} = 0$, Lemma 3.1 gives that $\mathrm{H}_*(\mathfrak{g}_{\mathrm{K}}, \mathrm{M}_{\mathrm{K}}) = 0$, and so the homology groups $\mathrm{H}_*(\mathfrak{g}, \mathrm{M})$ are in fact finite o_{K}-modules. Thus, since $\mathfrak{g}$ has rank at least 2 as an o_{K}-module, the Euler characteristic $\chi_{\mathrm{fin}}(\mathfrak{g}_{\mathrm{K}}, \mathrm{M}_{\mathrm{K}})$ is defined by Definition 2.4, and we want to show that it is 0. Since this Euler characteristic is unchanged under finite extensions of K, according to Definition 2.5, we can extend the field K so as to arrange that the reductive Lie algebra $\mathfrak{g}_{\mathrm{K}}$ has a Borel subalgebra $\mathfrak{b}_{\mathrm{K}}$ defined over K ([6], Ch. VIII, section 3). By Lemma 3.1 again, the assumption that $(\mathrm{M}_{\mathrm{K}})_{\mathfrak{g}_{\mathrm{K}}} = 0$ implies that M_{K} is an extension of nontrivial simple $\mathfrak{g}_{\mathrm{K}}$-modules, so it suffices to show that $\chi_{\mathrm{fin}}(\mathfrak{g}_{\mathrm{K}}, \mathrm{M}_{\mathrm{K}}) = 0$ when M_{K} is a nontrivial simple $\mathfrak{g}_{\mathrm{K}}$-module.

Let $\mathfrak{u}_{\mathrm{K}}$ be the commutator subalgebra of the Borel subalgebra $\mathfrak{b}_{\mathrm{K}}$, so that $\mathfrak{b}_{\mathrm{K}}/\mathfrak{u}_{\mathrm{K}} \cong \mathrm{K}^r$ where r is the rank of $\mathfrak{g}$. We are assuming that r is at least 2. Let $\mathfrak{g}$ and M be models over o_{K}, which we take to be finitely generated free as o_{K}-modules, for $\mathfrak{g}_{\mathrm{K}}$ and M_{K}. Our goal is to show that $\chi(\mathfrak{g}, \mathrm{M}) = 0$. Let $\mathfrak{b} = \mathfrak{g} \cap \mathfrak{b}_{\mathrm{K}}$ and $\mathfrak{u} = \mathfrak{g} \cap \mathfrak{u}_{\mathrm{K}}$; these are Lie subalgebras of $\mathfrak{g}$ over o_{K}. The quotient Lie algebra $\mathfrak{b}/\mathfrak{u}$ is isomorphic to $(o_{\mathrm{K}})^r$, where r is at least 2.

To analyze $\mathrm{H}_*(\mathfrak{g}, \mathrm{M})$, we use two spectral sequences, both of homological type in the sense that the differential d_r has bidegree $(-r, r-1)$. First, there is the spectral sequence defined by Koszul and Hochschild-Serre for any subalgebra of a Lie algebra, which we apply to the integral Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ ([20], Corollary to Theorem 2, p. 594):

$$\mathrm{E}^1_{kl} = \mathrm{H}_l(\mathfrak{b}, \mathrm{M} \otimes_{o_{\mathrm{K}}} \Lambda^k(\mathfrak{g}/\mathfrak{b})) \Rightarrow \mathrm{H}_{k+l}(\mathfrak{g}, \mathrm{M}).$$

In fact, Hochschild and Serre construct the analogous spectral sequence for cohomology rather than homology, and only over a field, but the same construction works for the homology of Lie algebras over a commutative ring R when (as here) the inclusion $\mathfrak{b} \subset \mathfrak{g}$ is R-linearly split. Next we have the Hochschild-Serre spectral sequence associated to the ideal $\mathfrak{u} \subset \mathfrak{b}$:

$$\mathrm{E}^2_{ij} = \mathrm{H}_i(\mathfrak{b}/\mathfrak{u}, \mathrm{H}_j(\mathfrak{u}, \mathrm{M} \otimes_{o_{\mathrm{K}}} \Lambda^k(\mathfrak{g}/\mathfrak{b}))) \Rightarrow \mathrm{H}_{i+j}(\mathfrak{b}, \mathrm{M} \otimes_{o_{\mathrm{K}}} \Lambda^k(\mathfrak{g}/\mathfrak{b})).$$

Combining these two spectral sequences gives a formula for the Euler characteristic $\chi(\mathfrak{g}, \mathrm{M})$ which is correct if the right-hand side is defined (that is, if the homology groups of $\mathfrak{b}/\mathfrak{u}$ acting on the modules shown are finite):

$$\chi(\mathfrak{g}, \mathrm{M}) = \sum_{j,k} (-1)^{j+k} \chi(\mathfrak{b}/\mathfrak{u}, \mathrm{H}_j(\mathfrak{u}, \mathrm{M} \otimes \Lambda^k(\mathfrak{g}/\mathfrak{b}))).$$

Since $\mathfrak{b}/\mathfrak{u} \cong o_K^r$ with $r \geqslant 2$, Proposition 4.1 says that the Euler characteristic $\chi(\mathfrak{b}/\mathfrak{u}, N)$ is defined and equal to 0 for any finitely generated o_K-module N with $\mathfrak{b}/\mathfrak{u}$-action such that $(N_K)_{\mathfrak{b}/\mathfrak{u}} = 0$. So to show that $\chi(\mathfrak{g}, M) = 0$, as we want, it suffices to prove the following statement, which fortunately follows from Kostant's theorem [22].

Proposition 5.2. — *Let* $\mathfrak{g}$ *be a reductive Lie algebra over a field* K *of characteristic zero which has a Borel subalgebra* $\mathfrak{b}$ *defined over* K, *and let* $\mathfrak{u}$ *be the commutator subalgebra of* $\mathfrak{b}$. *Let* M *be a nontrivial simple* $\mathfrak{g}$*-module. Then*

$$H_*(\mathfrak{u}, M \otimes_K \wedge^*(\mathfrak{g}/\mathfrak{b}))_{\mathfrak{b}/\mathfrak{u}} = 0.$$

Proof. — When we extend scalars from K to its algebraic closure, the $\mathfrak{g}$-module M becomes a direct sum of nontrivial simple modules. So it suffices to prove the proposition for K algebraically closed. In this case, the center of $\mathfrak{g}$ acts on M by scalars, and acts trivially by conjugation on $\mathfrak{g}/\mathfrak{b}$ and on $\mathfrak{u}$. If the center of $\mathfrak{g}$ acts nontrivially on M, then it acts nontrivially by scalars on $H_*(\mathfrak{u}, M \otimes \wedge^*(\mathfrak{g}/\mathfrak{b}))$, and so the coinvariants of $\mathfrak{b}/\mathfrak{u}$ (which includes the center of $\mathfrak{g}$) on these groups are 0. Thus we can assume that the center of $\mathfrak{g}$ acts trivially on M. Then, replacing $\mathfrak{g}$ by its quotient by the center, we can assume that the Lie algebra $\mathfrak{g}$ is semisimple. In this case, there is a canonical simply connected algebraic group G over K with Lie algebra $\mathfrak{g}$. Let B be the Borel subgroup of G with Lie algebra $\mathfrak{b}$, and let $U \subset B$ be its unipotent radical. Choosing a maximal torus $T \subset B$, we define the negative roots to be the weights of T acting on $\mathfrak{u}$.

Kostant's theorem, which can be viewed as a consequence of the Borel-Weil-Bott theorem, determines the weights of the torus $B/U \cong T$ acting on $H_*(\mathfrak{u}, M)$ for any simple $\mathfrak{g}$-module M in characteristic zero [22]. The result is that the total dimension of $H_*(\mathfrak{u}, M)$ is always equal to the order of the Weyl group W. More precisely, let $\lambda \in X(T) = \mathrm{Hom}(T, G_m)$ be the highest weight of M, in the sense that all other weights of M are obtained from λ by repeatedly adding negative roots. Then, for any weight μ, the μ-weight subspace of $H_j(\mathfrak{u}, M)$ has dimension 1 if there is an element $w \in W$ such that j is the length of w and $\mu = w \cdot \lambda$; otherwise the μ-weight subspace of $H_j(\mathfrak{u}, M)$ is 0. Here the notation $w \cdot \lambda$ refers to the dot action of W on the weight lattice X(T):

$$w \cdot \lambda := w(\lambda + \rho) - \rho,$$

where ρ denotes half the sum of the positive roots ([21], p. 179).

Since the weights of T occurring in $\mathfrak{u}$ are exactly the negative roots, the weights occurring in $\wedge^*(\mathfrak{u})$ are exactly the sums of some set of negative roots. It follows easily that the set S of weights occurring in $\wedge^*(\mathfrak{u})$ is invariant under the dot action of W on the weight lattice X(T).

Clearly the intersection of S with the cone $X(T)^+$ of dominant weights is the single weight 0. So if λ is any nonzero dominant weight, then λ is not in S. Since S is invariant under the dot action of W on X(T), it follows that $(W \cdot \lambda) \cap S$ is empty. By Kostant's theorem, as stated above, it follows that for any nontrivial simple $\mathfrak{g}$-module M, the T-module $H_*(\mathfrak{u}, M)$ has no weights in common with $\wedge^*(\mathfrak{u})$.

The weights of T that occur in the B-module $\mathfrak{g}/\mathfrak{b}$ are the negatives of those that occur in the B-module $\mathfrak{u}$. So the previous paragraph implies that

$$\begin{aligned} H_*(\mathfrak{u}, M \otimes \mu)_{B/U} &= (H_*(\mathfrak{u}, M) \otimes \mu)_{B/U} \\ &= 0 \end{aligned}$$

for any weight μ occurring in $\wedge^*(\mathfrak{g}/\mathfrak{b})$. Since the simple B-modules are the 1-dimensional B/U-modules, the B-module $\wedge^*(\mathfrak{g}/\mathfrak{b})$ has a filtration with graded pieces the weights μ as above. It follows that

$$H_*(\mathfrak{u}, M \otimes \wedge^*(\mathfrak{g}/\mathfrak{b}))_{B/U} = 0.$$

Since we are in characteristic zero, it is equivalent to say that the coinvariants of the Lie algebra $\mathfrak{b}/\mathfrak{u}$ are 0. This proves Proposition 5.2 and hence Theorem 5.1. □

6. Euler characteristics for $\mathbf{Z}_p$ and $\mathfrak{sl}_2\mathbf{Z}_p$

Since Theorem 5.1 proves the vanishing of the Euler characteristic we are considering for reductive Lie algebras of rank at least 2, it is natural to ask what happens in rank 1. If $\mathfrak{g}$ is a Lie algebra over $\mathbf{Z}_p$ such that $\mathfrak{g}_{\mathbf{Q}_p}$ is reductive of rank 1, then $\mathfrak{g}_{\overline{\mathbf{Q}}_p}$ is isomorphic to $\overline{\mathbf{Q}}_p$ or $\mathfrak{sl}_2\overline{\mathbf{Q}}_p$. In this section, we determine the Euler characteristic for all representations of such a Lie algebra $\mathfrak{g}$. In particular, for the irreducible representation of $\mathfrak{sl}_2\mathbf{Z}_p$ of a given highest weight, the Euler characteristic is 0 for all but finitely many prime numbers p. For example, for the standard representation $M = (\mathbf{Z}_p)^2$ of $\mathfrak{g} = \mathfrak{sl}_2\mathbf{Z}_p$, the Euler characteristic $\chi(\mathfrak{g}, M)$ is 0 for $p \neq 3$ and -2 for $p = 3$.

Let us first compute Euler characteristics for a Lie algebra $\mathfrak{g}$ over $\mathbf{Z}_p$ such that $\mathfrak{g}_{\overline{\mathbf{Q}}_p} \cong \overline{\mathbf{Q}}_p$. Clearly $\mathfrak{g}$ is isomorphic to $\mathbf{Z}_p$.

Proposition **6.1.** — *Let* $\mathfrak{g}$ *be the Lie algebra of rank 1 as a* $\mathbf{Z}_p$*-module with generator* x*. Let* M *be a finitely generated* $\mathbf{Z}_p$*-module with* $\mathfrak{g}$*-action. Then the homology groups* $H_*(\mathfrak{g}, M)$ *are finite if and only if* $x \in \mathrm{End}(M)$ *is invertible on* $M \otimes \mathbf{Q}_p$*. If this is so, then*

$$\chi(\mathfrak{g}, M) = \mathrm{ord}_p(\det x)$$

where we view x *as an endomorphism of* $M \otimes \mathbf{Q}_p$.

Proof. — The only homology groups for $\mathfrak{g} \cong \mathbf{Z}_p$ acting on M are $H_0(\mathfrak{g}, M) = M_{\mathfrak{g}}$ and $H_1(\mathfrak{g}, M)$, which is isomorphic to $M^{\mathfrak{g}} \otimes_{\mathbf{Z}_p} \mathfrak{g}$ by Poincaré duality (Lemma 1.2). For any $\mathfrak{g}$-module M, these two groups tensored with $\mathbf{Q}_p$ are 0 if and only if $(M \otimes \mathbf{Q}_p)_{\mathfrak{g}} = 0$ and $(M \otimes \mathbf{Q}_p)^{\mathfrak{g}} = 0$, which means precisely that x is invertible on $M \otimes \mathbf{Q}_p$.

Suppose that the $\mathfrak{g}$-module M is a finitely generated $\mathbf{Z}_p$-module and that x is invertible on $M \otimes \mathbf{Q}_p$. To prove that $\chi(\mathfrak{g}, M) = \operatorname{ord}_p(\det x)$, it suffices to prove it when M is a finitely generated free $\mathbf{Z}_p$-module. Indeed, we have $\chi(\mathfrak{g}, N) = 0$ for every finite $\mathfrak{g}$-module N by Proposition 2.2, so that

$$\begin{aligned}\chi(\mathfrak{g}, M) &= \chi(\mathfrak{g}, M_{\mathrm{tors}}) + \chi(\mathfrak{g}, M/M_{\mathrm{tors}})\\ &= \chi(\mathfrak{g}, M/M_{\mathrm{tors}}).\end{aligned}$$

Given that M is a finitely generated free $\mathbf{Z}_p$-module, with x invertible on $M \otimes \mathbf{Q}_p$, we have $M^{\mathfrak{g}} \subset (M \otimes \mathbf{Q}_p)^{\mathfrak{g}} = 0$, and so $H_1(\mathfrak{g}, M) = 0$. Thus

$$\begin{aligned}\chi(\mathfrak{g}, M) &= \operatorname{ord}_p |H_0(\mathfrak{g}, M)|\\ &= \operatorname{ord}_p |M/xM|\\ &= \operatorname{ord}_p \det x. \quad \square\end{aligned}$$

Now let $\mathfrak{g}$ be a Lie algebra over $\mathbf{Z}_p$ such that $\mathfrak{g}_{\overline{\mathbf{Q}}_p} \cong \mathfrak{sl}_2\overline{\mathbf{Q}}_p$. It follows that $\mathfrak{g}_{\mathbf{Q}_p}$ is isomorphic either to $\mathfrak{sl}_2\mathbf{Q}_p$ or to the Lie algebra $\mathfrak{sl}_1 D = [D, D] \subset D$ over $\mathbf{Q}_p$ associated to the nontrivial quaternion algebra D over $\mathbf{Q}_p$. In any case, Definition 2.5 shows that, since $\dim \mathfrak{g} \geqslant 2$, the integer $\chi(\mathfrak{g}, M)$ (assuming $H_*(\mathfrak{g}, M) \otimes \mathbf{Q}_p = 0$) only depends on the module $M_{\overline{\mathbf{Q}}_p}$ for $\mathfrak{sl}_2\overline{\mathbf{Q}}_p$. Since $\mathfrak{sl}_2\overline{\mathbf{Q}}_p$ is semisimple over a field of characteristic zero, every finite-dimensional $\mathfrak{sl}_2\overline{\mathbf{Q}}_p$-module is a direct sum of simple modules. The simple modules are the symmetric powers $S^a V_{\overline{\mathbf{Q}}_p}$ of the standard module $V_{\overline{\mathbf{Q}}_p} = \overline{\mathbf{Q}}_p^2$, $a \geqslant 0$, and Lemma 3.1 shows that $H_*(\mathfrak{sl}_2\overline{\mathbf{Q}}_p, S^a V_{\overline{\mathbf{Q}}_p}) = 0$ if and only if $a > 0$. So we only need to compute the Euler characteristic $\chi_{\mathrm{fin}}(\mathfrak{sl}_2\overline{\mathbf{Q}}_p, S^a V_{\overline{\mathbf{Q}}_p})$ for the integers $a > 0$. The answer is:

Proposition **6.2.** — *For any prime number p and any positive integer a,*

$$\chi_{\mathrm{fin}}(\mathfrak{sl}_2\overline{\mathbf{Q}}_p, S^a V_{\overline{\mathbf{Q}}_p}) = 2(\operatorname{ord}_p a - \operatorname{ord}_p(a+2)).$$

Thus, for a given $a > 0$, this Euler characteristic is 0 for almost all prime numbers p, in particular for all $p > a + 2$.

Proof. — By Definition 2.5, it suffices to compute the Euler characteristic for a single model over $\mathbf{Z}_p$ of the Lie algebra and the module. We will compute $\chi(\mathfrak{sl}_2\mathbf{Z}_p, S^a V)$ where $V = (\mathbf{Z}_p)^2$ is the standard representation of $\mathfrak{sl}_2\mathbf{Z}_p$. It is possible to compute the homology groups of $\mathfrak{sl}_2\mathbf{Z}_p$ acting on $S^a V$ explicitly, but the actual homology groups are considerably more complicated than the Euler characteristic. We will therefore use another approach which gives the simple formula for the Euler characteristic more directly.

We will imitate, as far as possible, the proof of Theorem 5.1 that these Euler characteristics are 0 for reductive Lie algebras of rank at least 2. The difference is that for an abelian Lie algebra over $\mathbf{Z}_p$ of rank at least 2 as a $\mathbf{Z}_p$-module, the Euler characteristic is 0 when it is defined (Proposition 4.1), whereas this is not true for the Lie algebra $\mathbf{Z}_p$. We can instead use Proposition 6.1 to compute Euler characteristics for the Lie algebra $\mathbf{Z}_p$ explicitly.

Let $\mathfrak{g} = \mathfrak{sl}_2\mathbf{Z}_p$ and $M = S^aV$, where $a > 0$. Let $\mathfrak{b}$ be the subalgebra of upper-triangular matrices in $\mathfrak{g}$, and $\mathfrak{u}$ the subalgebra of strictly upper-triangular matrices in $\mathfrak{g}$. Using the spectral sequences

$$E^1_{kl} = H_l(\mathfrak{b}, M \otimes_{\mathbf{Z}_p} \Lambda^k(\mathfrak{g}/\mathfrak{b})) \Rightarrow H_{k+l}(\mathfrak{g}, M)$$

and

$$E^2_{ij} = H_i(\mathfrak{b}/\mathfrak{u}, H_j(\mathfrak{u}, M \otimes_{\mathbf{Z}_p} \Lambda^k(\mathfrak{g}/\mathfrak{b}))) \Rightarrow H_{i+j}(\mathfrak{b}, M \otimes_{\mathbf{Z}_p} \Lambda^k(\mathfrak{g}/\mathfrak{b})),$$

we derive a formula for the Euler characteristic $\chi(\mathfrak{g}, M)$ which is correct if the right-hand side is defined (that is, if the homology groups of $\mathfrak{b}/\mathfrak{u}$ acting on the modules shown are finite):

$$\chi(\mathfrak{g}, M) = \sum_{j,k} (-1)^{j+k} \chi(\mathfrak{b}/\mathfrak{u}, H_j(\mathfrak{u}, M \otimes_{\mathbf{Z}_p} \Lambda^k(\mathfrak{g}/\mathfrak{b}))).$$

Here $\mathfrak{b}/\mathfrak{u}$, $\mathfrak{u}$, and $\mathfrak{g}/\mathfrak{b}$ all have rank 1 as $\mathbf{Z}_p$-modules, so the sum runs over $0 \leqslant j, k \leqslant 1$. A moment's calculation shows that $\mathfrak{u} \subset \mathfrak{sl}_2\mathbf{Z}_p$ acts trivially on $\mathfrak{g}/\mathfrak{b}$, so that the formula can be rewritten as:

$$\chi(\mathfrak{g}, M) = \sum_{j,k} (-1)^{j+k} \chi(\mathfrak{b}/\mathfrak{u}, H_j(\mathfrak{u}, M) \otimes_{\mathbf{Z}_p} \Lambda^k(\mathfrak{g}/\mathfrak{b})).$$

By Kostant's theorem (as in the proof of Proposition 5.2), the homology groups $H_j(\mathfrak{u}, M) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ are 1-dimensional, and the standard generator

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of the Lie algebra $\mathfrak{b}/\mathfrak{u} \cong \mathbf{Z}_p$ acts on $H_0(\mathfrak{u}, M) \otimes \mathbf{Q}_p$ by multiplication by $-a$ and on $H_1(\mathfrak{u}, M) \otimes \mathbf{Q}_p$ by multiplication by $a+2$, where $M = S^aV$. Also, the generator H of $\mathfrak{b}/\mathfrak{u}$ acts on $\mathfrak{g}/\mathfrak{b} \cong \mathbf{Z}_p$ by multiplication by -2. As a result, using that $a > 0$, Proposition 6.1

shows that the Euler characteristics in the previous paragraph's formula are defined, and gives the following result:

$$\begin{aligned}\chi(\mathfrak{g}, M) &= \mathrm{ord}_p(-a) - \mathrm{ord}_p(-a-2) - \mathrm{ord}_p(a+2) + \mathrm{ord}_p(a)\\ &= 2(\mathrm{ord}_p a - \mathrm{ord}_p(a+2)). \quad \square\end{aligned}$$

7. Euler characteristics for arbitrary Lie algebras

In this section we show that for any Lie algebra $\mathfrak{g}$ over $\mathbf{Z}_p$ in which the centralizer of every element has dimension at least 2, the Euler characteristic $\chi(\mathfrak{g}, M)$ is 0 whenever it is defined (Theorem 7.1). Conversely, for any Lie algebra $\mathfrak{g}$ over $\mathbf{Z}_p$ in which the centralizer of some element has dimension 1, we compute $\chi(\mathfrak{g}, M)$ here whenever it is defined, in particular observing that this Euler characteristic is nonzero for some M (Theorem 7.4).

Theorem **7.1.** — *Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{Z}_p$ such that the centralizer of every element has dimension at least 2. Then the Euler characteristic $\chi(\mathfrak{g}, M)$ is 0 whenever it is defined, that is, for all finitely generated $\mathbf{Z}_p$-modules M with $\mathfrak{g}$-action such that $H_*(\mathfrak{g}, M) \otimes \mathbf{Q}_p = 0$.*

Proof. — The Lie algebra $\mathfrak{g}_{\mathbf{Q}_p}$ obtained by tensoring $\mathfrak{g}$ with $\mathbf{Q}_p$ clearly also has the property that the centralizer of every element has dimension at least 2. Its structure is described well enough for our purpose by the following lemma.

Lemma **7.2.** — *Let $\mathfrak{g}$ be a Lie algebra over a field K of characteristic zero such that the centralizer of every element has dimension at least 2. Then $\mathfrak{g}$ satisfies at least one of the following three properties.*

1) *$\mathfrak{g}$ maps onto a semisimple Lie algebra $\mathfrak{r}$ of rank at least 2.*

2) *$\mathfrak{g}$ maps onto a semisimple Lie algebra $\mathfrak{r}$ of rank 1 with some kernel $\mathfrak{u}$, and there is an element $x \in \mathfrak{g}$ whose image spans a (1-dimensional) Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{r}$ and whose centralizer in $\mathfrak{u}$ is not 0.*

3) *$\mathfrak{g}$ maps onto a 1-dimensional Lie algebra $\mathfrak{r} = \mathfrak{h}$ with some kernel $\mathfrak{u}$, and there is an element $x \in \mathfrak{g}$ whose image spans $\mathfrak{h}$ and whose centralizer in $\mathfrak{u}$ is not 0.*

Proof. — The quotient of $\mathfrak{g}$ by its maximal solvable ideal, called the radical rad($\mathfrak{g}$), is semisimple ([4], Ch. 5, section 2 and Ch. 6, section 1). If $\mathfrak{r} := \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ has rank at least 2 then we have conclusion (1). Suppose it has rank 1. Then there is an element x in $\mathfrak{g}$ whose image spans a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{r}$, these being 1-dimensional. Since a Cartan subalgebra in a semisimple Lie algebra is its own centralizer, the centralizer of x in $\mathfrak{g}$ is contained in the inverse image of $\mathfrak{h}$ in $\mathfrak{g}$, an extension of $\mathfrak{h}$ by $\mathfrak{u} := \mathrm{rad}(\mathfrak{g})$. Since the centralizer of x in $\mathfrak{g}$ has dimension at least 2, the centralizer of x in $\mathfrak{u}$ is not 0, thus proving (2).

Otherwise, $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is 0, which means that $\mathfrak{g}$ is solvable. Since the centralizer of 0 in $\mathfrak{g}$ has dimension at least 2, $\mathfrak{g}$ is not 0, and so it maps onto a 1-dimensional Lie algebra $\mathfrak{h}$ in some way. Let $\mathfrak{u}$ be the kernel. Let x be any element of $\mathfrak{g}$ whose image spans $\mathfrak{h}$. Since the centralizer of x in $\mathfrak{g}$ has dimension at least 2, the centralizer of x in $\mathfrak{u}$ is not 0, and we have conclusion (3). □

To prove Theorem 7.1, it suffices to show that $\chi(\mathfrak{g}, \mathrm{M}) = 0$ if $\mathfrak{g}_{\mathbf{Q}_p}$ satisfies any of the three conditions of Lemma 7.2. We first need the following lemma.

Lemma **7.3.** — *Let* $\mathfrak{g}$ *be a Lie algebra over a field* K *of characteristic zero which maps onto a reductive Lie algebra* $\mathfrak{r}$. *Let* $\mathfrak{u}$ *be the kernel. If* M *is a finite-dimensional* $\mathfrak{g}$*-module such that* $\mathrm{H}_*(\mathfrak{g}, \mathrm{M}) = 0$, *then* $\mathrm{H}_*(\mathfrak{r}, \mathrm{H}_*(\mathfrak{u}, \mathrm{M})) = 0$. *In particular, the coinvariants of* $\mathfrak{r}$ *on* $\mathrm{H}_*(\mathfrak{u}, \mathrm{M})$ *are* 0.

Proof. — Consider the Hochschild-Serre spectral sequence

$$\mathrm{E}^2_{ij} = \mathrm{H}_i(\mathfrak{r}, \mathrm{H}_j(\mathfrak{u}, \mathrm{M})) \Rightarrow \mathrm{H}_{i+j}(\mathfrak{g}, \mathrm{M}).$$

We are assuming that the E_∞ term of the spectral sequence is 0, and we want to show that the E_2 term is also 0. Let l be the smallest integer, if any, such that $\mathrm{H}_0(\mathfrak{r}, \mathrm{H}_l(\mathfrak{u}, \mathrm{M}))$ is not 0. The differential d_r in the spectral sequence has bidegree $(-r, r-1)$, so all differentials are 0 on this group since they would map to homology in negative degrees. Moreover, Lemma 3.1 shows that all the homology groups of the characteristic-zero reductive Lie algebra $\mathfrak{r}$ acting on $\mathrm{H}_j(\mathfrak{u}, \mathrm{M})$ are 0 for $j < l$. So no differentials in the spectral sequence can go into or out of $\mathrm{H}_0(\mathfrak{r}, \mathrm{H}_l(\mathfrak{u}, \mathrm{M}))$, contradicting the assumption that the E_∞ term of the spectral sequence is 0. So in fact $\mathrm{H}_0(\mathfrak{r}, \mathrm{H}_j(\mathfrak{u}, \mathrm{M}))$ is 0 for all j, and by Lemma 3.1 again it follows that the whole E_2 term of the spectral sequence is 0. □

We now prove Theorem 7.1 for $\mathfrak{g}_{\mathbf{Q}_p}$ satisfying condition (1) in Lemma 7.2. Let $\mathfrak{u}_{\mathbf{Q}_p}$ be the kernel of $\mathfrak{g}_{\mathbf{Q}_p} \to \mathfrak{r}_{\mathbf{Q}_p}$. Let $\mathfrak{r}$ be the image of the integral Lie algebra $\mathfrak{g}$ in $\mathfrak{r}_{\mathbf{Q}_p}$, and let $\mathfrak{u}$ be the intersection of $\mathfrak{g}$ with $\mathfrak{u}_{\mathbf{Q}_p}$. It follows from Lemma 7.3 that the integral Hochschild-Serre spectral sequence,

$$\mathrm{H}_*(\mathfrak{r}, \mathrm{H}_*(\mathfrak{u}, \mathrm{M})) \Rightarrow \mathrm{H}_*(\mathfrak{g}, \mathrm{M}),$$

has finite E_2 term, and in particular that the coinvariants of $\mathfrak{r}_{\mathbf{Q}_p}$ on $\mathrm{H}_j(\mathfrak{u}, \mathrm{M}) \otimes \mathbf{Q}_p$ are 0 for all j. Since $\mathfrak{r}_{\mathbf{Q}_p}$ is reductive of rank at least 2, Theorem 5.1 says that the Euler characteristic $\chi(\mathfrak{r}, \mathrm{H}_j(\mathfrak{u}, \mathrm{M}))$ is 0 for all j. Then the spectral sequence implies that $\chi(\mathfrak{g}, \mathrm{M}) = 0$.

Cases (2) and (3) can be treated at the same time. In both cases, $\mathfrak{g}_{\mathbf{Q}_p}$ maps onto a reductive Lie algebra $\mathfrak{r}_{\mathbf{Q}_p}$ of rank 1 with some kernel $\mathfrak{u}_{\mathbf{Q}_p}$, and there is an element x of $\mathfrak{g}$ whose image in $\mathfrak{r}_{\mathbf{Q}_p}$ spans a Cartan subalgebra $\mathfrak{h}_{\mathbf{Q}_p}$ and whose centralizer in

$\mathfrak{u}_{\mathbf{Q}_p}$ is not 0. Let $\mathfrak{r}$ be the image of the integral Lie algebra $\mathfrak{g}$ in $\mathfrak{r}_{\mathbf{Q}_p}$ and let $\mathfrak{u}$ be the kernel of $\mathfrak{g}$ mapping to $\mathfrak{r}_{\mathbf{Q}_p}$.

By the Hochschild-Serre spectral sequence for the extension of $\mathfrak{r}$ by $\mathfrak{u}$, we have

$$\chi(\mathfrak{g}, \mathrm{M}) = \sum_j (-1)^j \, \chi(\mathfrak{r}, \mathrm{H}_j(\mathfrak{u}, \mathrm{M}))$$

provided that the right-hand side makes sense. We are assuming that $\mathrm{H}_*(\mathfrak{g}_{\mathbf{Q}_p}, \mathrm{M}_{\mathbf{Q}_p}) = 0$. By Lemma 7.3, it follows that $\mathrm{H}_*(\mathfrak{r}_{\mathbf{Q}_p}, \mathrm{H}_*(\mathfrak{u}_{\mathbf{Q}_p}, \mathrm{M}_{\mathbf{Q}_p})) = 0$, which means that the right-hand side in the above formula does make sense. Moreover, by Proposition 2.3(1), for a finitely generated $\mathbf{Z}_p$-module N with $\mathfrak{r}$-action such that $\mathrm{H}_*(\mathfrak{r}_{\mathbf{Q}_p}, \mathrm{N}_{\mathbf{Q}_p}) = 0$, the Euler characteristic $\chi(\mathfrak{r}, \mathrm{N})$ only depends on $\mathrm{N}_{\mathbf{Q}_p}$ as an $\mathfrak{r}_{\mathbf{Q}_p}$-module (since $\mathfrak{r}$ has rank at least 1 as an o_{K}-module). In fact, it only depends on the class of $\mathrm{N}_{\mathbf{Q}_p}$ in the Grothendieck group $\mathrm{Rep}_{\neq 1}(\mathfrak{r}_{\mathbf{Q}_p})$ of $\mathfrak{r}_{\mathbf{Q}_p}$-modules with all simple subquotients nontrivial. (This works even for $\mathfrak{r}$ of rank 1 as an o_{K}-module, so that Proposition 2.3(2) does not apply, because we are fixing $\mathfrak{r} \subset \mathfrak{r}_{\mathbf{Q}_p}$ and only considering the dependence of these Euler characteristics on N.)

Thus we have a well-defined homomorphism

$$\chi : \mathrm{Rep}_{\neq 1}(\mathfrak{r}_{\mathbf{Q}_p}) \to \mathbf{Z},$$

and the above formula for $\chi(\mathfrak{g}, \mathrm{M})$ says that $\chi(\mathfrak{g}, \mathrm{M})$ is the image of the alternating sum $\chi(\mathfrak{u}_{\mathbf{Q}_p}, \mathrm{M}_{\mathbf{Q}_p})$, as an element of $\mathrm{Rep}_{\neq 1}(\mathfrak{r}_{\mathbf{Q}_p})$, under this homomorphism. So Theorem 7.1 is proved if we can show that the element $\chi(\mathfrak{u}_{\mathbf{Q}_p}, \mathrm{M}_{\mathbf{Q}_p})$ is 0 in the Grothendieck group $\mathrm{Rep}_{\neq 1}(\mathfrak{r}_{\mathbf{Q}_p})$. Since this Grothendieck group injects into the usual Grothendieck group $\mathrm{Rep}(\mathfrak{r}_{\mathbf{Q}_p})$, it suffices to show that $\chi(\mathfrak{u}_{\mathbf{Q}_p}, \mathrm{M}_{\mathbf{Q}_p})$ is 0 in the latter group. The Grothendieck group of $\mathfrak{r}_{\mathbf{Q}_p}$ injects into that of the Cartan subalgebra $\mathfrak{h}_{\mathbf{Q}_p} \subset \mathfrak{r}_{\mathbf{Q}_p}$ spanned by the given element $x \in \mathfrak{g}_{\mathbf{Q}_p}$, so it suffices to show that $\chi(\mathfrak{u}_{\mathbf{Q}_p}, \mathrm{M}_{\mathbf{Q}_p})$ is 0 in $\mathrm{Rep}(\mathbf{Q}_p x)$.

But here we can use the standard complex that defines Lie algebra homology to see that

$$\chi(\mathfrak{u}_{\mathbf{Q}_p}, \mathrm{M}_{\mathbf{Q}_p}) = \Big(\sum_j (-1)^j \wedge^j \mathfrak{u}_{\mathbf{Q}_p}\Big)\mathrm{M}_{\mathbf{Q}_p}$$

in the representation ring $\mathrm{Rep}(\mathbf{Q}_p x)$. We are given that x has nonzero centralizer in $\mathfrak{u}_{\mathbf{Q}_p}$, so $\mathfrak{u}_{\mathbf{Q}_p}$ is equal in the representation ring of $\mathbf{Q}_p x$ to $1 + \mathrm{V}$ for some representation V. The operation

$$\wedge_{-1}\mathrm{V} := \sum_j (-1)^j \wedge^j(\mathrm{V})$$

takes a representation V to an element of the corresponding Grothendieck group, transforming sums into products. Since $\wedge_{-1} 1 = 0$, it follows that $\wedge_{-1}\mathfrak{u}_{\mathbf{Q}_p} = 0$ in $\mathrm{Rep}(\mathbf{Q}_p x)$. Therefore $\chi(\mathfrak{u}_{\mathbf{Q}_p}, \mathrm{M}_{\mathbf{Q}_p}) = 0$ in $\mathrm{Rep}(\mathbf{Q}_p x)$, as we needed. $\square$

Theorem **7.4.** — *Let* $\mathfrak{g}$ *be a Lie algebra over* $\mathbf{Z}_p$ *which contains an element with centralizer of dimension 1. There is a formula (below) for the Euler characteristic* $\chi(\mathfrak{g}, M)$ *whenever it is defined, that is, for all finitely generated* $\mathbf{Z}_p$*-modules* M *with* $\mathfrak{g}$*-action such that* $H_*(\mathfrak{g}, M) \otimes \mathbf{Q}_p = 0$. *It is nonzero for some* M.

Proof. — The structure of $\mathfrak{g}_{\mathbf{Q}_p}$ is described by the following lemma.

Lemma **7.5.** — *Let* $\mathfrak{g}$ *be a Lie algebra over a field* K *of characteristic zero which contains an element with centralizer of dimension 1. Then* $\mathfrak{g}$ *satisfies at least one of the following two properties.*

1) $\mathfrak{g}$ *maps onto a 1-dimensional Lie algebra* $\mathfrak{h}$ *with some kernel* $\mathfrak{u}$, *and there is an element* $x \in \mathfrak{g}$ *whose image spans* $\mathfrak{h}$ *and whose centralizer in* $\mathfrak{u}$ *is* 0.

2) $\mathfrak{g}$ *maps onto a semisimple Lie algebra* $\mathfrak{r}$ *of rank 1 with some kernel* $\mathfrak{u}$, *and there is an element* $x \in \mathfrak{g}$ *whose image spans a (1-dimensional) Cartan subalgebra* $\mathfrak{h}$ *in* $\mathfrak{r}$ *and whose centralizer in* $\mathfrak{u}$ *is* 0.

Proof. — Since the dimension of the centralizer is upper-semicontinuous in the Zariski topology on $\mathfrak{g}$, the general element of $\mathfrak{g}$ has centralizer of dimension 1.

If $\mathfrak{g}$ is solvable, then it maps onto a 1-dimensional Lie algebra $\mathfrak{h}$ with some kernel $\mathfrak{u}$. Let x be a general element of $\mathfrak{g}$ in the sense that the image of x spans $\mathfrak{h}$ and the centralizer of x in $\mathfrak{g}$ has dimension 1. Then the centralizer of x in $\mathfrak{u}$ is 0, proving statement (1).

Otherwise, $\mathfrak{g}$ maps onto some nonzero semisimple Lie algebra $\mathfrak{r}$. If $\mathfrak{r}$ has rank at least 2, then every element of $\mathfrak{r}$ has centralizer of dimension at least 2 in $\mathfrak{r}$. It follows that the linear endomorphism ad x of $\mathfrak{g}$ has rank at most $\dim(\mathfrak{g}) - 2$ for all $x \in \mathfrak{g}$. So every x has centralizer of dimension at least 2 in $\mathfrak{g}$, contrary to our assumption. So $\mathfrak{r}$ has rank 1. Let x be a general element of $\mathfrak{g}$ in the sense that the image of x spans a (1-dimensional) Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{r}$ and the centralizer of x in $\mathfrak{g}$ has dimension 1. Then we have statement (2). □

In fact, we need to strengthen Lemma 7.5 to say that, in both cases (1) and (2), any element of $\mathfrak{g}$ whose image spans $\mathfrak{h}$ has centralizer in $\mathfrak{u}$ equal to 0. This is a consequence of Lemma 7.6, as follows. (In case (2), we apply Lemma 7.6 with $\mathfrak{g}$ replaced by the inverse image of $\mathfrak{h}$ in $\mathfrak{g}$.)

Lemma **7.6.** — *Let* $\mathfrak{g}$ *be a Lie algebra over a field* K *of characteristic zero which maps onto a 1-dimensional Lie algebra* $\mathfrak{h}$ *with some kernel* $\mathfrak{u}$. *If there is an element* x *of* $\mathfrak{g}$ *whose image spans* $\mathfrak{h}$ *and whose centralizer in* $\mathfrak{u}$ *is* 0, *then every element* y *of* $\mathfrak{g}$ *whose image spans* $\mathfrak{h}$ *has centralizer in* $\mathfrak{u}$ *equal to* 0. *Moreover, for any such element* y, *the element* $\Lambda_{-1}\mathfrak{u} = \sum_i (-1)^i \Lambda^i \mathfrak{u}$ *in the Grothendieck group* $\mathrm{Rep}(\mathrm{K}y)$ *is not* 0.

Proof. — We first need the following elementary lemma.

Lemma **7.7.** — *Let $a_1, \ldots, a_n$ be elements of a field* K *of characteristic zero. Let* S_{even} *be the set of sums* $\sum_{i \in I} a_i \in K$ *for subsets* $I \subset \{1, \ldots, n\}$ *of even order, and let* S_{odd} *be the analogous set of odd sums, both sets being considered with multiplicities. Then* $S_{\text{even}} = S_{\text{odd}}$ *if and only if* $a_i = 0$ *for some* i.

Proof. — If $a_i = 0$ for some i, then the bijection from the set of even subsets of $\{1, \ldots, n\}$ to the set of odd subsets by adding or removing the element i does not change the corresponding sum of a_j's. To prove the converse, we use the following identity of formal power series:

$$\begin{aligned}(e^{x_1} - 1) \ldots (e^{x_n} - 1) &= (x_1 + \ldots) \ldots (x_n + \ldots) \\ &= x_1 \ldots x_n + \text{terms of higher degree.}\end{aligned}$$

We can also write

$$(e^{x_1} - 1) \ldots (e^{x_n} - 1) = \sum_{j=0}^{n} (-1)^{n-j} \sum_{1 \leqslant i_1 < \ldots < i_j \leqslant n} e^{x_{i_1} + \ldots + x_{i_j}}.$$

Equating terms in degree n, we find that

$$\sum_{j=0}^{n} (-1)^{n-j} \sum_{1 \leqslant i_1 < \ldots < i_j \leqslant n} (x_{i_1} + \ldots + x_{i_j})^n = n!\, x_1 \ldots x_n.$$

This is now an identity of polynomials with integer coefficients. Plugging in the values $a_1, \ldots, a_n \in K$, we find that the left-hand side is 0, since the set (with multiplicities) of sums of an even number of the a_i's is equal to the corresponding set of sums for an odd number of the a_i's. So the right-hand side is 0. Since K has characteristic zero, $n!$ is not 0 in K, and so one of the a_i's is 0. □ (Lemma 7.7)

We can now prove Lemma 7.6. Since we have an element x of $\mathfrak{g}$ whose image spans $\mathfrak{h}$ and whose centralizer in $\mathfrak{u}$ is 0, the eigenvalues of x on $\mathfrak{u}$ (in a suitable extension field of K) are all nonzero. By Lemma 7.7, the set with multiplicities of even sums of the eigenvalues of x on $\mathfrak{u}$ is not equal to the set of odd sums. Equivalently, $\wedge_{-1}\mathfrak{u}$ is not zero in the Grothendieck group $\operatorname{Rep}(Kx)$.

But the complex computing Lie algebra homology shows that the element $\chi(\mathfrak{u}) := \sum_i (-1)^i H_i(\mathfrak{u}, K)$ in $\operatorname{Rep}(\mathfrak{h})$ can be identified with $\wedge_{-1}\mathfrak{u}$ in $\operatorname{Rep}(Kx)$. The point is that the exact sequence of Lie algebras $0 \to \mathfrak{u} \to \mathfrak{g} \to \mathfrak{h} \to 0$ determines an action of $\mathfrak{h}$ on the homology of $\mathfrak{u}$, and hence an element $\chi(\mathfrak{u})$ of $\operatorname{Rep}(\mathfrak{h})$, whereas we need to choose an element x giving a splitting of the exact sequence in order to get an action of $Kx \cong \mathfrak{h}$ on $\mathfrak{u}$ itself and hence to define $\wedge_{-1}\mathfrak{u}$ in $\operatorname{Rep}(Kx)$. Since $\wedge_{-1}\mathfrak{u}$ is nonzero in $\operatorname{Rep}(Kx)$, the element $\chi(\mathfrak{u})$ is nonzero in $\operatorname{Rep}(\mathfrak{h})$. So $\wedge_{-1}\mathfrak{u}$ is nonzero in $\operatorname{Rep}(Ky)$ for any element y of $\mathfrak{g}$ whose image spans $\mathfrak{h}$. By the easy direction of Lemma 7.7, it follows that the eigenvalues of y on $\mathfrak{u}$ are all nonzero. Equivalently, the centralizer of y in $\mathfrak{u}$ is 0. □ (Lemma 7.6)

We return to the proof of Theorem 7.4. If $\mathfrak{g}$ has rank 1 as a $\mathbf{Z}_p$-module, then the theorem follows from Proposition 6.1, so we can assume that $\mathfrak{g}$ has rank at least 2 as a $\mathbf{Z}_p$-module. Let $\mathfrak{r}_{\mathbf{Q}_p} = \mathfrak{h}_{\mathbf{Q}_p}$ in case (1). Then, in both cases (1) and (2) of Lemma 7.5, let $\mathfrak{r}$ be the image of the integral Lie algebra $\mathfrak{g}$ in the reductive quotient $\mathfrak{r}_{\mathbf{Q}_p}$ and let $\mathfrak{u}$ be the kernel of $\mathfrak{g}$ mapping to $\mathfrak{r}_{\mathbf{Q}_p}$.

Given a finitely generated $\mathbf{Z}_p$-module M with $\mathfrak{g}$-action such that $H_*(\mathfrak{g}, M) \otimes \mathbf{Q}_p = 0$, Lemma 7.3 shows that $H_*(\mathfrak{r}, H_*(\mathfrak{u}, M)) \otimes \mathbf{Q}_p = 0$ and in particular that the coinvariants of $\mathfrak{r}_{\mathbf{Q}_p}$ on $H_*(\mathfrak{u}, M) \otimes \mathbf{Q}_p$ are 0. Therefore the Euler characteristic $\chi(\mathfrak{g}, M)$ is given by the formula

$$\chi(\mathfrak{g}, M) = \sum_j (-1)^j \chi(\mathfrak{r}, H_j(\mathfrak{u}, M)).$$

As in the proof of Theorem 7.1, let $\mathrm{Rep}_{\neq 1}(\mathfrak{r}_{\mathbf{Q}_p})$ denote the Grothendieck group of $\mathfrak{r}_{\mathbf{Q}_p}$-modules with all simple subquotients nontrivial. Then the above formula says that $\chi(\mathfrak{g}, M)$ is the image of the alternating sum $\chi(\mathfrak{u}_{\mathbf{Q}_p}, M_{\mathbf{Q}_p})$, as an element of $\mathrm{Rep}_{\neq 1}(\mathfrak{r}_{\mathbf{Q}_p})$, under a homomorphism

$$\chi : \mathrm{Rep}_{\neq 1}(\mathfrak{r}_{\mathbf{Q}_p}) \to \mathbf{Z}.$$

Now the Lie algebra $\mathfrak{r}_{\mathbf{Q}_p}$ is either 1-dimensional or else semisimple of rank 1, so we have computed the homomorphism $\chi : \mathrm{Rep}_{\neq 1}(\mathfrak{r}_{\mathbf{Q}_p}) \to \mathbf{Z}$ in Propositions 6.1 and 6.2.

Thus, to complete the calculation of $\chi(\mathfrak{g}, M)$, it suffices to compute the element $\chi(\mathfrak{u}_{\mathbf{Q}_p}, M_{\mathbf{Q}_p})$ in $\mathrm{Rep}_{\neq 1}(\mathfrak{r}_{\mathbf{Q}_p})$. As in the proof of Theorem 7.1, this Grothendieck group injects into the usual Grothendieck group $\mathrm{Rep}(\mathfrak{r}_{\mathbf{Q}_p})$, so it suffices to compute $\chi(\mathfrak{u}_{\mathbf{Q}_p}, M_{\mathbf{Q}_p})$ as an element of the latter group. We can choose a splitting of the Lie algebra extension

$$0 \to \mathfrak{u}_{\mathbf{Q}_p} \to \mathfrak{g}_{\mathbf{Q}_p} \to \mathfrak{r}_{\mathbf{Q}_p} \to 0,$$

since $\mathfrak{r}_{\mathbf{Q}_p}$ is either 1-dimensional or semisimple. Given such a splitting, $\mathfrak{r}_{\mathbf{Q}_p}$ acts on $\mathfrak{u}_{\mathbf{Q}_p}$. We can then compute the element $\chi(\mathfrak{u}_{\mathbf{Q}_p}, M_{\mathbf{Q}_p})$ in the Grothendieck group $\mathrm{Rep}(\mathfrak{r}_{\mathbf{Q}_p})$ using the definition of Lie algebra homology via the standard complex:

$$\begin{aligned}\chi(\mathfrak{u}_{\mathbf{Q}_p}, M_{\mathbf{Q}_p}) &= \Big(\sum_j (-1)^j \Lambda^j \mathfrak{u}_{\mathbf{Q}_p}\Big) M_{\mathbf{Q}_p} \\ &= (\Lambda_{-1} \mathfrak{u}_{\mathbf{Q}_p}) M_{\mathbf{Q}_p}\end{aligned}$$

in the representation ring $\mathrm{Rep}(\mathfrak{r}_{\mathbf{Q}_p})$. In particular, we see that $\chi(\mathfrak{u}_{\mathbf{Q}_p}, M_{\mathbf{Q}_p})$ and hence $\chi(\mathfrak{g}, M)$ only depend on $\mathfrak{u}_{\mathbf{Q}_p}$ and $M_{\mathbf{Q}_p}$ as $\mathfrak{r}_{\mathbf{Q}_p}$-modules.

We regard this as a calculation of $\chi(\mathfrak{g}, M)$. To complete the proof of Theorem 7.4, we need to show that there is some $\mathfrak{g}$-module M, finitely generated over $\mathbf{Z}_p$, such that $\chi(\mathfrak{g}, M)$ is defined but not equal to 0. We know that these properties only depend on $M_{\mathbf{Q}_p}$ as a $\mathfrak{g}_{\mathbf{Q}_p}$-module. We will take M to be a representation of the quotient Lie algebra $\mathfrak{r}$, in the notation we have been using, so that $\mathfrak{r}_{\mathbf{Q}_p}$ is either 1-dimensional or else semisimple of rank 1. It is enough to find a representation M_{o_K} of $\mathfrak{r}_{o_K}$ for some finite extension K of $\mathbf{Q}_p$ such that $\chi(\mathfrak{g}_{o_K}, M_{o_K})$ is not zero. Indeed, we can then view M_{o_K} as a representation of $\mathfrak{g}$ over $\mathbf{Z}_p$. We have

$$H_*(\mathfrak{g}, M_{o_K}) = H_*(\mathfrak{g}_{o_K}, M_{o_K})$$

by inspection of the standard complex defining Lie algebra homology, and so $\chi(\mathfrak{g}, M_{o_K})$ is not zero, giving a representation of $\mathfrak{g}$ over $\mathbf{Z}_p$ with nonzero Euler characteristic as we want.

We first consider case (1) of Lemma 7.5, where $\mathfrak{r}_{\mathbf{Q}_p}$ has dimension 1. In this case, I claim that there is an $\mathfrak{r}_{o_K}$-module M_{o_K} which is free of rank 1 over o_K, for some finite extension K of $\mathbf{Q}_p$, such that $\chi(\mathfrak{g}_{o_K}, M_{o_K})$ is defined and not 0; that will prove the theorem in this case. Let x be an element of $\mathfrak{g}$ which maps to a generator of $\mathfrak{r} = \mathrm{im}(\mathfrak{g} \to \mathfrak{r}_{\mathbf{Q}_p})$, which is isomorphic to $\mathbf{Z}_p$. For a finite extension K of $\mathbf{Q}_p$, an $\mathfrak{r}_{o_K}$-module of rank 1 is defined by an element $b \in o_K$, which gives the action of the generator x.

By the general description of how to compute $\chi(\mathfrak{g}_{o_K}, M_{o_K})$ which we have given, we have

$$\begin{aligned}\chi(\mathfrak{g}_{o_K}, M_{o_K}) &= \sum_j (-1)^j \chi(\mathfrak{r}_{o_K}, \Lambda^j \mathfrak{u}_{o_K} \otimes M_{o_K}) \\ &= \sum_j (-1)^j \operatorname{ord}_p \det(x \mid \Lambda^j \mathfrak{u}_K \otimes M_K),\end{aligned}$$

provided that x is invertible on $\Lambda^j \mathfrak{u}_K \otimes M_K$ for all j. Here we are using Proposition 6.1. We choose the p-adic field K to be any one which contains the eigenvalues $a_1, \ldots, a_n$ of x on $\mathfrak{u}$; these are all in the ring of integers o_K, because $\mathfrak{u}$ is a finitely generated $\mathbf{Z}_p$-module. Then the eigenvalues of x on $\Lambda^j \mathfrak{u}_K \otimes M_K$ are the numbers

$$b + a_{i_1} + \ldots + a_{i_j},$$

for $1 \leqslant i_1 < \ldots < i_j \leqslant n$. In particular, these eigenvalues are all nonzero, for all $0 \leqslant j \leqslant n$, if we choose b outside finitely many values, as we now decide to do. Then x acts invertibly on $\Lambda^j \mathfrak{u}_K \otimes M_K$, and so the above formula for $\chi(\mathfrak{g}_{o_K}, M_{o_K})$ is justified.

The above formula then gives, more explicitly:

$$\chi(\mathfrak{g}_{o_K}, M_{o_K}) = \operatorname{ord}_p \prod_{j=0}^{n} \prod_{1 \leqslant i_1 < \ldots < i_j \leqslant n} (b + a_{i_1} + \ldots + a_{i_j})^{(-1)^j}.$$

Let $f(b)$ be the rational function of b in this formula, whose p-adic order is $\chi(\mathfrak{g}_{o_K}, M_{o_K})$. By Lemmas 7.5 and 7.6, we know that $\wedge_{-1}\mathfrak{u}$ is not zero in the Grothendieck group $\mathrm{Rep}(\mathbf{Q}_p x)$. Equivalently, the set with multiplicities of even sums of $a_1, \ldots, a_n$ is not equal to the set of odd sums, and so the rational function $f(b)$ is not constant. The zeros and poles of this rational function are in the ring of integers o_K. Taking a $b \in o_K$ which is close but not equal to one of these zeros or poles, we can arrange that $f(b)$ is not a p-adic unit. That is, for the rank-1 $\mathfrak{g}_{o_K}$-module M associated to b, $\chi(\mathfrak{g}_{o_K}, M_{o_K})$ is not 0. As mentioned earlier, it follows that $\chi(\mathfrak{g}, M_{o_K})$ is not 0, where M_{o_K} is viewed as a $\mathbf{Z}_p$-module. Theorem 7.4 is proved in case (1) of Lemma 7.5.

We now prove Theorem 7.4 in case (2) of Lemma 7.5. Here $\mathfrak{g}_{\mathbf{Q}_p}$ is an extension of a semisimple Lie algebra $\mathfrak{r}_{\mathbf{Q}_p}$ of rank 1 by another Lie algebra $\mathfrak{u}_{\mathbf{Q}_p}$. As mentioned earlier, we can fix a splitting of this extension, and then $\mathfrak{r}_{\mathbf{Q}_p}$ acts on $\mathfrak{u}_{\mathbf{Q}_p}$. Let $\mathfrak{h}_{\mathbf{Q}_p}$ be the Cartan subalgebra given by case (2) of Lemma 7.5. No matter which splitting of the extension we have chosen, Lemma 7.6 shows that $\wedge_{-1}\mathfrak{u}_{\mathbf{Q}_p}$ is nonzero in the Grothendieck group of $\mathfrak{h}_{\mathbf{Q}_p}$-modules. A fortiori, it is nonzero in the Grothendieck group of $\mathfrak{r}_{\mathbf{Q}_p}$-modules.

We want to find a $\mathfrak{g}$-module M, finitely generated over $\mathbf{Z}_p$, such that $\chi(\mathfrak{g}, M)$ is defined and not 0. Let K be a finite extension of $\mathbf{Q}_p$ such that $\mathfrak{r}_K$ is isomorphic to $\mathfrak{sl}_2 K$. It suffices to find a $\mathfrak{g}_{o_K}$-module M_{o_K}, finitely generated over o_K, such that $\chi(\mathfrak{g}_{o_K}, M_{o_K})$ is defined and not 0, in view of the isomorphism $H_*(\mathfrak{g}, M_{o_K}) = H_*(\mathfrak{g}_{o_K}, M_{o_K})$. We will take M_{o_K} to be a module over the quotient Lie algebra $\mathfrak{r}_{o_K}$ (the image of $\mathfrak{g}_{o_K}$ in $\mathfrak{r}_K$). It suffices to find an $\mathfrak{r}_{o_K}$-module M, finitely generated over o_K, such that $(\wedge^i \mathfrak{u}_K)M_K$ has no trivial summands as an $\mathfrak{r}_K$-module for all $0 \leqslant i \leqslant n$ and $(\wedge_{-1}\mathfrak{u}_K)M_K$ has nonzero image under the homomorphism

$$\chi : \mathrm{Rep}_{\neq 1}(\mathfrak{r}_K) \to \mathbf{Z}.$$

It is somewhat difficult to construct $\mathfrak{r}_{o_K}$-modules, even though we know that $\mathfrak{r}_K$ is isomorphic to $\mathfrak{sl}_2 K$. (For example, let H, X, Y be the usual basis vectors for $\mathfrak{sl}_2\mathbf{Q}_2$. Then the integral form of the Lie algebra $\mathfrak{sl}_2\mathbf{Q}_2$ which is spanned by H/2, X, and Y has no action on $(\mathbf{Z}_2)^2$ which gives the standard representation of $\mathfrak{sl}_2\mathbf{Q}_2$ after tensoring with $\mathbf{Q}_2$.) The obvious example of an $\mathfrak{r}_{o_K}$-module is the adjoint representation $\mathfrak{r}_{o_K}$; by taking symmetric powers of $\mathfrak{r}_{o_K}$ and decomposing over K, we find that for every $m \geqslant 0$, there is an $\mathfrak{r}_{o_K}$-module M, finitely generated over o_K, such that M_K is a simple module over $\mathfrak{r}_K \cong \mathfrak{sl}_2 K$ of highest weight $2m$. We can get more if the $\mathfrak{r}_K$-module $\mathfrak{u}_K$ has a summand with odd highest weight, since we know that $\mathfrak{u}_K$ comes from an $\mathfrak{r}_{o_K}$-module $\mathfrak{u}_{o_K}$ which is finitely generated over o_K. In that case, by tensoring $\mathfrak{u}_{o_K}$ repeatedly with $\mathfrak{r}_{o_K}$ and decomposing over K (using the Clebsch-Gordan formula, as stated below), we find that every simple $\mathfrak{r}_K$-module comes from an $\mathfrak{r}_{o_K}$-module which is finitely generated over o_K. To sum up, let c be 1 if the $\mathfrak{r}_K$-module $\mathfrak{u}_K$ has a summand with odd highest weight, and 2 otherwise; then we have shown that for every $m \geqslant 0$ there is an $\mathfrak{r}_{o_K}$-module M_{o_K}, finitely generated over o_K, such that M_K is a simple module over $\mathfrak{r}_K \cong \mathfrak{sl}_2K$ of highest weight cm.

We have thereby reduced to the following question. Let V be the standard 2-dimensional representation of $\mathfrak{sl}_2K$. Given that $\Lambda_{-1}\mathfrak{u}_K = \sum_i (-1)^i \Lambda^i \mathfrak{u}_K$ is nonzero in the representation ring Rep($\mathfrak{sl}_2K$), find an integer $m \geqslant 0$ such that $(\Lambda^i\mathfrak{u}_K)S^{cm}V$ has no trivial summands for $0 \leqslant i \leqslant \dim \mathfrak{u}_K$ and $(\Lambda_{-1}\mathfrak{u}_K)S^{cm}V$ has nonzero image under the homomorphism

$$\chi : \mathrm{Rep}_{\neq 1}(\mathfrak{sl}_2K) \to \mathbf{Z}.$$

The Clebsch-Gordan formula for representations of $\mathfrak{sl}_2K$ says that

$$S^aV \cdot S^bV = S^{a-b}V + S^{a-b+2}V + \ldots + S^{a+b}V$$

for $0 \leqslant b \leqslant a$. This makes it clear that for $0 \leqslant i \leqslant \dim \mathfrak{u}_K$, $(\Lambda^i\mathfrak{u}_K)S^{cm}V$ has no trivial summands for m sufficiently large. Now let j be the largest natural number such that the multiplicity of S^jV in $\Lambda_{-1}\mathfrak{u}_K$ is not zero; there is such a j, since $\Lambda_{-1}\mathfrak{u}_K$ is not 0. If we can choose m such that $j + cm + 2$ is divisible by a sufficiently large power of p, then $(\Lambda_{-1}\mathfrak{u}_K)S^{cm}V$ has nonzero image under the homomorphism

$$\chi : \mathrm{Rep}_{\neq 1}(\mathfrak{sl}_2K) \to \mathbf{Z},$$

by the Clebsch-Gordan formula together with the formula for that homomorphism in Proposition 6.2:

$$\chi(S^aV) = 2(\mathrm{ord}_p a - \mathrm{ord}_p(a+2)).$$

There is no trouble choosing such an m if p is odd. If $p = 2$, we can do it unless j is odd and $c = 2$. But that cannot happen, since $c = 2$ means that the highest weights of $\mathfrak{r}_K \cong \mathfrak{sl}_2K$ on $\mathfrak{u}_K$ are all even, which would imply that the weights j occurring in $\Lambda_{-1}\mathfrak{u}_K$ were also even. So we can always find an m as needed, proving Theorem 7.4. □

8. Filtered and graded algebras

In section 9, we will explain how to relate Euler characteristics for Lie algebras over the p-adic integers to Euler characteristics for p-adic Lie groups. In this section we develop the homological algebra needed for that proof. In particular, we need the spectral sequence defined under various hypotheses by Serre ([26], p. II-17) and May [24], relating Tor over a filtered ring to Tor over the associated graded ring. We set up the spectral sequence here under fairly weak hypotheses. We also need a relative version of that spectral sequence.

We begin with some general homological definitions. For any ring S, we have the groups $\mathrm{Tor}_j^{\mathrm{S}}(\mathrm{A}, \mathrm{B})$ for any right S-module A and left S-module B. Given a ring homomorphism $\mathrm{R} \to \mathrm{S}$, we can view A and B as R-modules as well and consider the resulting Tor groups. Our first step is to define relative groups $\mathrm{Tor}_j^{\mathrm{S},\mathrm{R}}(\mathrm{A}, \mathrm{B})$ in this situation which fit into a long exact sequence

$$\mathrm{Tor}_j^{\mathrm{R}}(\mathrm{A}, \mathrm{B}) \to \mathrm{Tor}_j^{\mathrm{S}}(\mathrm{A}, \mathrm{B}) \to \mathrm{Tor}_j^{\mathrm{S},\mathrm{R}}(\mathrm{A}, \mathrm{B}) \to \mathrm{Tor}_{j-1}^{\mathrm{R}}(\mathrm{A}, \mathrm{B}).$$

To do this, let R_* be a free resolution of A as a right R-module, and let S_* be a free resolution of A as a right S-module. Since R_* is a complex of projective R-modules, there is an R-linear homomorphism of chain complexes $\mathrm{R}_* \to \mathrm{S}_*$, unique up to homotopy, which gives the identity map from $\mathrm{H}_0(\mathrm{R}_*) = \mathrm{A}$ to $\mathrm{H}_0(\mathrm{S}_*) = \mathrm{A}$. This homomorphism determines a $\mathbf{Z}$-linear homomorphism of chain complexes from $\mathrm{R}_* \otimes_{\mathrm{R}} \mathrm{B}$ to $\mathrm{S}_* \otimes_{\mathrm{S}} \mathrm{B}$. We define $\mathrm{Tor}_*^{\mathrm{S},\mathrm{R}}(\mathrm{A}, \mathrm{B})$ to be the homology of the mapping cone of the map of chain complexes $\mathrm{R}_* \otimes_{\mathrm{R}} \mathrm{B} \to \mathrm{S}_* \otimes_{\mathrm{S}} \mathrm{B}$. These groups $\mathrm{Tor}_*^{\mathrm{S},\mathrm{R}}(\mathrm{A}, \mathrm{B})$ fit into a long exact sequence as we wanted.

We now turn to the spectral sequence which relates Tor over filtered rings to Tor over the associated graded ring. We will need to modify this spectral sequence to apply to the above relative Tor groups. The proof below is essentially Serre's argument in [26], p. II-17.

Proposition **8.1.** — *Let Ω be a complete filtered commutative ring, $\Omega = \Omega^0 \supset \Omega^1 \supset \ldots$, with* gr Ω *noetherian. Let* R *be a complete filtered Ω-algebra,* $\mathrm{R} = \mathrm{R}^0 \supset \mathrm{R}^1 \supset \ldots$, *with* gr R *right noetherian. Let* A *be a complete filtered right* R*-module with* gr A *finitely generated over* gr R, *and let* B *be a complete filtered left* R*-module with* gr B *finitely generated as a* gr Ω*-module (not just as a* gr R*-module). Then there is a spectral sequence*

$$\mathrm{E}_{ij}^1 = \mathrm{Tor}_{i+j}^{\mathrm{gr}\,\mathrm{R}}(\mathrm{gr}\ \mathrm{A}, \mathrm{gr}\ \mathrm{B})_{\mathrm{degree}\ -i} \Rightarrow \mathrm{Tor}_{i+j}^{\mathrm{R}}(\mathrm{A}, \mathrm{B}).$$

This is a homological spectral sequence, meaning that the differential d_r has bidegree $(-r, r-1)$ for $r \geqslant 1$. The groups $\mathrm{Tor}_*^{\mathrm{R}}(\mathrm{A}, \mathrm{B})$ *are finitely generated Ω-modules, complete with respect to a filtration whose associated graded groups are the* E_∞ *term of the spectral sequence.*

In our applications, Ω will be $\mathbf{Z}_p$, and gr A and gr B will both be finitely generated gr Ω-modules. We state Proposition 8.1 under the above weaker (asymmetrical) assumptions only because the proof happens to work that way.

Proof. — Since gr R is noetherian and gr A is finitely generated over gr R, there is a resolution G_* of the graded module gr A over gr R by finitely generated free graded modules. We will use completeness of R and A to lift G_* to a filtered free resolution R_* of the filtered module A over R. Serre's Lemma V.2.1.1, p. 545 in [23], is closely related, but we will prove what we need directly. For each $i \geqslant 0$, let R_i be a filtered free R-module with generators in degrees so that gr $R_i \cong G_i$. The surjection $G_0 \to$ gr A lifts to a filtered R-linear map $R_0 \to A$ by freeness of R_0. It is surjective, by the following lemma.

Lemma **8.2.** — *Let* $f\colon A \to B$ *be a homomorphism of filtered abelian groups,* $A = A^0 \supset A^1 \supset \dots$ *and* $B = B^0 \supset B^1 \supset \dots$. *Suppose that* A *is complete* ($A \to \varprojlim A/A^n$ *is an isomorphism),* B *is separated* ($B \to \varprojlim B/B^n$ *is injective) and* gr A $\to$ gr B *is surjective. Then* $A \to B$ *is surjective and* B *is complete.*

Proof (repeated from [23], Prop. I.2.3.13, p. 415). — For any $b \in \varprojlim B/B^n$, we can use surjectivity of gr A $\to$ gr B to define an element $a \in \varprojlim A/A^n$, step by step, which maps to b in $\varprojlim B/B^n$. That is, $\varprojlim A/A^n \to \varprojlim B/B^n$ is surjective. Since $A \to \varprojlim A/A^n$ is an isomorphism, the map $A \to \varprojlim B/B^n$ is surjective. Therefore $B \to \varprojlim B/B^n$ is surjective as well as injective, so B is complete. It also follows that $A \to B$ is surjective. □

We continue the proof of Proposition 8.1. By Lemma 8.2, the lift $R_0 \to A$ is surjective. Suppose, inductively, that we have defined an exact sequence of filtered R-modules

$$R_i \to \dots \to R_0 \to A \to 0$$

which lifts the exact sequence

$$G_i \to \dots \to G_0 \to \mathrm{gr}\ A \to 0.$$

Let $K_j = \ker(R_j \to R_{j-1})$ for $0 \leqslant j \leqslant i$, with its filtration as a submodule of R_j, and let $K_{-1} = A$ with its given filtration. Then the natural map gr $R_i \to$ gr K_{i-1} is surjective; this is clear for $i = 0$, and for $i > 0$ it follows from injectivity of the map gr $K_{i-1} \to$ gr R_{i-1} and surjectivity of the map

$$\mathrm{gr}\ R_i \to \ker(\mathrm{gr}\ R_{i-1} \to \mathrm{gr}\ R_{i-2}).$$

We have an exact sequence of filtered R-modules,

$$0 \to K_i \to R_i \to K_{i-1} \to 0.$$

Here K_i has the filtration induced from R_i by definition. Moreover, surjectivity of gr $R_i \to$ gr K_{i-1} implies that the filtration of K_{i-1} is also the one induced from R_i, that is, that $(R_i)^j \to (K_{i-1})^j$ is surjective for all j; use Lemma 8.2 to prove this, noting that R_i is complete since it is a finitely generated free filtered R-module. It follows that the sequence

$$0 \to \mathrm{gr}\ K_i \to \mathrm{gr}\ R_i \to \mathrm{gr}\ K_{i-1} \to 0$$

is exact. Since $\mathrm{gr}\ K_{i-1} \subset \mathrm{gr}\ R_{i-1}$ by definition of the filtration on K_{i-1}, it follows that $\mathrm{gr}\ K_i = \ker(\mathrm{gr}\ R_i \to \mathrm{gr}\ R_{i-1})$. So we have a surjection $\mathrm{gr}\ R_{i+1} \to \mathrm{gr}\ K_i$, which we can lift to a filtered R-linear map $R_{i+1} \to K_i$. This map is surjective by Lemma 8.2. So we have an exact sequence

$$R_{i+1} \to R_i \to \dots \to R_0 \to A \to 0$$

of filtered R-modules, lifting the exact sequence

$$G_{i+1} \to G_i \to \dots \to G_0 \to \mathrm{gr}\ A \to 0.$$

This completes the induction. Thus, we have shown that G_* lifts to a filtered free resolution R_* of the complete filtered right R-module A.

In Proposition 8.1, we are also given a complete filtered left R-module B with gr B finitely generated as a gr Ω-module. Then $R_* \otimes_R B$ is a filtered complex of Ω-modules, with homology equal to $\mathrm{Tor}_*^R(A, B)$. Its associated graded complex is $G_* \otimes_{\mathrm{gr}\, R} \mathrm{gr}\ B$, which has homology equal to $\mathrm{Tor}_*^{\mathrm{gr}\, R}(\mathrm{gr}\ A, \mathrm{gr}\ B)$. We define the spectral sequence of Proposition 8.1 to be the spectral sequence associated to the filtered complex $R_* \otimes_R B$. The strong assumption on B is used to guarantee the convergence of the spectral sequence of this filtered complex, via the following lemma. □ (Proposition 8.1)

Lemma **8.3.** — *Let* $\Omega = \Omega^0 \supset \Omega^1 \supset \dots$ *be a complete filtered ring with* gr Ω *noetherian,* M_* *a homological complex (meaning that* d *has degree* -1*) of complete filtered* Ω*-modules with* gr M_j *finitely generated over* gr Ω *for each* $j \in \mathbf{Z}$. *Then the spectral sequence of this filtered complex converges:*

$$E^0_{ij} = \mathrm{gr}^{-i} M_{i+j} \Rightarrow H_{i+j} M.$$

This is a homological spectral sequence, meaning that the differential d_r *has bidegree* $(-r, r-1)$ *for* $r \geqslant 0$. *The groups* $H_k M$ *are finitely generated* Ω*-modules, complete with respect to a filtration whose associated graded groups are the* E_∞ *term of the spectral sequence.*

Proof. — We refer to Cartan-Eilenberg [11], Chapter XV, as a reference for the spectral sequence of a filtered complex, although the gradings there (for a cohomological complex) are the negatives of ours. For each $i, j \in \mathbf{Z}$, we have subgroups

$$0 \subset B^1_{ij} \subset B^2_{ij} \subset \dots \subset Z^2_{ij} \subset Z^1_{ij} \subset E^0_{ij} = \mathrm{gr}^{-i} M_{i+j},$$

with $E^r_{ij} = Z^r_{ij}/B^r_{ij}$. Explicitly,

$$Z^r_{ij} = \mathrm{im}(\{x \in M^{-i}_{i+j} : dx \in M^{-i+r}_{i+j-1}\} \to \mathrm{gr}^{-i} M_{i+j})$$
$$B^r_{ij} = \mathrm{im}(\{dx \in M^{-i}_{i+j} : x \in M^{-i-r+1}_{i+j+1}\} \to \mathrm{gr}^{-i} M_{i+j}).$$

Moreover, for each $k \in \mathbf{Z}$, $E^0_{*,k-*} = \mathrm{gr}\ M_k$ is a finitely generated module over gr Ω, and the B^r's and Z^r's are all submodules. Since gr Ω is noetherian, the increasing sequence of submodules

$$B^1_{*,k-*} \subset B^2_{*,k-*} \subset \dots \subset \mathrm{gr}\ M_k$$

eventually terminates. That is, all differentials into total degree k are 0 after the rth term of the spectral sequence, for some $r < \infty$ depending on k. By the same statement for $k-1$, it follows that all differentials out of total degree k are also 0 after some point. So there is an $r = r(k) < \infty$ such that $E^r_{*,k-*} = E^\infty_{*,k-*}$.

Under the weaker assumption that for each $i, j \in \mathbf{Z}$ there is an r such that all differentials starting with d_r are zero on E^r_{ij}, together with completeness of the M_k's, Boardman shows that the filtration induced by each group M_k on its subquotient H_kM is complete, with associated graded groups equal to the E_∞ term of the spectral sequence, in [1], Theorem 7.1, the remark after it, and Theorem 9.2. Since gr H_kM is a subquotient of H_kM for each k and gr Ω is noetherian, gr H_kM is a finitely generated gr Ω-module. It follows that H_kM is a finitely generated Ω-module. □

Now we set up the relative version of the above spectral sequence, the last general homological result we need here. Let Ω, R, A, B be as in Proposition 8.1. Suppose that we also have a homomorphism $R \to S$ of complete filtered Ω-algebras such that gr S is noetherian and A and B are S-modules.

Proposition **8.4.** — *There is a spectral sequence*

$$E^1_{ij} = \mathrm{Tor}^{\mathrm{gr}\,S,\,\mathrm{gr}\,R}_{i+j}(\mathrm{gr}\ A, \mathrm{gr}\ B)_{\mathrm{degree}\ -i} \Rightarrow \mathrm{Tor}^{S,R}_{i+j}(A, B).$$

Here the groups $\mathrm{Tor}^{S,R}_*(A, B)$ *are finitely generated* Ω*-modules, complete with respect to a filtration whose associated graded groups are the* E_∞ *term of the spectral sequence.*

Proof. — Start with a graded finitely generated free resolution G_* of gr A as a gr R-module and a graded finitely generated free resolution H_* of gr A as a gr S-module. As in the definition of relative Tor groups, above, there is a graded gr R-linear homomorphism $G_* \to H_*$, unique up to homotopy, which gives the identity map from $H_0(G_*) = \mathrm{gr}\ A$ to $H_0(H_*) = \mathrm{gr}\ A$.

As in the construction of this spectral sequence for a single ring (Proposition 8.1), we can lift G_* to a filtered free resolution R_* of A as an R-module and H_* to a filtered free resolution S_* of A as an S-module. The new point here, using completeness again, is that the homomorphism $G_* \to H_*$ of complexes of graded gr R-modules lifts to a homomorphism $R_* \to S_*$ of complexes of filtered R-modules. (We can argue as in the proof of Proposition 8.1, or we can just refer to [23], Lemma V.2.1.5, p. 548.) Then $\mathrm{Tor}^{S,R}_*(A, B)$ is defined as the homology of the mapping cone of the map of chain complexes $R_* \otimes_R B \to S_* \otimes_S B$. This mapping cone is now a filtered complex, with associated graded complex being the mapping cone of the map of chain complexes $G_* \to H_*$. The homology of the latter mapping cone is therefore $\mathrm{Tor}_*^{\mathrm{gr}\, S,\, \mathrm{gr}\, R}(\mathrm{gr}\ A, \mathrm{gr}\ B)$, and the spectral sequence we want is the usual spectral sequence of a filtered complex. It converges in the required sense by Lemma 8.3. □

9. Relating groups and Lie algebras

We now explain how the results so far about Euler characteristics for Lie algebras over the p-adic integers imply analogous results for a large class of p-adic Lie groups, what Lazard called p-valued groups. For example, the group $GL_n\mathbf{Z}_p$ is not of this type, but any closed subgroup of the congruence subgroup $\ker(GL_n\mathbf{Z}_p \to GL_n(\mathbf{Z}/p))$ for p odd, or of $\ker(GL_n\mathbf{Z}_2 \to GL_n(\mathbf{Z}/4))$ for $p=2$, is p-valued. Groups of this type are in particular torsion-free pro-p groups.

For completeness, we recall Lazard's definition of p-valued groups. First ([23], p. 428), define a filtration ω of a group G to be a function

$$\omega : G \to (0, \infty]$$

such that, for $x, y \in G$,

$$\omega(xy^{-1}) \geqslant \min(\omega(x), \omega(y))$$
$$\omega(x^{-1}y^{-1}xy) \geqslant \omega(x) + \omega(y).$$

It follows in particular that $G_\nu := \{x \in G : \omega(x) \geqslant \nu\}$ and $G_{\nu+} := \{x \in G : \omega(x) > \nu\}$ are normal subgroups of G. A filtered group G is said to be complete if $G = \varprojlim G/G_\nu$. For a fixed prime number p, a filtration ω of a group G is called a valuation (and G is called p-valued) if

$$\omega(x) < \infty \text{ for all } x \neq 1 \text{ in } G$$
$$\omega(x) > (p-1)^{-1}$$
$$\omega(x^p) = \omega(x) + 1$$

for $x \in G$ ([23], p. 465). Then $\mathrm{gr}\ G := \oplus G_\nu/G_{\nu+}$ is a Lie algebra over the graded ring $\Gamma := \mathrm{gr}\ \mathbf{Z}_p = \mathbf{F}_p[\pi]$ with π in degree 1 ([23], pp. 464–465). The action of π on gr G corresponds to taking the pth power of an element of G. The Lie algebra gr G is torsion-free, hence free, as a Γ-module. The dimension of a p-valued group G is defined to be the rank of the free Γ-module gr G. In this paper, p-valued groups will be assumed to be complete and of finite dimension. Such a group is automatically a p-adic Lie group ([23], Theorem III.3.1.7, p. 478).

Let $v : \mathbf{Z}_p \to [0, \infty]$ be the standard valuation, which we sometimes call ord_p, so that $v(p) = 1$. By definition, a valuation on a $\mathbf{Z}_p$-module M is a function w from M to $[0, \infty]$ such that

$$w(x) < \infty \text{ for all } x \neq 0 \text{ in M}$$
$$w(x - y) \geqslant \min(w(x), w(y))$$
$$w(ax) = v(a) + w(x)$$

for $a \in \mathbf{Z}_p$ and $x, y \in$ M ([23], Def. I.2.2.2, p. 409). We define a valuation on a $\mathbf{Q}_p$-vector space V to be a function w from M to $(-\infty, \infty]$ which satisfies the same three properties; this definition generalizes to vector spaces over any p-adic field K using the standard valuation $v = \mathrm{ord}_p$ on K. A valuation on a $\mathbf{Z}_p$-module M extends to a valuation on the vector space $\mathrm{M} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ in a natural way, and we define

$$\mathrm{div}\ \mathrm{M} = \{x \in \mathrm{M} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p : w(x) \geqslant 0\}.$$

Let Sat M be the completion of div M with respect to the filtration w. A valued $\mathbf{Z}_p$-module M is called saturated if the natural homomorphism M $\rightarrow$ Sat M is an isomorphism. For a p-valued group G with valuation ω, we say that an action of G on a valued $\mathbf{Z}_p$-module or $\mathbf{Q}_p$-vector space M is compatible with the valuations if

$$w((g - 1)x) \geqslant \omega(g) + w(x)$$

for $g \in$ G and $x \in$ M.

Here is the basic theorem. Corollary 9.3 gives the main applications of this statement.

Theorem **9.1.** — *Let* G *be a p-valued group. Suppose that the given valuation of* G *takes rational values, and that* G *has dimension at least 2. Let* M *be a finitely generated free* $\mathbf{Z}_p$*-module with* G*-action. Suppose that* M *admits a valuation with rational values, compatible with the valuation of* G, *and that* M *is saturated for this valuation. The Lie algebra* $\mathfrak{g}_{\mathbf{Q}_p}$ *of* G *over* $\mathbf{Q}_p$ *acts on* $\mathrm{M} \otimes \mathbf{Q}_p$; *let* $\mathfrak{g}$ *be any Lie algebra over* $\mathbf{Z}_p$ *such that* $\mathfrak{g} \otimes \mathbf{Q}_p = \mathfrak{g}_{\mathbf{Q}_p}$ *and such that* $\mathfrak{g}$ *acts on* M. *Then the homology groups* $\mathrm{H}_*(\mathrm{G}, \mathrm{M})$ *are finite in all degrees if and only if the groups* $\mathrm{H}_*(\mathfrak{g}, \mathrm{M})$ *are finite in all degrees, and if this is so, then*

$$\chi(\mathrm{G}, \mathrm{M}) = \chi(\mathfrak{g}, \mathrm{M}).$$

The proof relies on Proposition 2.3, which says that for a Lie algebra over a discrete valuation ring Γ whose rank as a Γ-module is at least 2, the Euler characteristics we are considering do not change upon passage from one Lie algebra to an open Lie subalgebra. The idea here is to think of both G and the Lie algebra $\mathfrak{g}$ as "subgroups of finite index" in the same thing, a ring which Lazard calls the saturation of the group ring of G. We relate Tor groups over this ring to Tor groups over its associated graded ring, which is essentially the universal enveloping algebra of a Lie algebra over the polynomial ring $\Gamma = \mathbf{F}_p[\pi]$. Once we reduce to a question about such Lie algebras, we can apply Proposition 2.3, since Γ is a discrete valuation ring as a graded ring.

Proof. — Let G be a p-valued group. Let $\mathbf{Z}_p G$ denote the completed group ring of G,

$$\mathbf{Z}_p G := \varprojlim \mathbf{Z}_p [G/U]$$

where U runs over the open normal subgroups of G. (Lazard uses the name Al G for this ring.) Then the given valuation of G determines a complete filtration of the ring $\mathbf{Z}_p G$ which is also a valuation on $\mathbf{Z}_p G$ as a $\mathbf{Z}_p$-module. The associated graded ring of $\mathbf{Z}_p G$ is the universal enveloping algebra of the Lie algebra gr G over the graded ring $\Gamma := \mathrm{gr}\ \mathbf{Z}_p = \mathbf{F}_p[\pi]$, by Theorem III.2.3.3, p. 471, in [23]. Explicitly, Lazard first defines a filtration on the naive group ring $\mathbf{Z}_p[G]$ as the infimum w of all filtrations of $\mathbf{Z}_p[G]$ as a $\mathbf{Z}_p$-algebra which satisfy

$$w(g-1) \geqslant \omega(g)$$

for all $g \in G$. He then identifies the completed group ring $\mathbf{Z}_p G$ in the sense defined above with the completion of $\mathbf{Z}_p[G]$ with respect to this filtration.

If M is a finitely generated free $\mathbf{Z}_p$-module with G-action and with a valuation compatible with that on G, then M is a filtered $\mathbf{Z}_p G$-module. To check this, observe that M induces a filtration w_M on the naive group ring $\mathbf{Z}_p[G]$ by

$$w_M(f) = \inf_{x \in M - \{0\}} [w(f(x)) - w(x)] .$$

Then the above filtration w on $\mathbf{Z}_p[G]$ clearly satisfies $w \leqslant w_M$, which says exactly that M is a filtered $\mathbf{Z}_p[G]$-module. If M is a finitely generated free $\mathbf{Z}_p$-module with G-action and a valuation compatible with that on G, then M is complete for its filtration and hence is a filtered module over the completed ring $\mathbf{Z}_p G$.

Under the assumptions of Theorem 9.1, gr G and gr M are finitely generated free Γ-modules with all degrees rational. So they are concentrated in degrees $(1/e)\mathbf{Z}$ for some positive integer e.

The spectral sequence of Proposition 8.1, applied to the $\mathbf{Z}_p G$-modules $\mathbf{Z}_p$ and M, has the form

$$\mathrm{Tor}_*^{U\mathrm{gr}\, G}(\Gamma, \mathrm{gr}\ M) \Rightarrow \mathrm{Tor}_*^{\mathbf{Z}_p G}(\mathbf{Z}_p, M).$$

In that proposition, we assumed that the rings and modules were filtered by the integers, but we can apply the proposition to filtrations in $(1/e)\mathbf{Z}$, as here, by rescaling the filtrations. By Brumer ([9], Lemma 4.2, p. 455, and Remark 1, p. 452), the homology of a compact p-adic Lie group G with coefficients in a pseudocompact $\mathbf{Z}_p G$-module M is equal to $\mathrm{Tor}_*^{\mathbf{Z}_p G}(\mathbf{Z}_p, M)$, since $\mathbf{Z}_p G$ is noetherian by [23], Prop. V.2.2.4, p. 550. So the spectral sequence can be rewritten as:

$$H_*(\mathrm{gr}\ G, \mathrm{gr}\ M) \Rightarrow H_*(G, M).$$

Here the initial term is the homology of gr G as a Lie algebra over Γ. Under our assumptions, gr G and gr M are finitely generated free Γ-modules. This spectral sequence also appears in the paper by Symonds and Weigel [31] in the case of $\mathbf{F}_p$G-modules M.

We use the spectral sequence to compute the cohomology with nontrivial coefficients of congruence subgroups in Theorem 10.1. It is strange that no such direct relation is known between integral homology for p-adic Lie groups and for Lie algebras over $\mathbf{Z}_p$; we have instead a relation between p-adic Lie groups and Lie algebras over $\Gamma = \mathbf{F}_p[\pi]$.

There are many cases in which the spectral sequence can be used to compute the Euler characteristic $\chi(G, M)$, assuming that the homology groups $H_*(G, M)$ are finite. It does not work in the generality of our assumptions here, however, because it is possible for $H_*(G, M)$ to be finite while $H_*(\mathrm{gr}\ G, \mathrm{gr}\ M)$ is not. For example, if G is a p-valued open subgroup of $SL_n\mathbf{Z}_p$ and $M = (\mathbf{Z}_p)^n$ is the standard module, with the standard valuations on G and M as in the proof of Corollary 9.3, then $H_*(G, M)$ is always finite for $n \geqslant 2$, but $H_*(\mathrm{gr}\ G, \mathrm{gr}\ M)$ is finite if and only if p does not divide $n-1$ or $n+1$.

So we consider instead the more general spectral sequence of Proposition 8.4, for the homomorphism $\mathbf{Z}_p G \to \mathrm{Sat}\ \mathbf{Z}_p G$ of complete filtered rings, where the saturation of a valued $\mathbf{Z}_p$-module such as $\mathbf{Z}_p G$ is defined before the statement of Theorem 9.1. Since M has a valuation compatible with the action of G, the ring div $\mathbf{Z}_p G$ acts on div M, compatibly with the filtrations, and so the completion Sat $\mathbf{Z}_p G$ acts on Sat M. Since we assume M is saturated, the action of $\mathbf{Z}_p G$ on M extends to Sat $\mathbf{Z}_p G$.

For any valued $\mathbf{Z}_p$-module N, it is easy from the definition of Sat N to check that

$$\mathrm{gr\ Sat\ N} = (\mathrm{gr\ N} \otimes_{\mathbf{F}_p[\pi]} \mathbf{F}_p[\pi, \pi^{-1}])_{\mathrm{degrees} \geqslant 0}.$$

If the valuation on G takes integer values, then one can show that gr Sat $\mathbf{Z}_p G$ is the universal enveloping algebra of a Lie algebra over Γ; in general, one can draw a similar conclusion after extending scalars as follows.

We know that gr G is concentrated in degrees $(1/e)\mathbf{Z}$. Let K be a finite extension of $\mathbf{Q}_p$ with the same residue field $\mathbf{F}_p$ such that the maximal ideal of the ring of integers o_K is generated by an element π_K with valuation $v(\pi_K) = 1/e$. We have $\mathrm{gr}\ o_K = \mathbf{F}_p[\pi_K]$ where π_K has degree $1/e$, and there is a natural inclusion $\mathrm{gr}\ \mathbf{Z}_p = \mathbf{F}_p[\pi] \subset \mathbf{F}_p[\pi_K]$. The definitions of valued $\mathbf{Z}_p$-modules and their saturations extend to o_K-modules in a natural way. We therefore have

$$\begin{aligned}\mathrm{gr\ Sat\ } o_K G &= (\mathrm{gr\ } o_K G \otimes_{\mathbf{F}_p[\pi_K]} \mathbf{F}_p[\pi_K, \pi_K^{-1}])_{\geqslant 0}\\ &= (\mathrm{gr\ } \mathbf{Z}_p G \otimes_{\mathbf{F}_p[\pi]} \mathbf{F}_p[\pi_K, \pi_K^{-1}])_{\geqslant 0}\\ &= (U(\mathrm{gr\ G}) \otimes_{\mathbf{F}_p[\pi]} \mathbf{F}_p[\pi_K, \pi_K^{-1}])_{\geqslant 0}.\end{aligned}$$

Defining a graded Lie algebra $\mathfrak{s}$ over $\mathbf{F}_p[\pi_K]$ by tensoring gr G up from $\mathbf{F}_p[\pi]$ to $\mathbf{F}_p[\pi_K]$, we can say that

$$\text{gr Sat } o_K G = (U\mathfrak{s} \otimes_{\mathbf{F}_p[\pi_K]} \mathbf{F}_p[\pi_K, \pi_K^{-1}])_{\geqslant 0}.$$

Let $\mathfrak{t}$ be the saturation of $\mathfrak{s}$, defined by

$$\mathfrak{t} = (\mathfrak{s} \otimes_{\mathbf{F}_p[\pi_K]} \mathbf{F}_p[\pi_K, \pi_K^{-1}])_{\geqslant 0}.$$

Since $\mathfrak{s}$ is a graded free $\mathbf{F}_p[\pi_K]$-module of rank n, so is $\mathfrak{t}$. Since $\mathfrak{s}$ is concentrated in degrees $(1/e)\mathbf{Z}$ and π_K has degree $1/e$, the generators of $\mathfrak{t}$ are all in degree 0. Finally, $\mathfrak{t}$ is a Lie algebra over $\mathbf{F}_p[\pi_K]$ in an obvious way. It follows that gr Sat o_K is the universal enveloping algebra of $\mathfrak{t}$.

As a result, the spectral sequence of Proposition 8.4 has the form

$$\mathrm{Tor}_*^{U\mathfrak{t}, U\mathfrak{s}}(\mathbf{F}_p[\pi_K], \text{gr } M_{o_K}) \Rightarrow \mathrm{Tor}_*^{\text{Sat } o_K G, o_K G}(o_K, M_{o_K}).$$

Again, in the proposition, we assumed that the filtrations were indexed by the integers, but we can apply the proposition when the filtrations are indexed by $(1/e)\mathbf{Z}$, as here, by rescaling the filtrations. In a somewhat simpler notation, we can rename the groups in this spectral sequence as:

$$H_*(\mathfrak{t}, \mathfrak{s}; \text{gr } M_{o_K}) \Rightarrow H_*(\text{Sat } o_K G, G; M_{o_K}).$$

Here, for the augmented algebra Sat $o_K G$ over o_K, we write $H_*(\text{Sat } o_K G, M_{o_K})$ to mean $\mathrm{Tor}_*^{\text{Sat } o_K G}(o_K, M_{o_K})$, by analogy with the definitions of group homology and Lie algebra homology. The homomorphism of Lie algebras $\mathfrak{s} \to \mathfrak{t}$ over $\mathbf{F}_p[\pi_K]$ is an injection from one free $\mathbf{F}_p[\pi_K]$-module of finite rank to another, and the $\mathfrak{t}$-module gr M_{o_K} is also a free $\mathbf{F}_p[\pi_K]$-module of finite rank. Since G has dimension at least 2, the Lie algebras $\mathfrak{s}$ and $\mathfrak{t}$ have rank at least 2 as free $\mathbf{F}_p[\pi_k]$-modules. By Proposition 2.3, the relative Lie algebra homology groups $H_*(\mathfrak{t}, \mathfrak{s}; \text{gr } M_{o_K})$ are finite, and the resulting Euler characteristic is 0. Then the above spectral sequence shows that the groups $H_*(\text{Sat } o_K G, G; M_{o_K})$ are also finite and that the resulting Euler characteristic is 0. These groups are just the analogous groups over $\mathbf{Z}_p$ tensored up to o_K, so we deduce the same conclusions for the groups $H_*(\text{Sat } \mathbf{Z}_p G, G; M)$. So $H_*(G, M)$ is finite if and only if $H_*(\text{Sat } \mathbf{Z}_p G, M)$ is finite; and if either condition holds, then

$$\chi(G, M) = \chi(\text{Sat } \mathbf{Z}_p G, M).$$

To analyze the homology of Lie algebras over $\mathbf{Z}_p$ by the above methods, which as written apply to complete rings, we first need the following lemma.

Lemma **9.2.** — *Let* $\mathfrak{g}$ *be a valued Lie algebra over* $\mathbf{Z}_p$. *That is,* $\mathfrak{g}$ *is a filtered Lie algebra over* $\mathbf{Z}_p$ *which is valued as a* $\mathbf{Z}_p$*-module. As always, assume that* $\mathfrak{g}$ *is free of finite rank over* $\mathbf{Z}_p$.

For any complete filtered $\mathfrak{g}$*-module* M, *we can view* M *as a module over the completion* $\mathrm{U}\mathfrak{g}^\wedge$ *of the universal enveloping algebra, and we have*

$$\mathrm{H}_*(\mathfrak{g}, \mathrm{M}) = \mathrm{H}_*(\mathrm{U}\mathfrak{g}^\wedge, \mathrm{M}).$$

Proof. — Let

$$\to \mathrm{U}\mathfrak{g} \otimes_{\mathbf{Z}_p} \wedge^2\mathfrak{g} \to \mathrm{U}\mathfrak{g} \otimes_{\mathbf{Z}_p} \mathfrak{g} \to \mathrm{U}\mathfrak{g} \to \mathbf{Z}_p \to 0$$

be the standard resolution of $\mathbf{Z}_p$ as a $\mathrm{U}\mathfrak{g}$-module. Clearly these modules are filtered in a natural way. The point is that this is a resolution in the filtered sense, meaning that not only this complex but also the subcomplexes of elements of filtration $\geqslant \nu$, for all real numbers ν, are exact. Indeed, the $\mathbf{Z}_p$-linear homotopies that prove exactness of the standard complex are compatible with the filtrations (V.1.3.7, p. 545, in [23]).

It follows that these $\mathbf{Z}_p$-linear homotopies are defined on the completion of this complex, and so this completion is exact. It clearly has the form

$$\to \mathrm{U}\mathfrak{g}^\wedge \otimes_{\mathbf{Z}_p} \wedge^2\mathfrak{g} \to \mathrm{U}\mathfrak{g}^\wedge \otimes_{\mathbf{Z}_p} \mathfrak{g} \to \mathrm{U}\mathfrak{g}^\wedge \to \mathbf{Z}_p \to 0.$$

So, for any $\mathrm{U}\mathfrak{g}^\wedge$-module M, $\mathrm{H}_*(\mathfrak{g}, \mathrm{M})$ and $\mathrm{H}_*(\mathrm{U}\mathfrak{g}^\wedge, \mathrm{M})$ are computed by the same complex

$$\to \wedge^2\mathfrak{g} \otimes_{\mathbf{Z}_p} \mathrm{M} \to \mathfrak{g} \otimes_{\mathbf{Z}_p} \mathrm{M} \to \mathrm{M} \to 0. \quad \square$$

If $\mathfrak{g}$ is a sufficiently small open Lie subalgebra over $\mathbf{Z}_p$ of the Lie algebra of G over $\mathbf{Q}_p$, then $\mathfrak{g}$ inherits a valuation from G, and we have $\mathrm{Sat}\ \mathrm{U}\mathfrak{g} = \mathrm{Sat}\ \mathbf{Z}_p\mathrm{G}$ by the proof of Theorem V.2.4.9, p. 562 in [23]. In particular, we have a homomorphism from $\mathrm{U}\mathfrak{g}^\wedge$ to $\mathrm{Sat}\ \mathbf{Z}_p\mathrm{G}$. As in the argument for groups, let o_K be an extension of $\mathbf{Z}_p$ such that a uniformizer π_K has valuation $1/e$. After tensoring up to o_K, the graded homomorphism associated to $\mathrm{U}\mathfrak{g}^\wedge \to \mathrm{Sat}\ \mathbf{Z}_p\mathrm{G}$ maps the universal enveloping algebra of one graded Lie algebra over $\mathbf{F}_p[\pi_\mathrm{K}]$, $\mathfrak{r} := \mathrm{gr}\ \mathfrak{g} \otimes_{\mathbf{F}_p[\pi]} \mathbf{F}_p[\pi_\mathrm{K}]$, to that of another, the saturation $\mathfrak{t}$ of $\mathfrak{r}$ as above. The Lie algebra homomorphism $\mathfrak{r} \to \mathfrak{t}$ is again an injection from one finitely generated free $\mathbf{F}_p[\pi_\mathrm{K}]$-module to another. So the argument for groups applies, again using that the dimension is at least 2, to show that $\mathrm{H}_*(\mathrm{U}\mathfrak{g}^\wedge, \mathrm{M})$ is finite if and only if $\mathrm{H}_*(\mathrm{Sat}\ \mathbf{Z}_p\mathrm{G}, \mathrm{M})$ is finite, and if either condition holds then

$$\chi(\mathrm{U}\mathfrak{g}^\wedge, \mathrm{M}) = \chi(\mathrm{Sat}\ \mathbf{Z}_p\mathrm{G}, \mathrm{M}).$$

By Lemma 9.2, we can replace $\mathrm{H}_*(\mathrm{U}\mathfrak{g}^\wedge, \mathrm{M})$ in these statements by $\mathrm{H}_*(\mathfrak{g}, \mathrm{M})$.

Having related Euler characteristics for both the group G and the Lie algebra $\mathfrak{g}$ to those for $\mathrm{Sat}\ \mathbf{Z}_p\mathrm{G}$, we have the relation between $\mathfrak{g}$ and G that we wanted. We had to assume above that the Lie algebra $\mathfrak{g}$ was sufficiently small, but that implies the same result for any open Lie subalgebra over $\mathbf{Z}_p$ of the Lie algebra of G over $\mathbf{Q}_p$ which acts on M, by Proposition 2.3. Theorem 9.1 is proved. $\square$

Corollary **9.3.** — *Let p be any prime number. Let* G *be a compact p-adic Lie group of dimension at least 2, and let* M *be a finitely generated free $\mathbf{Z}_p$-module with* G*-action. Suppose that the image of* G *in Aut*(M) *is sufficiently small in the sense that either (1) this image is a pro-p group and $p >$* rank(M) + 1, *or else (2)* G *acts trivially on* M/p *if p is odd, or on* M/4 *if $p = 2$. Also assume that there is some faithful* G*-module (which could be* M*), finitely generated and free as a $\mathbf{Z}_p$-module, which satisfies (1) or (2).*

Let $\mathfrak{g}$ be any Lie algebra over $\mathbf{Z}_p$ such that $\mathfrak{g} \otimes \mathbf{Q}_p$ is the Lie algebra $\mathfrak{g}_{\mathbf{Q}_p}$ of G *and such that $\mathfrak{g}$ acts on* M. *Then the homology groups* $\mathrm{H}_*(\mathrm{G}, \mathrm{M})$ *are finite if and only if the groups* $\mathrm{H}_*(\mathfrak{g}, \mathrm{M})$ *are finite, and if either condition holds then*

$$\chi(\mathrm{G}, \mathrm{M}) = \chi(\mathfrak{g}, \mathrm{M}).$$

Proof. — According to Theorem 9.1, it suffices to show that G has a valuation and that M has a compatible valuation which is saturated, both taking rational values. If M is faithful as well as satisfying (1) or (2), then Lazard constructed the required valuations of G and M; we will recall his definitions in the following paragraphs. In general, if there is some faithful module N which satisfies (1) or (2), then the minimum of the filtrations of G associated to M and N is a valuation of G which is compatible with the valuation of M, as we want. (To be precise, if M is not faithful, the filtration of G associated to M alone satisfies all the properties of a valuation, as defined before Theorem 9.1, except that it takes the value ∞ on the kernel of $\mathrm{G} \to \mathrm{Aut}(\mathrm{M})$. The minimum just mentioned is a genuine valuation of G.)

For (2), we use the obvious integral valuations. That is, after choosing a basis for M, G becomes a subgroup of $\mathrm{GL}_n\mathbf{Z}_p$ whose image in $\mathrm{GL}_n\mathbf{Z}/p$ is trivial, and we define

$$\omega(a) = \min_{1 \leqslant i,j \leqslant n} v(a_{ij} - \delta_{ij})$$

for $a \in \mathrm{G}$ and

$$w(x) = \min_{1 \leqslant i \leqslant n} v(x_i)$$

for $x \in \mathrm{M} = (\mathbf{Z}_p)^n$. The stronger assumption for $p = 2$ is needed to ensure that ω is a valuation of G (see the definition before Theorem 9.1).

In case (1), use the rational valuation of G defined in section III.3.2.7, pp. 484-486, of [23]. We will generalize this construction to groups other than GL_n in Proposition 12.1. Namely, since G is a pro-p subgroup of $\mathrm{GL}_n\mathbf{Z}_p$, it is conjugate to a subgroup of the Sylow p-subgroup $\mathrm{Iw}_u \subset \mathrm{GL}_n\mathbf{Z}_p$ of matrices whose image in $\mathrm{GL}_n\mathbf{Z}/p$ is strictly upper-triangular. (We call this subgroup Iw_u since it is the pro-p, or pro-unipotent, radical of an Iwahori subgroup of $\mathrm{GL}_n\mathbf{Q}_p$, the group of matrices in $\mathrm{GL}_n\mathbf{Z}_p$ whose image in $\mathrm{GL}_n\mathbf{Z}/p$ is non-strictly upper triangular.) So it is enough to define valuations on the group Iw_u and its standard module $\mathrm{M} = (\mathbf{Z}_p)^n$. Since we assume

$n < p-1$, there is a rational number α such that $\alpha > (p-1)^{-1}$ and $(n-1)\,\alpha < 1-(p-1)^{-1}$. Choose such an α and a finite extension K of $\mathbf{Q}_p$ with an element $a \in$ K such that $v(a) = \alpha$. Let D denote the diagonal matrix $d_{ij} = a^{i-n}\delta_{ij}$; this differs from Lazard's definition by a constant factor, which makes no difference in defining the valuation on Iw_u. Namely, let w be the standard valuation of the algebra $\mathrm{M}_n o_{\mathrm{K}}$, $w(\mathrm{X}) = \min v(x_{ij})$, and define a valuation of Iw_u by

$$\omega(\mathrm{X}) = w(\mathrm{D}^{-1}\mathrm{X}\,\mathrm{D} - 1).$$

Lazard shows that this is a valuation of Iw_u. Similarly, let w be the standard valuation on $\mathrm{M} = (\mathbf{Z}_p)^n$, $w(m) = \min v(m_i)$, and define a new valuation w' of M by

$$w'(m) = w(\mathrm{D}^{-1}m).$$

It is immediate that this valuation is compatible with that on G, in the sense defined before Theorem 9.1. Also, from our choice of D, M is saturated for this valuation. □

Theorem 0.1 follows from Theorem 7.1, Theorem 7.4, and Corollary 9.3 when the G-module M is a free $\mathbf{Z}_p$-module. To include arbitrary finitely generated $\mathbf{Z}_p$-modules M in Theorem 0.1, we use that $\chi(\mathrm{G}, \mathrm{A}) = 0$ for all p-adic Lie groups G of positive dimension which are pro-p groups and all finite $\mathbf{Z}_p$G-modules A, by [28], I.4.1, exercise (e).

10. Cohomology of congruence subgroups

In this section we show how to use the spectral sequence arising in the proof of Theorem 9.1 to compute the whole homology with nontrivial coefficients of certain congruence subgroups, not just an Euler characteristic. See Corollary 10.2 for the special case of congruence subgroups of $\mathrm{SL}_n\mathbf{Z}_p$. In contrast to the results on Euler characteristics, we need to assume that p does not divide $n-1$ or $n+1$ in Corollary 10.2.

Theorem **10.1.** — *Let* $\mathfrak{g}$ *be a Lie algebra over* $\mathbf{Z}_p$, M *a finitely generated free* $\mathbf{Z}_p$*-module on which* $\mathfrak{g}$ *acts. Suppose that the homology groups* $\mathrm{H}_*(\mathfrak{g}_{\mathbf{F}_p}, \mathrm{M}_{\mathbf{F}_p})$ *are* 0. *This holds for example if* $\mathfrak{g}_{\mathbf{Q}_p}$ *is semisimple,* $\mathrm{M}_{\mathbf{Q}_p}$ *is a nontrivial simple* $\mathfrak{g}_{\mathbf{Q}_p}$*-module, and* p *does not divide the eigenvalue of the Casimir operator, scaled to lie in the* $\mathbf{Z}_p$*-algebra* $\mathrm{U}\mathfrak{g}$, *on* $\mathrm{M}_{\mathbf{Q}_p}$.

Let G_r *be the group defined by the Baker-Campbell-Hausdorff formula from the Lie algebra* $p^r\mathfrak{g}$, *where* $r \geqslant 1$ *if* p *is odd and* $r \geqslant 2$ *if* $p = 2$. *Then the abelian group* $\mathrm{H}_i(\mathrm{G}_r, \mathrm{M})$ *is isomorphic to the direct sum of* $\binom{n-1}{i}\mathrm{rank}(\mathrm{M})$ *copies of* $\mathbf{Z}/p^r$, *where* n *is the rank of* $\mathfrak{g}$ *as a free* $\mathbf{Z}_p$*-module. Also,* $\mathrm{H}_i(p^r\mathfrak{g}, \mathrm{M})$ *is isomorphic to the same group.*

Moreover, for any group H *which acts compatibly on* $\mathfrak{g}$ *and* M, *we have*

$$H_i(G_r, M) = r\sum_{j=0}^{i}(-1)^{i-j}(\Lambda^j \mathfrak{g}_{\mathbf{F}_p} \otimes_{\mathbf{F}_p} M_{\mathbf{F}_p})$$

in the Grothendieck group of finite p-torsion H*-modules.*

Proof. — Since $r \geqslant 1$ if p is odd and $r \geqslant 2$ if $p=2$, G_r is a p-valued group by Lazard [23], IV.3.2.6, pp. 518-519. The valuation is defined by: $\omega(x)=a$ if $x \in G_r$ corresponds to an element of $p^a\mathfrak{g} - p^{a+1}\mathfrak{g}$. Since M is a $\mathfrak{g}$-module, the standard saturated valuation on M, where $w(x)=a$ if $x \in M$ lies in $p^aM - p^{a+1}M$, is compatible with the valuation of G_r.

In the proof of Theorem 9.1, we defined a spectral sequence

$$H_*(\mathrm{gr}\ G_r, \mathrm{gr}\ M) \Rightarrow H_*(G_r, M).$$

Here gr G_r is the Lie algebra $\pi^r \mathrm{gr}\ \mathfrak{g}$ over $\Gamma = \mathrm{gr}\ \mathbf{Z}_p = \mathbf{F}_p[\pi]$. The complex for computing the Lie algebra homology $H_*(\pi^r \mathrm{gr}\ \mathfrak{g}, \mathrm{gr}\ M)$ has the form

$$\to \pi^r \mathrm{gr}\ \mathfrak{g} \otimes_\Gamma \mathrm{gr}\ M \to \mathrm{gr}\ M \to 0.$$

It can be identified in an obvious way with the complex defining $H_*(\mathrm{gr}\ \mathfrak{g}, \mathrm{gr}\ M)$,

$$\to \mathrm{gr}\ \mathfrak{g} \otimes_\Gamma \mathrm{gr}\ M \to \mathrm{gr}\ M \to 0,$$

but with the differentials multiplied by π^r.

Clearly the Lie algebra gr $\mathfrak{g} \otimes_\Gamma \mathbf{F}_p$ is equal to $\mathfrak{g}_{\mathbf{F}_p}$ and gr $M \otimes_\Gamma \mathbf{F}_p$ is equal to $M_{\mathbf{F}_p}$. We assumed that $H_*(\mathfrak{g}_{\mathbf{F}_p}, M_{\mathbf{F}_p}) = 0$, and it follows that $H_*(\mathrm{gr}\ \mathfrak{g}, \mathrm{gr}\ M \otimes_\Gamma \mathbf{F}_p) = 0$ (since these homology groups are defined by the same complex). By the universal coefficient theorem, using that $H_*(\mathrm{gr}\ \mathfrak{g}, \mathrm{gr}\ M)$ is a finitely generated Γ-module, it follows that $H_*(\mathrm{gr}\ \mathfrak{g}, \mathrm{gr}\ M) = 0$.

So the complex defining $H_*(\mathrm{gr}\ G_r, \mathrm{gr}\ M)$ is obtained from an exact complex by multiplying all the differentials by π^r. Since the Γ-modules in the complex are torsion-free, multiplying by π^r does not change the kernels, but the images are multiplied by π^r. Thus we have a canonical isomorphism

$$H_i(\mathrm{gr}\ G_r, \mathrm{gr}\ M) \cong \ker(d_i) \otimes_\Gamma \Gamma/\pi^r,$$

where $d_i : \Lambda^i \mathrm{gr}\ \mathfrak{g} \otimes_\Gamma \mathrm{gr}\ M \to \Lambda^{i-1} \mathrm{gr}\ \mathfrak{g} \otimes_\Gamma \mathrm{gr}\ M$ is a differential in the complex defining $H_*(\mathrm{gr}\ \mathfrak{g}, \mathrm{gr}\ M)$. By exactness of the latter complex, we have

$$\ker(d_i) = \sum_{j=0}^{i}(-1)^{i-j}(\Lambda^j \mathrm{gr}\ \mathfrak{g} \otimes_\Gamma \mathrm{gr}\ M)$$

in the Grothendieck group of finitely generated Γ-modules. So we know the rank of the free Γ-module $\ker(d_i)$, and it follows that $H_i(\mathrm{gr}\ G_r, \mathrm{gr}\ M)$ is isomorphic to the direct sum of $\binom{n-1}{i}\mathrm{rank}(M)$ copies of Γ/π^r, where n is the rank of $\mathfrak{g}$ as a free $\mathbf{Z}_p$-module. This is a graded Γ-module with generators in degree ir, as we see by going through the above identifications. We can also describe these homology groups less precisely but more canonically. Any group H which acts compatibly on $\mathfrak{g}$ and M automatically preserves the filtrations of $\mathfrak{g}$ and M, so it acts on the above spectral sequence, and we have

$$H_i(\mathrm{gr}\ G_r, \mathrm{gr}\ M) = r\sum_{j=0}^{i}(-1)^{i-j}(\Lambda^j \mathfrak{g}_{\mathbf{F}_p} \otimes_{\mathbf{F}_p} M_{\mathbf{F}_p})$$

in the Grothendieck group of finite H-modules.

The differentials d_k in the spectral sequence

$$E^1_{ij} = H_{i+j}(\mathrm{gr}\ G_r, \mathrm{gr}\ M)_{\mathrm{degree}-i} \Rightarrow H_{i+j}(G_r, M)$$

have bidegree $(-k, k-1)$. Since $H_i(\mathrm{gr}\ G_r, \mathrm{gr}\ M)$ is concentrated in degrees from ir to $ir + r - 1$, the spectral sequence degenerates at E_1. It follows that the abelian group $H_i(G_r, M)$ is isomorphic to the direct sum of $\binom{n-1}{i}\mathrm{rank}(M)$ copies of $\mathbf{Z}/p^r$. Again, for any group H which acts compatibly on $\mathfrak{g}$ and M, it follows from the above results that

$$H_i(G_r, M) = r\sum_{j=0}^{i}(-1)^{i-j}(\Lambda^j \mathfrak{g}_{\mathbf{F}_p} \otimes_{\mathbf{F}_p} M_{\mathbf{F}_p})$$

in the Grothendieck group of finite H-modules.

The proof of Theorem 9.1 gives a similar spectral sequence

$$H_*(\pi^r \mathrm{gr}\ \mathfrak{g}, \mathrm{gr}\ M) \Rightarrow H_*(p^r\mathfrak{g}, M)$$

which degenerates by the same argument. So we get the same description of $H_*(p^r\mathfrak{g}, M)$. □

Corollary **10.2.** — *Let* $G_r = \ker(SL_n\mathbf{Z}_p \to SL_n\mathbf{Z}/p^r)$, *where* $r \geqslant 1$ *if* p *is odd and* $r \geqslant 2$ *if* $p = 2$. *Let* $M = (\mathbf{Z}_p)^n$ *be the standard representation of* G_r. *Suppose that* $p \nmid (n-1)$ *and* $p \nmid (n+1)$. *Then the abelian group* $H_i(G_r, M)$ *is isomorphic to the direct sum of* $\binom{n^2-2}{i}n$ *copies of* $\mathbf{Z}/p^r$. *Moreover, the group* $SL_n\mathbf{Z}/p^r$ *acts on* $H_*(G_r, M)$ *in a natural way, and we have*

$$H_i(G_r, M) = r\sum_{j=0}^{i}(-1)^{i-j}(\Lambda^i \mathfrak{sl}_n\mathbf{F}_p \otimes_{\mathbf{F}_p} M_{\mathbf{F}_p})$$

in the Grothendieck group of finite p*-torsion* $SL_n\mathbf{Z}/p^r$*-modules.*

Proof. — To deduce the first statement from Theorem 10.1, we need to check that $H_*(\mathfrak{sl}_n\mathbf{F}_p, M_{\mathbf{F}_p}) = 0$ if $p \nmid n-1$ and $p \nmid n+1$.

Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}_n\mathbf{Z}$, with its standard module $M = \mathbf{Z}^n$. The Casimir operator c in the center of the enveloping algebra $U\mathfrak{g}_{\mathbf{Q}}$ acts on $M_{\mathbf{Q}}$ by multiplication by $(n^2-1)/2n^2$, say by formula (25.14) in Fulton and Harris [17], p. 418. Writing out c in terms of a basis for $\mathfrak{g} = \mathfrak{sl}_n\mathbf{Z}$ shows that $c' := 2n^2c$ lies in the integral enveloping algebra $U\mathfrak{g}$. Clearly it acts by n^2-1 on M. It is also clear that c' maps to an element in the center of $U\mathfrak{g}_{\mathbf{F}_p}$, which acts by a nonzero scalar on $M_{\mathbf{F}_p}$ if $n^2 - 1 \not\equiv 0 \pmod p$, that is, if $p \nmid (n-1)$ and $p \nmid (n+1)$. It follows that $H_*(\mathfrak{g}_{\mathbf{F}_p}, M_{\mathbf{F}_p}) = 0$ if $p \nmid (n-1)$ and $p \nmid (n+1)$, as claimed.

So Theorem 10.1 applies, and we have the computation of $H_*(G_r, M)$ as an abelian group. The theorem also computes $H_i(G_r, M)$ as an element in the Grothendieck group $\mathrm{Rep}(SL_n\mathbf{Z}_p)$ of finite p-torsion $SL_n\mathbf{Z}_p$-modules, since $SL_n\mathbf{Z}_p$ acts compatibly on $\mathfrak{sl}_n\mathbf{Z}_p$ and on $M_{\mathbf{Z}_p}$. Since $H_i(G_r, M)$ and the expression on the right are in fact $SL_n\mathbf{Z}/p^r$-modules, we deduce the same equality in the Grothendieck group $\mathrm{Rep}(SL_n\mathbf{Z}/p^r)$ of finite p-torsion $SL_n\mathbf{Z}/p^r$-modules, because the restriction map

$$\mathrm{Rep}(SL_n\mathbf{Z}/p^r) \to \mathrm{Rep}(SL_n\mathbf{Z}_p)$$

is injective. Indeed, $\mathrm{Rep}(SL_n\mathbf{Z}/p^r) \cong \mathrm{Rep}(\mathbf{F}_p[SL_n\mathbf{Z}/p^r])$ is detected by restriction to cyclic subgroups of order prime to p by Brauer [16], and these all lift to $SL_n\mathbf{Z}_p$ since the kernel of $SL_n\mathbf{Z}_p \to SL_n\mathbf{Z}/p^r$ is a pro-p group. □

11. Euler characteristics for p-adic Lie groups which are not pro-p groups

Here at last we prove the vanishing of the Euler characteristics we have been considering for some p-adic Lie groups such as $SL_n\mathbf{Z}_p$ which are not pro-p groups. See Corollary 11.6 for some more explicit consequences of the following theorem.

Theorem **11.1.** — *Let $G_{\mathbf{Q}_p}$ be a connected reductive algebraic group whose rank over $\overline{\mathbf{Q}_p}$ is at least 2, and let $M_{\mathbf{Q}_p}$ be a finite-dimensional $G_{\mathbf{Q}_p}$-module with no trivial summands. Let G be a compact open subgroup of $G(\mathbf{Q}_p)$ and let M be a G-invariant lattice in $M_{\mathbf{Q}_p}$. Suppose that there is a Sylow p-subgroup $G_p \subset G$ with a valuation and that M has a compatible saturated valuation, both taking rational values. Then the homology groups $H_*(G, M)$ are finite and the resulting Euler characteristic $\chi(G, M)$ is 0.*

Proof. — Since $G_{\mathbf{Q}_p}$ is a reductive group in characteristic zero, representations of $G_{\mathbf{Q}_p}$ are completely reducible, and so the assumption on $M_{\mathbf{Q}_p}$ implies that the coinvariants of $G_{\mathbf{Q}_p}$ on $M_{\mathbf{Q}_p}$ are 0. Since $G_{\mathbf{Q}_p}$ is connected, it follows that the coinvariants of its Lie algebra $\mathfrak{g}_{\mathbf{Q}_p}$ on $M_{\mathbf{Q}_p}$ are 0. It follows that $H_*(K, M)$ is finite for all open subgroups K of G, by Lemma 3.1 and Lazard's theorem that $H_*(K, M) \otimes \mathbf{Q}_p$ injects into $H_*(\mathfrak{g}_{\mathbf{Q}_p}, M_{\mathbf{Q}_p})$ ([23], V.2.4.10, pp. 562-563).

For any finite group F, let $a(\mathbf{Z}_p F)$ denote the Green ring, the free abelian group on the set of isomorphism classes of indecomposable $\mathbf{Z}_p F$-modules that are finitely generated and free over $\mathbf{Z}_p$. Conlon's induction theorem says that for any finite group F, there are rational numbers a_K such that

$$\mathbf{Z}_p = \sum_{K} a_K \mathbf{Z}_p[F/K]$$

in $a(\mathbf{Z}_p G) \otimes \mathbf{Q}$, where K runs over the set of p-hyperelementary subgroups of F, that is, extensions of a cyclic group of order prime to p by a p-group ([16], Theorem 80.51). In fact, although we do not need it here, there is an explicit formula for the rational numbers a_K, using Gluck's formula for the idempotents in the Burnside ring tensored with the rationals [19]:

$$\mathbf{Z}_p = \sum_{H} \frac{1}{|N_F(H)|} \sum_{K \subset H} |K| \mu(K, H) \mathbf{Z}_p[F/K].$$

Here the first sum runs over the conjugacy classes of p-hyperelementary subgroups $H \subset F$, and μ denotes the Möbius function on the partially ordered set of subgroups of F. Boltje's paper [2] uses Gluck's formula for similar purposes in Proposition VI.1.2 and the remarks afterward. For example, for the group $F = S_3$ and $p = 2$, the above formula gives the identity

$$\mathbf{Z}_2 = -\frac{1}{2}\mathbf{Z}_2 S_3/1 + \mathbf{Z}_2 S_3/\langle (12) \rangle + \frac{1}{2}\mathbf{Z}_2 S_3/\langle (123) \rangle.$$

Returning to the p-adic Lie group G, we know that there is an open normal subgroup H of G contained in the given Sylow p-subgroup G_p, for example the intersection of the conjugates of G_p. We apply Conlon's induction theorem to the finite group G/H to get an equality

$$\mathbf{Z}_p = \sum_{K} a_K \mathbf{Z}_p[G/K]$$

in the Green ring of G/H-modules, where K runs over the p-hyperelementary subgroups of G containing H (that is, K is an extension of a cyclic group of order prime to p by a pro-p group). If we multiply this equation by a suitable positive integer and move terms with a_K negative to the other side of the equation, it states the existence of an isomorphism between two explicit G/H-modules, which we can view as an isomorphism between the same groups viewed as G-modules.

It follows that, for the given $\mathbf{Z}_p$G-module M and all $j \geqslant 0$, we have

$$\begin{aligned} H_j(G, M) &= \sum_K a_K H_j(G, M \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[G/K]) \\ &= \sum_K a_K H_j(K, M). \end{aligned}$$

This is an equality in the Grothendieck group tensored with $\mathbf{Q}$ of finite abelian groups with respect to direct sums. It follows that

$$\chi(G, M) = \sum_K a_K \chi(K, M).$$

So, to show that the Euler characteristic $\chi(G, M)$ is 0, it suffices to show that $\chi(K, M) = 0$ for all open p-hyperelementary subgroups K of G. Since such a subgroup satisfies all the properties we assumed of G, we can assume from now on that G is itself p-hyperelementary. That is, the Sylow p-subgroup G_p is normal in G and the quotient group $Z := G/G_p$ is cyclic of order prime to p, and we want to show that $\chi(G, M) = 0$.

The extension

$$1 \to G_p \to G \to Z \to 1$$

splits, and so G is a semidirect product $Z \ltimes G_p$. Also, $H_*(G, M)$ is equal to the coinvariants of Z acting on $H_*(G_p, M)$, and taking the coinvariants of Z is an exact functor on $\mathbf{Z}_p$Z-modules. So it suffices to show that $\chi(G_p, M) = 0$ in Rep(Z), the Grothendieck group of finite p-torsion Z-modules. We are given that M is a $(Z \ltimes G_p)$-module and that G_p and M have compatible valuations. Replace the given valuation of G_p by the minimum of its conjugates under the action of Z on G_p. This is again a valuation of G_p, now Z-invariant, and still compatible with the given valuation of M since it is less than or equal to the original valuation of G_p.

Let $\mathfrak{g}$ be any Z-invariant Lie subalgebra over $\mathbf{Z}_p$ of the Lie algebra $\mathfrak{g}_{\mathbf{Q}_p}$ such that $\mathfrak{g} \otimes \mathbf{Q}_p = \mathfrak{g}_{\mathbf{Q}_p}$. To see that one exists, start with any Z-invariant $\mathbf{Z}_p$-lattice in $\mathfrak{g}_{\mathbf{Q}_p}$, and then multiply it by a big power of p. Propositions 11.2 and 11.4 will imply that $\chi(G_p, M) = 0$ in Rep(Z), thus proving Theorem 11.1.

Proposition **11.2.** — *In the above notation, we have* $\chi(\mathfrak{g}, M) = 0$ *in the Grothendieck group* Rep(Z) *of finite p-torsion* Z*-modules.*

Proof. — Let g be a generator of the finite cyclic group Z. Since $G = Z \ltimes G_p$ is an open subgroup of the connected reductive algebraic group $G(\mathbf{Q}_p)$, g is an element of finite order in $G(\mathbf{Q}_p)$, hence a semisimple element. So g is contained in some maximal torus $T_{\mathbf{Q}_p}$, not necessarily split. Over some finite extension K of $\mathbf{Q}_p$, T_K is contained in a Borel subgroup B_K. Therefore, when g acts on the Lie algebra $\mathfrak{g}_K$, it acts trivially on the Cartan subalgebra $\mathfrak{t}_K$ and maps the Borel subalgebra $\mathfrak{b}_K$ into itself. We will show that $\chi(\mathfrak{g}_{o_K}, M_{o_K}) = 0$ in the Grothendieck group of finite p-torsion Z-modules, which implies the statement of the proposition.

Briefly, the proofs of Proposition 4.1 and Theorem 5.1 work Z-equivariantly. The Z-equivariant analogue of Proposition 4.1 which we need is the following lemma.

Lemma **11.3.** — *Let $\mathfrak{h}$ be an abelian Lie algebra of the form $(o_K)^r$ for some $r \geqslant 2$. Let* M *be a finitely generated o_K-module with $\mathfrak{h}$-action such that $M_{\mathfrak{h}} \otimes K = 0$. Let Z be a group which acts trivially on $\mathfrak{h}$ and acts compatibly on* M *(in an obvious terminology,* M *is a* $(Z \times \mathfrak{h})$*-module). Then the homology groups* $H_*(\mathfrak{h}, M)$ *are finite and the resulting Euler characteristic* $\chi(\mathfrak{h}, M)$ *in the Grothendieck group* Rep(Z) *of finite* Z*-modules is* 0.

Proof. — First, we show that $\chi(\mathfrak{h}, M) = 0$ in Rep(Z) for any abelian Lie algebra $\mathfrak{h}$ of rank at least 1 as an o_K-module and any finite $(Z \times \mathfrak{h})$-module M. Indeed, in Rep(Z),

$$
\begin{aligned}
\chi(\mathfrak{h}, M) &= \sum_i (-1)^i \wedge^i \mathfrak{h} \otimes_{o_K} M \\
&= \operatorname{rank}\Big(\sum_i (-1)^i \wedge^i \mathfrak{h}\Big) M \\
&= 0 \cdot M \\
&= 0,
\end{aligned}
$$

where the first equality follows from the complex that computes Lie algebra homology, the second equality holds because Z acts trivially on $\mathfrak{h}$, and the third is because $\mathfrak{h}$ has rank at least 1 as an o_K-module.

Now suppose that $\mathfrak{h}$ has rank at least 2 as an o_K-module and that M is finitely generated over o_K, with $M_{\mathfrak{h}} \otimes K = 0$. We know that the Z-modules $H_*(\mathfrak{h}, M)$ are finite by the corresponding nonequivariant statement, Proposition 4.1. The previous paragraph implies that the Euler characteristic $\chi(\mathfrak{h}, M)$ in Rep(Z) only depends on the $(Z \times \mathfrak{h}_K)$-module $M_K := M \otimes_{o_K} K$, so we can use the notation $\chi_{\text{fin}}(\mathfrak{h}_K, M_K) := \chi(\mathfrak{h}, M)$ in Rep(Z), generalizing Definition 2.4. Also, we can extend scalars as in Definition 2.5, so it suffices to show that $\chi_{\text{fin}}(\mathfrak{h}_{\overline{\mathbf{Q}}_p}, M_{\overline{\mathbf{Q}}_p}) = 0$ in Rep(Z) for all $(Z \times \mathfrak{h}_{\overline{\mathbf{Q}}_p})$-modules $M_{\overline{\mathbf{Q}}_p}$ such that the coinvariants of $\mathfrak{h}_{\overline{\mathbf{Q}}_p}$ on $M_{\overline{\mathbf{Q}}_p}$ are 0.

The simple $(Z \times \mathfrak{h}_{\overline{\mathbf{Q}}_p})$-modules are 1-dimensional by Schur's lemma, and the assumption that the coinvariants of $\mathfrak{h}_{\overline{\mathbf{Q}}_p}$ on $M_{\overline{\mathbf{Q}}_p}$ are 0 means that all the simple subquotients of $M_{\overline{\mathbf{Q}}_p}$ as an $\mathfrak{h}_{\overline{\mathbf{Q}}_p}$-module are nontrivial, by Lemma 3.1. So it suffices to show that $\chi_{\text{fin}}(\mathfrak{h}_{\overline{\mathbf{Q}}_p}, M_{\overline{\mathbf{Q}}_p}) = 0$ in Rep(Z) for a 1-dimensional $(Z \times \mathfrak{h}_{\overline{\mathbf{Q}}_p})$-module $M_{\overline{\mathbf{Q}}_p}$ which is nontrivial as an $\mathfrak{h}_{\overline{\mathbf{Q}}_p}$-module. That is, it suffices to show that $\chi(\mathfrak{h}, M) = 0$ in Rep(Z) for any abelian Lie algebra $\mathfrak{h}$ of rank at least 2 over a p-adic ring of integers o_K and any $(Z \times \mathfrak{h})$-module M of rank 1 which is nontrivial as an $\mathfrak{h}$-module.

Since $\mathfrak{h}$ has rank at least 2 as an o_K-module, there is an o_K-submodule $\mathfrak{l} \subset \mathfrak{h}$ such that $\mathfrak{h}/\mathfrak{l} \cong o_K$ and $\mathfrak{l}$ acts nontrivially on M. Then

$$\chi(\mathfrak{h}, M) = \sum_i (-1)^i \chi(\mathfrak{h}/\mathfrak{l}, H_i(\mathfrak{l}, N)).$$

Since Z acts trivially on $\mathfrak{h}$, it preserves $\mathfrak{l}$, and so this is an equality in Rep(Z). We have arranged that $H_i(\mathfrak{l}, N)$ is a finite $(Z \times \mathfrak{h}/\mathfrak{l})$-module for all i, so the individual terms in this sum are 0 in Rep(Z) by the first paragraph of this proof. So $\chi(\mathfrak{h}, M) = 0$ in Rep(Z) as we want. □

To complete the proof of Proposition 11.2, we need to show that $\chi(\mathfrak{g}_{o_K}, M_{o_K}) = 0$ in Rep(Z). We know that M is a $(Z \ltimes \mathfrak{g}_{o_K})$-module and that Z acts trivially on a Cartan subalgebra $\mathfrak{t}_K$ and preserves a Borel subalgebra $\mathfrak{b}_K$ containing $\mathfrak{t}_K$. Then it is clear that the following formula from the proof of Theorem 5.1 holds in Rep(Z):

$$\chi(\mathfrak{g}, M) = \sum_{j,k} (-1)^{j+k} \chi(\mathfrak{b}/\mathfrak{u}, H_j(\mathfrak{u}, M \otimes \Lambda^k(\mathfrak{g}/\mathfrak{b}))).$$

We are assuming in Theorem 11.1 that the algebraic group $G_{\mathbf{Q}_p}$ has rank at least 2 over $\overline{\mathbf{Q}}_p$, so the Lie algebra $\mathfrak{g}_K$ has rank at least 2. That is, $\mathfrak{b}_K/\mathfrak{u}_K$ has dimension at least 2. Also, the group Z acts trivially on $\mathfrak{b}_K/\mathfrak{u}_K \cong \mathfrak{t}_K$, so we can apply Lemma 11.3 to show that all the terms in this sum are 0 in Rep(Z). So $\chi(\mathfrak{g}, M) = 0$ in Rep(Z). □ (Proposition 11.2)

Proposition **11.4.** — *In the notation defined before Proposition 11.2, we have* $\chi(\mathfrak{g}, M) = \chi(G_p, M)$ *in the Grothendieck group of finite p-torsion* Z*-modules.*

Proof. — Since the valuation of G_p is Z-invariant, the proof of Theorem 9.1 works Z-equivariantly. The only point which is not obvious is that Proposition 2.3(2) works Z-equivariantly, given that the Lie algebra $\mathfrak{g}$ over a discrete valuation ring Γ in the proposition has $(\mathfrak{g} \otimes_\Gamma F)^Z$ of dimension at least 2 over the field $F = \Gamma[\pi^{-1}]$, as the following lemma asserts. That hypothesis will be valid for our Lie algebra gr G over the graded discrete valuation ring $\Gamma = \mathbf{F}_p[\pi]$ because $\mathfrak{g}_{\mathbf{Q}_p}^Z$ has dimension at least 2 over $\mathbf{Q}_p$. Indeed, since the reductive algebraic group $G_{\mathbf{Q}_p}$ has rank at least 2 over $\overline{\mathbf{Q}}_p$, every element of $G(\mathbf{Q}_p)$ (in particular, a generator of the cyclic group $Z \subset G(\mathbf{Q}_p)$) has centralizer of dimension at least 2.

Lemma **11.5.** — *Let* Γ *be a discrete valuation ring with uniformizer* π. *Let* $\mathfrak{g}$ *be a Lie algebra over* Γ *which is a finitely generated free* Γ*-module, and* $\mathfrak{h} \subset \mathfrak{g}$ *a Lie subalgebra of the same rank as a free* Γ*-module. Let* M *be a finitely generated free* Γ*-module with* $\mathfrak{g}$*-action. Finally, let* Z *be a group which acts compatibly on* $\mathfrak{g}$, $\mathfrak{h}$, *and* M *such that the trivial* Z*-module over the field* $F = \Gamma[\pi^{-1}]$

occurs with multiplicity at least 2 in $\mathfrak{g} \otimes F$. *Then the relative Lie algebra homology groups* $H_*(\mathfrak{g}, \mathfrak{h}; M)$ *have finite length as* Γ*-modules, and the corresponding Euler characteristic* $\chi(\mathfrak{g}, \mathfrak{h}; M)$ *is* 0 *in the Grothendieck group of* $\Gamma[Z]$*-modules of finite length over* Γ.

Proof. — It suffices to consider the case where $\pi\mathfrak{g} \subset \mathfrak{h} \subset \mathfrak{g}$. We apply that special case to the sequence of Z-invariant Lie subalgebras of $\mathfrak{g}$,

$$\mathfrak{g} \supset \pi\mathfrak{g} + \mathfrak{h} \supset \pi^2\mathfrak{g} + \mathfrak{h} \supset \ldots,$$

which eventually equals $\mathfrak{h}$.

For $\pi\mathfrak{g} \subset \mathfrak{h} \subset \mathfrak{g}$, let $B = \mathfrak{g}/\mathfrak{h}$ and $A = \ker(\mathfrak{g}/\pi \to \mathfrak{g}/\mathfrak{h})$. These are representations of the group Z over the field Γ/π which form an exact sequence

$$0 \to A \to \mathfrak{g}/\pi \to B \to 0.$$

We compute that, in the Grothendieck group Rep(Z) of $\Gamma[Z]$-modules of finite length over Γ,

$$\Lambda^i\, \mathfrak{g}/\Lambda^i\, \mathfrak{h} = \sum_{j=0}^{i} j(\Lambda^{i-j} A \otimes_{\Gamma/\pi} \Lambda^j B),$$

This is proved using a canonical filtration of the finite-length Γ-module $\Lambda^i\, \mathfrak{g}/\Lambda^i\, \mathfrak{h}$ with quotients vector spaces over Γ/π. It follows that, in Rep(Z),

$$\begin{aligned}\sum_i (-1)^i \Lambda^i\, \mathfrak{g}/\Lambda^i\, \mathfrak{h} &= \sum_{j \leqslant i} (-1)^i j(\Lambda^{i-j} A)(\Lambda^j B) \\ &= \sum_{i,j} (-1)^i \Lambda^i (A) \cdot j(-1)^j \Lambda^j B \\ &= F_1(A) F_2(B),\end{aligned}$$

where we define

$$\begin{aligned}F_1(A) &= \sum_i (-1)^i \Lambda^i A \\ F_2(A) &= \sum_i (-1)^i i \Lambda^i A.\end{aligned}$$

The operations F_1 and F_2 take a representation A of Z over Γ/π to an element of the corresponding Grothendieck group. For A of dimension n, the operation $F_1(A) = \Lambda_{-1}(A)$ is related to the top gamma operation γ_n (the top Chern class with values in K-theory) and $F_2(A)$ is related to the operation γ_{n-1}, in the terminology of λ-rings [18]. We do not need that terminology, but only the elementary properties that

$$F_1(A+B) = F_1(A)F_1(B)$$

and

$$F_2(A+B) = F_1(A)F_2(B) + F_2(A)F_1(B).$$

Also, $F_1(1)=0$, so $F_1(A+1)=0$ for all representations A, and $F_2(1) = -1$ and $F_2(2)=0$, so $F_2(A+2)=0$ for all representations A.

The relative Lie algebra homology $H_*(\mathfrak{g}, \mathfrak{h}; M)$ is computed by a chain complex with Z-action, with ith group equal to $(\Lambda^i \mathfrak{g}/\Lambda^i \mathfrak{h}) \otimes_\Gamma M$. From this it is immediate that the Γ-modules $H_*(\mathfrak{g}, \mathfrak{h}; M)$ have finite length. Moreover, the previous paragraph shows that, in the Grothendieck group Rep(Z) of finite-length $\Gamma[Z]$-modules,

$$\begin{aligned}\chi(\mathfrak{g}, \mathfrak{h}; M) &= \sum_i (-1)^i (\Lambda^i \mathfrak{g}/\Lambda^i \mathfrak{h}) \otimes_\Gamma M \\ &= F_1(A)F_2(B)M/\pi,\end{aligned}$$

where $B = \mathfrak{g}/\mathfrak{h}$ and $A = \ker(\mathfrak{g}/\pi \rightarrow \mathfrak{g}/\mathfrak{h})$.

We are assuming that the trivial Z-module over the field F occurs with multiplicity at least 2 in $\mathfrak{g} \otimes F$. It follows easily that the trivial Z-module over the field Γ/π occurs with multiplicity at least 2 in $\mathfrak{g}/\pi$. So it occurs either with multiplicity at least 1 in A or with multiplicity at least 2 in B. By the properties of the operations F_1 and F_2 listed above, either $F_1(A)=0$ or $F_2(B)=0$ in Rep(Z). So $\chi(\mathfrak{g}, \mathfrak{h}; M)=0$ in Rep(Z), as we want. □ (Lemma 11.5 and hence Proposition 11.4).

Theorem 11.1 follows from Propositions 11.2 and 11.4, together with the analysis before Proposition 11.2. □

Corollary **11.6.** — *Let* $G_{\mathbf{Q}_p}$ *be a connected reductive algebraic group, of rank over* $\overline{\mathbf{Q}_p}$ *at least 2. Suppose that p is greater than the dimension of some faithful* $G_{\mathbf{Q}_p}$*-module plus 1. (For* $G_{\mathbf{Q}_p}$ *semisimple, it suffices to assume instead the lower bound for p given in Proposition 12.1.) Let* $M_{\mathbf{Q}_p}$ *be a* $G_{\mathbf{Q}_p}$*-module with no trivial summands. Let* G *be a compact open subgroup of* $G(\mathbf{Q}_p)$*, and let* M *be a* G*-invariant* $\mathbf{Z}_p$*-lattice in* $M_{\mathbf{Q}_p}$*. Then the homology groups* $H_*(G, M)$ *are finite and the resulting Euler characteristic* $\chi(G, M)$ *is* 0.

Proof. — The bound on p in Proposition 12.1 will imply that any Sylow p-subgroup G_p of G admits a valuation, and that the vector space $M_{\mathbf{Q}_p}$ has a compatible valuation. Let us now prove the same statements when p is greater than the dimension of some faithful $G_{\mathbf{Q}_p}$-module plus 1.

Let $N_{\mathbf{Q}_p}$ be a faithful $G_{\mathbf{Q}_p}$-module with $p > \dim(N_{\mathbf{Q}_p}) + 1$. Since G is compact, it preserves some $\mathbf{Z}_p$-lattice N in $N_{\mathbf{Q}_p}$. Any Sylow p-subgroup G_p of G is a subgroup of some Sylow p-subgroup $\mathrm{Iw}_u \subset \mathrm{GL(N)}$. Since $p > \mathrm{rank(N)} + 1$, the proof of Corollary 9.3 shows that Iw_u has a valuation, which we can restrict to G_p, and that N has a compatible saturated valuation.

Since the group $G_{\mathbf{Q}_p}$ is reductive, any $G_{\mathbf{Q}_p}$-module $M_{\mathbf{Q}_p}$ is a direct summand of some direct sum of tensor products $N_{\mathbf{Q}_p}^{\otimes a} \otimes (N_{\mathbf{Q}_p}^*)^{\otimes b}$ for $a, b \geqslant 0$, by [25], II.4.3.2(a), p. 156. The valuation of N induces a vector space valuation

$$w : N_{\mathbf{Q}_p}^{\otimes a} \otimes (N_{\mathbf{Q}_p}^*)^{\otimes b} \to (\infty, \infty],$$

which we can restrict to the subspace $M_{\mathbf{Q}_p}$. Let

$$M_0 = \{x \in M_{\mathbf{Q}_p} : w(x) \geqslant 0\}.$$

Then w is a saturated valuation on M_0. Since the valuation of G_p is compatible with that of N, it is compatible with that of M_0. By Theorem 11.1, given that $M_{\mathbf{Q}_p}$ has no trivial summand, we have $\chi(G, M_0) = 0$. To deduce that $\chi(G, M) = 0$ for all G-invariant lattices M in $M_{\mathbf{Q}_p}$, we use Serre's theorem that $\chi(G, A) = 0$ for all finite $\mathbf{Z}_p$G-modules A, since G is an open subgroup of a connected algebraic group over $\mathbf{Q}_p$ of dimension greater than zero ([29], Corollary to Theorem C). □

For example, Corollary 11.6 implies that $\chi(G, M) = 0$ for all open subgroups G of $\mathrm{SL}_n\mathbf{Z}_p$ when $M = (\mathbf{Z}_p)^n$ is the standard module, $n \geqslant 3$, and $p > n + 1$.

12. Construction of valuations on pro-p subgroups of a semisimple group

In this section, we will improve the bound on p in Corollary 11.6, which says that for p sufficiently large, the Euler characteristics are zero for all compact open subgroups of a reductive group of rank at least 2. By the proof of Corollary 11.6, all we need is to give a weaker sufficient condition on a group G_K so that every closed pro-p subgroup of G(K) is p-valued. Proposition 12.1 will give such a weaker sufficient condition when the group G_K is semisimple. It may be interesting for other purposes to know that every closed pro-p subgroup of G(K) has a valuation; in particular, it follows that G(K) has no p-torsion. The proof combines the Bruhat-Tits structure theory of p-adic groups [32] with a generalization of Lazard's construction of a valuation for pro-p subgroups of GL_nK (given in the proof of Corollary 9.3, above).

For the group $\mathrm{SL}_n\mathbf{Q}_p$, the bound in Corollary 11.6 is optimal: every closed pro-p subgroup of $\mathrm{SL}_n\mathbf{Q}_p$ is p-valued if $p > n+1$. Indeed, if $p = n+1$ and $n \geqslant 2$, then the cyclic group $\mathbf{Z}/p$ imbeds in $\mathrm{SL}_n\mathbf{Z}$ and hence in $\mathrm{SL}_n\mathbf{Q}_p$. For other groups, however, we can do better. For example, let D be a division algebra of degree n (that is, of dimension n^2) over $\mathbf{Q}_p$. Then the proof of Corollary 11.6 shows that every pro-p subgroup of SL_1D is p-valued if $p > n^2 + 1$, but in fact $p > n+1$ is enough, as the following proposition gives, using that SL_1D becomes isomorphic to SL_n over some unramified extension of $\mathbf{Q}_p$.

The need for an improvement in Corollary 11.6 is most apparent for the exceptional groups. For example, $E_8(\mathbf{Q}_p)$ has p-torsion if and only if $p = 2, 3, 5, 7, 11, 13, 19$, or 31 by [29], p. 492, but the proof of Corollary 11.6 shows only that every pro-p subgroup of $E_8(\mathbf{Q}_p)$ is p-valued (hence $E_8(\mathbf{Q}_p)$ has no p-torsion) if $p > 248 + 1$. The following proposition gives the optimal estimate that every pro-p subgroup of $E_8(\mathbf{Q}_p)$ is p-valued if $p > 30 + 1$, since 30 is the Coxeter number of E_8.

To state a sharp bound even for non-split groups, we need to define a generalization of the Coxeter number. Let G_K be an absolutely simple quasi-split group over a field K. (Quasi-split means that G_K has a Borel subgroup defined over K [3].) Such a group is described by its Dynkin diagram over the separable closure $\overline{K}$ of K, of type A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, or G_2, together with an action of the Galois group $\mathrm{Gal}(\overline{K}/K)$ on the Dynkin diagram. That is, the Galois group maps into the automorphism group of the Dynkin diagram, which has order 1 or 2, except that the automorphism group of the Dynkin diagram D_4 is isomorphic to the symmetric group S_3. Equivalently, the Galois group acts on the root system, preserving the set of positive roots. Let $\alpha_1, \ldots, \alpha_n$ be the simple roots. We define the generalized Coxeter number $h(G_K)$ to be the maximum over all positive roots $\alpha = \sum r_i \alpha_i$ of the numbers $(1 + \sum r_i)|\mathrm{Gal}(\overline{K}/K)\alpha|$, unless the Dynkin diagram is of type A_n with n even and the Galois action is nontrivial (so that the universal cover of G_K is a unitary group $SU_{n+1}K$ with $n+1$ odd); in that case, we define $h(G_K)$ to be $2(n+1)$. If G_K is split, meaning that the Galois action is trivial, then $h(G_K)$ is the Coxeter number as defined in Bourbaki [5], VI.1, Proposition 31. Using the notation ${}^iX_n$ to denote a quasi-split group with Dynkin diagram of type X_n where the image of the Galois group has order i, we tabulate the numbers $h(G_K)$ below.

G_K	$h(G_K)$
${}^1A_n$	$n+1$
${}^2A_n$	$2n$ if n is odd
${}^2A_n$	$2(n+1)$ if n is even
B_n	$2n$
C_n	$2n$
${}^1D_n$	$2n-2$
${}^2D_n$	$2n$
${}^3D_4$	12
${}^6D_4$	12
${}^1E_6$	12
${}^2E_6$	18
E_7	18
E_8	30
F_4	12
G_2	6

Proposition **12.1.** — *Let* G_K *be a connected semisimple algebraic group over a p-adic field* K. *Let* e_K *be the absolute ramification degree of* K. *By Steinberg [30],* G_K *becomes quasi-split over some finite unramified extension* E *of* K. *Let* E *be any such extension. The universal covering of* G_E *is a product of restrictions of scalars of absolutely simple quasi-split groups* H_M *over finite extensions* M *of* E. *Suppose that*

$$p > h(H_M)e_M + 1$$

for all the simple factors H_M, *where* $h(H_M)$ *is the generalized Coxeter number as listed above. Then every pro-p subgroup* G_p *of the original group* G(K) *admits a valuation. Moreover, every representation of the algebraic group* G_K *admits a valuation compatible with that of* G_p. *Both valuations take rational values.*

Proof. — We first reduce to the case of a simply connected group. Let $\widetilde{G}_K \to G_K$ be the universal covering of the semisimple group G_K, and let Z be its kernel, which is a finite subgroup of the center of $\widetilde{G}_K$. From the above table, and the known centers of the simple algebraic groups, we see that any prime number p that divides the order of the center of a simply connected semisimple group is at most the maximum of the Coxeter numbers of its simple factors over the algebraic closure. So our assumption on p is more than enough to ensure that Z has order prime to p. By the exact sequence

$$Z(K) \to \widetilde{G}(K) \to G(K) \to H^1(K, Z(\overline{K})),$$

where the groups on the ends are abelian groups in which every element has order prime to p, we see that every pro-p subgroup of G(K) is the isomorphic image of some pro-p subgroup of $\widetilde{G}(K)$. Also, any representation of G_K can be viewed as a representation of $\widetilde{G}_K$. So it suffices to prove the proposition with G_K replaced by $\widetilde{G}_K$, that is, for G_K simply connected.

For a simply connected semisimple group G_K over a p-adic field K, Bruhat and Tits, generalizing earlier work by Iwahori-Matsumoto and Hijikata, defined a conjugacy class of compact open subgroups of G(K), the Iwahori subgroups Iw. A convenient reference is [32], 3.7. Write k for the finite residue field of K. For example, the group of matrices in $SL_n o_K$ whose image in $SL_n k$ is upper-triangular is an Iwahori subgroup of $SL_n K$. Bruhat and Tits showed that every compact subgroup of G(K) is contained in a maximal compact subgroup, and they classified the maximal compact subgroups C. In particular, every maximal compact subgroup contains an Iwahori subgroup. Moreover, using that G_K is simply connected, each maximal compact subgroup C is an extension of the k-points of some connected group $G(k)$ over the finite residue field k by a pro-p group, by [32], 3.5.2. The inverse image of a Borel subgroup $B(k) \subset G(k)$ in C is an Iwahori subgroup, by [32], 3.7. Let Iw_u denote the inverse image of a Sylow p-subgroup $U(k) \subset B(k) \subset G(k)$ in C; then Iw_u is a pro-p group. We see that Iw_u is a Sylow p-subgroup of C. Also, all these subgroups Iw_u in different maximal compact subgroups are conjugate in G(K), since they are all Sylow p-subgroups of Iwahori subgroups. It is natural to call Iw_u the pro-p radical of an Iwahori subgroup, since it is normal in Iw and $\mathrm{Iw}/\mathrm{Iw}_u$ is a finite group of order prime to p. It follows that any pro-p subgroup in the whole group G(K) is contained in some subgroup conjugate to Iw_u.

As a result, to prove the proposition, it suffices to define a valuation on one subgroup $\mathrm{Iw}_u \subset G(K)$, and to show that every representation of the algebraic group G_K admits a valuation compatible with that on Iw_u. Generalizing Lazard's definition of a valuation on $\mathrm{Iw}_u \subset GL_n K$ (given in the proof of Corollary 9.3, above), the idea is to find an element $a \in G(L)$ for some finite extension L of K and a Chevalley group G_{o_L} extending G_L such that $a^{-1}(\mathrm{Iw}_u)a$ is contained in the subgroup of elements $g \in G(o_L)$ with $g \equiv 1 \pmod{m}$ for some $m \in o_L$ with $\mathrm{ord}_p(m) > (p-1)^{-1}$. Then, taking a faithful representation V of G_{o_L}, $a^{-1}(\mathrm{Iw}_u)a$ is contained in the subgroup of elements $g \in GL(V)$ with $g \equiv 1 \pmod{m}$ for some $m \in o_L$ with $\mathrm{ord}_p(m) > (p-1)^{-1}$. So the group $a^{-1}(\mathrm{Iw}_u)a$ has the valuation

$$\omega(g) = \mathrm{ord}_p(g-1) \in (1/e_L)\mathbf{Z}.$$

Moreover, this valuation is compatible with the obvious valuation w on V. It follows that $\mathrm{Iw}_u \subset G(K)$ has a valuation defined by

$$\omega'(g) = \omega(a^{-1}ga),$$

and this is compatible with the valuation of V_L defined by

$$w'(x) = w(a^{-1}x).$$

Every representation of G_L (in particular, any representation of G_K tensored up to L) is a direct summand of a direct sum of tensor products of V_L and V_L^*, by [25], II.4.3.2(a), p. 156. It follows that every representation of G_K has a valuation compatible with that of Iw_u, as we want. Thus, the proposition is proved if we can find an element $a \in G(L)$ as above.

For this purpose, as mentioned in the proposition, we can choose a finite unramified extension E of K such that G_E is quasi-split, by Steinberg [30], Corollary 10.2(a), applied to the maximal unramified extension of K. The original pro-p subgroup Iw_u of G(K) is contained in the analogous subgroup of G(E), so it suffices to prove the same statement for G_E in place of G_K. Since G_E is simply connected and quasi-split, it is a product of restrictions of scalars of absolutely simple quasi-split groups H_M over finite extensions M of E, by [3], 6.21(ii). The pro-p subgroup Iw_u of G(E) is the product of the analogous subgroups for the simple factors, so it suffices to consider the simple factors.

That is, writing G_K in place of H_M, we are given an absolutely simple quasi-split group G_K over a p-adic field K such that $p > h(G_K)e_K + 1$. Write Iw_u for the pro-p radical of an Iwahori subgroup of G(K), as above. To prove the proposition, it suffices to find an element $a \in G(L)$ for some finite extension L of K and a Chevalley group G_{o_L} extending G_L such that $a^{-1}(\mathrm{Iw}_u)a$ is contained in the subgroup of elements $x \in G(o_L)$ with $x \equiv 1 \pmod{m}$ for some $m \in o_L$ with $\mathrm{ord}_p(m) > (p-1)^{-1}$.

For G_K of type ${}^2A_n$ with n even, so that G_K is the unitary group $SU_{n+1}K$ associated to a quadratic extension L/K, we defined $h(G_K) = 2(n+1)$, while $h(G_L)$ is the Coxeter number of $SL_{n+1}L$, that is, $n+1$. The assumption that $p > h(G_K)e_K + 1$ implies that $p > h(G_L)e_L + 1$. Thus, if we can prove the above statement for G_L in place of G_K, then the statement for G_K follows. So we can assume from now on that G_K is not of type ${}^2A_n$ with n even. Equivalently, the relative root system Φ of G_K (defined below, or see Borel-Tits [3]) is reduced; that is, there is no root a such that $2a$ is also a root. (If G_K is split, its relative root system is reduced. Otherwise, G_K is of type ${}^2A_{2m}$, ${}^2A_{2m-1}$, ${}^2D_n$, ${}^2E_6$, ${}^3D_4$, or ${}^6D_4$, and then the relative root system is of type BC_m, C_m, B_{n-1}, F_4, G_2, or G_2, respectively, of which only BC_m is non-reduced.)

To prove the above statement, we need a more explicit description of an Iwahori subgroup of G(K), following [8], section 4. Let S be a maximal split torus in G_K. Since G_K is quasi-split, the centralizer T of S is a maximal torus in G_K, and there is a Borel subgroup B defined over K that contains T. Let $\Phi \subset X^*(S)$ be the set of roots of G_K relative to S, and let $U_a \subset G(K)$ be the unipotent subgroup corresponding to $a \in \Phi$. We know that the root system Φ is reduced because we have arranged that G_K is not of type ${}^2A_n$ with n even.

Also, let $\widetilde{K}$ be the Galois extension of K which corresponds to the kernel of the action of the Galois group $\mathrm{Gal}(\overline{K}/K)$ on $X^*(T)$, let $\widetilde{\Phi} \subset X^*(T)$ be the set of roots of $G_{\widetilde{K}}$ relative to T, and let $\widetilde{U}_\alpha \subset G(K)$ be the unipotent subgroup corresponding to $\alpha \in \widetilde{\Phi}$. We can choose isomorphisms $x_\alpha : K \to \widetilde{U}_\alpha$ which satisfy the compatibility conditions with the action of the Galois group Gal_K on $\widetilde{\Phi}$ needed to form a "Chevalley-Steinberg system" by [8], 4.1.3. These define a valuation of the root datum $(T(\widetilde{K}), \widetilde{U}_\alpha : \alpha \in \widetilde{\Phi})$, meaning a set of functions $\widetilde{\varphi}_\alpha : \widetilde{U}_\alpha \to (-\infty, \infty]$ satisfying certain properties, by

$$\widetilde{\varphi}_\alpha(x_\alpha(u)) = \mathrm{ord}_p u.$$

In particular, the subsets $\widetilde{U}_{\alpha,c} = \widetilde{\varphi}_\alpha^{-1}([c,\infty])$ and $\widetilde{U}_{\alpha,c+} = \widetilde{\varphi}_\alpha^{-1}((c,\infty])$ are subgroups of $\widetilde{U}_\alpha$ for all real numbers c.

The Chevalley system of the split group $G_{\widetilde{K}}$ determines a model G_{o_K} of G_K over the ring of integers $o_{\widetilde{K}}$ which is a Chevalley group; see [8], proof of 4.6.15. By the construction, the obvious integral model $\widetilde{\mathscr{U}}_{\alpha,0} \cong o_{\widetilde{K}}$ of $\widetilde{U}_\alpha \cong K$ is a closed subgroup of G_{o_K}.

The Chevalley-Steinberg system also determines a valuation of the root datum $(\mathrm{T}, \mathrm{U}_a : a \in \Phi)$, by [8], 4.2. The definition is simplest in the case we need here, where Φ is reduced. Namely, any element u of U_a can be written uniquely as a product

$$u = \prod_{\alpha \in \mathrm{A}} \widetilde{u}_\alpha$$

where α runs over the set A of roots in $\widetilde{\Phi}$ that restrict to a, ordered in some fixed way. The functions $\varphi_a : \mathrm{U}_a \to (-\infty, \infty]$ are defined by

$$\varphi_a(u) = \inf_{\alpha \in \mathrm{A}} \widetilde{\varphi}_\alpha(\widetilde{u}_\alpha).$$

Moreover, since Φ is reduced, the Chevalley-Steinberg system of $\mathrm{G}(\widetilde{\mathrm{K}})$ induces an isomorphism

$$x_a : \mathrm{L}_a \to \mathrm{U}_a$$

for every root $a \in \Phi$, where $\mathrm{L}_a \subset \widetilde{\mathrm{K}}$ is the extension field of K corresponding to the subgroup of $\mathrm{Gal}(\widetilde{\mathrm{K}}/\mathrm{K})$ which fixes some root $\alpha \in \widetilde{\Phi}$ that restricts to a. By [8], 4.2.2, the valuation of U_a is given in terms of this isomorphism by

$$\varphi_a(x_a(u)) = \mathrm{ord}_p u$$

for $u \in \mathrm{L}_a$. Combining the two descriptions, we can say that the subgroup $\mathrm{U}_{a,0} \subset \mathrm{G}(\mathrm{K})$ is contained in the subgroup of $\mathrm{G}(\widetilde{\mathrm{K}})$ generated by $\mathrm{U}_{\alpha,0}$ for roots $\alpha \in \widetilde{\Phi}$ restricting to a. Likewise, $\mathrm{U}_{a,0+} \subset \mathrm{G}(\mathrm{K})$ is contained in the subgroup of $\mathrm{G}(\widetilde{\mathrm{K}})$ generated by $\mathrm{U}_{\alpha,(c(\alpha)e_{\mathrm{K}})^{-1}}$ for roots $\alpha \in \widetilde{\Phi}$ restricting to a, where $c(\alpha) := |\mathrm{Gal}(\widetilde{\mathrm{K}}/\mathrm{K})\alpha| = 1, 2$, or 3. This uses that L_a is an extension of degree $c(\alpha)$ of K. We should add that the roots α of $\widetilde{\Phi}$ which restrict to a given element of Φ form a single orbit under the Galois group, by Borel and Tits [3], 6.4(2).

The reason for the above comments is that we can define an Iwahori subgroup Iw of G(K) as the subgroup generated by the subgroups $\mathrm{U}_{a,0}$ for all positive roots $a \in \Phi$, $\mathrm{U}_{a,0+}$ for all negative roots $a \in \Phi$, and the maximal compact subgroup H of T(K), by [7], 6.4.2 and 7.2.6 (where we take x to be the origin of the affine space A corresponding to the given valuation φ). Since $\mathrm{U}_{a,0}$ and $\mathrm{U}_{a,0+}$ are pro-p groups, they map trivially into the quotient group $\mathrm{Iw}/\mathrm{Iw}_u$, of order prime to p. So the pro-p radical Iw_u of Iw is the subgroup of G(K) generated by the subgroups $\mathrm{U}_{a,0}$ for positive roots $a \in \Phi$, $\mathrm{U}_{a,0+}$ for negative roots $a \in \Phi$, and the maximal pro-p subgroup H_u of T(K). By the previous paragraph, it follows that Iw_u is contained in the subgroup of $\mathrm{G}(\widetilde{\mathrm{K}})$ generated by $\widetilde{\mathrm{U}}_{\alpha,0}$ for positive roots $\alpha \in \widetilde{\Phi}$, $\widetilde{\mathrm{U}}_{\alpha,(c(\alpha)e_{\mathrm{K}})^{-1}}$ for negative roots $\alpha \in \widetilde{\Phi}$, and the group $\widetilde{\mathrm{H}}_u := \{x \in \mathrm{T}(\widetilde{\mathrm{K}}) : x \equiv 1 \pmod{\pi_{\widetilde{\mathrm{K}}}}\}$.

Let $\alpha_1, \ldots, \alpha_n$ be the simple roots of $\widetilde{\Phi}$. We have assumed that $p-1 > h(G_K)e_K$, which means (since G_K is not of type ${}^2A_n$ with n even) that $p-1 > (1+\sum r_i)c(\alpha)$ for all positive roots $\alpha = \sum r_i\alpha_i$ in $\widetilde{\Phi}$. Equivalently, for all positive roots $\alpha = \sum r_i\alpha_i$,

$$\frac{\sum r_i}{p-1} < \frac{1}{c(\alpha)e_K} - \frac{1}{p-1}.$$

It follows that there is an element $x \in X_*(T) \otimes \mathbf{Q}$ such that $\langle x, \alpha_i\rangle > 1/(p-1)$ for all the simple roots α_i and $\langle x, \alpha\rangle < (c(\alpha)e_K)^{-1} - (p-1)^{-1}$ for all positive roots α.

We can find an element $a \in T(L)$ for some finite extension L of $\widetilde{K}$ whose absolute value is $x \in X_*(T) \otimes \mathbf{Q}$. (We identify a cocharacter of T, $f\colon G_m \to T$, with $|f(p)|$.) It follows that $a^{-1}(\mathrm{Iw}_u)a$ is contained in the subgroup of $G(L)$ generated by the subgroups $U_{\alpha, \langle x, \alpha\rangle}$ for positive roots $\alpha \in \widetilde{\Phi}$, $U_{-\alpha, (c(\alpha)e_K)^{-1} - \langle x, \alpha\rangle}$ for negative roots $-\alpha$, and $\widetilde{H}_u := \{x \in T(\widetilde{K}) : x \equiv 1 \pmod{\pi_{\widetilde{K}}}\}$. The inequalities on x imply that the first two subgroups are contained in the subgroups $U_{\alpha, 1/(p-1)^+}$ for all roots $\alpha \in \widetilde{\Phi}$. We check immediately from the table of values of $h(G_K)$ that $h(G_K) \geqslant [\widetilde{K} : K]$, so that the assumption $p-1 > h(G_K)e_K$ implies that $p-1 > e_{\widetilde{K}}$, or in other words $e_{\widetilde{K}}^{-1} > (p-1)^{-1}$. It follows that the subgroup $\widetilde{H}_u$ is contained in $\{x \in T(\widetilde{K}) : \mathrm{ord}_p(x-1) > (p-1)^{-1}\}$.

In terms of the Chevalley group G_{o_K} extending G_K discussed earlier, these statements say that $a^{-1}(\mathrm{Iw}_u)a$ is contained in the subgroup of elements $x \in G(o_L)$ with $x \equiv 1 \pmod{m}$ for some $m \in o_L$ with $\mathrm{ord}_p(m) > (p-1)^{-1}$. This completes the proof, as explained earlier. □

13. Open subgroups of $SL_2\mathbf{Z}_p$

We will now show that the assumption that $G_{\mathbf{Q}_p}$ has rank at least 2 in Theorem 11.1 and Corollary 11.6 is essential. Some examples of nonzero Euler characteristics for open subgroups of $SL_2\mathbf{Z}_p$ follow already from Proposition 6.2, combined with Corollary 9.3. Those examples involve prime numbers p which are small compared to the representation considered. For example, if G is an open pro-p subgroup of $SL_2\mathbf{Z}_p$, $p \geqslant 5$, and M is the standard module $M = (\mathbf{Z}_p)^2$, then those results just say that $\chi(G, M) = 0$. In this section, we will show that for any prime $p \geqslant 5$, there is an open subgroup G in $SL_2\mathbf{Z}_p$, necessarily not a pro-p group, such that $\chi(G, M)$ is not zero. We do this by computing all the homology groups $H_*(G, M)$ for a natural class of subgroups of $SL_2\mathbf{Z}_p$.

Proposition **13.1.** — *Let $p \geqslant 5$ be a prime number, and G be the inverse image in $SL_2\mathbf{Z}_p$ of some subgroup Q of $SL_2\mathbf{Z}/p$. Let $M = (\mathbf{Z}_p)^2$ be the standard representation of G. Then the homology groups $H_*(G, M)$ are zero unless Q is either the trivial group, a cyclic group $\mathbf{Z}/3$, a*

Sylow p-subgroup $\mathbf{Z}/p$, *or a semidirect product* $\mathbf{Z}/3 \ltimes \mathbf{Z}/p$. *In those four cases, the homology groups* $H_*(G, M)$ *are* $\mathbf{F}_p$*-vector spaces, zero except in degrees* 0, 1, 2, *of dimensions*

$$\begin{aligned} &2, 4, 2 \text{ if } Q = 1 \\ &0, 2, 0 \text{ if } Q \cong \mathbf{Z}/3 \\ &1, 2, 1 \text{ if } Q \cong \mathbf{Z}/p \\ &0, 2, 0 \text{ if } Q \cong \mathbf{Z}/3 \ltimes \mathbf{Z}/p. \end{aligned}$$

So the Euler characteristic $\chi(G, M)$ *is* 0 *unless* Q *is isomorphic to* $\mathbf{Z}/3$ *or* $\mathbf{Z}/3 \ltimes \mathbf{Z}/p$, *in which case it is* -2.

In particular, for every $p \geqslant 5$, the group $SL_2\mathbf{Z}/p$ contains a subgroup Q of order 3, and the Proposition implies that the inverse image G of Q in $SL_2\mathbf{Z}_p$ has $\chi(G, M)$ equal to -2, not zero. This is the counterexample described above. For $p \equiv 1 \pmod 3$, $SL_2\mathbf{Z}/p$ also has a subgroup isomorphic to $\mathbf{Z}/3 \ltimes \mathbf{Z}/p$, giving another counterexample.

Proof. — We first prepare to analyze subgroups Q of $SL_2\mathbf{Z}/p$ of order prime to p. Let G_1 be the congruence subgroup

$$\ker(SL_2\mathbf{Z}_p \to SL_2\mathbf{Z}/p).$$

By Corollary 10.2, the homology groups $H_*(G_1, M)$ are $\mathbf{F}_p$-vector spaces, zero except in degrees 0, 1, 2, of dimensions 2, 4, 2. Moreover, the action of $SL_2\mathbf{Z}/p$ on these groups is given by

$$\begin{aligned} H_0(G_1, M) &= M_{\mathbf{F}_p} \\ H_1(G_1, M) &= \mathfrak{sl}_2\mathbf{F}_p - M_{\mathbf{F}_p} \\ H_2(G_1, M) &= \Lambda^2\, \mathfrak{sl}_2\mathbf{F}_p - \mathfrak{sl}_2\mathbf{F}_p + M_{\mathbf{F}_p} \end{aligned}$$

in the Grothendieck group of finite p-torsion $SL_2\mathbf{Z}/p$-modules. Using the representation theory of SL_2, we compute that these homology groups are $M_{\mathbf{F}_p}$, $S^3M_{\mathbf{F}_p}$, $M_{\mathbf{F}_p}$ in this Grothendieck group. In fact, these $SL_2\mathbf{Z}/p$-modules are simple, since $p \geqslant 5$, and so the homology groups $H_*(G_1, M)$ are actually isomorphic to these $SL_2\mathbf{Z}/p$-modules. The restrictions of these modules to the diagonal torus $(\mathbf{Z}/p)^*$ in $SL_2\mathbf{Z}/p$ have the form

$$\begin{aligned} H_0(G_1, M) &= L^{-1} + L \\ H_1(G_1, M) &= L^{-3} + L^{-1} + L + L^3 \\ H_2(G_1, M) &= L^{-1} + L, \end{aligned}$$

where L is the standard 1-dimensional representation over $\mathbf{F}_p$ of the group $(\mathbf{Z}/p)^*$. The restrictions of these modules to a non-split torus $\ker(\mathbf{F}_{p^2}^* \to \mathbf{F}_p^*)$ in $SL_2\mathbf{Z}/p$ have the same form, after extending scalars from $\mathbf{F}_p$ to $\mathbf{F}_{p^2}$.

Let Q be a subgroup of $SL_2\mathbf{Z}/p$ of order prime to p, and let G be its inverse image in $SL_2\mathbf{Z}_p$. Then the homology groups $H_*(G, M)$ are the coinvariants of Q acting on $H_*(G_1, M)$. Since every element of $SL_2\mathbf{Z}/p$ of order prime to p belongs to some torus, possibly non-split, the calculation of $H_*(G_1, M)$ shows that $H_*(G, M) = 0$ if Q contains any elements of order not equal to 1 or 3. The Sylow 3-subgroup of $SL_2\mathbf{Z}/p$ is cyclic, so this leaves only the cases $Q = 1$ and $Q \cong \mathbf{Z}/3$. We read off from the calculation of $H_*(G_1, M)$ that the $\mathbf{F}_p$-vector spaces $H_*(G, M)$ have dimension 2, 4, 2 for $Q = 1$ and 0, 2, 0 for $Q \cong \mathbf{Z}/3$, as we want.

Next, let Iw_u be the inverse image in $SL_2\mathbf{Z}_p$ of the strictly upper-triangular matrices in $SL_2\mathbf{Z}/p$. Since $p \geqslant 5$, the group Iw_u has a valuation as described in the proof of Corollary 9.3, and M has a compatible saturated valuation. So we have a spectral sequence

$$H_*(\mathrm{gr}\ Iw_u, \mathrm{gr}\ M) \Rightarrow H_*(Iw_u, M)$$

as in the proof of Theorem 9.1. Moreover, the diagonal subgroup $\mathbf{Z}_p^* \subset SL_2\mathbf{Z}_p$ normalizes Iw_u, preserving its valuation, and acts compatibly on M, so it acts on this spectral sequence. The Lie algebra homology is easy to compute, and the spectral sequence degenerates for degree reasons. The result is that the groups $H_*(Iw_u, M)$ are $\mathbf{F}_p$-vector spaces of dimensions 1, 2, 1, on which $\mathbf{Z}_p^*$ acts by

$$\begin{aligned} H_0(Iw_u, M) &= L^{-1} \\ H_1(Iw_u, M) &= L^{-3} + L^3 \\ H_2(Iw_u, M) &= L. \end{aligned}$$

Here L is the standard 1-dimensional representation over $\mathbf{F}_p$ of the quotient group $(\mathbf{Z}/p)^*$ of $\mathbf{Z}_p^*$.

Let Q be any subgroup of $SL_2\mathbf{Z}/p$ of order a multiple of p. By conjugating Q, we can assume that it contains the Sylow p-subgroup $U \cong \mathbf{Z}/p$ of strictly upper-triangular matrices. The normalizer of U in $SL_2\mathbf{Z}/p$ is the Borel subgroup $B = (\mathbf{Z}/p)^* \ltimes \mathbf{Z}/p$. Let G be the inverse image of Q in $SL_2\mathbf{Z}_p$; then $H_*(G, M)$ is a quotient of the coinvariants of $Q \cap (\mathbf{Z}/p)^*$ on $H_*(Iw_u, M)$. Using the calculation of $H_*(Iw_u, M)$, it follows that $H_*(G, M) = 0$ unless $Q \cap (\mathbf{Z}/p)^*$ has order 1 or 3. So suppose that $Q \cap (\mathbf{Z}/p)^*$ has order 1 or 3. Then Q is contained in the normalizer B of U in $SL_2\mathbf{Z}/p$, since any subgroup of $SL_2\mathbf{Z}/p$ which contains U but is not contained in B must contain two distinct subgroups of order p, hence the subgroup they generate, which is the whole group $SL_2\mathbf{Z}/p$. Thus, either $Q = U$ or $Q \cong \mathbf{Z}/3 \ltimes U$, and we can read off $H_*(G, M)$ in these two cases as the coinvariants of Q/U on $H_*(Iw_u, M)$. The dimensions of the $\mathbf{F}_p$-vector spaces $H_*(G, M)$ are 1, 2, 1 for $Q = U$ and 0, 2, 0 for $Q \cong \mathbf{Z}/3 \ltimes U$. □

REFERENCES

[1] M. Boardman, Conditionally convergent spectral sequences. In *Homotopy invariant algebraic structures* (Baltimore, 1998), ed. J.-P. Meyer, J. Morava, and W. S. Wilson, AMS (1999), 49-84.

[2] R. Boltje, Mackey functors and related structures in representation theory and number theory. Habilitation-Thesis, Universitat Augsburg (1995). Also at: http://www.math.uni-augsburg.de/~boltje/publications.html

[3] A. Borel and J. Tits, Groupes réductifs. *Publ. Math. IHES* **27** (1965), 55-150.

[4] N. Bourbaki, *Groupes et algèbres de Lie. Chapitre 1.* Hermann (1960).

[5] N. Bourbaki, *Groupes et algèbres de Lie. Chapitres 4, 5, et 6.* Hermann (1968).

[6] N. Bourbaki, *Groupes et algèbres de Lie. Chapitres 7 et 8.* Diffusion C.C.L.S. (1975).

[7] F. Bruhat and J. Tits, Groupes réductifs sur un corps local, I: Données radicielles valuées. *Publ. Math. IHES* **41** (1972), 5-251.

[8] F. Bruhat and J. Tits, Groupes réductifs sur un corps local, II: Schémas en groupes. Existence d'une donnée radicielle valuée. *Publ. Math. IHES* **60** (1984), 5-184.

[9] A. Brumer, Pseudocompact algebras, profinite groups and class formations. *J. Alg.* **4** (1966), 442–470.

[10] E. Cartan, Sur les invariants intégraux de certains espaces homogènes. *Ann. Société Pol. de Math.* **8** (1929), 181-225.

[11] H. Cartan and S. Eilenberg, *Homological algebra.* Princeton University Press (1956).

[12] C. Chevalley and S. Eilenberg, Cohomology theory of Lie groups and Lie algebras. *Trans. AMS* **63** (1948), 85-124.

[13] J. Coates, Fragments of the GL_2 Iwasawa theory of elliptic curves without complex multiplication. In *Arithmetic theory of elliptic curves* (Cetraro, 1997), ed. C. Viola, Lecture Notes in Mathematics 1716, Springer (1999).

[14] J. Coates and S. Howson, Euler characteristics and elliptic curves II. *J. Math. Soc. Japan,* to appear.

[15] J. Coates and R. Sujatha, Euler-Poincaré characteristics of abelian varieties. *C. R. Acad. Sci. Paris* **329** (1999), 309-313.

[16] C. Curtis and I. Reiner, *Methods of representation theory,* v. 2. John Wiley (1987).

[17] W. Fulton and J. Harris, *Representation theory.* Springer (1991).

[18] W. Fulton and S. Lang, *Riemann-Roch algebra.* Springer (1985).

[19] D. Gluck, Idempotent formula for the Burnside algebra with applications to the p-subgroup complex. *Ill. J. Math.* **25** (1981), 63-67.

[20] G. Hochschild and J.-P. Serre, Cohomology of Lie algebras. *Ann. Math.* **57** (1953), 591–603. Also in J.-P. Serre, *Œuvres,* Springer (1986), v. 1, 152-164.

[21] J. Jantzen, *Representations of algebraic groups.* Academic Press (1984).

[22] B. Kostant, Lie algebra homology and the generalized Borel-Weil theorem. *Ann. Math.* **74** (1961), 329-387.

[23] M. Lazard, Groupes analytiques p-adiques. *Publ. Math. IHES* **26** (1965), 389-603.

[24] J.-P. May, The cohomology of restricted Lie algebras and of Hopf algebras. *J. Alg.* **3** (1966), 123-146.

[25] N. Saavedra Rivano, *Catégories tannakiennes.* Lecture Notes in Mathematics 265, Springer (1972).

[26] J.-P. Serre, *Algèbre locale. Multiplicités.* Lecture Notes in Mathematics 11, Springer (1965).

[27] J.-P. Serre, Sur la dimension cohomologique des groupes profinis. *Topology* **3** (1965), 413-420.

[28] J.-P. Serre, *Cohomologie galoisienne.* Lecture Notes in Mathematics 5, 5th edition, Springer (1994).

[29] J.-P. Serre, La distribution d'Euler-Poincaré d'un groupe profini. In *Galois representations in arithmetic algebraic geometry* (Durham, 1996), ed. A. Scholl and R. Taylor, Cambridge University Press (1998), 461-493.

[30] R. Steinberg, Regular elements of semi-simple algebraic groups. *Publ. Math. IHES* **25** (1965), 49-80.

[31] P. SYMONDS and T. WEIGEL, Cohomology for p-adic analytic groups. To appear in *New horizons in pro-p groups*, ed. M. du Sautoy, D. Segal, and A. Shalev, Birkhauser.

[32] J. TITS, Reductive groups over local fields. In *Automorphic forms, representations, and L-functions* (Corvallis, 1977), ed. A. Borel and W. Casselman, AMS (1979), 29-69.

B. T.
DPMMS, University of Cambridge, Wilberforce Road,
Cambridge CB3 0WB, England
b.totaro@dpmms.cam.uk

Manuscrit reçu le 9 novembre 1999.