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Reflections and twists

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REFLECTIONS AND TWISTS*

by Arthur JAFFE

I. Life as a student

My extraordinary first-hand introduction to the Feldverein took place 35 years ago, during the year I spent at the IHÉS. The path I followed from Princeton to Bures-sur-Yvette had a definite random element, so please bear with a bit of autobiographical perspective. I began as an undergraduate majoring in experimental chemistry, originally thinking of going into medicine, but eventually moving toward the goal to become a theoretical chemist. With luck, I received a Marshall Scholarship to do just that in Cambridge, and upon advice from my undergraduate mentor Charles Gillispie, I applied and was admitted to Clare College. But the spring before sailing with the other Marshall scholars to Southampton on the QE II, second thoughts surfaced about continuing in chemistry. After some soul searching, I decided to follow the constant encouragement of my other mentor, Don Spencer, and to delve into the fascinating but less scrutable world of mathematics and physics.

I had to explain this to J.F. Foster, the man who administered the Marshall scholarships, and whose letterhead indicated he was the Director of the "Association of Universities of the British Commonwealth". A kindly man, Foster eventually replied with advice that went something like "I recommend that you pursue your new interests at Clare, though formal approval by the Marshall Commission for this change will probably take some time. In fact I would not be surprised if permission for your plan comes only after the completion of your studies". I decided that it was good not to worry about details over which I had no control; at least I did have a definite short-term goal, and the tutor John Northam at Clare had no problem with my plan.

So for two years at Cambridge I studied mathematics, with an emphasis on mathematical physics; friends of mine steered me during that time toward reading papers of Lehmann, Symanzik, Zimmermann, and Glaser, as well as some others by Arthur Wightman. I found them fascinating, and I felt certain that I wanted to get involved in this type of research;

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the fundamental nature of the work attracted me (1). Arthur Wightman was a name that I had never encountered as a Princeton undergraduate, and this discovery eventually meant that I set my sights on returning to Princeton. A similar thought two years earlier would have horrified me, for in those days Princeton undergraduates had exaggerated opinions on life at the graduate college. I arrived back at Princeton a bit older than the other entering students, but I had in fact seen (and tried to learn) a good deal.

Wightman was on leave my first year, so I had a fair amount of time to myself. During the spring, after corresponding with a couple of former Cambridge compatriots, we formulated plans to attend a school at Robert College in Istanbul. This not only gave me an opportunity to see several friends, but it also presented a chance to learn some different things, and to meet a new set of people. Louis Michel gave a course on group theory in physics at Robert College, and he asked me to assist him to edit his lecture notes. I am not sure how helpful I actually was, but I did learn some group theory and also thoroughly enjoyed discussions with Michel. At the end of that summer, I was back in Princeton.

The person who organized the lives of Arthur Wightman's students was his strong-minded wife Anna-Greta, who never hid her likes nor her dislikes. Anna-Greta often formulated ambitious plans for photography, sports, excursions, or social occasions, and these projects occasionally involved us students. Though she was intimidating, we liked her. Anna-Greta gave us all advice, about Princeton as well as about life in general. One bit of advice that Anna-Greta tried to give me, on more than one occasion, was the message that I was too shy for my own good. This I found hard to change and didn't try. I was pleasantly surprised when Arthur one day explained that he planned to spend the following year on sabbatical at Bures-sur-Yvette, and moreover, if I were interested, he would inquire whether (my good friend) Oscar Lanford and I could spend the year with him. What a wonderful present! It took no thought at all before I graciously accepted.

II. The Valley of the Chevreuse

I wish that I could remember the first moment I entered the IHÉS. But although the swinging wrought iron gates at the main entrance on the Route de Chartres remain riveted in my mind, I cannot even recall whether I flew directly from New York to Paris, or whether I first went to pick up the small red car that would return with me to Princeton one year later. I do recall that the airlines planned to charge a good deal to transport the amount of luggage that Anna-Greta and I thought would be needed for a year abroad. So we pooled resources: the Wightmans took my luggage along with theirs by oceanliner and I flew to Europe, while I returned with the luggage on the Normandie.

Somehow I became installed in the Résidence de Chevreuse in Orsay, where IHÉS visitors stayed in the days between the IHÉS moving to Bures and the era of L'Ormaille. The Wightmans, the Artins visiting for the year from MIT (and who had a piano that I

(1) During my second year at Cambridge, I worked for several months at the Cavendish on a problem concerning magneto resistance. It turned out that my frequent tea companion during that time, the student Brian Josephson, also was pursuing fundamental work.

occasionally tried to play), the Lanfords ([1]), after some time the Glasers from Geneva, other IHÉS visitors and professors, some Orsay researchers, and many non-academic persons all lived in the Résidence de Chevreuse. My home was on the ground floor, in a small but attractive maid's room, one of three along a corridor with a shared shower, etc. The other two maid's rooms in my building housed Sergio Doplicher and the Glaser's maid. Sergio had an expresso maker, and if I were patient he would make a morning coffee for us, over which he was especially fond of discussing some aspect of von Neumann algebras, or papers by Araki, Haag, or Kastler. Then we often went in my car to the institute, stopping in the village of Orsay along the way to see if the Corriere della Serra could be found.

When there was a respite in work, I took my car to explore the countryside, both in the Valley of the Chevreuse, and on the hills that surrounded it. I especially enjoyed visiting nearby villages, some of which had country houses and gardens open to the public. On a couple of occasions, I ventured as far as Chartres. A favorite local trip led to the hill just above the IHÉS, which was covered by strawberry farms (now Les Ulis). For a few francs, one could buy a huge basket of delicious, ripe berries. But they would not last, generally becoming covered by mold the next day. That was not my worst encounter with nature in Orsay. I quickly learned to take precautions against the mosquitoes, as they seemed unusually attracted to me. During the mosquito season, I rarely opened the window in my room. I also enjoyed other sports; finding a parking space in Paris, or adventuring around various traffic circles, from small ones in Bures to the grand challenge of the Étoile. Later in the evening Paris offered movies, plays, and concerts. I recall going to the Théâtre des Champs-Élysées to attempt to enter a sold-out recital by Joan Sutherland, then at the pinnacle of her operatic career. An elegantly dressed Frenchman standing near the box-office observed my disappointment and approached. He handed me a third row center ticket with a smile, refusing any payment, saying only that he hoped he would be able to give the ticket to someone who appreciated the concert.

I have vivid recollections of Annie Roland visiting my room in the Résidence. She first came to ensure that everything was in order. I believe that I was the first IHÉS occupant of that room; Annie explained that she had chosen the furniture and had arranged it herself. Whenever there actually was a problem, small or large, Annie always had it rectified promptly. And when I got the flu, she made sure that I had food and drink or flowers or something to read to cheer me up. My memories of that year at the IHÉS center about the wonderful atmosphere for work, on the one hand, and about Annie, the spirit of the IHÉS, on the other. While Léon Motchane had little interaction with the students, Annie played a pivotal role for us in "running" the Institute.

I got to know her in stages. Eventually she invited me to a wonderful evening at her apartment in the Résidence Gratien. Not only did the other guests and I enjoy her company and hospitality, but the food was a veritable feast, both for the eye as well as the stomach. I

([1]) Oscar and Regina had an infant Lizabeth, who later was my advisee as a Harvard student, and now is a doctor.

wanted to reciprocate, but had not spent much time visiting restaurants beyond the Croix de Bures (which was good and a place I often had dinner), or the university cafeteria on top of the hill across the valley in Orsay (where I went on occasion to see a friend, but not because of the food). The IHÉS had an excellent cook, and by having a full lunch, including a good deal of bread and dessert, I did not always want an evening meal, or I could make do with some bread and cheese. After a while, I did find a Paris restaurant that seemed sufficiently interesting to invite Annie. It was housed in a basement on rue Laplace, near the Panthéon, and has disappeared years ago. The evening pleased Annie tremendously, and I was proud of her telling how she recommended the place to others. Annie often had recommendations about Paris. She was a wealth of information about things to see or to do, and about books to read, if one had time. She would also be interested in my reactions to her recommendations, and she would always inquire if the right moment arose.

Annie was not just interested in our being happy while visiting her institute. It appeared that she genuinely wanted to know of our personal lives. She had a great curiosity about each of us, about our aspirations and hopes. What was my route to mathematical physics? What was it like to be a student in Princeton? Did I plan to become a professor? Where? Did I hope to be at Princeton? It was clear that Annie was proud of each visitor who came to Bures, and she prided herself on sharing with them. Years later, I was extremely happy for her when she moved to Aix and married Léon. Even though I met them only on a few occasions from that time until they returned to Paris, they remained in my heart.

In January 1993, my wife Sarah and I, along with another friend, invited Annie for dinner at a restaurant we like on the Ile St. Louis. Again we had a wonderful time. A day or two afterward, Sarah and I went for tea at Annie's apartment at 69 rue Claude-Bernard. She had sold her place in the rue Tournefort, next to the eccentric Hotel Parisiana, and moved into a one-room apartment where her mother spent the last years of her life. Sarah and I compared notes after our visit, remarking how Annie used her limited space with great efficiency. Everything appeared perfectly in place; though her room overflowed with books, it appeared much more spacious than it was. I remember encouraging Annie to visit in Boston and to think of taking a job. We planned to discuss that after a while. Sarah and I were returning to the US a few days later, but we did have another trip to Paris scheduled; so we made plans to see Annie at that time. That meeting never happened.

III. The IHÉS

The academic year 1963–64 was an especially active one for mathematical physics in Bures. Louis Michel was very much in evidence. Harry Lehmann was also a professor there, though he oscillated between Paris and Hamburg. In addition to Arthur Wightman, there was a set of other outstanding visitors: Hans Borchers, Jurko Glaser, Henri Epstein, and Jean Lascoux. Furthermore, Jacques Bros from Saclay spent a good deal of time at the IHÉS, collaborating with Epstein and Glaser, on their program that established analyticity in quantum field theory. Another frequent visitor was Dimitri Fotiadi. I enjoyed luncheon conversation with François Lurcat. I recall that Raymond Stora was present occasionally and had an apartment in Orsay. Although David Ruelle had already been appointed as a member of the faculty, early in years but not in accomplishment, David was still visiting in Princeton.

When I had met Louis Michel in Istanbul his student Eduardo de Raphael was with him. What a great pleasure it was to visit Eduardo for dinner, after which he might play a Beethoven piano sonata on the upright in their living room. I was amazed by Eduardo's musical talent; it also discouraged me from playing the Artin's piano. I spent a good deal of time with the Lanfords that year. There were a handful of other students at Bures, and each of them played an important part in creating the atmosphere. Klaus Pohlmeyer was interested in quantum field theory. Through Princeton acquaintances, I received introductions to several persons working at Orsay, so even though there were only a handful of students at Bures, I had no shortage of compatriots.

Early in the fall, Res Jost and his student Klaus Hepp came to visit from the ETH. I had spent long periods of time reading Jost's work on scattering theory, and I had also read a fascinating review article on the PCT theorem in field theory written in German. The problem was that Res' expression in German was so elegant, it made my job of translation quite difficult. Jost probably coined the name Feldverein as a joke. I very much looked forward to meeting him in person.

Klaus and I had encountered each other two summers earlier at a summer school in Hercegnovi, just before I returned from Cambridge to Princeton, and too long a story to tell here. I liked Klaus a great deal, and appreciated his energy and spirit, as well as his quick thinking. The Wightmans and Res Jost took advantage of our reunion and invited Oscar, Klaus and myself to dinner in Paris. They chose the Café Procope, an ancient restaurant in the 6$^{\text{th}}$, that in 1963 still retained a great deal of charm. We had a happy, lively evening followed by a long walk by the Seine. Res Jost was in high spirits, and I recall his encouraging the study of self adjointness. Eventually we became good friends ([1]).

When I arrived at Bures, the institute had three buildings: the main building, the cafeteria, and the library; the present bâtiment scientifique did not exist. Almost all the offices were in the main building, and on reflection I cannot reconstruct how all the people actually fit. Most of the professors and some visitors were on the upstairs floor facing the garden, some were downstairs facing the cafeteria. Somehow, Wightman, Borchers, Michel, Lehmann, Artin, and some others were in the main building, that also housed the tea room and the Director, in their present locations, and somewhere the small staff had their place too. I do not recall where Grothendieck or Dieudonné had offices. I believe that Thom had an office over the cafeteria. Some visitors like Jean Lascoux also had desks in that area. All these locations now house staff. Everyone else, myself included, worked at their home or in the library. I liked coming to the library. It was in the same building as today, with the wonderful windows across the front looking over the neatly manicured lawns and gardens.

([1]) Five years later Jost and Hepp were my hosts for a ten-week visit in Zürich, during which I met Konrad Osterwalder, Robert Schrader, and many others. This began an ETH/Harvard collaboration that continues today. Jürg Fröhlich tells me that he was discouraged from attending my course, the assistants describing it as "too advanced" for an undergraduate. I got to know Jürg two years later in Les Houches.

Even though about half the library served as the seminar room, there were long tables against the windows that provided an excellent place to think and work, inspired by the view. I could also easily walk down to Bures for a coffee. Occasionally I interacted with other visitors in the library, but the big opportunities for discussion were at lunch and at tea.

I began the year in Paris proof-reading Arthur's book with Streater. Oscar Lanford and I had a small competition over who would find the most corrections. During this period, we were shocked by John Kennedy's assassination, and for a long time discussion at lunch revolved about this topic.

There were two distinct groups of persons at the institute, and they seemed to behave quite differently. The mathematical physicists were in evidence all the time. We liked to talk with each other, perhaps too much. The other group was to a large extent made up of algebraic geometers or number theorists. They tended to keep to themselves; at least they were much less in evidence. However, once a week they and their Paris colleagues descended in large numbers upon the library for the Grothendieck seminar. Also on that day, there was a special table in the cafeteria, (I seem to remember it seated at most four) where Dieudonné and Grothendieck had lunch with their guests, occasionally Serre or Atiyah. On the day of the mathematics seminar, I sometimes remained, but often I made other plans, either going to visit friends at the University in Orsay, or exploring Paris. I did not realize at the time the relevance for mathematical physics of their discussions.

Another regular trip I made to Paris involved following the lecture course of Laurent Schwartz. Five or six of us from Bures attended his course on topological vector spaces at the Institut Henri Poincaré. For me, they were also wonderful lessons in French. We marveled at Schwartz' style, where everything in his course appeared totally simple, obvious, and easy to reconstruct – until one left the room. We wondered if the students understood much.

Shortly after I arrived at the IHÉS, work began on the construction of a new office building, the first of the three spokes of the present scientific office space. After several months one floor was ready – the *upper* floor – and for the rest of the academic year the visitors (myself included) moved into these offices. The lower floor remained a shell of steel girders holding the floor and roof above. It did seem unusual to me that the work on the lower floor ceased just then, and it did not recommence before I left in July. But if the construction had continued, the noise would have made working above quite difficult. Walking by, I often peered into this empty structure with a certain expectation and worry that some new event might happen; it never did. The new offices led to more interaction, more discussion and talk, and perhaps less time to think. But I was learning a great deal.

It was in the new building that I interacted a good deal with Henri Epstein and Jurko Glaser, and we ended up writing a short paper on the energy density in quantum field theory [2]. In fact, my year at the IHÉS surpassed all expectations, for at the IHÉS I would work on my thesis in a stimulating environment, I would get to know a number of extraordinary persons, and the interactions begun in Bures would influence the rest of my life.

IV. Constructive QFT and inessential self-adjointness

During the year in Bures, I began my first work on examples of cutoff bosonic field theories, and a year later this became my thesis [4]. While I studied bosonic self-interactions, Oscar worked on Yukawa-type boson-fermion interactions. In both cases, we approximated the interactions. The first question that arose, was to determine whether the Hamiltonian H, defined on a suitable regular domain $\mathcal{D}$, uniquely determines the one-parameter Schrödinger group $e^{-it\mathrm{H}}$, yielding the time evolution of the system. This is the mathematical question of whether a densely defined, unbounded symmetric operator H has a unique self-adjoint extension, as well as the question about how the general results apply to specific examples. If the closure of the operator H is self-adjoint (and thus the unique self-adjoint extension), then the operator H is said to be *essentially self-adjoint*. At that time I had studied the theory of defect indices, how to determine abstractly whether an operator has a self-adjoint extension, and how to parameterize such extensions. This text-book study remained theoretical, so faced with a concrete problem I liked to begin with understanding some examples. Arthur Wightman was wrestling with the same question, and he had a bunch of simple illustrative examples that have served ever since as my guide to thinking about this problem ([1]). Perhaps the simplest of these examples concerned the operator $\mathrm{D} = i\frac{d}{dt}$ defined on the domain $\mathcal{D} = \mathrm{C}_0^\infty$ of smooth, compactly supported functions $f(t)$ defined on the unit interval, in the Hilbert space $\mathrm{L}^2([0,1], dt)$. The operator D is symmetric, but not essentially self adjoint. The one-parameter family D_θ of self-adjoint extensions of D have a simple interpretation. Each D_θ is the infinitesimal generator of a unitary translation group $\mathrm{U}_\theta(t) = e^{it\mathrm{D}_\theta}$. Acting on a compactly supported function $f(t')$, all the translations agree, $(\mathrm{U}_\theta(t)f)(t') = f(t'-t)$, at least for t sufficiently small so that the support of the translated function $f(t'-at)$ remains bounded away from either endpoint of the interval, for all $0 \leqslant a \leqslant 1$. In order for the translation to preserve the norm of f as t increases, the part of f translated past an endpoint must reappear at the other endpoint, where it may be multiplied by a phase $e^{-i\theta}$. The choice of the one-parameter group $\mathrm{U}_\theta(t)$ (or of its generator D_θ) corresponds to choice of the twist $e^{-i\theta}$.

There is an alternate description of this phenomenon, from the point of view of boundary conditions on the differential operator D. Instead of defining the derivative D on the domain C_0^∞, one can define D on the larger domain $\mathcal{S}_\theta$ of smooth functions on $[0,1]$ satisfying

$$f(1) = e^{-i\theta} f(0)\ . \tag{IV.1}$$

This domain includes the dense set of eigenfunctions of D, namely $e^{it(2\pi n-\theta)}$, $n \in \mathbb{Z}$, and on this domain D is essentially self-adjoint with closure D_θ.

In understanding this and many other examples – including a number of quantum field theory Hamiltonians – we had a lot of fun, even if this understanding did not

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([1]) These examples can be found in the helpful notes from the Cargèse summer school held the following year [8].

immediately lead to a solution of our original problem. In fact our enthusiasm at Bures spilled over into many interesting tea-room discussions, some of them also quite a lot of fun. Harry Lehmann used to find a way to turn the most serious discussion into a joke; he had the role of chief skeptic in the Feldverein. One day after his returning from a trip to Hamburg, I recall Lehmann coming up as I entered the tea room to ask with a laugh, "Any new progress on inessential self-adjointness?" He was fond of plays on words. A few years later while having lunch with Curt Callan at the Harvard Faculty Club, we got into a somewhat philosophical discussion. At one point Curt remarked: "You mathematical physicists like to solve problems; particle physicists try to avoid them." This brought back to me the memory of Harry Lehmann's joke.

But variations of the Hamiltonian method did ultimately lead to the first existence proofs of two-dimensional, Lorentz-invariant quantum fields [3, 1]. And Kurt Symanzik began his famous 1968 Varenna lectures [7] with a chart comparing the advantages and disadvantages of Hamiltonian methods versus his own Euclidean methods, based on probability measures on random fields. In fact, his comparison came down too far on the side of giving the advantage to the Hamiltonian methods. Unfortunately this was something the author took to heart, and the Varenna lectures mark Symanzik's last major contribution to constructive quantum field theory. Only later were the Hamiltonian and Euclidean random field methods reconciled mathematically (through the work of Nelson and the Osterwalder-Schrader construction) to the point that today they are complementary tools within one approach.

V. Old friends

Mathematical physicists are impressed how they may learn some fact, result, method, relation, or even an entire subject, and think it interesting though irrelevant at the time. Then, after months or even years, this same fact emerges as centrally relevant to them in another context, helping to solve a problem they are investigating. And they might even discover the solution to their current question through their familiarity with such material from an earlier time. I have been fascinated by this phenomenon since I was a student and discuss this on occasion with my own students or friends. Often this topic arises when a student advisee asks what courses he or she needs to take in order to become a mathematical physicist. But the topic also arises in correspondence with former students remarking on the relevance of some course material. One might say that one experiences a twist in our expectations.

I can report that I recently came across several old friends from my first visit to Bures: twists, self-adjoint differential operators, translations, Schwartz spaces, and interacting quantum fields. They come together in an interesting way: consider a cylindrical, $(s+1)$-dimensional space-time, $(x, t) \in \mathcal{C} = \mathbb{T}^s \times [0, \beta]$. Here $\mathbb{T}^s$ denotes an s-torus with periods $\{\ell_j\}$, with $1 \leqslant j \leqslant s$. Let $\gamma = \{\beta, \tau, \theta\} \in \mathbb{R}_+ \times \mathbb{T}^s \times \mathbb{R}$ denote an $(s+2)$-dimensional parameter that summarizes a good deal of information about the boundary conditions on functions.

Let $\mathcal{S}_\gamma(\mathcal{C})$ denote functions on $\mathcal{C}$ that have Fourier series of the form

$$f(x,t) = \sum_{\substack{\mathrm{E}\in\frac{2\pi}{\beta}\mathbb{Z}\\ k\in\hat{\mathbb{T}}^s}} \hat{f}(k,\mathrm{E})\, e^{ikx+i\mathrm{E}t} e^{i(\theta+k\tau)t/\beta}. \tag{V.1}$$

Here the set $\hat{\mathbb{T}}^s$ denotes the lattice $\prod_{j=1}^s \frac{2\pi}{\ell_j}\mathbb{Z}$ dual to the torus $\mathbb{T}^s$, and the expression $k\tau$ is expressed in components as $k\tau = \sum_{j=1}^s k_j\tau_j$. We assume that the coefficients $\hat{f}(k,\mathrm{E})$ decrease faster than the inverse of any polynomial function of $k^2+\mathrm{E}^2$. Endowed with the topology given by the countable set of norms

$$\|f\|_n = \sup_{k,\,\mathrm{E}} (1+|k|^2+\mathrm{E}^2)^n |\hat{f}(k,\mathrm{E})|, \quad \text{for } n\in\mathbb{Z}_+, \tag{V.2}$$

the space $\mathcal{S}_\gamma(\mathcal{C})$ is a Schwartz space of C^∞ functions, that is dense in the Hilbert space $\mathrm{L}^2(\mathcal{C})$.

These functions (V.1) satisfy the boundary condition

$$f(x,\beta) = e^{-i\theta} f(x+\tau, 0), \tag{V.3}$$

relating the two ends of the cylinder ([1]). Furthermore, the representation (V.1) shows that functions $f\in\mathcal{S}_\gamma(\mathcal{C})$ extend from $\mathcal{C}=\mathbb{T}^s\times[0,\beta]$ to smooth functions on $\mathbb{T}^s\times\mathbb{R}$, that satisfy the *twist relation*

$$f(x,t+\beta) = e^{-i\theta} f(x-\tau, t). \tag{V.4}$$

The operator $\Delta_\gamma = \dfrac{\partial^2}{\partial t^2} + \displaystyle\sum_{i=1}^s \frac{\partial^2}{\partial x_i^2}$, defined on the domain $\mathcal{S}_\gamma(\mathcal{C})\subset\mathrm{L}^2(\mathcal{C})$, is essentially self adjoint. In fact, each function in $\mathcal{S}_\gamma(\mathcal{C})$ of the form (V.1) with $\hat{f}(k,\mathrm{E}) = \delta_{k,\,k'}\delta_{\mathrm{E},\,\mathrm{E}'}$, namely $e^{ik'x+i\mathrm{E}'t}e^{-i(\theta+k'\tau)t/\beta}$, is an eigenfunction. We denote the self adjoint closure also by Δ_γ, and we call it the *twisted Laplace operator.* Let $\mathrm{C}_\gamma = (-\Delta_\gamma + m^2\mathrm{I})^{-1}$ be its resolvent. For $m>0$, the operator C_γ is compact, and strictly positive. This operator acts as multiplication on the Fourier series for f by the function $((\mathrm{E}-\theta/\beta-k\tau/\beta)^2+k^2+m^2)^{-1}$. Hence C_γ transforms $\mathcal{S}_\gamma(\mathcal{C})$ into itself, and it is continuous in the topology of $\mathcal{S}_\gamma(\mathcal{C})$ determined by the countable set of norms above. If $m=0$, then for γ on the complement of a singular set Υ_{sing}, we can also bound C_γ; we do not discuss that possibility further here.

Denote an element of the dual space by $\Phi_\gamma(x,t)\in\mathcal{S}'_\gamma(\mathcal{C})$, where functions in the dual space pair with functions $f\in\mathcal{S}_\gamma(\mathcal{C})$ by $\Phi_\gamma(f) = \int_{\mathcal{C}}\Phi_\gamma(x,t)f(x,t)\,dxdt$. With $\gamma' = \{\beta,\tau,-\theta\}$, note that $\mathcal{S}_{\gamma'}(\mathcal{C})\subset\mathcal{S}'_\gamma(\mathcal{C})$. Define the adjoint C_γ^+ of C_γ by $(\mathrm{C}_\gamma^+\Phi_\gamma)(f) = \Phi_\gamma(\mathrm{C}_\gamma f)$. Thus C_γ^+ is a continuous transformation of $\mathcal{S}'_\gamma(\mathcal{C})$ into itself. The integral kernel $\mathrm{C}_\gamma^+(x-y,t-s)$ of C_γ^+ satisfies the twist relation

$$\mathrm{C}_\gamma^+(x-y,t-s+\beta) = e^{i\theta}\mathrm{C}_\gamma^+(x-y-\tau,t-s). \tag{V.5}$$

([1]) In the case $\beta=1$, $t=0$, $\tau=0$, and ignoring x, we recover the twisted boundary condition (IV.1).

As a consequence, functions $\Phi_\gamma \in \mathcal{S}'_\gamma(\mathcal{C})$ satisfy the twist relation

$$\Phi_\gamma(x, t+\beta) = e^{i\theta}\Phi_\gamma(x-\tau, t), \tag{V.6}$$

and this identity extends appropriately to generalized functions in $\mathcal{S}'_\gamma(\mathcal{C})$.

The operator C_γ is the covariance operator for a Gaussian probability measure $d\mu_\gamma$ on $\mathcal{S}'_\gamma(\mathcal{C})$. We take this measure to have mean zero, and to have the second moments

$$\int_{\mathcal{S}'_\gamma} \Phi_\gamma(f)^2\, d\mu_\gamma = 0, \qquad \text{and} \qquad \langle f, C_\gamma g\rangle_{L^2(\mathcal{C})} = \int_{\mathcal{S}'_\gamma} \overline{\Phi_\gamma(f)}\Phi_\gamma(g)\, d\mu_\gamma. \tag{V.7}$$

The measure extends to functionals of $\Phi_\gamma(f)$, where $f(x,t)$ is C^∞ in $x \in \mathbb{T}^s$ and concentrated at one time. On functionals of such paths $\Phi_\gamma(h,t)$, with $h \in C^\infty$, the measure $d\mu_\gamma$ is concentrated on functions of t that are Hölder continuous with exponent $\alpha < \frac{1}{2}$. Let V denote a real functional of $\Phi_\gamma(h,t)$ that is twist-invariant, and that also is invariant under translations of the torus. Suppose also that $e^{-\int_0^\beta V dt}$ is $d\mu_\gamma$-integrable. Define the statistical mechanics *partition function* by

$$\mathfrak{Z}_\gamma^V = \int_{\mathcal{S}'_\gamma(\mathcal{C})} e^{-\int_0^\beta V dt} d\mu_\gamma, \tag{V.8}$$

and the probability measure $d\mu_\gamma^V$ by

$$d\mu_\gamma^V = \frac{1}{\mathfrak{Z}_\gamma^V} e^{-\int_0^\beta V dt} d\mu_\gamma. \tag{V.9}$$

Surprisingly, the family of measures $d\mu_\gamma^V$ define a Feynman-Kac representation for a quantum field theory that lives on a cylindrical space-time $\mathbb{T}^s \times \mathbb{R}$. This field theory has the Hamiltonian H, and momentum operator P, that are the infinitesimal generators of time translations and of space translations, on $\mathbb{R}$ and on $\mathbb{T}^s$ respectively. These generators mutually commute, and they also commute with a 1-parameter symmetry group that we denote $U(\theta) = e^{i\theta J}$. The Hilbert space of this field theory is the Fock space $\mathfrak{H}$ of a bosonic field theory on the torus $\mathbb{T}^s$. Let H_0 and P denote the Hamiltonian and momentum operator respectively for a mass-m free field on $\mathfrak{H}$. The form of H is $H = H_0 + V$. The translation and twist symmetries act on the field $\varphi(x,t)$ as

$$e^{itH - ixP + i\theta J}\varphi(x', t')e^{-itH + ixP - i\theta J} = e^{i\theta}\varphi(x+x', t+t'). \tag{V.10}$$

The measure $d\mu_\gamma^V$ defines a twisted Gibbs functional on time ordered products of field operators. The field theory partition functions are related to the statistical mechanics partition functions above by

$$\mathfrak{Z}_\gamma^V = \frac{\mathrm{Tr}_{\mathfrak{H}}\left(e^{i\tau P - i\theta J - \beta H}\right)}{\mathrm{Tr}_{\mathfrak{H}}\left(e^{i\tau P - i\theta J - \beta H_0}\right)} = \int_{\mathcal{S}'_\gamma(\mathcal{C})} e^{-\int_0^\beta V dt} d\mu_\gamma. \tag{V.11}$$

The twisted Gibbs expectation functional is also related to the Feynman-Kac family of measures by

$$\langle \cdot \rangle_\gamma = \frac{\mathrm{Tr}_{\mathfrak{H}}\left(\cdot\, e^{i\tau \mathrm{P} - i\theta \mathrm{J} - \beta \mathrm{H}}\right)}{\mathrm{Tr}_{\mathfrak{H}}\left(e^{i\tau \mathrm{P} - i\theta \mathrm{J} - \beta \mathrm{H}}\right)} = \int_{\mathcal{S}'_\gamma(C)} \cdot\, d\mu^{\mathrm{V}}_\gamma . \tag{V.12}$$

As is the case with a usual Feynman-Kac representation, the expectation on the left is applied on time-ordered products of functions of imaginary time fields, $e^{-s_j \mathrm{H}} \varphi(h_j)\, e^{s_j \mathrm{H}}$ with $0 \leqslant s_j \leqslant \beta$, and $e^{-s_j \mathrm{H}} \varphi(h_j)^* e^{s_j \mathrm{H}}$. On the right-hand side, the integral is taken over the same function of random fields $\Phi_\gamma(h_j, s_j)$ and their complex conjugates $\overline{\Phi_\gamma(h_j, s_j)}$.

For proofs of what I have described here, see [5], where one can also find further elaboration on the connection between twists and Feynman-Kac representations. The systems under discussion have a ground state vector Ω_{vacuum} for H that is unique (up to a phase). Therefore Ω_{vacuum} is an eigenvector of the symmetry transformation $e^{i\tau \mathrm{P} - i\theta \mathrm{J}}$, with an eigenvalue of modulus one. Let us normalize each element of the group $e^{i\tau \mathrm{P} - i\theta \mathrm{J}}$ by the inverse of this eigenvalue, so that

$$e^{i\tau \mathrm{P} - i\theta \mathrm{J}}\, \Omega_{\text{vacuum}} = \Omega_{\text{vacuum}}. \tag{V.13}$$

With this choice, we obtain partition functions that are positive for all $\tau \in \mathbb{T}^s$, $\theta \in \mathbb{R}$, and $\beta \in \mathbb{R}_+$, namely

$$\mathrm{Tr}_{\mathfrak{H}}\left(e^{i\tau \mathrm{P} - i\theta \mathrm{J} - \beta \mathrm{H}}\right) > 0. \tag{V.14}$$

I call this (rather surprising) property *twist positivity* [5]. It is intimitely connected with the existence of the probability measure $d\mu^{\mathrm{V}}{}_\gamma$. With the existence of this measure, one has the tools of constructive field theory to study functionals with twists. For example, one can use L^p inequalities and harmonic analysis to establish properties of integrals of random fields that occur in the representation of twisted Gibbs functionals. One can also investigate and take advantage of certain regularity of the measure $d\mu^{\mathrm{V}}_\gamma$ for classes of potentials V, or for certain τ, θ, in the limit as $m \to 0$.

Afterward

Reflections on 1963–64 in Bures remain fascinating and fruitful. But I am sad to no longer have the possibility of sharing my enthusiasm with some special friends from those early days. I dedicate this essay to the memory of two persons to whom we owe a great debt, Annie Roland and Léon Motchane. The love and energy with which they raised their “child” created a unique attractor, the Bois Marie.

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