PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

NORBERT A'CAMPO

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Publications mathématiques de l'I.H.É.S., tome 88 (1998), p. 171-180
<http://www.numdam.org/item?id=PMIHES_1998__88__171_0>

PLANAR TREES, SLALOM CURVES AND HYPERBOLIC KNOTS

by NORBERT A'CAMPO

1. Introduction

An embedded tree B in the unit disk D, such that the intersection $B \cap \partial D$ consists of one terminal vertex r of B, is called a rooted planar tree. For a rooted planar tree B there exists an immersed copy $P_B \subset D$ of the interval $[0, 1]$ with the following properties:

(i) The immersion is relative, i.e. the endpoints are embedded in ∂D.

(ii) The immersion is generic, i.e. there are only transversal crossing points, only the endpoints lie on ∂D and the immersion is transversal to ∂D.

(iii) The double points of P_B lie in the interior of the edges of B, and the local branches are transversal to the corresponding edge of B.

(iv) Each connected component of $D \setminus P_B$ contains exactly one vertex of B.

(v) The only intersection points of P_B with B are the double points of P_B.

The immersed curve P_B is well defined up to regular relative isotopy and is called the slalom curve or slalom divide of the rooted planar tree B, see Fig. 1, 2, 4.

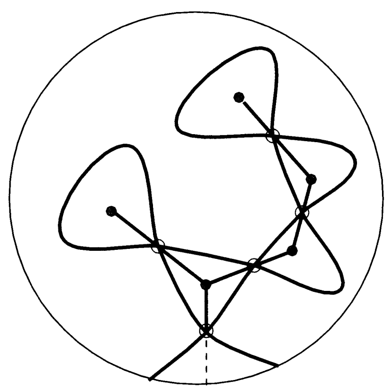

FIG. 1. – Rooted planar tree, its Dynkin diagram E_{10} and slalom

The slalom curve P_B is a divide to which corresponds a classical knot K_B in S^3, which we call a slalom knot. The complement of the slalom knot K_B admits a fibration over the circle S^1, see [AC4] and Section 2 for basic definitions and properties. The Dynkin diagram Δ_B of the divide P_B is deduced from the rooted tree B as follows: First make a new tree B′ by subdividing each edge of B with a new vertex, which is placed at the crossing point of P_B on the edge; next, remove from B' the root vertex r and the terminal edge of B' pointing to r. In Fig. 1 the tree B has the shape of the classical Dynkin diagram D_6 but the Dynkin diagram Δ_B of P_B, which we can denote by E_{10} has 10 vertices. The Dynkin diagram Δ_B of a rooted tree B is a bicolored rooted tree with an embedding in the plane. The root is the new vertex which lies on the edge of B originating from the root point of B and the bicoloring is such that the new vertices are of the same color. Moreover, the Dynkin diagram Δ_B has the property that the terminal vertices of Δ_B different from the root, are never new. The purpose of this paper is to prove the following theorem.

Theorem **1.** — *Let* B *be a rooted tree. The complement of the slalom knot* K_B *admits a complete hyperbolic metric of finite volume, if and only if the Dynkin diagram* Δ_B *is neither the diagram* A_{2k}, $1 \leqslant k$, *nor the diagram* E_6 *or* E_8.

If the Dynkin diagram Δ_B is among A_{2k}, $1 \leqslant k$, E_6, E_8, the knot K_B is the torus knot $(2, 2k+1)$, $(3, 4)$ or $(3, 5)$ and appears as local knot of a simple plane curve singularity [AC1]; the monodromy diffeomorphism (with free boundary) of the knot K_B can be chosen to be of finite order in those cases and its complement does not carry a complete hyperbolic metric. We only need to prove the if part of the theorem.

From the above theorem we get many examples of hyperbolic fibered knots, whose monodromy diffeomorphism and gordian number are known explicitly. The monodromy diffeomorphism of a slalom knot can be realized as the product of right Dehn twists of a system of simple closed curves on the fiber surface, such that the union of the curves is a spline in the fiber surface and the dual graph of the system is the Dynkin diagram of the rooted tree; the gordian number of a slalom knot equals the number of crossings of the slalom divide [AC4]. We call (see section 3) the isotopy class of the monodromy diffeomorphism of the slalom knot of a rooted tree the Coxeter diffeomorphism of the Dynkin diagram of the rooted tree. It follows from Theorem 1 that a Coxeter diffeomorphism of a Dynkin diagram of a rooted tree is pseudo-Anosov, if and only if the Dynkin diagram is not a classical Dynkin diagram (see Theorem 3). We do not know the lattice in $iso(H^3) = PSL(2, \mathbf{C})$ of the hyperbolic uniformization for the complement of the hyperbolic slalom knots K_B. I wish to thank Makoto Sakuma for explaining to me his joint work with Jeff Weeks on hyperberbolic 2-bridge links [S-W], which indicates a road leading to a description of the uniformization lattice and the canonical decomposition in ideal hyperbolic simplices of the complement of hyperbolic slalom knots.

2. Divide and knot of a planar rooted tree.

Let B be a rooted planar tree in the unit disk $D \subset \mathbf{R}^2$ and let P_B be its divide. The knot K_B of the tree B is the knot of its divide P_B (see [AC3-4]), i.e.

$$K_B := \{(x, u) \in T(P_B) \mid \|(x, u)\| = 1\} \subset S(T(\mathbf{R}^2)) = S^3$$

where $T(P_B) \subset T(\mathbf{R}^2)$ is the subspace of tangent vectors to the divide P_B in the space of tangent vectors to the plane $\mathbf{R}^2$. A tangent vector of the plane $(x, u) \in T(\mathbf{R}^2) = \mathbf{R}^2 \times \mathbf{R}^2$, is represented by its foot $x \in \mathbf{R}^2$ and its linear part $u \in T_x(\mathbf{R}^2) = \mathbf{R}^2$. The norm $\|(x, u)\|$ is the usual euclidean norm of $\mathbf{R}^4$. In Fig. 5 is shown a computer drawing of the knot of the divide Lys (see Fig. 4). This knot can be presented with 11 crossings and its gordian number equals the number of crossing points of the divide, i.e. 4. Since a slalom divide is connected, the complement of the knot K_B of a rooted tree fibers over the circle [AC4]. A model for the fiber surface and monodromy diffeomorphism can be read by a graphical algorithm from the divide P_B as follows: replace each crossing point of P_B by a square, which has its vertices on the local branches of P_B at the crossing point, and get a trivalent graph Γ embedded in the disk D; the fiber is diffeomorphic to the interior of the surface with boundary F obtained from a thickening of the graph Γ. The thickening corresponds to the cyclic ordering of the edges of Γ at each vertex of Γ, which alternatingly agrees or disagrees with an orientation of the ambient plane. The graph Γ has only circuits of even length, so the alternating cyclic ordering of the edges at the vertices of Γ exists. For each of its squares and for each region of the divide P_B the graph Γ has a circuit, which surrounds the square or region. To these circuits of Γ correspond simple closed curves on the surface F. The monodromy T is the product of the right Dehn twists along those closed curves.

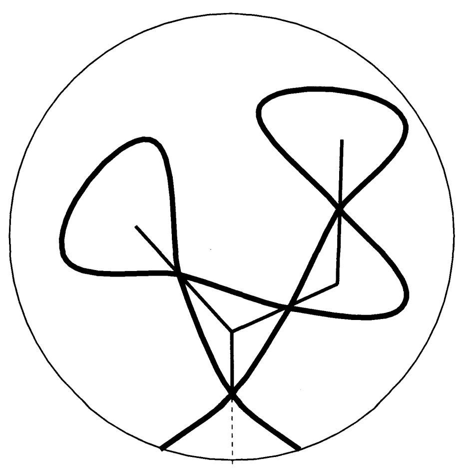

FIG. 2. – The slalom E_8

This product is well defined up to conjugacy in the relative mapping class group of the surface $(F, \partial F)$ since the non-commutation graph of this set of Dehn twists is precisely the Dynkin diagram Δ_B, which is a tree ([B], Fascicule XXXIV, Chap. 4, par. 6, lemme 1). The graph Γ with its cyclic orientation of the edges at the vertices allows us to give a combinatorial description of the diffeomorphism T of the surface F, which can be used as input to the Bestvina-Handel algorithm, [B-H] see also [L] . In practice, we use the Cayley code for rooted trees, see [S-Wh], and a maple program to deduce from the Cayley code the combinatorial description of T, which was finally the input to the program TRAINS, written by Tobi Hall, doing the Bestvina-Handel algorithm. This way we get extra stimulating evidence for Theorem 1 and 3. I would like to thank Tobi Hall for allowing me to use his program TRAINS.

3. Conway spheres and Bonahon-Siebenmann decomposition for slalom knots.

Let B be a rooted tree with slalom divide $P_B \subset D$ and slalom knot K_B. Let $f_B : D \to \mathbf{R}$ be a Morse function for the divide P_B as in the proof of the fibration theorem of [AC4], i.e. a generic C^∞ function, such that P_B is its 0-level and that each interior region has exactly one non-degenerate minimum and that each region which meets the boundary has exactly one non-degenerate maximum or minimum on the intersection of the region with ∂D. The underlying tree of the slalom divide can be reconstructed up to isotopy as the closure of the union of the gradient lines of f_B, which lie in $\{f_B < 0\}$ and which contain a saddle point in their closure. The singular gradient lines L of f_B in $\{f_B > 0\}$ give enough Conway spheres to build the Bonahon-Siebenmann decomposition [B-S] of the knot K_B, see [K]. For a singular gradient line L of f_B in $\{f_B > 0\}$ we define $C(L) := \{(x, u) \in T(D) \mid x \in L, ||x||^2 + ||u||^2 = 1\}$. Observe that such a gradient line passes through a saddle point of f_B and both end points of L are on ∂D. It follows that C(L) is a smooth embedded 2-sphere in S^3. Each sphere C(L) is invariant under the involution $(x, u) \mapsto (x, -u)$.

We state without proof:

Theorem **2.** — *Let* B *be a rooted tree and* f_B *its Morse function. The spheres* C(L) *of the singular gradient* L *lines of* f_B *in* $\{f_B > 0\}$ *are Conway spheres for the slalom knot* K_B. *The spheres* C(L) *which correspond to edges of the tree* B, *with at least one endpoint of valency* $\geqslant 3$ *or equal to the root vertex, give the Bonahon-Siebenmann decomposition of the slalom knot.*

We wish to mention here that the knot of the slalom divide of the tree [0,1,2,2] is the knot 10_{139} of the table of Rolfson's book [R], which is equivalent to the Montesinos knot $M(1, (3, 1), (3, 1), (4, 1))$, see [Ka]. The gordian number of 10_{139} is shown to be 4 by Tomomi Kawamura [Kaw].

I am grateful to Mikami Hirasawa for explaining to me his method of constructing a knot diagram for slalom knots directly from the slalom divide. He first

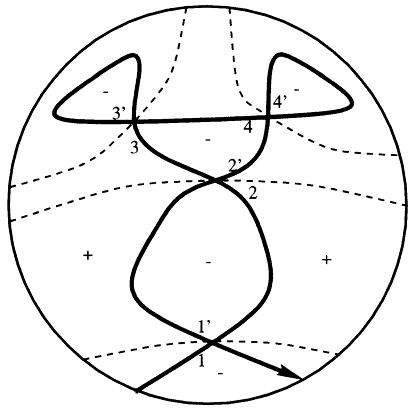

FIG. 3. – The slalom divide of the tree [0, 1, 2, 2] and singular gradient lines.

doubles the slalom divide and then changes according to local rules the doubled divide to a knot diagram. It follows from his construction that slalom knots are arborescent knots in the sense of Bonahon and Siebenmann [B-S]. The notation as arborescent knot for the slalom knot P_B is the Dynkin diagram Δ_B with the weighting 2 at each vertex.

3. Trees, forms, plumbings.

Let A be a tree with vertex set $\{v_1, v_2, \ldots, v_n\}$. We will choose the numbering of the vertices such that for some m, $1 \leqslant m \leqslant n$, there are no pairs of vertices v_i and v_j connected by an edge of A with $i \leqslant m$ and $j \leqslant m$ or with $m < i$ and $m < j$. The chosen numbering corresponds to a bicoloring of the vertices of the tree. The real vector space V_A generated by the set of vertices of A carries a quadratic form q_A, whose matrix is $q_A(v_i, v_i) = -2$, $1 \leqslant i \leqslant n$, and $q_A(v_i, v_j) = 1$, if and only if, the vertices v_i and v_j are connected by an edge of A. To each vertex v_i corresponds an isometry R_i of (V_A, q_A)

$$R_i(v_j) := v_j + q_A(v_i, v_j)v_i,$$

which is a reflection. Since the non-commutation graph of the set $\{R_1, R_2, \ldots, R_n\}$ is a tree the product of the reflections R_i does not depend up to conjugacy on the order in which the product is evaluated [B] and is called the Coxeter element C_A of the tree A. The vector space V_A also carries a skew form sq_A, whose matrix is $sq_A(v_i, v_j) = 1$ or $sq_A(v_i, v_j) = -1$ if and only if the vertices v_i and v_j are connected by an edge of A. If $i \leqslant m$ then $sq_A(v_i, v_j) = 1$ else if $j \leqslant m$ then $sq_A(v_i, v_j) = -1$. To each vertex v_i corresponds an endomorphism T_i of (V_A, sq_A)

$$T_i(v_j) := v_j + sq_A(v_i, v_j)v_j,$$

which is a transvection. The product of the transvections T_i, is equal to $-C_A$, and we call its conjugacy class in the group of the form sq_A, well defined by [B], the skew Coxeter element sC_A of the tree A.

From [AC2] we recall the following (see also [Hu]). If the tree A is not among the diagrams A_k, D_{k+3}, $\tilde{D}_{k+3}$, $1 \leqslant k$, E_6, $\tilde{E}_6$, E_7, $\tilde{E}_7$, E_8, $\tilde{E}_8$, the endomorphism C_A has a real eigenvalue $\lambda_{max} > 1$ with multiplicity 1, such that for any eigenvalue λ of C_A we have $|\lambda| < |\lambda_{max}|$ unless $\lambda = \lambda_{max}$. We call λ_{max} the dominating eigenvalue of C_A and $-\lambda_{max}$ the dominating eigenvalue of sC_A.

Let A be a planar tree with vertex set $\{v_1, v_2, \ldots, v_n\}$. To the tree A corresponds a surface S_A by the following plumbing. First realize the planar tree A by a planar circle packing with small overlappings. Each vertex v_i is represented by an oriented circle c_i. As orientation we choose the counterclockwise orientation. The circles c_i, c_j are disjoint if the vertices v_i and v_j are not connected in B and touch each other from the outside with a small overlap, if v_i and v_j are connected in A. Let C_i be a tubular neighborhood in the plane of c_i, which is an oriented cylinder. The surface S_A is obtained by plumbing the cylinders C_i and C_j at one of the intersection points of c_i and c_j and making an overcrossing at the other intersection point, if the vertices v_i and v_j are connected in A. The choice at which intersection point the plumbing takes place, is made such that on the surface S_A the cycles c_i and c_j have the intersection number $sq_A(v_i, v_j)$. The surface S_A is naturally immersed in the plane. Let D_i be the right Dehn twist with core the curve c_i of S_A. Let $T_A : S_A \to S_A$ be the composition $D_1 \circ D_2 \circ \ldots \circ D_n$, which we call the Coxeter diffeomorphism of the planar tree A.

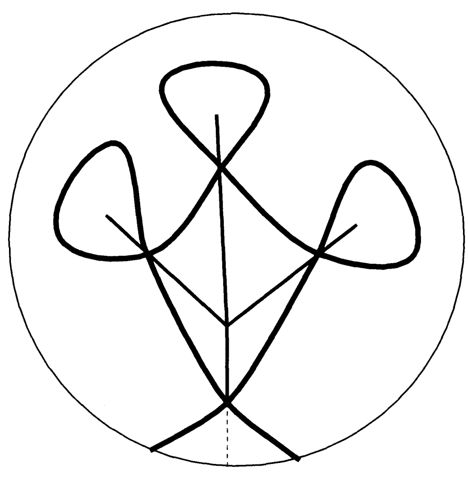

FIG. 4. – The slalom *Lys*

The numerical function $A \mapsto \lambda_{H_1}(T_A)$ on trees is monotone for the inclusion of trees [AC2]. Question: is the function $A \mapsto \lambda_{\pi_1}(T_A)$ monotone?

4. Trees and hyperbolic knots.

We now give the proof of the if part of Theorem 1:

Proof. — Let $B \subset D$ be a rooted tree, such that the Dynkin diagram Δ_B is neither the diagram A_{2k}, $1 \leqslant k$, nor E_6 or E_8. We will show that the isotopy class of the monodromy of the fibered knot K_B is pseudo-Anosov. The geometric monodromy T of the knot K_B is up to conjugacy the diffeomorphism $T_A : S_A \to S_A$, where we put $A := \Delta_B$, see [AC4]. The action of T_A on the first homology of S_A is conjugate to the skew Coxeter element $s C_A$ of the tree A. It follows from [AC2] that the biggest absolute value s of an eigenvalue of the action of T_A on the first homology of S_A strictly exceeds 1. So for the homological entropy we have $\lambda_{H_1}(T) = \log(s) > 0$. By the entropy inequality, we deduce for the isotopical entropy $\lambda_{\text{isotop}}(T)$ the inequalities

$$0 < \lambda_{H_1}(T) \leqslant \lambda_{\pi_1}(T) \leqslant \lambda_{\text{isotop}}(T) \leqslant \lambda_{\text{top}}(T)$$

where $\lambda_{\text{isotop}}(T)$ is the minimum of the topological entropy $\lambda_{\text{top}}(T)$ over the relative isotopy class of T. Since the isotopical entropy of T is positive, we conclude that in the decomposition of Thurston [T1] in quasi-finite and pseudo-Anosov pieces of the diffeomorphism T at least one pseudo-Anosov piece occurs. So, to prove that the isotopy class of the diffeomorphism T is pseudo-Anosov, we need to prove that T is irreducible. A reduction of the diffeomorphism T would give an essential torus in the complement of the knot K_B. Since the knot K_B is an arborescent knot, as shown by the construction of Mikami Hirasawa, we conclude with the proposition 2.1 of [B-Z], see also [O], that the complement of the knot K_B does not have an essential torus. So, the diffeomorphism T is irreducible and hence pseudo-Anosov. We can conclude with a celebrated Theorem of W. Thurston [T2], see [O], that the mapping torus of the diffeomorphism T, which is diffeomorphic to the complement of the knot K_B, admits a complete hyperbolic metric. □

The knot of the slalom curve E_8 of the rooted tree with Cayley code [0,1,1,2] is not hyperbolic (see Fig. 2). The knots of the slalom curve Lys of the tree with code [0,1,1,1] and of the slalom curve E_{10} of the tree [0,1,1,2,4] are hyperbolic (see Fig. 1, 4).

In fact, for a diffeomorphism T of surfaces the equality $\lambda_{\pi_1}(T) = \lambda_{\text{isotop}}(T)$ holds, and moreover, for pseudo-Anosov diffeomorphisms the equality $\lambda_{\pi_1}(T) = \lambda_{\text{top}}(T)$ holds. It would be very interesting to compute the hyperbolic volume of the knot of the rooted tree E_{10} and to relate it with $\lambda_{\text{isotop}}(T_{E_{10}})$.

The knots K_B for B such that the Dynkin Δ_B diagram equals A_{2n}, E_6 or E_8, are links of singularities and the corresponding monodromies are irreducible and of finite order. From this fact and from the proof of the theorem we deduce that Coxeter diffeomorphisms of rooted trees have in general a pseudo-Anosov isotopy class. More precisely, with the notation of section 2 we have:

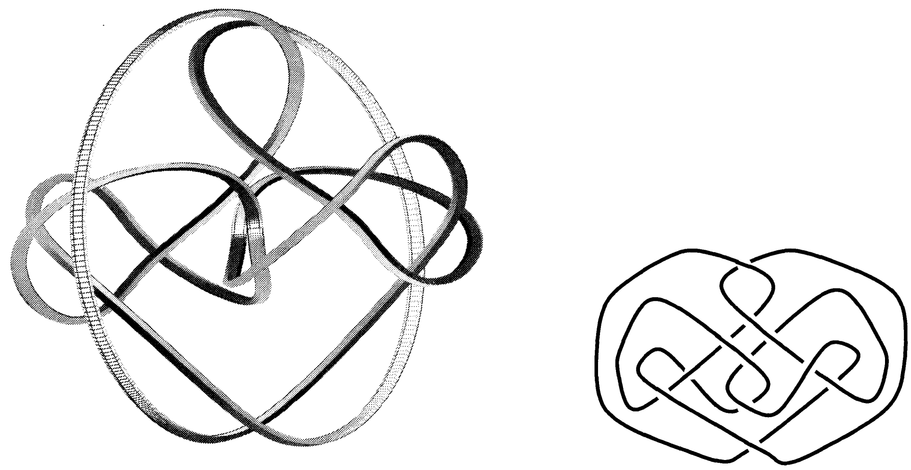

Fig. 5. – The knot of the slalom Lys

Theorem **3.** — *Let* A ⊂ D *be a rooted, bicolored, tree embedded in the plane such that the root is a terminal vertex and that no other terminal vertex has the color of the root. The Coxeter diffeomorphism* $T_A : S_A \rightarrow S_A$ *is irreducible. Moreover, if* A *is not among* A_{2n}, E_6, E_8, *the diffeomorphism is pseudo-Anosov.*

The pseudo-Anosov diffeomorphisms given by this theorem are products of Dehn twists, which belong to the same conjugacy class in the mapping class group of a surface with one boundary component and the union of the cores of the Dehn twists of the product decomposition is a spline in the surface. The pseudo-Anosov diffeomorphisms, which we obtain here, differ from the examples of R. C. Penner [P], see also [F], since all Dehn twists in the product belong to the same conjugacy class. The diffeomorphism is pseudo-Anosov, if and only if the Dynkin diagram of the intersection of the core curves is not a classical Dynkin diagram of a finite Coxeter group. A finite tree can be realized as Dynkin diagram of a slalom divide of a (disk wide) web, which we define as an embedded finite tree B in the unit disk D such that the intersection $B \cap \partial D$ is a set of terminal vertices of B, which are called root vertices of B. The definition of a slalom curve remains unchanged, except for the slalom of a web without root vertices, where we consider an immersion of the circle instead of the interval. For instance the extended Dynkin diagram $\widetilde{E}_8$ with 9 vertices is the Dynkin diagram of the slalom of Fig. 6, which is the slalom of a web with 2 root vertices.

The link of the slalom divide $\widetilde{E}_8$ has 2 components; it is the Montesinos link M(0,(2,1),(3,1),(6,1)). The extended Dynkin diagram $\widetilde{D}_4$ with 5 vertices corresponds to the slalom of the web with 4 root vertices and a single vertex of valency 4 and its link is the Montesinos link M(0,(2,1),(2,1),(2,1),(2,1)). It is interesting to observe that both links are in the list *b*). of proposition 2.1 of [B-Z]. The Coxeter diffeomorphism of a Dynkin diagram, which we suppose to be a tree here, is always the monodromy diffeomorphism

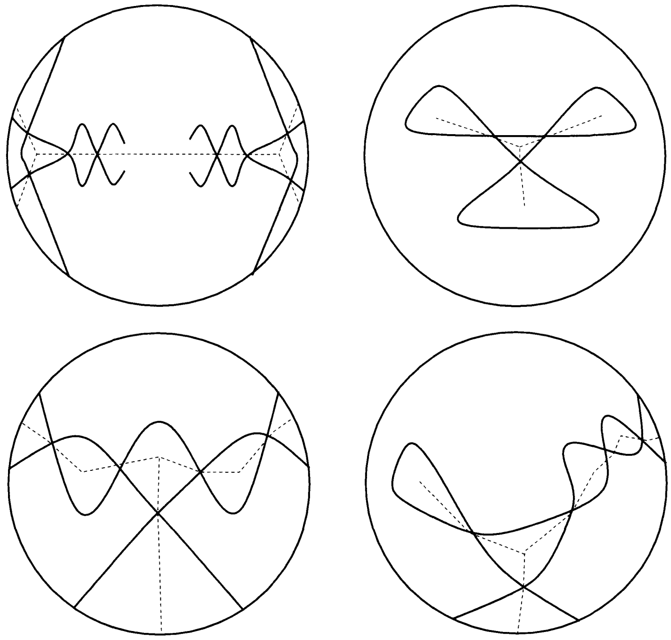

FIG. 6. – The slalom of $\tilde{D}_n$, $n \geqslant 4$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$.

of a fibered link by the fibration theorem of [AC4]. The Dynkin diagram $\tilde{D}_4$ is realized as the Dynkin diagram of a complete intersection curve singularity by Marc Giusti [G] and the Coxeter diffeomorphism of $\tilde{D}_4$ appears as monodromy in the unfolding of this singularity. The fibered link of the web with $n+1$ vertices and n root vertices, $n \geqslant 5$, is a chain with n links. This link is studied in the lecture notes of W. Thurston and is hyperbolic.

Remark. — The complexity $C_{\pi_1}(\phi)$ of an orientation preserving isotopy class ϕ of diffeomorphisms of a surface can be defined as the minimum of the quantity $a+b$ over all the product decompositions of ϕ as product of Dehn twists, where a is the length of the product and where b is the number of intersection points of the core curves of the twists involved in the product decomposition. The corresponding homological complexity is the complexity $C_{H_1}(\phi)$ where we minimize the quantity $a+h$, where h stands for the sum of the absolute values of the mutual homological intersection numbers of the core curves.

The homological complexity of the monodromy T of a non trivial fibered knot is estimated from below by $C_{H_1}(T) \geqslant 4\delta - 1$, where δ is the genus of the fiber. So, we can observe that both complexities coincide and are minimal with respect to this estimation by the genus for monodromies of knots of slalom curves. It would be nice to deduce from this observation that the homological and isotopical entropy of monodromies of knots of slalom curves coincide.