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VARIATION OF GEOMETRIC INVARIANT THEORY QUOTIENTS

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ABSTRACT

Geometric Invariant Theory gives a method for constructing quotients for group actions on algebraic varieties which in many cases appear as moduli spaces parameterizing isomorphism classes of geometric objects (vector bundles, polarized varieties, etc.). The quotient depends on a choice of an ample linearized line bundle. Two choices are equivalent if they give rise to identical quotients. A priori, there are infinitely many choices since there are infinitely many isomorphism classes of linearized ample line bundles. Hence several natural questions arise. Is the set of equivalence classes, and hence the set of non-isomorphic quotients, finite? How does the quotient vary under change of the equivalence class? In this paper we give partial answers to these questions in the case of actions of reductive algebraic groups on nonsingular projective algebraic varieties. We shall show that among ample line bundles which give projective geometric quotients there are only finitely many equivalence classes. These classes span certain convex subsets (chambers) in a certain convex cone in Euclidean space, and when we cross a wall separating one chamber from another, the corresponding quotient undergoes a birational transformation which is similar to a Mori flip.

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0. INTRODUCTION

0.1. Motivation

Consider a projective algebraic variety X acted on by a reductive algebraic group G, both defined over the field of complex numbers. In general the orbit space X/G does not exist in the category of separated algebraic varieties. One of the reasons for this is the presence of non-closed orbits. A solution is suggested by Geometric Invariant Theory ([MFK]). It begins with a choice of a G-equivariant embedding of X in a projective space and then introduces the algebraic quotient $X//G$ as the projective spectrum of the subring of G-invariant elements in the projective coordinate ring of X. This quotient comes with a canonical rational map $X \to X//G$ whose domain of definition X^{ss} is the set of semistable points. The regular map $f: X^{ss} \to X//G$ is a good categorical quotient $X^{ss} \to X^{ss}//G$. It defines a bijective correspondence between closed orbits in X^{ss} and points of $X^{ss}//G$. Also there exists a maximal G-invariant Zariski open subset X^s of stable points such that the restriction of f to X^s is the quotient map $X^s \to X^s/G \subset X^{ss}//G$ for the orbit space X^s/G. An implicit ingredient of this construction is the G-equivariant projective embedding which allows one to linearize the action. It is defined by a choice of a very ample G-linearized line bundle L. Actually, it is enough to take any ample G-linearized bundle L and define the embedding by taking sufficiently large tensor power of L. The previous construction will depend only on L but not on the choice of a tensor power. The subsets of semistable and stable points can be described as follows:

$$X^{ss}(L) = \{ x \in X : \exists\, \sigma \in \Gamma(X, L^{\otimes n})^G \quad \text{such that } \sigma(x) \neq 0 \},$$

$$X^{s}(L) = \{ x \in X^{ss}(L) : G\cdot x \text{ is closed in } X^{ss}(L) \text{ and the stabilizer } G_x \text{ is finite} \}.$$

There are infinitely many isomorphism classes of ample G-linearized line bundles L. So it is natural to ask the following questions:

(i) Is the set of non-isomorphic quotients $X^{ss}(L)//G$ finite? Describe this set.

(ii) How does the quotient $X^{ss}(L)//G$ change if we vary L in the group $\mathrm{Pic}^G(X)$ of isomorphism classes of G-linearized line bundles?

The fundamental comparison problem of different GIT quotients was apparently first addressed by M. Goresky and R. MacPherson in the paper [GM], where they pioneered the use of natural morphisms among quotients.

This problem is analogous to the problem of the variation of symplectic reductions of a symplectic manifold M with respect to an action of a compact Lie group K. Recall that if $K \times M \to M$ is a Hamiltonian action with a moment map $\Phi : M \to \mathrm{Lie}(K)^*$, then for any point $p \in \Phi(M)$, the orbit space $\Phi^{-1}(K\cdot p)/K$ is the symplectic reduction of M by K with respect to the point p. If K is a torus, $M = X$ with the symplectic form defined by the Chern form of L, and K acts on X via the restriction of an algebraic action of its complexification T, then the choice of a rational point $p \in \Phi(X)$ corresponds to the choice of a T-linearization on L, and the symplectic reduction $Y_p = \Phi^{-1}(p)/K$ is isomorphic to the GIT quotient $X^{ss}(L)//T$. It turns out that in this case, if we let p vary in a connected component C (chamber) of the set of regular values of the moment map, the symplectic reductions Y_p are all homeomorphic (in fact, diffeomorphic away from singularities) to the same orbifold Y_C. However if we let p cross a wall separating one connected component from another, the reduction Y_p undergoes a very special surgery which is similar to a birational transformation known as a flip. This was shown in a work of V. Guillemin and S. Sternberg [GS] (under the assumption that the action is quasi-free). In a purely algebraic setting an analogous result was proved independently by M. Brion and C. Procesi [BP] and the second author [Hu1].

Our results extend the previous facts to the situation when T is replaced by any reductive group and we allow L itself as well as its linearization to vary. We would like to point out that our results are new even for torus actions, considering that we vary linearizations as well as their underlying ample line bundles in a single setting (the G-ample cone).

0.2. Main results

To state our partial answers to the questions raised earlier, we give the following main definition:

0.2.1. *Definition.* — *The* G-*ample cone* $C^G(X)$ *(for the action of* G *on* X) *is the convex cone in* $NS^G(X) \otimes \mathbf{R}$ *spanned by ample* G-*linearized line bundles* L *with* $X^{ss}(L) \neq \emptyset$, *where* $NS^G(X)$ *is the (Néron-Severi) group of* G-*linearized line bundles modulo homological equivalence.*

0.2.2. One of the key ideas in our project is to introduce certain walls in $C^G(X)$. The philosophy is that a polarization (induced by a G-linearized ample line bundle) lies on a wall if and only if it possesses a point that is semistable but not stable. To this end, the Hilbert-Mumford numerical criterion for stability is the key clue. For any point $x \in X$, the Hilbert-Mumford numerical criterion for stability allows one to introduce a function

$$\mathrm{Pic}^G(X) \to \mathbf{Z}, \qquad L \mapsto M^L(x),$$

such that $x \in X^{ss}(L)$ if and only if $M^L(x) \leqslant 0$, with strict inequality for stable points. We show that this function can be extended to a *lower convex* function $M^\bullet(x)$ on $C^G(X)$.

Using the notion of Kähler quotients one can give meaning to the sets $X^s(l)$, $X^{ss}(l)$ for points $l \in C^G(X)$ not necessarily coming from the classes of G-linearized ample line bundles. The function $M^\bullet(x)$ can be used to give a criterion for a point x to belong to the sets $X^s(l)$, $X^{ss}(l)$.

Next we define a *wall* in $C^G(X)$ as the set $H(x)$ of zeroes of the function $M^\bullet(x)$, where the stabilizer of the point x is of positive dimension. The class $l \in C^G(X)$ belongs to the union of walls if and only if $X^{ss}(l) \neq X^s(l)$. A non-empty connected component of the complement of the union of walls is called a *chamber*. Any chamber is an equivalence class with respect to the equivalence relation $l \sim l' \Leftrightarrow X^{ss}(l) = X^{ss}(l')$. Other equivalence classes are contained in walls and each one is a finite disjoint union of connected subsets called *cells*.

0.2.3. *Theorem.* — (i) *There are only finitely many chambers, walls and cells.*
(ii) *Each wall is a closed convex cone.*
(iii) *The closure of a chamber is a rational polyhedral cone inside of* $C^G(X)$.

0.2.4. To answer the second main question, we describe how the quotient changes when the G-linearized ample line bundle moves from one chamber to another by crossing a wall. Assuming some conditions on the wall we prove that under this change the GIT quotient undergoes a very special birational transformation which is similar to a Mori flip. To state the main result in this direction we need a few definitions. For any chamber C we denote by $X^s(C)$ the set $X^s(l)$, where $l \in C$. Likewise, for any cell F we denote by $X^{ss}(F)$ (resp. $X^s(F)$) the set $X^{ss}(l)$ (resp. $X^s(l)$), where $l \in F$. Any point $x \in X^{ss}(F) \backslash X^s(F)$ whose orbit is closed in $X^{ss}(F)$ is called a *pivotal point* of F. Its stabilizer is a reductive subgroup of G. A cell F is said to be *truly faithful* if it lies in the closure of a chamber and the stabilizer subgroup G_x of any pivotal point x of F is a one-dimensional diagonalizable group. For example, any cell which is not contained in another cell is always truly faithful when $G = T$ or when we replace X by $X \times G/B$ (G/B is the flag variety of G) and consider the natural diagonal action of G on the product. The latter will assure that our theory applies to symplectic reductions at general coadjoint orbits. We say that two chambers C_1 and C_2 are *relevant* with respect to a cell F if both of them contain a point $x \in F$ in their closures and there exists a straight segment through x with points in C_1, C_2. We say that a cell is *rational* if it contains a point from $NS^G(X) \otimes \mathbf{Q}$. Let F be a truly faithful rational cell F. Then $X^{ss}(F) \backslash X^s(F)$ has a *canonical* stratification by the so-called orbital type: two points x and y of $X^{ss}(F) \backslash X^s(F)$ are said to have the same orbital type if G_x and G_y are conjugate to each other. This stratification induces a stratification of $(X^{ss}(F) /\!/ G) \backslash (X^s(F) /\!/ G)$.

0.2.5. *Variation Theorem.* — *Assume* X *is nonsingular. Let* C^+ *and* C^- *be a pair of chambers relevant to a truly faithful rational cell* F. *Then there are two natural birational morphisms*

$$f_+ : X^s(C^+) /\!/ G \to X^{ss}(F) /\!/ G$$

and

$$f_- : X^s(C^-) /\!/ G \to X^{ss}(F) /\!/ G$$

such that, setting Σ_0 *to be* $(X^{ss}(F) /\!/ G) \backslash (X^s(F) /\!/ G)$, *we have*

(i) f_+ *and* f_- *are isomorphisms over the complement of* Σ_0;
(ii) *the fibers of the maps* $f_\pm$ *over each connected component* Σ'_0 *of a stratum of* Σ_0 *are the quotients of weighted projective spaces of some dimension* $d_\pm$ *by the finite group* $\pi_0(G_x)$ *where* x *is some pivotal point of* F;
(iii) $d_+ + d_- + 1 = \operatorname{codim} \Sigma'_0$.

Note that for any two cells F and F' such that $F \cap \overline{F}' \neq \emptyset$ there is a natural map $f_{F',F} : X^{ss}(F') // G \to X^{ss}(F) // G$. Our result covers only a very special case of possible variation theorems which should describe the fibers of the maps $f_{F',F}$. For example, we have no results in the case when there are no non-empty chambers although one may still pose the question of variations of quotients in this case. Neither do we have results describing the variation of quotients when one moves from one chamber to another by crossing an arbitrary cell.

0.3. Symplectic reductions

The theory of moment maps sets up a remarkable link between quotients in Algebraic Geometry and quotients in Symplectic Geometry. Let (X, ω) be a compact symplectic manifold with a compact Lie group K acting by symplectomorphisms. Let $\Phi : X \to \mathrm{Lie}(K)^*$ be a moment map. Choose a point $p \in \Phi(X)$, the orbit space $\Phi^{-1}(K\cdot p)/K$ is, by definition, the Marsden-Weinstein symplectic reduction at p. When p is a regular value of Φ and the action of K on $\Phi^{-1}(K\cdot p)$ is free, $\Phi^{-1}(K\cdot p)/K$ inherits the structure of a symplectic manifold from (X, ω). It is known that for a fixed Weyl chamber $\mathfrak{h}^*_+$, the intersection $\Phi(X) \cap \mathfrak{h}^*_+$ is a convex polytope. The connected components of the regular values of Φ in $\mathfrak{h}^*_+$ form top chambers in $\Phi(X) \cap \mathfrak{h}^*_+$. The set of critical values of Φ in $\mathfrak{h}^*_+$ forms walls. If p stays in a chamber, the differential structure of $\Phi^{-1}(K\cdot p)/K$ remains the same (the symplectic form, however, has to change). But when we cross a wall, the diffeomorphism type of $\Phi^{-1}(K\cdot p)/K$ undergoes a "flip" (see [GS]). In the situation when X is an embedded projective variety and the symplectic form is the Fubini-Study symplectic form, the variation of the symplectic reductions can be seen as the variation of GIT-quotients. This can be achieved by using the so-called *shifting trick*; one can identify, for generic rational point p, $\Phi^{-1}(K\cdot p)/K$ with a GIT-quotient of $X \times G/B$ by the diagonal action of the group G (when p is on the boundary of $\mathfrak{h}^*_+$, one should consider G/P instead of G/B). Here K is a maximal compact subgroup of G and the linearization of the action is obtained by the Segre embedding of $X \times G/B$ by using the ample bundle on G/B corresponding to the character of B defined by the point p.

0.4. Hilbert, limit quotients, and moduli problems

The quotients $X^{ss}(F)//G$ and the morphisms $f_{F',F}$ form a projective system. So by taking the projective limit of this system we obtain a variety which dominates all GIT quotients of X. We call this variety the *limit quotient* of X by G. The notion of the limit quotient is closely related to some earlier constructions using the Chow or Hilbert scheme parameterizing the generic orbit closures and their limits (see [BBS], [Li], [Ka], [KSZ], and [Hu1]).

There are numerous examples of birational variations of moduli spaces in the case when the notion of stability of geometric objects depends on a parameter. In many of these cases the variation can be explained as a variation of a geometric invariant quotient. This basic observation was probably made first by M. Thaddeus (cf. [Re],

[Th1]). In this paper we are not discussing the applications of our theory to the moduli problems and the limit quotients; we plan to return to this and produce more examples in subsequent publications.

After the first preprint version of our paper had appeared, we saw a preprint of Thaddeus (now the paper [Th2]) where our finiteness theorem and some results in section 4 were reproved by different methods. For example, by using Luna's slice theorem, he obtains some further information about the algebraic structure of the flip maps in Theorem 0.2.5. A similar approach to the proof of Theorem 0.2.5 was independently proposed by Charles Walter.

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1. THE NUMERICAL FUNCTION $M^{\bullet}(x)$ AND SOME FINITENESS THEOREMS

Throughout the paper we freely use the terminology and the basic facts from Geometric Invariant Theory ([MFK]). We shall be working over the field $\mathbf{C}$ of complex numbers, although most of what follows is valid over an arbitrary algebraically closed field.

1.1. The function $M^L(x)$

1.1.1. Let $\sigma : G \times X \to X$ be an algebraic action of a connected reductive linear algebraic group G on an irreducible projective algebraic variety X. Let $L \in \mathrm{Pic}^G(X)$ be a G-linearized line bundle over X. For every $x \in X$ and any 1-parameter subgroup $\lambda : \mathbf{C}^* \to G$, the subgroup $\lambda(\mathbf{C}^*)$ acts on the fiber L_{x_0} over the point $x_0 = \lim_{t \to 0} \lambda(t) . x$ via a character $t \to t^{\mu^L(x, \lambda)}$. The number $\mu^L(x, \lambda)$ satisfies the following properties:

(i) $\mu^L(g \cdot x, g \cdot \lambda \cdot g^{-1}) = \mu^L(x, \lambda)$ for any $g \in G$;

(ii) for fixed x and λ, the map $L \mapsto \mu^L(x, \lambda)$ is a homomorphism $\mathrm{Pic}^G(X) \to \mathbf{Z}$;

(iii) $\mu^L(x, g \cdot \lambda \cdot g^{-1}) = \mu^L(x, \lambda)$ for any $g \in P(\lambda)$, where $P(\lambda)$ is a certain parabolic subgroup of G associated with λ (see 1.2.1 below);

(iv) $\mu^L(\lim_{t \to 0} \lambda(t) \cdot x, \lambda) = \mu^L(x, \lambda)$.

The numbers $\mu^L(x, \lambda)$ are used to give a numerical criterion for stability of points in X with respect to an *ample* G-linearized line bundle L:

$$x \in X^{ss}(L) \Leftrightarrow \mu^L(x, \lambda) \leqslant 0 \quad \text{for all one-parameter subgroups } \lambda,$$

$$x \in X^{s}(L) \Leftrightarrow \mu^L(x, \lambda) < 0 \quad \text{for all one-parameter subgroups } \lambda.$$

Here we use the definitions of the sets $X^{ss}(L)$ and $X^{s}(L)$ given in 0.1. As usual, $X^{us}(L) := X\setminus X^{ss}(L)$ denotes the set of unstable points. For future use we let $X^{sss}(L)$ denote the complement $X^{ss}(L)\setminus X^{s}(L)$. Its points are called *strictly semistable*.

1.1.2. It follows from the numerical criterion of stability that

$$X^{ss}(L) = \bigcap_{T \text{ maximal torus}} X^{ss}(L_T), \qquad X^{s}(L) = \bigcap_{T \text{ maximal torus}} X^{s}(L_T),$$

where L_T denotes the image of L under the restriction map $\mathrm{Pic}^G(X) \to \mathrm{Pic}^T(X)$.

1.1.3. Let T be a maximal torus of G and $W = N_G(T)/T$ be its Weyl group. We denote by $\mathscr{X}_*(G)$ the set of one-parameter subgroups of G. We have $\mathscr{X}_*(G) = \bigcup_{g\in G} \mathscr{X}_*(gTg^{-1})$. Let us identify $\mathscr{X}_*(T)\otimes \mathbf{R}$ with $\mathbf{R}^n$, where $n = \dim T$, and consider a W-invariant Euclidean norm $\|\ \|$ in $\mathbf{R}^n$. Then we can define for any $\lambda \in \mathscr{X}_*(G)$, $\|\lambda\| := \|\mathrm{Int}(g)\circ\lambda\|$ where $\mathrm{Int}(g)$ is an inner automorphism of G such that $\mathrm{Int}(g)\circ\lambda \in \mathscr{X}_*(T)$.

Now let

$$\bar{\mu}^L(x, \lambda) := \frac{\mu^L(x, \lambda)}{\|\lambda\|}, \quad M^L(x) := \sup_{\lambda\in\mathscr{X}_*(G)} \bar{\mu}^L(x, \lambda).$$

The function $M^{\bullet}(x) : \mathrm{Pic}^G(X) \to \mathbf{R}$ will play a key role in the rest of the paper. We shall show in Proposition 1.1.6 that $M^L(x)$ is always finite. To this end, we first have

1.1.4. *Lemma.* — *Assume* L *is ample. Let* T *be a maximal torus of* G *and* $r_T : \mathrm{Pic}^G(X) \to \mathrm{Pic}^T(X)$ *be the restriction map. Then for any* $x \in X$, *the set* $\{ M^{r_T(L)}(g\cdot x), g \in G \}$ *is finite and* $M^L(x) = \max_{g\in G} M^{r_T(L)}(g\cdot x)$.

Proof. — See [Ne1], Lemma 3.4. □

1.1.5. Assume L is ample. Then we can give the following interpretation of the number $M^L(x)$. Replacing L with some positive power $L^{\otimes n}$, we may assume that L is very ample. Choose a G-equivariant embedding of X in a projective space $\mathbf{P}(V)$ such that G acts on X via its linear representation in V. Let us assume that G is the n-dimensional torus $(\mathbf{C}^*)^n$. Then its group of characters $\mathscr{X}(G)$ is isomorphic to $\mathbf{Z}^n$. The isomorphism is defined by assigning to any $(m_1, \ldots, m_n) \in \mathbf{Z}^n$ the homomorphism $\chi : G \to \mathbf{C}^*$ defined by the formula

$$\chi(t_1, \ldots, t_n) = t_1^{m_1} \ldots t_n^{m_n}.$$

Any one-parameter subgroup $\lambda : \mathbf{C}^* \to G$ is given by the formula

$$\lambda(t) = (t^{r_1}, \ldots, t^{r_n})$$

for some $(r_1, \ldots, r_n) \in \mathbf{Z}^n$. In this way we can identify the set $\mathscr{X}_*(G)$ with the group $\mathbf{Z}^n$. Let $\lambda \in \mathscr{X}_*(G)$ and $\chi \in \mathscr{X}(G)$; the composition $\chi\circ\lambda$ is a homomorphism $\mathbf{C}^* \to \mathbf{C}^*$, hence is defined by an integer m. We denote this integer by $\langle \lambda, \chi \rangle$. It is clear that the pairing

$$\mathscr{X}_*(G) \times \mathscr{X}(G) \to \mathbf{Z}, \qquad (\lambda, \chi) \mapsto \langle \lambda, \chi \rangle,$$

is isomorphic to the natural dot-product pairing $\mathbf{Z}^n \times \mathbf{Z}^n \to \mathbf{Z}$. Now let

$$V = \bigoplus_{\chi \in \mathscr{X}(G)} V_\chi,$$

where $V_\chi = \{ v \in V : g \cdot v = \chi(g) \cdot v \}$. For any $v \in V$ we can write $v = \Sigma_\chi v_\chi$, where $v_\chi \in V_\chi$. The group G acts on the vector v by the formula $g \cdot v = \Sigma_\chi \chi(g) \cdot v_\chi$. Let $x \in \mathbf{P}(V)$ be represented by a vector v in V. We set

$$\mathrm{st}(x) = \{ \chi : v_\chi \neq 0 \} \text{ (the state set of } x),$$

$$\mathrm{Conv}(\mathrm{st}(x)) = \text{convex hull of } \mathrm{st}(x) \text{ in } \mathscr{X}(G) \otimes \mathbf{R}.$$

Then $\mu^L(x, \lambda) = \min_{\chi \in \mathrm{st}(x)} \langle \lambda, \chi \rangle$. In particular,

$$x \in X^{ss}(L) \Leftrightarrow 0 \in \mathrm{Conv}(\mathrm{st}(x))$$

$$x \in X^{s}(L) \Leftrightarrow 0 \in \mathrm{Conv}(\mathrm{st}(x))^0,$$

where the upper "0" means taking the interior.

In fact, $\dfrac{\mu^L(x, \lambda)}{\| \lambda \|}$ is equal to the signed distance from the origin to the boundary of the projection of $\mathrm{Conv}(\mathrm{st}(x))$ to the positive ray spanned by the vector λ. Then $| M^L(x) |$ is equal to the distance from the origin to the boundary of $\mathrm{Conv}(\mathrm{st}(x))$. Now if G is any reductive group we can fix a maximal torus T in G and apply Lemma 1.1.4. This will give us the interpretation of the function $M^L(x)$ in the general case.

Now we are ready to show

1.1.6. *Proposition. — For any* $L \in \mathrm{Pic}^G(X)$ *and* $x \in X$, $M^L(x)$ *is finite.*

Proof. — It follows from the previous discussion that $M^L(x)$ is finite if L is ample. It is known that for any $L \in \mathrm{Pic}(X)$ and an ample $L_1 \in \mathrm{Pic}(X)$ the bundle $L \otimes L_1^{\otimes N}$ is ample for sufficiently large N. This shows that any $L \in \mathrm{Pic}^G(X)$ can be written as a difference $L_1 \otimes L_2^{-1}$ for ample $L_1, L_2 \in \mathrm{Pic}^G(X)$. Replacing L with sufficiently high power $L^{\otimes n}$ we may assume that L_1 and L_2 are very ample. We have

$$M^{L_2^{-1}}(x) = \sup_\lambda \frac{\mu^{L_2^{-1}}(x, \lambda)}{\| \lambda \|} = \sup_\lambda \frac{-\mu^{L_2}(x, \lambda)}{\| \lambda \|} = - \inf_\lambda \frac{\mu^{L_2}(x, \lambda)}{\| \lambda \|}.$$

If G is a torus, then it follows from 1.1.5 that the function $\lambda \to \mu^{L_2}(x, \lambda)$ is piecewise linear, hence continuous. This implies that the function $\dfrac{\lambda}{\| \lambda \|} \to \dfrac{\mu^{L_2}(x, \lambda)}{\| \lambda \|}$ is continuous, and hence bounded on the sphere of radius 1. Hence it is bounded from below and from above. If G is any reductive group, and T is a maximal torus of G, we use that

$\dfrac{\mu^{L_2}(x, \lambda)}{\|\lambda\|} = \dfrac{\mu^{r_T(L_2)}(g\cdot x, \lambda_1)}{\|\lambda_1\|}$ for some $g \in G$ and some $\lambda_1 \in \mathscr{X}_*(T)$. Since the set of all possible state sets $\mathrm{st}_T(g\cdot x)$ is finite, we obtain that

$$\inf_\lambda \frac{\mu^{L_2}(x, \lambda)}{\|\lambda\|} = \inf_g \inf_{\lambda_1} \frac{\mu^{r_T(L_2)}(g\cdot x, \lambda_1)}{\|\lambda_1\|}$$

is finite. We now have

$$M^L(x) = \sup_\lambda \left(\frac{\mu^{L_1}(x, \lambda)}{\|\lambda\|} + \frac{\mu^{L_2^{-1}}(x, \lambda)}{\|\lambda\|}\right) \leqslant \sup_\lambda \frac{\mu^{L_1}(x, \lambda)}{\|\lambda\|} + \sup_\lambda \frac{\mu^{L_2^{-1}}(x, \lambda)}{\|\lambda\|}.$$

This implies that $M^L(x)$ is finite because $M^L(x)$ is obviously bounded from below. □

1.1.7. One can restate the numerical criterion using the function $M^L(x)$: For any ample $L \in \mathrm{Pic}^G(X)$ on a complete algebraic variety X

$$X^{ss}(L) = \{ x \in X : M^L(x) \leqslant 0 \}, \quad X^s(L) = \{ x \in X : M^L(x) < 0 \}.$$

1.2. Adapted one-parameter subgroups

The main references here are [Ne2], [Ke], [Ki1].

1.2.1. For every one-parameter subgroup λ one defines a subgroup $P(\lambda) \subseteq G$ by

$$P(\lambda) := \{ g \in G : \lim_{t\to 0} \lambda(t)\cdot g\cdot \lambda(t)^{-1} \text{ exists in } G \}.$$

This is a parabolic subgroup of G; λ is contained in its radical. The set

$$L(\lambda) := \{ \bar{g} = \lim_{t\to 0} \lambda(t)\cdot g\cdot\lambda(t)^{-1} : g \in P(\lambda) \}$$

is the subgroup of $P(\lambda)$ which centralizes λ. The set of g's such that the limit equals 1 forms the unipotent radical $U(\lambda)$ of $P(\lambda)$ (see [MFK], Prop. 2.6). We have

$$P(\lambda) = U(\lambda) \rtimes L(\lambda).$$

Fix a maximal torus T containing λ. Let $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ be the root decomposition for the Lie algebra $\mathfrak{g}$ of G, where $\mathfrak{t}$ is the Lie algebra of T. Then

$$\mathrm{Lie}(L(\lambda)) = \mathfrak{t} \oplus \bigoplus_{\langle\lambda,\alpha\rangle=0} \mathfrak{g}_\alpha, \quad \mathrm{Lie}(U(\lambda)) = \bigoplus_{\langle\lambda,\alpha\rangle>0} \mathfrak{g}_\alpha.$$

1.2.2. *Definition. — Let* $L \in \mathrm{Pic}^G(X)$, $x \in X$. *A one-parameter subgroup* λ *is called* adapted *to x with respect to* L *if*

$$M^L(x) = \frac{\mu^L(x, \lambda)}{\|\lambda\|}.$$

The set of primitive (i.e. not divisible by any positive integer) adapted one-parameter subgroups will be denoted by $\Lambda^L(x)$.

The following three statements are important for the later use.

1.2.3. *Theorem. — Assume* L *is ample and* $x \in X^{us}(L)$. *Then*

(i) *there exists a parabolic subgroup* $P(L)_x \subseteq G$ *such that* $P(L)_x = P(\lambda)$ *for all* $\lambda \in \Lambda^L(x)$;
(ii) *all elements of* $\Lambda^L(x)$ *are conjugate to each other by elements from* $P(\lambda)$;
(iii) *if* T *is a maximal torus contained in* $P(L)_x$, *then* $\Lambda^L(x) \cap \mathscr{X}_*(T)$ *forms one orbit with respect to the action of the Weyl group.*

Proof. — This is a theorem of G. Kempf [Ke]. □

1.2.4. *Corollary. — Assume* $x \in X^{us}(L)$. *Then for all* $g \in G$

(i) $\Lambda^L(g \cdot x) = \mathrm{Int}(g)\, \Lambda^L(x)$;
(ii) $P(L)_{g \cdot x} = \mathrm{Int}(g)\, P(L)_x$;
(iii) $G_x \subset P(L)_x$.

1.2.5. *Theorem. — Assume that* $L \in \mathrm{Pic}^G(X)$ *is ample and* $x \in X^{us}(L)$. *Let* $\lambda \in \Lambda^L(x)$ *and* $y = \lim_{t \to 0} \lambda(t) \cdot x$. *Then*

(i) $\lambda \in \Lambda^L(y)$;
(ii) $M^L(x) = M^L(y)$.

Proof. — This is Theorem 9.3 from [Ne2]. □

1.3. Stratification of the set of unstable points

Following Kempf [Ke] and Hesselink [He], we introduce the following algebraic stratification of the set $X^{us}(L)$, which will serve as a basic tool in § 3 and § 4.

1.3.1. For each $d > 0$ and conjugacy class $\langle \tau \rangle$ of a one-parameter subgroup τ of G, we set

$$S^L_{d,\langle\tau\rangle} = \{ x \in X : M^L(x) = d,\ \exists\, g \in G \text{ such that } \mathrm{Int}(g) \circ \tau \in \Lambda^L(x) \}.$$

Let $\mathscr{T}$ be the set of conjugacy classes of one-parameter subgroups. For any ample L

$$X = X^{ss}(L) \cup \bigcup_{d>0,\, \langle\tau\rangle \in \mathscr{T}} S^L_{d,\langle\tau\rangle}$$

is a finite stratification of X into Zariski locally closed G-invariant subvarieties of X. Observe that property (ii) of Theorem 1.2.3 ensures that the subsets $S_{d,\langle\tau\rangle}$, $d > 0$, are disjoint.

1.3.2. Let $x \in S^L_{d,\langle\tau\rangle}$. For each $\lambda \in \Lambda(x)$ we can define the point $y = \lim_{t \to 0} \lambda(t) \cdot x$. Clearly λ fixes y. By Theorem 1.2.5, $y \in S^L_{d,\langle\tau\rangle}$. Set

$$Z^L_{d,\langle\tau\rangle} := \{ x \in S^L_{d,\langle\tau\rangle} : \lambda(\mathbf{C}^*) \subset G_x \text{ for some } \lambda \in \langle \tau \rangle \}.$$

Let us further subdivide each stratum $S^L_{d,\langle\tau\rangle}$ by putting for each $\lambda \in \langle \tau \rangle$

$$S^L_{d,\lambda} := \{ x \in S^L_{d,\langle\tau\rangle} : \lambda \in \Lambda^L(x) \}.$$

Hesselink calls these subsets *blades*. If

$$Z^L_{d,\lambda} := \{ x \in S^L_{d,\lambda} : \lambda(\mathbf{C}^*) \subset G_x \},$$

then we have the map

$$p_{d,\lambda} : S^L_{d,\lambda} \to Z^L_{d,\lambda}, \qquad x \mapsto y = \lim_{t\to 0} \lambda(t)\cdot x.$$

The following result will be frequently used later. Its proof can be found in [Kil], § 13.

1.3.3. *Proposition.* — (i) $S^L_{d,\lambda} = \{ x \in X : \lim_{t\to 0} \lambda(t)\cdot x \in Z^L_{d,\lambda} \} = p^{-1}_{d,\lambda}(Z^L_{d,\lambda})$;

(ii) *for each connected component* $Z^L_{d,\lambda,i}$ *of* $Z^L_{d,\lambda}$ *the restriction of the map* $p_{d,\lambda} : S^L_{d,\lambda} \to Z^L_{d,\lambda}$ *over* $Z^L_{d,\lambda,i}$ *is a vector bundle with the zero section equal to* $Z^L_{d,\lambda,i}$, *assuming in addition that* X *is smooth*;

(iii) $S^L_{d,\lambda}$ *is* $P(\lambda)$*-invariant,* $Z^L_{d,\lambda,i}$ *is* $L(\lambda)$*-invariant; if* $\bar{g}$ *denotes the projection of* $g \in P(\lambda)$ *to* $L(\lambda)$, *then for each* $x \in S^L_{d,\lambda}$, $p_{d,\lambda}(g\cdot x) = \bar{g}.p_{d,\lambda}(x)$;

(iv) *there is a surjective finite morphism* $G \times_{P(\lambda)} S^L_{d,\lambda} \to S^L_{d,\langle\tau\rangle}$. *It is bijective if* $d > 0$ *and is an isomorphism if* $S^L_{d,\langle\tau\rangle}$ *is normal.*

1.3.4. Let $X^\lambda = \{ x \in X : \lambda(\mathbf{C}^*) \subset G_x \}$. By definition, $Z^L_{d,\lambda} \subset X^\lambda$. As usual we may assume that G acts on X via its linear representation in the space $V = \Gamma(X, L)^*$. Let $V = \bigoplus_i V_i$, where $V_i = \{ v \in V : \lambda(t)\cdot v = t^i v \}$. Then $X^\lambda = \bigcup_i X^\lambda_i$, where $X^\lambda_i = \mathbf{P}(V_i) \cap X$. For any $x \in Z^L_{d,\lambda}$, $d = M^L(x) = \mu^L(x,\lambda)/\|\lambda\|$. If v represents x in V, then, by definition of $\mu^L(x,\lambda)$, we have $v \in V_{d\|\lambda\|}$. Therefore, $Z^L_{d,\lambda} \subset X^\lambda_{d\|\lambda\|}$. Since $L(\lambda)$ centralizes λ, each λ-eigensubspace V_i is stable with respect to $L(\lambda)$. By Proposition 1.3.3 (iii), the group $L(\lambda)$ leaves $Z^L_{d,\lambda}$ invariant. The subgroup $\lambda(\mathbf{C}^*)$ of $L(\lambda)$ acts trivially on $\mathbf{P}(V_{d\|\lambda\|})$, hence we get the action of $L(\lambda)' = L(\lambda)/\lambda(\mathbf{C}^*)$ on $\mathbf{P}(V_{d\|\lambda\|})$ leaving $Z^L_{d,\lambda}$ invariant. Let $\mathcal{O}_{X^\lambda_{d\|\lambda\|}}(1)$ be the very ample $L(\lambda)$-linearized line bundle obtained from the embedding of $X^\lambda_{d\|\lambda\|}$ into $\mathbf{P}(V_{d\|\lambda\|})$. (Note that $L(\lambda)' = L(\lambda)/\lambda(\mathbf{C}^*)$ may not admit any induced linearization.) Then:

1.3.5. *Proposition.* — *We have* $Z^L_{d,\lambda} = (X^\lambda_{d\|\lambda\|})^{ss}(\mathcal{O}_{X^\lambda_{d\|\lambda\|}}(1))$.

Proof. — It follows from [Ne2], Theorem 9.4. □

The lemma below paves a way to our main finiteness results in the sequel.

1.3.6. *Lemma.* — *Let* $\Pi(X^\lambda)$ *be the (finite) set of connected components of* X^λ *for a fixed* $\lambda \in \mathscr{X}_*(G)$. *Then* G *acts naturally on the set* $\bigcup_{\lambda\in\mathscr{X}_*(G)} \Pi(X^\lambda)$ *and has finitely many orbits in it.*

Proof. — For any $g \in G$, we have $g(\Pi(X^\lambda)) = \Pi(X^{g\cdot\lambda\cdot g^{-1}})$. Since any λ is conjugate to some one-parameter subgroup of a maximal torus T, it is enough to show that the set $\bigcup_{\lambda \in \mathscr{X}_*(T)} \Pi(X^\lambda)$ is finite. Choose a G-equivariant projective embedding $X \hookrightarrow \mathbf{P}(V)$. Let $V = \bigoplus_\chi V_\chi$ be the decomposition into the direct sum of eigenspaces with respect to the action of T. For each integer i and λ, let $V_{i,\lambda} = \bigoplus_{\langle\chi,\lambda\rangle = i} V_\chi$. Then $X^\lambda = \bigcup_i \mathbf{P}(V_{i,\lambda}) \cap X$. Since the number of non-trivial direct summands V_χ of V is finite, there are only finitely many different subvarieties of the form X^λ. Each of them consists of finitely many connected components. □

1.3.7. For each connected component X_i^λ of X^λ, one can define its contracting set $X_i^+ = \{x \in X : \lim_{t\to 0} \lambda(t)\cdot x \in X_i^\lambda\}$. When X is nonsingular, by a theorem of Bialynicki-Birula [B-B], this set has the structure of a vector bundle with respect to the natural map $x \mapsto \lim_{t\to 0}\lambda(t)\cdot x$. The decomposition $X = \bigcup_i X_i^+$ is the so-called Bialynicki-Birula decomposition induced by λ. Although there are infinitely many one-parameter subgroups of G, the number of their corresponding Bialynicki-Birula decompositions is finite up to the action of the group G. This follows from the next proposition which is proven in [Hu2].

1.3.8. *Proposition.* — *Let* T *be a fixed maximal torus of* G. *Then there are only finitely many Bialynicki-Birula decompositions induced by all* $\lambda \in \mathscr{X}_*(T)$.

In fact, there is a decomposition of $\mathscr{X}_*(T)$ into a finite union of rational cones such that two one-parameter subgroups give rise to the same Bialynicki-Birula decomposition if and only if they lie in the same cone. This is Theorem 3.5 of [Hu2]. A simple example of this theorem is the case of the flag variety G/B acted on by a maximal torus T. In this case, the cone decomposition of $\mathscr{X}_*(T)$ is just the fan formed by the Weyl chambers. Each Weyl chamber gives a Bialynicki-Birula decomposition (there are $|W|$ of them, where W is the Weyl group). Other Bialynicki-Birula decompositions correspond to the faces of the Weyl chambers.

One can also prove this proposition without appealing to the cone decomposition of $\mathscr{X}_*(T)$. Let $W = \bigcup_i W_i$ be the union of the connected components of the fixed point set of T. Two points x and y of X are called equivalent if whenever $\overline{T.x} \cap W_i \neq \emptyset$, then $\overline{T.y} \cap W_i \neq \emptyset$, and vice versa. This gives a decomposition of X into a finite union of equivalence classes $X = \bigcup_E X^E$, where E ranges over all equivalence classes (X^E are called torus strata in [Hu2]). It is proved in Lemma 6.1 of [Hu2] that every Bialynicki-Birula stratum X_i^+ is a (finite) union of X^E. This implies that there are only finitely many Bialynicki-Birula decompositions.

Now we can state and prove our main finiteness theorem.

1.3.9. *Theorem.* — (i) *The set of locally closed subvarieties* S *of* X *which can be realized as the stratum* $S^L_{d,\langle\tau\rangle}$ *for some ample* $L \in \mathrm{Pic}^G(X)$, $d > 0$ *and* $\tau \in \mathscr{X}_*(G)$ *is finite.*

(ii) *The set of possible open subsets of* X *which can be realized as the set of semistable points with respect to some ample* G*-linearized line bundle is finite.*

Proof. — The second assertion obviously follows from the first one. We prove both by using induction on the dimension of X. The assertion is obvious if $\dim X = 0$. Assume that the statement is true for varieties of dimensions less than dim X. Consider an arbitrary one-parameter subgroup λ such that $S^{L}_{d,\lambda} X^{\lambda} \neq X$. In particular, for such a one-parameter subgroup, the dimension of each connected component X^{λ}_{i} of X^{λ} is less than X. We now apply induction to each X^{λ}_{i} acted upon by the group $G' = L(\lambda)$ and obtain the statements (i) and (ii) for this action. (Here, we remark that we do not need to require that G'-linearized line bundles are induced from the restrictions of some G-line bundles; we just apply induction to the space X^{λ}_{i} which is acted upon by the group G'). By Proposition 1.3.5 and Lemma 1.3.6, this implies that the set of locally closed subsets of X which can be realized as the subsets $Z^{L}_{d,\langle\tau\rangle}$ is finite. Now, by Proposition 1.3.3 (i) and Proposition 1.3.8, out of the finitely many $Z^{L}_{d,\langle\tau\rangle}$, one can only construct finitely many $S^{L}_{d,\langle\tau\rangle}$, and we are done. □

1.3.10. *Remark.* — As was pointed out by A. Bialynicki-Birula, it is enough to prove assertion (ii) of Theorem 1.3.9 in the case when G is a torus. This immediately follows from 1.1.2. Then one uses the fact that the set of semi-stable points is a union of torus strata.

2. MOMENT MAP AND SYMPLECTIC REDUCTIONS

Here we explain the relationship between the geometric invariant theory quotients (briefly GIT quotients) and the symplectic reductions. The main references are [Ki1], [Ne2]. The discussion is necessary for extending geometric invariant theory to the Kähler case, throughout § 3, although it is not absolutely necessary for our approaches.

2.1. Moment map

2.1.1. Let M be a compact symplectic manifold, i.e. a compact smooth manifold equipped with a non-degenerate smooth closed 2-form ω on M (a symplectic form). Let K be a compact Lie group which acts symplectically on M. This means that K acts smoothly on M and for any $g \in K$, $g^{*}(\omega) = \omega$. We denote by $\mathfrak{k}$ the Lie algebra of K. We shall consider any $\xi \in \mathfrak{k}$ as a linear function on the dual space $\mathfrak{k}^{*}$. Each point $x \in M$ defines a map $K \to M$, $g \mapsto g \cdot x$. For any $\xi \in \mathfrak{k}$, the differential of this map at the identity element of K sends ξ to a tangent vector $\xi^{\#}_{x} \in T(K \cdot x)_{x} \subset T(M)_{x}$. Thus each ξ in $\mathfrak{k}$ defines a vector field $\xi^{\#} \in T(M)$, $x \mapsto \xi^{\#}_{x}$. The non-degeneracy condition on ω allows one to define an isomorphism $\imath_{\omega}$ from the space T(M) of smooth vector fields on M to the space $T^{*}(M)$ of smooth 1-forms on M.

2.1.2. *Definition. — A moment map for the action of* K *on* M *is a smooth map* $\Phi : M \to \mathfrak{k}^{*}$ *satisfying the following two properties*:

(i) Φ *is equivariant with respect to the action of* K *on* M *and the co-adjoint action of* K *on* $\mathfrak{k}^{*}$;
(ii) *for any* $\xi \in \mathfrak{k}$, $\imath_{\omega}(\xi^{\#}) = d(\xi \circ \Phi)$.

If a moment map exists then it is defined uniquely up to the addition of a constant from $\mathfrak{k}^{*}$ fixed by the coadjoint action. In particular, it is unique if K is semi-simple.

2.1.3. We will be using the moment maps in the following situation. Let G be a reductive algebraic group over the field of complex numbers $\mathbf{C}$ acting on a projective nonsingular algebraic variety $X \subset \mathbf{P}(V)$ via a linear representation on V. We consider G as a complex Lie group which is the complexification of a maximal compact subgroup K of G. One can choose a positive definite Hermitian inner product $\langle\ ,\ \rangle$ on V such that K acts on V by means of a homomorphism $\rho : K \to U(V)$, where $U(V)$ is the unitary group of V. Let ω be the real 2-form corresponding to a Fubini-Study metric on $\mathbf{P}(V)$ determined by the inner product on V. We normalize it by requiring that, for any local holomorphic lifting $z : U \to V \setminus \{0\}$,

$$\omega = 2i\, \partial\bar{\partial} \log \| z \|^2.$$

The form ω is a symplectic form and K acts on $(\mathbf{P}(V), \omega)$ symplectically.

There is a moment map $\mathbf{P}(V) \to \mathrm{Lie}(U(V))^*$ defined by

$$\Phi(x)(\xi) = \frac{\langle x^*, \xi^\# x^* \rangle}{2\pi i \| x^* \|^2}$$

for all $\xi \in \mathfrak{k}$, where x^* is the vector of projective coordinates of x and we identify the vector field $\xi^\#$ with an element of the Lie algebra of $U(V)$ represented by a skew-Hermitian operator. It induces a canonical moment map $\Phi : X \to \mathfrak{k}^*$. Note that this map depends on the choice of the representation $K \to U(V)$ which is the choice of a linearization on the line bundle $\mathcal{O}_X(1)$.

This construction allows one to associate to any very ample G-linearized line bundle L on X a unique moment map $\Phi^L : X \to \mathrm{Lie}(U(V))$. If L is not necessarily very ample but ample, we can define Φ^L by $\frac{1}{n}\Phi^{L^{\otimes n}}$, where $L^{\otimes n}$ is very ample. We shall call this moment map the *Fubini-Study moment map* associated to L. If no confusion arises we shall drop the superscript L from its notation.

2.1.4. We want to describe the image $\Phi(X) \subset \mathfrak{k}^*$. Since Φ is K-equivariant, $\Phi(X)$ consists of the union of co-adjoint orbits. We use some non-degenerate K-invariant quadratic form on $\mathfrak{k}$ to identify $\mathfrak{k}^*$ with $\mathfrak{k}$ (in the case that K is semisimple, one simply uses the Killing form). This changes the moment map to a map $\Phi^* : X \to \mathfrak{k}$. Let $\mathfrak{h}$ be a Cartan algebra of $\mathfrak{k}$ and let H be the maximal torus in K whose Lie algebra is $\mathrm{Lie}(H) = \mathfrak{h}$. Via derivations any character $\chi : H \to U(1)$ of H is identified with a linear function on $\mathfrak{h}$. The set of such functions is a lattice $\mathfrak{h}^*_{\mathbf{Z}}$ in $\mathfrak{h}^*$. Under the isomorphism $\mathfrak{h} \to \mathfrak{h}^*$ defined by the non-degenerate quadratic form, we obtain a $\mathbf{Q}$-vector subspace $\mathfrak{h}_{\mathbf{Q}}$ of $\mathfrak{h}$ which defines a rational structure on $\mathfrak{h}$. It is known that each adjoint orbit intersects $\mathfrak{h}$ at a unique orbit of the Weyl group acting on $\mathfrak{h}$. If we fix a positive Weyl chamber $\mathfrak{h}_+$, then we obtain a bijective correspondence between adjoint orbits and points of $\mathfrak{h}_+$. This defines *the reduced moment map*:

$$\Phi_{\mathrm{red}} : X \to \mathfrak{h}_+, \quad x \mapsto \Phi^*(K \cdot x) \cap \mathfrak{h}_+.$$

2.1.5. A co-adjoint orbit O is called *rational* if under the correspondence

$$\text{co-adjoint orbits} \leftrightarrow \mathfrak{h}_+$$

it is defined by an element α in $\mathfrak{h}_{\mathbf{Q}}$. We can write α in the form $\frac{\chi}{n}$ for some integer n and $\chi \in \mathscr{X}(\mathrm{T})$, where T is a fixed maximal torus of G. Elements of $\mathfrak{h}_+$ of the form $\frac{\chi}{n}$ will be called rational. We denote the set of such elements by $(\mathfrak{h}_+)_{\mathbf{Q}}$. By the Borel-Weil theorem, χ determines an irreducible representation $\mathrm{V}(\chi)$ which can be realized in the space of sections of an ample line bundle L_χ on the homogeneous space G/B, where B is a Borel subgroup containing T. Its highest weight is equal to χ.

2.1.6. *Theorem. — Let* $\Phi_{\mathrm{red}} : \mathbf{P}(\mathrm{V}) \to \mathfrak{h}_+$ *be the reduced moment map for the action of* K *on* $\mathbf{P}(\mathrm{V})$. *Then* $\Phi_{\mathrm{red}}(\mathrm{X})$ *is a compact convex subpolyhedron of* $\Phi_{\mathrm{red}}(\mathbf{P}(\mathrm{V}))$ *with vertices at rational points. Moreover*

$$\Phi_{\mathrm{red}}(\mathrm{X}) \cap (\mathfrak{h}_+)_{\mathbf{Q}} = \left\{ \frac{\chi}{n} : \mathrm{V}(\chi)^* \textit{ is a direct summand of } \Gamma(\mathrm{X}, \mathrm{L}^{\otimes n}) \textit{ as a } \mathrm{G}\textit{-module} \right\}.$$

This result is due to D. Mumford (see the proof in [Ne2], Appendix, or in [Br]). Note that the assertion is clear in the situation of 1.1.5. In that case the image of the moment map is equal to the convex hull Conv(St(V)) of the set $\mathrm{St}(\mathrm{V}) = \{ \chi : \mathrm{V}_\chi \neq \{ 0 \} \}$. This convexity result was first observed by M. Atiyah [At] in the setting of symplectic and Kähler geometry.

The following result from [Ne2] relates the moment map to the function $\mathrm{M}^{\bullet}(x)$.

2.1.7. *Theorem. — For any* $x \in \mathrm{X}$, $\mathrm{M}^{\mathrm{L}}(x)$ *is equal to the signed distance from the origin to the boundary of* $\Phi(\overline{\mathrm{G}.x})$. *If* $x \in \mathrm{X}^{\mathrm{us}}(\mathrm{L})$, *then the following three properties are equivalent*:

(i) $\mathrm{M}^{\mathrm{L}}(x) = \| \Phi(x) \|$;

(ii) *x is a critical point for* $\| \Phi \|^2$ *with nonzero critical value*;

(iii) *for all* $t \in \mathbf{R}$, $\exp(t\Phi^*(x)) \in \mathrm{G}_x$, *and the complexification of* $\exp(\mathbf{R}\Phi^*(x))$ *is adapted for x with respect to* L.

2.2. Relationship with geometric invariant theory

We shall consider the situation of 2.1.3. Let L be the restriction of $\mathcal{O}_{\mathbf{P}(\mathrm{V})}(1)$ to X and $\Phi : \mathrm{X} \to \mathfrak{k}^*$ be the moment map as defined there. Recall that the stability of a point $x \in \mathrm{X}$ with respect to L is determined by the function $\mathrm{M}^{\mathrm{L}}(x)$ (see 1.1.7). The following result follows from [KN] and is explicitly stated in Theorem 2.2 of [Ne2] and § 7 of [Kil] (in the general Kähler setting).

2.2.1. *Theorem.* — (i) $\mathrm{X}^{\mathrm{ss}}(\mathrm{L}) = \{ x \in \mathrm{X} : \overline{\mathrm{G}\cdot x} \cap \Phi^{-1}(0) \neq \emptyset \}$;

(ii) *the inclusion of* $\Phi^{-1}(0)$ *into* $\mathrm{X}^{\mathrm{ss}}(\mathrm{L})$ *induces a homeomorphism*

$$\Phi^{-1}(0)/\mathrm{K} \to \mathrm{X}^{\mathrm{ss}}(\mathrm{L})//\mathrm{G};$$

(iii) $\mathrm{X}^{\mathrm{s}}(\mathrm{L}) = \mathrm{X}^{\mathrm{ss}}(\mathrm{L})$ *if and only if* 0 *is not a critical value of* Φ.

2.2.2. The orbit space $\Phi^{-1}(0)/K$ is called the *symplectic reduction* or *Marsden-Weinstein reduction* of X by K. It has a natural structure of a symplectic manifold provided that 0 is not a critical value of Φ and K acts freely on $\Phi^{-1}(0)$.

2.2.3. Let $p \in \mathfrak{h}_+$ be a point in the image of the moment map $\Phi_{\rm red}$, and $O^p \subset \mathfrak{k}^*$ be the corresponding co-adjoint orbit. It comes equipped with a canonical symplectic structure defined as follows. Given a co-adjoint orbit $O \subset \mathfrak{k}^*$ and a point $q \in O$, we identify the tangent space $T(O)_q$ of O at q with a subspace of $\mathfrak{k}^*$. Now the skew-symmetric bilinear map $(a, b) \mapsto q([a, b])$ is an element ω_x of $\Lambda^2(T(O)_q^*)$.

Let $\bar{O}^p$ denote the symplectic manifold obtained from the symplectic manifold O^p by replacing its symplectic form ω with $-\omega$. Then the product symplectic manifold $X \times \bar{O}^p$ admits a moment map Φ_p defined by the formula $\Phi_p(x, q) = \Phi(x) - q$. Now the set $\Phi_p^{-1}(0)$ becomes identified with the set $\Phi^{-1}(O^p)$ and

$$\Phi_p^{-1}(0)/K \cong \Phi^{-1}(p)/K_p \cong \Phi^{-1}(O^p)/K,$$

where K_p is the isotropy subgroup of K at p. This quotient space is called the *Marsden-Weinstein reduction* or *symplectic reduction* of X with respect to p. Evidently it depends only on the orbit of p.

The following theorem more or less follows from the arguments given in [Ne2], Appendix by D. Mumford. We state it without proof.

2.2.4. *Theorem. — Let* $\alpha = \chi/n$ *be a rational element of* $\mathfrak{h}_+$ *and* L *be an ample* G*-linearized line bundle on* X. *Denote by* $L(\chi, n)$ *the line bundle on* $X \times G/B$ *equal to the tensor product of the pull-backs of the bundles* $L^{\otimes n}$ *and* L_χ *under the projection maps. Let* $\Phi^{L^{\otimes n}}$ *and* Φ^{L_χ} *be the moment map associated to* $L^{\otimes n}$ *and* L_χ, *respectively. Then for n sufficiently large, the moment map* Φ_α *defined by* $L(\chi, n)$ *is given by* $\Phi_\alpha(x, gB) = \Phi^{L^{\otimes n}}(x) + \Phi^{L_\chi}(gB)$. *Consequently,*

$$(X \times G/B)^{ss}(L(\chi, n))//G \cong \Phi_\alpha^{-1}(0)/K = (\Phi^{L^{\otimes n}})^{-1}(O^{-\alpha})/K.$$

2.3. Homological equivalence for G-linearized line bundles

We assume X to be a projective variety. This section bears no relation with moment maps except for Theorem 2.3.8.

2.3.1. Recall the definition of the Picard variety $\mathrm{Pic}(X)_0$ and the Néron-Severi group $NS(X)$. We consider the Chern class map $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbf{Z})$ and put

$$\mathrm{Pic}(X)_0 = \mathrm{Ker}(c_1), \quad NS(X) = \mathrm{Im}(c_1).$$

One way to define c_1 is to choose a Hermitian metric on $L \in \mathrm{Pic}(X)$, and set $c_1(L)$ to be equal to the cohomology class of the curvature form of this metric. Thus elements of $\mathrm{Pic}(X)_0$ are isomorphism classes of line bundles which admit a Hermitian metric with exact curvature form Θ. If Θ is given locally as $\frac{i}{2\pi} d' d'' \log(\rho_U)$, then Θ is exact if and only if there exists a global function $\rho(x)$ such that $d' d'' \log(\rho_U/\rho) = 0$ for each U. This implies that $\rho_U/\rho = |\varphi_U|$ for some holomorphic function φ_U on U. Replacing the transition functions σ_{UV} of L with $\varphi_U \sigma_{UV} \varphi_V^{-1}$ we find an isomorphic bundle L′ whose transition functions σ'_{UV} satisfy $|\sigma'_{UV}| = 1$. Since σ'_{UV} are holomorphic, we obtain that the transition functions of L′ are constants of modulus 1.

2.3.2. Now assume L is a G-linearized line bundle, where G is a complex Lie group acting holomorphically on X. Let K be a maximal compact subgroup of G. By averaging over X (here we use that K and X are compact) we can find a Hermitian structure on L such that K acts on L preserving this structure (i.e. the maps $L_x \to L_{gx}$ are unitary maps). There is a unique unitary connection on L compatible with its holomorphic structure. Its curvature form is a K-invariant 2-form Θ of type (1, 1). If L is ample, $\omega = \frac{i}{2\pi}\Theta$ is a symplectic form equal to the imaginary part of a Kähler metric on X compatible with the holomorphic structure on X. A different choice of a K-invariant Hermitian structure on L replaces ω with $\omega' = \omega + \frac{i}{2\pi} d' d'' \log(\rho)$ where ρ is a K-invariant positive-valued smooth real function on X. This can be seen as follows. Obviously $\Theta - \Theta'$ is a K-invariant 2-form $d' d'' \log \rho$ for some global function ρ. By the invariance, we get for any $g \in K$, $d' d'' \log(\rho(g \cdot x)/\rho(x)) = 0$. This implies that $\rho(g \cdot x)/\rho(x) = |c_g(x)|$ for some holomorphic function $c_g(x)$ on X. Since X is compact, this function must be constant, and the map $g \mapsto |c_g|$ is a continuous homomorphism from K to $\mathbf{R}^*$. Since K is compact and connected, it must be trivial. This gives $|c_g| = \rho(g \cdot x)/\rho(x) = 1$, hence $\rho(x)$ is K-invariant.

Taking the cohomology class of Θ we get a homomorphism:

$$c : \mathrm{Pic}^G(X) \to H^2(X, \mathbf{Z}).$$

2.3.3. *Proposition. — The restriction to* Ker(c) *of the canonical forgetful homomorphism*

$$\mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$$

has an image equal to $\mathrm{Pic}_0(X)$ *and a kernel isomorphic to* $\mathscr{X}(G)$. *Moreover there is a natural section* $s : \mathrm{Pic}_0(X) \to \mathrm{Pic}^G(X)$ *defining an isomorphism*

$$\mathrm{Ker}(c) \cong \mathscr{X}(G) \times \mathrm{Pic}(X)_0.$$

Proof. — It is known (see [KKV]) that the cokernel of the canonical forgetful homomorphism $\mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$ is isomorphic to Pic(G). Since the latter group is finite and $\mathrm{Pic}(X)_0$ is divisible, it is easy to see that Ker(c) is mapped surjectively to $\mathrm{Pic}(X)_0$ with a kernel isomorphic to the group $\mathscr{X}(G)$. To prove the assertion it suffices to construct a section $s : \mathrm{Pic}(X)_0 \to \mathrm{Pic}^G(X)$. Now let $L \in \mathrm{Pic}(X)_0$. We can find an open trivializing cover $\{ U_i \}_{i \in I}$ of L such that L is defined by constant transition functions σ_{UV}. For each $g \in G$ the cover $\{ g(U_i) \}_{i \in I}$ has the same property. Moreover,

$$\sigma_{g(U)\, g(V)} = (g^{-1})^*(\sigma_{UV}) = \sigma_{UV}.$$

This shows that we can define the *(trivial)* action of G on the total space **L** of L by the formula $g.(x, t) = (g.x, t)$ which is well-defined and is a G-linearization on L. This allows one to define the section s and check that $s(\mathrm{Pic}(X)_0) \cap \mathscr{X}(G) = \{ 1 \}$. □

2.3.4. *Definition. — A* G*-linearized line bundle* L *is called* homologically trivial *if* $L \in \mathrm{Ker}(c)$ *and the image of* L *in* $\mathscr{X}(G)$ *is the identity. In other words,* L *is homologically trivial if it belongs to* $\mathrm{Pic}(X)_0$ *as a line bundle, and the* G*-linearization on* L *is trivial (in the sense of the previous proof). The subgroup of* $\mathrm{Pic}^G(X)$ *formed by homologically trivial* G*-linearized line bundles is denoted by* $\mathrm{Pic}^G(X)_0$. *Two elements of* $\mathrm{Pic}^G(X)$ *defining the same element of the factor group* $\mathrm{Pic}^G(X)/\mathrm{Pic}^G(X)_0$ *are called* homologically equivalent.

2.3.5. *Lemma. — Let* L *and* L′ *be two homologically equivalent* G*-linearized line bundles. Then for any* $x \in X$ *and any one-parameter subgroup* λ, $\mu^L(x, \lambda) = \mu^{L'}(x, \lambda)$.

Proof. — It is enough to verify that for any homologically trivial G-linearized line bundle L and any one-parameter subgroup λ we have $\mu^L(x, \lambda) = 0$. But this follows immediately from the definition of the trivial G-linearization on L. □

2.3.6. *Proposition. — Let* L, L′ *be two ample* G*-linearized line bundles. Suppose they are homologically equivalent. Then* $X^{ss}(L) = X^{ss}(L')$.

Proof. — This follows from the numerical criterion of stability and Lemma 2.3.5. □

2.3.7. *Remark.* — One should compare this result with Corollary 1.20 from [MFK]. Under the assumption that $\mathrm{Hom}(G, \mathbf{C}^*) = \{ 1 \}$ it asserts that $X^s(L) = X^s(L')$ for any G-linearized line bundles defining the same element in $\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}(X)_0$.

2.3.8. *Theorem. — Let* L *and* L′ *be two ample* G*-linearized bundles. Suppose that* L *is homologically equivalent to* L′. *Then the moment maps* Φ^L *and* $\Phi^{L'}$ *are equal.*

Proof. — There is nothing to prove if G is trivial. So, assume that $G \neq \{ \mathrm{id} \}$. Without loss of generality we may assume that L and L′ are very ample. Choose the Fubini-Study symplectic forms ω and ω' of L and L′, respectively. By assumption, $\omega' = \omega + \frac{i}{2\pi} d' d'' \log(\rho)$ for some positive-valued K-invariant function ρ (see 2.3.2). Thus we can write $\omega' = \omega + d\theta$, where θ is a K-invariant 1-form of type (0, 1). By definition of the moment map, we have for each $\xi \in \mathfrak{k}$

$$d(\xi \circ \Phi') = \iota_{\omega'}(\xi^\#) = \iota_{\omega + d\theta}(\xi^\#) = d(\xi \circ \Phi) + L(\xi^\#)\, \theta - d\langle \xi^\#, \theta \rangle,$$

where $L(\xi^\#)$ denotes the Lie derivative along the vector field $\xi^\#$. Here $\Phi = \Phi^L$ and $\Phi' = \Phi^{L'}$. Since θ is K-invariant, we get $L(\xi^\#)\,\theta = 0$. Because G (hence also K) acts on X holomorphically the vector field $\xi^\#$ is holomorphic. But θ is of type (0, 1), so $< \xi^\#, \theta > = 0$. This shows that $d(\xi \circ \Phi') = d(\xi \circ \Phi)$ and hence $\Phi - \Phi'$ is an ad(K)-invariant constant c in $\mathfrak{k}^*$.

We need to show that this constant c is zero. Pick a maximal torus H in K such that $c \in \mathrm{Lie}(H)^*$. Let T be the complexification of H and Φ_T be the moment map for the action of T. Since the moment map for the action of T is the moment map for the action of G followed by the projection $\mathfrak{k}^* \to \mathrm{Lie}(H)^*$, by the choice of H, we have $\Phi'_T = \Phi_T - c$. Since both $\Phi'_T(X)$ and $\Phi_T(X)$ are rational polytopes (Theorem 2.1.6), we see that c must be rational.

Assume $c \neq 0$. Consider the one-parameter (algebraic) subgroup λ generated by the vector c and let Φ_λ and Φ'_λ be the moment maps for the action of λ (with respect to the obvious linearizations induced from L and L', respectively). Clearly, $\Phi'_\lambda = \Phi_\lambda - c$. Recall here that $\bar{\mu}^\bullet(x, \lambda)$ is the (signed) distance from 0 to the moment map image of $\overline{\lambda \cdot x}$. If λ acts trivially on X, the moment map must be constant. From 1.1.5 and Lemma 2.3.5, we see that $\Phi'_\lambda = \Phi_\lambda$. Thus $c = 0$, a contradiction. If λ does not act trivially on X, then the moment map image $\Phi_\lambda(X)$ must be a one-dimensional closed interval. Obviously we can find a vertex F such that $\mathrm{dist}(0, F) \neq \mathrm{dist}(c, F) = \mathrm{dist}(0, F - c)$. Note that the vertex F must be equal to $\Phi_\lambda(x)$ for some λ-fixed point $x = \overline{\lambda . x}$. Using that $\Phi'_\lambda = \Phi_\lambda - c$, we conclude $d(0, \Phi_\lambda(\overline{\lambda . x})) \neq d(0, \Phi'_\lambda(\overline{\lambda . x}))$. That is, $\bar{\mu}^{r_\lambda(L)}(x, \lambda) \neq \bar{\mu}^{r_\lambda(L')}(x, \lambda)$, where $r_\lambda : \mathrm{Pic}^G(X) \to \mathrm{Pic}^\lambda(X)$ is the restriction map, thus a contradiction to Lemma 2.3.5. □

2.3.9. Let $NS^G(X) = \mathrm{Pic}^G(X)/\mathrm{Pic}^G(X)_0$. By Proposition 2.3.3, we have an exact sequence

$$0 \to \mathscr{X}(G) \to NS^G(X) \to NS(X) \to A \to 0,$$

where A is a finite group. It is known that the Néron-Severi group NS(X) is a finitely generated abelian group. Its rank is called the *Picard number* of X and is denoted by $\rho(X)$. From this we infer that $NS^G(X)$ is a finitely generated abelian group of rank $\rho^G(X)$ equal to $\rho(X) + t(G)$, where $t(G)$ is the dimension of the radical R(G) of G. Let

$$NS^G(X)_{\mathbf{R}} = NS^G(X) \otimes \mathbf{R}$$

be the finite-dimensional real vector space generated by $NS^G(X)$.

2.4. Stratification of the set of unstable points via moment map

Here we need to compare the Hesselink stratification of the set $X \backslash X^{ss}(L)$ with the Ness-Kirwan stratification of the same set using the Morse theory for the moment map.

2.4.1. Let X be a compact symplectic manifold and K be a compact Lie group acting symplectically on it with a moment map $\Phi : X \to \mathfrak{k}^*$. We choose a K-invariant inner product on $\mathfrak{k}$ and identify $\mathfrak{k}$ with $\mathfrak{k}^*$. Let $f(x) = \| \Phi(x) \|^2$. It is obvious that the critical points of Φ are critical points of $f: X \to \mathbf{R}$. But some minimal critical points of f need not be critical for Φ (consider the case when 0 is a regular point of Φ). For any $\beta \in \mathfrak{k}$ we denote by Φ_β the composition of Φ and $\beta : \mathfrak{k}^* \to \mathbf{R}$. Let Z_β be the set of critical points of Φ_β with critical value equal to $\| \beta \|^2$. This is a symplectic submanifold of X (possibly disconnected) fixed by the subtorus T_β which is equal to the closure in H of the real one-parameter subgroup $\exp \mathbf{R}\beta$, where H is a suitable maximal torus of K.

2.4.2. Now assume that the symplectic structure on X is defined by the imaginary part of a Kähler metric on X. Let G be the complexification of K. This is a reductive complex algebraic group. We assume that the action of K on X is the restriction of an action of G on X which preserves the Kähler structure of X. Then we have a stratification $\{ S_\beta \}$ chosen with respect to the Kähler metric on X. In this case we can describe the stratum S_β as follows:

$$S_\beta = \{ x \in X : \beta \text{ is the unique closest point to 0 of } \Phi_{\mathrm{red}}(\overline{G.x}) \}.$$

Let $Z_\beta^{\min}$ denote the minimal Morse stratum of Z_β associated to the function $\| \Phi - \beta \|^2$ restricted to Z_β. Note that Φ_β is the moment map for the symplectic manifold Z_β with respect to the stabilizer group Stab_β of β under the adjoint action of K. Let $Y_\beta^{\min}$ be the pre-image of $Z_\beta^{\min}$ under the natural map $Y_\beta \to Z_\beta$.

For any $\beta \in \mathbf{B}$ let

$$P(\beta) = \{ g \in G : \exp(it\beta) \cdot g \cdot \exp(-it\beta) \text{ has a limit in } G \}.$$

It is a parabolic subgroup of G, and is the product $B.\mathrm{Stab}_\beta$ where B is a suitable Borel subgroup of G.

2.4.3. *Theorem.* — (i) *If* $x \in Y_\beta^{\min}$ *then* $\{ g \in G \mid g \cdot x \in Y_\beta^{\min} \} = P(\beta)$.
(ii) *There is an isomorphism* $S_\beta \cong G \times_{P(\beta)} Y_\beta^{\min}$.

Proof. — This is Theorem 6.18 and Lemma 6.15 from [Kil]. □

The relationship between the stratification $\{ S_\beta \}$ and the stratification described in section 1.3 is as follows. First of all we need to assume that X is a nonsingular projective algebraic variety and the Kähler structure on X is given by the curvature form of an ample L on X. In this case each β is a rational vector in $\mathfrak{h}_+ = \mathrm{Lie}(H)_+$. For any such β we can find a positive integer n such that $n\beta$ is a primitive integral vector of $\mathfrak{h}_+$ and hence defines a primitive one-parameter subgroup λ_β of G.

2.4.4. *Theorem.* — *There is a bijective correspondence between the moment map strata* $\{ S_\beta \}_{\beta \neq 0}$ *and the strata* $\{ S^L_{d,\langle\tau\rangle} \}$ *given by* $S_\beta \to S^L_{\|\beta\|, \langle\lambda_\beta\rangle}$. *Under this correspondence*:

(i) $P(\beta) = P(\lambda_\beta)$, $\mathrm{Stab}_\beta = L(\lambda_\beta)\ (\mathbf{R})$;
(ii) $Z_\beta = X^{\lambda_\beta}_{\|\beta\| \, \|\lambda_\beta\|}$, $Z_\beta^{\min} = Z^L_{\|\beta\|, \lambda_\beta}$;
(iii) $Y_\beta^{\min} = S^L_{\|\beta\|, \lambda_\beta}$.

Proof. — See [Kil], § 12, or [Ne2]. □

There is the following analog of Theorem 1.3.9:

2.4.5. *Theorem.* — (i) *The set of locally closed subvarieties* S *of* X *which can be realized as the stratum* S_β *for some* K*-equivariant Kähler symplectic structure and* $\beta \in \mathrm{Lie}(H)^*$ *is finite.*

(ii) *The set of possible open subsets of* X *which can be realized as the stratum* S_0 *for some Kähler symplectic structure on* X *is finite.*

Proof. — The second assertion obviously follows from the first one. We prove the first assertion by using induction on the rank of K. The assertion is obvious if $K = \{1\}$. Applying induction to the case when X is equal to a connected component of Z_β and $K = \mathrm{Stab}_\beta/T_\beta$, we obtain that the set of open subsets of some connected component of Z_β which can be realized as the connected component of the set $Z_\beta^{\min}$ is finite. The finiteness of the set of subsets that can be realized as Z_β follows from Lemma 1.3.6. This implies that the set of locally closed subsets of X which can be realized as the subsets $Z_\beta^{\min}$ is finite. Now each $Z_\beta^{\min}$ determines $Y_\beta^{\min}$, and by Theorem 2.4.3, the stratum S_β. □

2.5. Kähler quotients

We can extend many notions of GIT to the Kähler category using a moment map $\Phi : X \to \mathfrak{k}^*$ with respect to a Kähler symplectic form ω on X, i.e. the imaginary part of a Kähler Hermitian metric on X (see [Ki1], § 7).

2.5.1. As in Definition 2.2 of [Sj], one possible way to go for a Kähler quotient is to set

$$X^{ss}(\Phi) = \{ x \in X : \overline{G.x} \cap \Phi^{-1}(0) \neq \emptyset \}.$$

If K acts quasi-freely on $\Phi^{-1}(0)$, Kirwan ([Ki1]) proved that $\Phi^{-1}(0)/K$ is naturally homeomorphic to the orbit space $X^{ss}(\Phi)/G$ which is Hausdorff and has an induced complex analytic structure. This correspondence was extended to the case when 0 is a singular value of Φ (Theorem 2.5, [Sj]).

2.5.2. To relate to the numerical function $M^\bullet(x)$, we can give an equivalent way to define notions of Kähler stability and quotient as follows. (The equivalence follows from Proposition 2.4 of [Sj].) For any $x \in X$ and $\lambda \in \mathscr{X}_*(G)$ we define

$$\mu^\Phi(x, \lambda) = \| \lambda \| \, d_\lambda(0, \Phi(\overline{G.x})),$$

where $d_\lambda(0, A)$ denotes the signed distance from the origin to the boundary of the projection of the set A to the positive ray spanned by λ. Thus we can define

$$M^\Phi(x) = \sup_\lambda d_\lambda(0, \Phi(\overline{G.x}))$$

so that $M^{\Phi}(x)$ is equal to the signed distance from the origin to the boundary of $\Phi(\overline{G.x})$. Then we define

$$X^{ss}(\Phi) = \{ x \in X : M^{\Phi}(x) \leqslant 0 \}, \qquad X^{s}(\Phi) := \{ x \in X : M^{\Phi}(x) < 0 \},$$

and $X^{sss}(\Phi) = X^{ss}(\Phi) \backslash X^{s}(\Phi)$. The quotient $X^{ss}(\Phi)//G$ (i.e. modulo the relation $x \sim y \Leftrightarrow \overline{G.x} \cap \overline{G.y} \cap X^{ss}(\Phi) \neq \emptyset$) exists as a Hausdorff complex analytic space and is homeomorphic to the symplectic reduction $\Phi^{-1}(0)/K$, as proved by Kirwan [Ki1] and Sjamaar [Sj]. These agree with the previous notions when ω is induced from a G-linearized line bundle L and Φ is the Fubini-Study moment map.

As in the case of Hodge Kähler structures we can define the set of primitive adapted one-parameter subgroups $\Lambda^{\Phi}(x)$. There are analogs of Theorems 1.2.3 and 1.2.5 in our situation.

3. THE G-AMPLE CONE

3.1. G-effective line bundles

We will assume that X is projective and normal, possibly singular; X will be assumed to be smooth whenever moment maps are used, explicitly or implicitly. We want to point out that even in such cases, smoothness assumption may not be essential and may be removed by some rationality results about chambers or walls.

We first recall some standard definitions from the geometric invariant theory. For any G-linearized line bundle L (not necessarily ample) and a section $\sigma \in \Gamma(X, L^{\otimes n})^G$ for some positive integer n, set

$$X_\sigma = \{ x \in X \mid \sigma(x) \neq 0 \},$$

$$X^{ss}(L) = \{ x \in X \mid \exists\, n > 0 \text{ and } \sigma \in \Gamma(X, L^{\otimes n})^G \text{ such that } x \in X_\sigma \text{ and } X_\sigma \text{ is affine} \},$$

and

$$X^{s}(L) = \{ x \in X^{ss}(L) \mid G_x \text{ is finite and } G \cdot x \text{ is closed in } X^{ss}(L) \}.$$

When L is ample, the above definitions are the same as described in 0.1.

3.1.1. *Definition. — A* G-*linearized line bundle* L *on* X *is called* G-*effective if* $X^{ss}(L) \neq \emptyset$. *A* G-*effective ample* G-*linearized line bundle is called* G-*ample.*

3.1.2. *Proposition. — Let* L *be an ample* G-*linearized line bundle. The following assertions are equivalent*:

(i) L *is* G-*effective*;
(ii) $\Gamma(X, L^{\otimes n})^G \neq \{ 0 \}$ *for some* $n > 0$;
(iii) *if* $\Phi : X \to \mathfrak{k}^*$ *is the moment map associated to a* G-*equivariant embedding of* X *into a projective space given by some positive tensor power of* L, *then* $\Phi^{-1}(0) \neq \emptyset$.

Proof. — (i) $\Leftrightarrow$ (ii). Follows from the definition of semistable points.
(ii) $\Leftrightarrow$ (iii). Follows from Theorem 2.2.1. $\square$

3.1.3. *Proposition. — If* L *and* M *are two* G-*effective* G-*linearized bundles then* $L \otimes M$ *is* G-*effective.*

Proof. — Let $x \in X^{ss}(L) \cap X^{ss}(M)$. Then there exists $\sigma \in \Gamma(X, L^{\otimes n})^G$ and $\sigma' \in \Gamma(X, M^{\otimes m})^G$, for some $n, m > 0$ such that $\sigma(x) \neq 0$, $\sigma'(x) \neq 0$. Moreover X_σ and $X_{\sigma'}$ are both affine. By taking suitable tensor powers if necessary, we may assume that $m = n$. Hence $X_{\sigma\otimes\sigma'} = X_\sigma \cap X_{\sigma'}$ is also affine and the G-invariant section $\sigma \otimes \sigma'$ of $L^{\otimes n} \otimes M^{\otimes n}$ does not vanish at x. This shows that $x \in X^{ss}(L \otimes M)$, hence $L \otimes M$ is G-effective. □

3.1.4. An element of $NS^G(X)$ will be called G-*ample* if it can be represented by a G-effective ample line bundle. By Proposition 2.3.6, all ample G-linearized line bundles in the same G-ample homological equivalence class are G-effective.

Let $NS^G(X)^+$ denote the subset of G-ample homological equivalence classes. Using Proposition 3.1.3, one checks that it is a semigroup in $NS^G(X)$.

Here comes our main definition:

3.2. The G-ample cone

Here comes our main definition :

3.2.1. *Definition. — The* G-*ample cone (for the action of* G *on* X*) is the convex cone in* $NS^G(X)_{\mathbf{R}}$ *spanned by the subset* $NS^G(X)^+$. *It is denoted by* $C^G(X)$.

3.2.2. Let X be a compact Kähler manifold. The subset of the space $H^{1,1}(X, \mathbf{R})$ formed by the classes of Kähler forms is an open convex cone. It is called the *Kähler cone*. Its integral points are the classes of Hodge Kähler forms. By a theorem of Kodaira, each such class is the first Chern class of an ample line bundle L. The subcone of the Kähler cone spanned by its integral points is called the *ample cone* and is denoted by $A^1(X)^+$. It is not empty if and only if X is a projective algebraic variety. It spans the subspace $A^1(X)^+_{\mathbf{R}}$ of $H^{1,1}(X, \mathbf{R})$ formed by the cohomology classes of algebraic cycles of codimension 1. The dimension of this subspace is equal to the Picard number of X. The closure of the ample cone consists of the classes of numerically effective (nef) line bundles [K1]. Recall that a line bundle is called *nef* if its restriction to any curve is an effective line bundle. Under the forgetful map $NS^G(X)_{\mathbf{R}} \to NS(X)_{\mathbf{R}}$, the G-ample cone is mapped to the ample cone. Summing up we conclude that

$$C^G(X) = EF^G(X) \cap \alpha^{-1}(A^1(X)^+_{\mathbf{R}}),$$

where $\alpha : NS^G(X)_{\mathbf{R}} \to NS(X)_{\mathbf{R}} \to H^{1,1}(X, \mathbf{R})$ is the composition map, and $EF^G(X)$ is the convex cone in $NS^G(X)_{\mathbf{R}}$ spanned by G-effective G-linearized line bundles.

3.2.3. *Remark.* — There could be no G-effective G-linearized line bundles, so $C^G(X)$ could be empty. The simplest example is any homogeneous space $X = G/P$, where P is a parabolic subgroup of a reductive group G.

3.2.4. Let us recall some standard terminology and elementary facts about convex cones. The minimal dimension of a linear subspace containing a convex cone S is called the *dimension* of S and is denoted by dim(S). The *relative interior* ri(S) of S is the interior of S in the sense of the topology of the minimal linear subspace containing S. It is not-empty if S is of positive dimension. A closed convex cone is called *polyhedral* if it is equal to the intersection of a finite number of closed half-spaces. A linear hyperplane L is called a *supporting hyperplane* at a point $x \in \partial(S)$ if $x \in L$ and S lies in one of the two half-spaces defined by this hyperplane. This half-space is called the *supporting half-space* at x. Each point in the boundary of a convex cone belongs to a supporting hyperplane at this point. A *face* of a convex cone S is a subset of S equal to the intersection of S with a supporting hyperplane. Each face is a closed convex cone. The closure of a convex cone is a convex cone. The boundary of a closed convex cone is equal to the union of its faces. A function $f: V \to \mathbf{R} \cup \{\infty\}$ is called *lower convex* if

$$f(x+y) \leqslant f(x) + f(y)$$

for any $x, y \in V$. It is called *positively homogeneous* if $f(\lambda x) = \lambda f(x)$ for any nonnegative λ. If f is a lower convex positively homogeneous function, then the set $\{x \in V : f(x) \leqslant 0\}$ is a closed convex cone.

Now we can go back to our convex cone $C^G(X)$.

3.2.5. *Lemma.* — *For each* $x \in X$ *the function* $\mathrm{Pic}^G(X) \to \mathbf{R}$, $L \mapsto M^L(x)$ *factors through* $NS^G(X)$ *and can be uniquely extended to a positively homogeneous lower convex function* $M^{\bullet}(x) : NS^G(X)_{\mathbf{R}} \to \mathbf{R}$.

Proof. — By Lemma 2.3.5, $M^L(x) = M^{L'}(x)$ if L is homologically equivalent to L'. This shows that we can descend the function $L \to M^L(x)$ to the factor group $NS^G(X)$. By using 1.1.1 (ii), we find $M^{L^{\otimes n}}(x) = nM^L(x)$ for any nonnegative integer n, and

$$M^{L\otimes L'}(x) = \sup_{\lambda} \frac{\mu^{L\otimes L'}(x, \lambda)}{\|\lambda\|} = \sup_{\lambda} \left(\frac{\mu^{L}(x, \lambda) + \mu^{L'}(x, \lambda)}{\|\lambda\|}\right)$$

$$\leqslant \sup_{\lambda} \frac{\mu^{L}(x, \lambda)}{\|\lambda\|} + \sup_{\lambda} \frac{\mu^{L'}(x, \lambda)}{\|\lambda\|} = M^L(x) + M^{L'}(x).$$

Let us denote by [L] the class of $L \in \mathrm{Pic}^G(X)$ in $NS^G(X)$. For any positive integer n, set

$$M^{\frac{1}{n}[L]}(x) = \frac{1}{n} M^L(x).$$

This enables us to extend the function $L \to M^L(x)$ to a unique function on

$$NS^G(X)_{\mathbf{Q}} = NS^G(X) \otimes \mathbf{Q}$$

satisfying

$$M^{l+l'}(x) \leqslant M^l(x) + M^{l'}(x), \qquad M^{\alpha l}(x) = \alpha M^l(x)$$

for any $l, l' \in \mathrm{NS}^{G}(X)_{\mathbf{Q}}$ and any non-negative rational number α. Now set

$$S = \{ (l, r) \in \mathrm{NS}^{G}(X)_{\mathbf{Q}} \times \mathbf{R} : M^{l}(x) \leqslant r \}.$$

The closure of S in $\mathrm{NS}^{G}(X)_{\mathbf{R}} \times \mathbf{R}$ is a convex cone. It is easy to see that the boundary of this cone is equal to the graph of a positively homogeneous lower convex function. This function is the needed extension of $M^{L}(x)$ to $\mathrm{NS}^{G}(X)_{\mathbf{R}}$. □

3.2.6. Let $\rho = \dim C^{G}(X)$. Fix ρ linearized very ample line bundles $L_1, \ldots, L_\rho$ whose images (still denoted by L_i, $i = 1, \ldots, \rho$) in $C^{G}(X)$ form a basis over $\mathbf{R}$. Let ω_i ($i = 1, \ldots, \rho$) be the Fubini-Study Kähler form defined by L_i. By Theorem 2.3.8, there is a unique moment map Φ^{L_i} for each $1 \leqslant i \leqslant \rho$. Now for any

$$l = a_1 L_1 + \ldots + a_\rho L_\rho \in C^{G}(X),$$

one verifies directly that $\Phi^{l} = a_1 \Phi^{L_1} + \ldots + a_\rho \Phi^{L_\rho}$ is a moment map with respect to the Kähler form $\omega = a_1 \omega_1 + \ldots + a_\rho \omega_\rho$. By 2.5, this moment map defines a Kähler quotient $X^{ss}(l)//G$. The following proposition will imply that $X^{ss}(l)//G$ does not depend on the choice of the $\mathbf{R}$-basis $L_1, \ldots, L_\rho$.

3.2.7. *Proposition. — For each* $x \in X$ *the restriction of the function* $M^{\bullet}(x)$ *to* $C^{G}(X)$ *coincides with the function* $[\omega] \to M^{\omega}(x)$, *where* ω *is the representative of* $[\omega]$ *as in* 3.2.6 *and* $M^{\omega}(x)$ *is* $M^{\Phi}(x)$ *as defined in* 2.5.2 *using the moment map described above.*

Proof. — The two functions coincide on the dense set of rational points in $C^{G}(X)$. □

For convenience, we fix an $\mathbf{R}$-basis $L_1, \ldots, L_\rho$ of $C^{G}(X)$ once and for all. In what follows, we shall always use the moment maps as described in 3.2.6. The so-defined Kähler quotients $X^{ss}(l)//G$, however, do not depend on these choices by Proposition 3.2.7.

Next, what can we say about the boundary of $C^{G}(X)$? It is clear that any integral point in it is represented either by an ample line G-bundle or by a nef but non-ample G-bundle L.

3.2.8. *Proposition. — Let* L *be an ample* G*-linearized line bundle which belongs to the boundary of* $C^{G}(X)$. *Then* $L \in C^{G}(X)$ *and* $X^{s}(L) = \emptyset$.

Proof. — First let us show that $X^{s}(L) = \emptyset$. Suppose $x \in X^{s}(L)$, then $M^{L}(x) < 0$ and hence the intersection of the open set $\{ l : M^{l}(x) < 0 \}$ with the open cone $\alpha^{-1}(A^{1}(X)^{+}_{\mathbf{R}})$ is an open neighborhood of $[L]$ contained in $C^{G}(X)$. This contradicts the assumption that $[L]$ is on the boundary.

Now let us show that $L \in C^{G}(X)$, i.e. $X^{ss}(L) \neq \emptyset$. Suppose $X^{ss}(L) = \emptyset$. This means that $M^{L}(x) > 0$ for all $x \in X$. By Theorem 1.3.9 the set $\mathscr{S}$ of open subsets U of X which can be realized as the set $X^{ss}(M)$ for some ample G-linearized line bundle M is finite. Choose a point x_U from each such U. Then $X^{ss}(L) = \emptyset$ if and only if $M^{L}(x_U) > 0$

for each U. This shows that [L] is contained in the intersection of the open set $\{ l : M^l(x_U) > 0, U \in \mathscr{S} \}$ with the open cone $\alpha^{-1}(A^1(X)^+_{\mathbf{R}})$. Thus [L] belongs to the complement of the closure of $C^G(X)$, contradicting the assumption on L. □

3.2.9. *Corollary. — Assume that there exists an ample* G*-linearized line bundle* L *with* $X^s(L) \neq \emptyset$. *Then the interior of* $C^G(X)$ *in* $NS^G(X)_{\mathbf{R}}$ *is not empty.*

3.3. Walls and chambers

3.3.1. *Definition. — A subset* H *of* $C^G(X)$ *is called a* wall *if there exists a point* $x \in X_{(>0)} := \{ x : \dim G_x > 0 \}$ *such that* $H = H(x) := \{ l \in C^G(X) : M^l(x) = 0 \}$. *A connected component of the complement of the union of walls in* $C^G(X)$, *if non-empty, is called a* chamber.

3.3.2. *Theorem. — Let* l, l' *be two points from* $C^G(X)$.

(i) l *belongs to some wall if and only if* $X^{sss}(l) \neq \emptyset$;

(ii) l *and* l' *belong to the same chamber if and only if* $X^s(l) = X^{ss}(l) = X^{ss}(l') = X^s(l')$;

(iii) *each chamber* C *is a convex cone, and is of the form*

$$C = \cap_{x \in X^s(C)} \{ l : M^l(x) < 0 \}$$

where $X^s(C) := X^s(l)$ *for any* $l \in C$.

Proof. — (i) If l belongs to some wall then there exists a point $x \in X_{(>0)}$ such that $M^l(x) = 0$. By 2.5.2, $x \in X^{sss}(l)$. Conversely, if $x \in X^{sss}(l)$ then the closure of $G \cdot x$ contains a closed orbit $G \cdot y$ in $X^{sss}(l)$ with stabilizer G_y of positive dimension. Thus l lies on the wall $H(y)$.

(ii) First assume that l and l' belong to the same chamber C. By (i) we have $X^s(l) = X^{ss}(l)$ and $X^{ss}(l') = X^s(l')$. Note that by definition $X^s(l) = \{ x \in X : M^l(x) < 0 \}$. The function $M^\bullet(x)$ does not change sign in the interior of C. Because if it did, we would find a point l_0 in C such that $M^{l_0}(x) = 0$. This would mean, by (i), that l_0 belongs to a wall, a contradiction. This shows that the set $X^s(l)$ does not depend on $l \in C$, in particular, $X^s(l) = X^s(l')$. We shall denote $X^s(l)$ by $X^s(C)$.

Let us prove the converse. Assume that $l \in C$ and $l' \in C'$ where C and C′ are two chambers. By assumption, $X^s(C) = X^s(C')$. We obviously have

$$C, C' \subset \bigcap_{x \in X^s(C)} \{ M^\bullet(x) < 0 \}.$$

Pick any l'' in $\bigcap_{x \in X^s(C)} \{ M^\bullet(x) < 0 \}$. Then $X^s(C) \subset X^s(l'')$, hence we get

$$X^s(C)/G \subset X^s(l'')/G.$$

By (i), the first quotient is compact. Therefore the second quotient is compact and $X^s(l'') = X^{ss}(l'')$. This shows that l'' does not belong to the union of walls. That is,

$\bigcap_{x \in X^s(C)} \{ M^\bullet(x) < 0 \}$ consists of points from chambers. However, it is easy to see that $\bigcap_{x \in X^s(C)} \{ l : M^l(x) < 0 \}$ is convex, and hence connected. Therefore it must coincide with C by the definition of a chamber and the inclusion displayed above. This shows that $C = C'$ (that is, the set $X^s(C)$ determines the chamber C uniquely).

(iii) By the proof of the property (ii), any chamber C is of the form

$$C = \bigcap_{x \in X^s(C)} \{ l : M^l(x) < 0 \}.$$

Now the convexity follows from the fact that the function $M^\bullet(x)$ is positively homogeneous lower convex. □

3.3.3. *Theorem.* — *There are only finitely many walls.*

Proof. — For any wall H let $X(H) = \{ x \in X_{(>0)} : H \subset H(x) \}$ where $X_{(>0)}$ is the set of points in X with positive-dimensional stabilizer. We first check that

$$X(H) = \bigcap_{l \in H} (X^{ss}(l) \cap X_{(>0)}).$$

For any point $x \in X(H)$, we have $H \subset H(x)$. Thus $M^\bullet(x)$ is identically zero on H. This implies that $x \in \bigcap_{l \in H}(X^{ss}(l) \cap X_{(>0)})$. On the other hand, if $x \in \bigcap_{l \in H}(X^{ss}(l) \cap X_{(>0)})$, then $M^l(x) = 0$ for all $l \in H$. This implies that $H \subset H(x)$. Hence $x \in X(H)$.

Now by Theorem 2.4.5, we can find a finite set of points $l_1, \ldots, l_N$ in $C^G(X)$ such that for any $l \in C^G(X)$, the set $X^{ss}(l)$ equals one of the sets $X^{ss}(l_i)$. Using the above description of X(H) we obtain that there are only finitely many subsets of X which are of the form X(H) for some wall H. However, it is straightforward to check that two walls H, H′ are equal if and only if $X(H) = X(H')$. This proves the assertion. □

3.3.4. *Proposition.* — *Each wall is a convex cone which is closed in* $C^G(X)$.

Proof. — Every wall is of the form $H = \{ l \in C^G(X) : M^l(x) = 0 \}$ for some $x \in X_{(>0)}$. Since for any such x the conditions $M^l(x) = 0$ and $M^l(x) \leqslant 0$ are equivalent, we obtain that $H = \{ l \in C^G(X) : M^l(x) \leqslant 0 \}$. The latter set is obviously a convex cone which is closed in $C^G(X)$. □

Since a wall is a convex cone we can speak about its dimension and codimension.

3.3.5. *Proposition.* — *Assume all walls are of positive codimension. Then* $X^s(l) \neq \emptyset$ *for any* l *in the interior of* $C^G(X)$.

Proof. — Assume l is contained in an open subset U of $C^G(X)$. In particular, we assume that the interior of $C^G(X)$ is not empty. By Theorem 3.3.3, the union of walls is a proper closed subset of $C^G(X)$ contained in a finite union of hyperplanes. If l does not belong to any wall, the assertion is obvious. If it does, we choose a line segment through l with two end-points $l_1, l_2 \in U$ which do not lie in the union of walls. Then $X^s(l_1) \cap X^s(l_2) \neq \emptyset$ so that $M^{l_1}(x) < 0$ and $M^{l_2}(x) < 0$ for some $x \in X$. By convexity, $M^l(x) < 0$, hence $x \in X^s(l)$. □

Together with Proposition 3.2.8 this gives the following: *Assume all walls are of positive codimension. Then the subset of the boundary of* $C^G(X)$ *consisting of classes of* G*-linearized ample line bundles equals the set* $\{ l \in C^G(X) : X^s(l) = \emptyset \}$.

3.3.6. *Remark.* — Walls of codimension 0 could exist. For example, the whole $C^G(X)$ could be a wall. In this case $X^{sss}(L) \neq \emptyset$ for any G-linearized ample line bundle L with $X^{ss}(L) \neq \emptyset$. This happens for actions of semi-simple groups G on $X = \mathbf{P}^n$. Less trivial are examples of proper walls of codimension 0. A first example of this sort was communicated to us by N. Ressayre (see [Res] and the appendix to the present paper). Note, however, that in some important applications such as the moduli spaces of parabolic vector bundles over curves and the moduli spaces of vector bundles over surfaces, walls are all linear and of positive codimensions.

3.3.7. *Lemma.* — (i) *If* $y \in \overline{G\cdot x} \cap X^{sss}(l)$, *then* $x \in X^{sss}(l)$.
(ii) *If* $y \in \overline{G\cdot x} \cap X^{ss}(l)$, *then* $x \in X^{ss}(l)$.

Proof. — (i) Since $y \in X^{sss}(l)$, 0 lies on the boundary of $\Phi^l(\overline{G\cdot y})$. This implies that 0 lies on the boundary of $\Phi^l(\overline{G\cdot x})$ because $\overline{G\cdot y} \subset \overline{G\cdot x}$. That is, $x \in X^{sss}(l)$.

(ii) If $y \in \overline{G\cdot x} \cap X^{ss}(l)$, then either $y \in G\cdot x$ or $y \in \overline{G\cdot x}\setminus G\cdot x$. The former implies trivially that $x \in X^{ss}(l)$, while the latter, by (i), implies that $x \in X^{sss}(l)$.

3.3.8. *Lemma.* — *Let* $l, l' \in C^G(X)$. *The following properties are equivalent*:

(i) $X^{sss}(l) = X^{sss}(l')$;
(ii) $X^{sss}(l) \cap X_{(>0)} = X^{sss}(l') \cap X_{(>0)}$;
(iii) $\{ x \in X_{(>0)} : l \in H(x) \} = \{ x \in X_{(>0)} : l' \in H(x) \}$.

Proof. — (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i) Note that $X^{sss}(l) \cap X_{(>0)} = \emptyset$ if and only if $X^{sss}(l) = \emptyset$. In fact, the stabilizer of any point $x \in X^{sss}(l)$ with closed orbit in $X^{ss}(l)$ is of positive dimension. So we may assume that $X^{sss}(l) \cap X_{(>0)} \neq \emptyset$. Let $x \in X^{sss}(l)$, then we can always find $y \in \overline{G\cdot x} \cap X_{(>0)} \cap X^{ss}(l)$. By assumption, $y \in X^{sss}(l')$. By Lemma 3.3.7, $x \in X^{sss}(l')$. This shows that $X^{sss}(l) \subset X^{sss}(l')$. The opposite inclusion follows in the same way.

(ii) $\Leftrightarrow$ (iii) Follows immediately from the definition of $H(x)$. □

3.3.9. *Definition.* — *We say that* $l, l' \in C^G(X)$ *are* wall equivalent *if one of the equivalent properties from the previous lemma holds. A connected component of a wall equivalence class, if not a chamber, is called a* cell.

We point out that any two points $l, l' \in C^G(X)$ away from walls (i.e. in chambers) are always equivalent with respect to the wall equivalence relation. It corresponds to the case when all the subsets in Lemma 3.3.8 are empty.

For a point $l \in \mathcal{C}^G(X)$ that lies on some walls, it is clear, by Lemma 3.3.8 (iii), that the equivalence class of l with respect to the wall equivalence is of the form

$$\bigcap_{i \in I} H_i \setminus \bigcup_{j \in J} H_j$$

where $\{H_i\}_{i \in I}$ are the walls that contain l and $\{H_j\}_{j \in J}$ are the walls that do not contain l. (When $I = \emptyset$, $\bigcap_{i \in I} H_i \setminus \bigcup_{j \in J} H_j$ is the union of all the chambers if we agree that $\bigcap_{\emptyset} H_i = \mathcal{C}^G(X)$.) Since the set of walls is finite, we see that the set of cells is finite. Also it is obvious that any wall is a union of cells.

3.3.10. *Lemma.* — *Let* F *be a cell. For any* $l, l' \in F$ *one has* $X^{ss}(l) = X^{ss}(l')$. *Denote this set by* $X^{ss}(F)$. *Then*

$$X^{ss}(F') \subsetneq X^{ss}(F) \quad \textit{if } F \cap \bar{F}' \neq \emptyset, \quad F \neq F'.$$

Proof. — We already know that $X^{sss}(l) = X^{sss}(l')$. For any $x \in X$ the function $M^{\bullet}(x)$ either vanishes identically on F or does not take the value zero at any point of F (see Lemma 3.3.8 (iii)). This implies that $X^s(l) = X^s(l')$ and hence $X^{ss}(l) = X^{ss}(l')$.

Now, if $x \in X^{ss}(F')$, then $M^l(x) \leqslant 0$ for any $l \in F'$. By continuity, $M^{l'}(x) \leqslant 0$ for any $l' \in \bar{F}'$. This shows that $x \in X^{ss}(l')$ for any $l' \in \bar{F}'$, in particular, for $l' \in F \cap \bar{F}' \neq \emptyset$. Hence $X^{ss}(F') \subset X^{ss}(F)$. If $F \cap \bar{F}' \neq \emptyset$ and $F \neq F'$, then F and F′ are connected components of different wall equivalence classes. Hence $X^{sss}(F') \neq X^{sss}(F)$ by Lemma 3.3.8 (iii). □

3.3.11. *Proposition-Definition.* — *Let* F *be a cell. There exists a point* $x \in X$ *satisfying the following properties*:

(i) $x \in X_{(>0)}$ *and the orbit of* x *is closed in* $X^{ss}(F)$;
(ii) $F \subset H(x)$.

A point x satisfying the above properties is called a pivotal point *of* F.

Proof. — By assumption, $X^{sss}(F) \cap X_{(>0)} \neq \emptyset$. Pick $x \in X^{sss}(F) \cap X_{(>0)}$ such that $G.x$ is closed in $X^{ss}(F)$. Then $M^{\bullet}(x)$ is identically zero on F. That is, $F \subset H(x)$. □

3.3.12. *Lemma.* — *If* $G.x$ *is a closed orbit in* $X^{ss}(l)$ *for some* $l \in \mathcal{C}^G(X)$, *then* G_x *is a reductive algebraic group.*

Proof. — This is equivalent to Lemma 2.5 of [Ki2]. When l is rational, a proof goes as follows. Let $\pi : X^{ss}(l) \to X^{ss}(l)//G$ be the quotient projection. Its fibers are affine, and hence $G.x$ is a closed subset of an affine variety. Thus $G.x = G/G_x$ is an affine variety. Now the assertion follows from the well-known Matsushima's criterion. □

3.3.13. *Proposition.* — *Let x be a pivotal point for a cell* F. *Then the stabilizer subgroup* G_x *is a reductive subgroup of* G.

Proof. — This immediately follows from the previous lemma and the definition of a pivotal point. □

3.3.14. *Definition.* — *Let $\mathcal{S}$ be the set of all possible moment map stratifications of* X *defined by points $l \in C^G(X)$. There is a natural partial order in $\mathcal{S}$ by refinement. We say that s refines s' if any stratum of s' is a union of strata of s. In this case we write $s < s'$ if $s \neq s'$.*

Let C be a subset of $C^G(X)$. For any $s \in \mathcal{S}$, let C_s be the set of elements $l \in C$ which define the stratification s. Then $C = \bigcup_{s \in \mathcal{S}} C_s$. Recall that by Theorem 1.3.9 the set $\mathcal{S}$ is finite.

The next result implies that s refines any element in the closure of C_s.

3.3.15. *Proposition.* — *Let $l_n \in C^G(X)$ be a sequence of points in $C^G(X)$ that induce the same stratification $s \in \mathcal{S}$. Assume that $l_n \to l \in C^G(X)$. Then s refines the stratification s' induced by l.*

Proof. — Let $S_{\beta(l_n)}$ be a stratum in s. Since $S_{\beta(l_n)} = GY^{\min}_{\beta(l_n)}$, it suffices to show that $Y^{\min}_{\beta(l_n)}$ is contained in a stratum of s'. Let $\lambda(l_n)$ be the one-parameter subgroup generated by $\beta(l_n)$. Then, by Theorem 2.4.4, we can also write $Y^{\min}_{\beta(l_n)}$ as $S_{d_n, \lambda(l_n)}$, where $d_n = \| \beta(l_n) \|$. By passing to a subsequence, we may assume that $d_n \to d$, $\beta(l_n) \to \beta(l)$. Let $\lambda(l)$ be the one-parameter subgroup generated by $\beta(l)$. For any point $x \in Y^{\min}_{\beta(l_n)}$, we have $M^{l_n}(x) = \bar{\mu}^{l_n}(x, \lambda(l_n)) = d_n$ (see Theorem 2.1.7 (iii)). By taking limits, we find that $M^l(x) = \bar{\mu}^l(x, \lambda(l)) = d$. If $d = 0$, we obtain that $Y^{\min}_{\beta(l_n)} \subset X^{ss}(l)$. If $d \neq 0$, we obtain that $Y^{\min}_{\beta(l_n)} \subset S_{d, \langle\lambda(l)\rangle}$. This completes the proof. □

The following will play a key role in the sequel.

3.3.16. *Lemma.* — *Let l_0 be in the closure of a chamber* C. *Then there is a sequence of rational points l_n in* C *which induce the same stratification s and such that $l_n \to l_0$. In particular, $X^{ss}(l_0)$ is a union of strata of s.*

Proof. — Take a basis $U_1 \supset U_2 \supset \ldots \supset U_n \supset \ldots$ of open subsets containing l_0. We have that $U_n \cap C$ is an open non-empty subset of C. Write $\mathcal{S} = \{ s_1, \ldots, s_m \}$. If s_1 occurs as the induced stratification for some rational point l_n in $U_n \cap C$ for every n, then we are done. Suppose it does not. Then there is n_1 such that $(U_{n_1} \cap C)_{s_1} = \emptyset$. Consider s_2. If s_2 occurs as the induced stratification for some rational point l_n in $U_n \cap C$ for every $n \geqslant n_1$, then we are done. Suppose it does not. Then there is $n_2 \geqslant n_1$ such that $(U_{n_2} \cap C)_{s_2} = \emptyset$. The next step is to consider s_3, and so on. Since the set $\mathcal{S} = \{ s_1, \ldots, s_m \}$ is finite and $U_1 \cap C \supset U_2 \cap C \supset \ldots \supset U_n \cap C \supset \ldots$ is infinite, there must be an $s_i \in \mathcal{S}$ such that s_i occurs as the induced stratification for some rational point l_n in $U_n \cap C$ for every $n \geqslant n_{i-1}$.

Then $\{ l_n \}$ $(n \geqslant n_{i-1})$ is the desired sequence. □

3.3.17. For any $x \in X$ and $L \in \mathrm{Pic}^G(X)$ let $\rho_x(L) : G_x \to GL(\mathbf{L}_x) \cong \mathbf{C}^*$ be the isotropy representation. For any $\lambda \in \mathcal{X}_*(G_x)$ the composition

$$\langle \lambda, \rho_x(L) \rangle = \lambda \circ \rho_x^L : \mathbf{C}^* \to \mathbf{C}^*$$

is equal to the map $t \to t^{\mu^{L}(x,\lambda)}$. The correspondence $L \to \rho_x(L)$ defines a homomorphism

$$\rho_x : \mathrm{Pic}^G(X) \to \mathscr{X}_*(G_x).$$

Obviously $\rho_x(L)$ is trivial if L is G-homologically trivial. Thus we can define a homomorphism

$$\bar{\rho}_x : \mathrm{NS}^G(X) \to \mathscr{X}(G_x), \quad L \to \rho_x(L).$$

By linearity we can extend $\bar{\rho}_x$ to a linear map

$$\bar{\rho}_x^{\mathbf{R}} : \mathrm{NS}^G(X)_{\mathbf{R}} \to \mathscr{X}(G_x) \otimes \mathbf{R}.$$

For any $l \in \mathrm{NS}^G(X)_{\mathbf{R}}$ and $\lambda \in \mathscr{X}_*(G_x)$ we have, by continuity,

$$\langle \lambda, \bar{\rho}_x^{\mathbf{R}}(l) \rangle (t) = t^{\mu^{l}(x,\lambda)}.$$

We denote by $\mathbf{E}_x$ the kernel of the map $\bar{\rho}_x^{\mathbf{R}}$.

3.3.18. *Proposition. — Let* F *be a cell with a pivotal point* x. *Then* $H(x) \subset \mathbf{E}_x$. *Moreover, if* F *has a non-empty intersection with the closure of a chamber, then* $\mathbf{E}_x \neq \mathrm{NS}^G(X)_{\mathbf{R}}$.

Proof. — Take any $l \in H(x)$. Since $x \in X^{ss}(l)$, for any one-parameter subgroup λ in G_x we have $\mu^l(x, \lambda) = 0$. In fact, otherwise $\mu^l(x, \lambda)$ or $\mu^l(x, \lambda^{-1})$ is positive, and then $x \in X^{us}(l)$. This shows that the composition of $\lambda : \mathbf{C}^* \to G_x$ with $\bar{\rho}_x^{\mathbf{R}}(l) : G_x \to \mathbf{C}^*$ is trivial. Hence $l \in \mathbf{E}_x$ and $H(x) \subset \mathbf{E}_x$.

Let us prove the second assertion which is less trivial. Take a point $l \in F$. By assumption we can find an open neighborhood U of l in $\mathrm{NS}^G(X)_{\mathbf{R}}$ such that it contains a rational point $l' \in C^G(X)$ which is not contained in any wall and the stratification induced by l' refines the stratification induced by l (see Lemma 3.3.16). We can also find a continuous path in U which starts from l', ends at l and does not cross any walls. Since $M^{\bullet}(x)$ is a continuous function, $M^{l'}(y) < 0$ implies $M^{l}(y) \leqslant 0$, and $M^{l}(y) < 0$ implies $M^{l'}(y) < 0$. This shows that $X^s(l') \subset X^{ss}(l)$, $X^s(l) \subset X^s(l')$.

In particular, $X^{us}(l) \subset X^{us}(l')$. Let us now use the moment map stratifications for X with respect to l and l'. We can write

$$X = X^{ss}(l) \cup X^{us}(l) = X^s(l') \cup S_{\beta_1}(l') \cup \ldots \cup S_{\beta_k}(l') \cup X^{us}(l).$$

Here, we use the fact that the strata of $X^{us}(l)$ coincide with the union of some strata of X with respect to l'. Now the point x must belong to some stratum $S_{\beta_i}(l')$ since it is semistable with respect to l but unstable with respect to l'. By substituting x with another point in the orbit $G \cdot x$ if necessary, we may assume that $x \in Y^{\min}_{\beta_i}(l')$. Let us denote by λ the one-parameter subgroup generated by β_i. Let $y = \lim_{t \to 0} \lambda(t) \cdot x$. It is contained in $Y^{\min}_{\beta_i}(l') \subset X^{ss}(l)$ (see 1.3.2) and is obviously fixed by $\lambda(\mathbf{C}^*)$. Because $G \cdot x$ is closed in $X^{ss}(l)$, y must belong to the orbit of x. Since $x \in Y^{\min}_{\beta_i}(l')$, we have that

$\lambda \in \Lambda^{l'}(x)$ (see Theorem 2.4.4 (iii) and 1.3.2). Now by Theorem 1.2.5, λ belongs to $\Lambda^{l'}(y)$. Hence

$$M^{l'}(y) = \frac{\mu^{l'}(y, \lambda)}{\| \lambda \|} > 0.$$

Let $y = g \cdot x$ (g actually belongs to $\mathbf{P}(\lambda)$ by Theorem 2.4.3 (i)). Applying the known properties of the function $\mu^{l'}(x, \lambda)$ (see 1.1.1), we obtain

$$\mu^{l'}(y, \lambda) = \mu^{l'}(g \cdot x, \lambda) = \mu^{l'}(x, g^{-1} \circ \lambda \circ g) > 0.$$

Hence $l' \notin \mathbf{E}_x$ because $\lambda' = g^{-1} \circ \lambda \circ g \subset G_x$. □

3.3.19. *Corollary.* — *Let* F *be a cell which has nonempty intersection with the boundary of a chamber. Let* x *be a pivotal point of* F. *Then*
(i) G_x *is a reductive group whose radical* $R(G_x)$ *is of positive dimension;*
(ii) *if* codim $H(x) = 1$ *then* $H(x)$ *equals the closure of its set of rational points.*

Proof. — By Proposition 3.3.13, G_x is reductive. Then by Proposition 3.3.18, the first assertion follows from the fact that $\mathbf{E}_x \neq NS^G(X)$ implies $\mathscr{X}(G_x) \neq \{ 1 \}$.

To prove (ii) we use that, by the previous proposition, $H(x)$ is contained in $\mathbf{E}_x$. Since $\mathbf{E}_x$ is a proper subspace, $H(x)$ spans it. This implies that the relative interior $\mathrm{ri}(H(x))$ of $H(x)$ is an open subset of $\mathbf{E}_x$. Since $\mathbf{E}_x$ can be defined by a linear equation over $\mathbf{Q}$, this open subset contains a dense subset of rational points. □

3.3.20. *Proposition.* — *Any wall in the interior of* $C^G(X)$ *is contained in a wall of codimension* $\leqslant 1$.

Proof. — Since the union of walls is a closed subset of $C^G(X)$, each chamber is an open subset of $C^G(X)$. Suppose there is a wall H of codimension $\geqslant 2$ which is not contained in any wall of codimension $\leqslant 1$. Let l be a point in the relative interior of H. Then there exists an open subset U of $C^G(X)$ containing l such that $U \backslash H \cap U$ is contained in the complement of the union of walls. Since H is of codimension $\geqslant 2$, the set $U \backslash H \cap U$ is connected. Hence there exists a chamber C containing this set. Let $l_1, l_2 \in C$ be such that l lies on the line segment joining l_1 with l_2. As in the proof of Proposition 3.3.5,

$$X^s(l_1) \cap X^s(l_2) \subset X^s(l).$$

By Theorem 3.3.2 (ii), we see that the left-hand side is equal to $X^{ss}(l_1)$. Since the quotient $X^{ss}(l_1)//G = X^s(l_1)/G$ is compact, $X^s(l)/G$ is compact, and hence $X^{ss}(l) = X^s(l)$. But this contradicts the assumption that l belongs to a wall. □

3.3.21. *Theorem.* — *Let* C *be a chamber. Its boundary intersects* $C^G(X)$ *in a union of finitely many rational supporting hyperplanes.*

Proof. — This basically follows from the proof of Proposition 3.3.18. Let l be a point of $C^G(X)$ which belongs to the boundary of a chamber C and F be a cell that contains l. In the proof of Proposition 3.3.18, we showed that we can pick a suitable pivotal point x of F and a one-parameter subgroup $\lambda' \subset G_x$ such that the rational linear function $\mu^\bullet(x, \lambda')$ vanishes on the wall $H(x)$ and is positive on the chamber C. Thus the set of zeroes of $\mu^\bullet(x, \lambda')$ defines a rational supporting hyperplane of C at the point l. Since the number of cells is finite, we are done. □

3.3.22. *Remark.* — Theorem 3.3.21 implies that each chamber is a *rational convex polyhedron* away from the boundary of $C^G(X)$. There is no reason to expect that the boundary of $C^G(X)$ is polyhedral. This is not true even when G is trivial. However, some assumptions on X, for example, X is a Fano variety (the dual of the canonical line bundle is ample) may imply that the closure of $C^G(X)$ is a convex polyhedral cone.

3.3.23. *Example.* — Let G be an n-dimensional torus $(\mathbf{C}^*)^n$, acting on $X = \mathbf{P}(V)$ via a linear representation $\rho : G \to GL(V)$. Then $NS^G(X) \cong Pic(X) \times \mathscr{X}(G) \cong \mathbf{Z}^{n+1}$. The splitting is achieved by fixing the G-linearization on $\mathcal{O}_X(1)$ defined by the linear representation ρ. In addition, $NS^G(X)^+ = \mathbf{Z}_0 \times NS^G(X)_1$, where $NS^G(X)_1$ is the group of G-linearizations on the line bundle $\mathcal{O}_X(1)$ identified with $\mathscr{X}(G)$. Thus $L_0 = (\mathcal{O}_X(1), \rho)$ corresponds to the zero in $\mathscr{X}(G)$. By 1.1.5, $X^{ss}(L_0) \neq \emptyset$ (resp. $X^s(L_0) \neq \emptyset$) if and only if $0 \in Conv(St(V))$ (resp. $0 \in Conv(St(V))^0$), where $St(V) = \{ \chi : V_\chi \neq \{ 0 \} \}$. If L_χ is defined by twisting the linearization of L_0 by a character χ, we obtain that $X^{ss}(L_\chi) \neq \emptyset$ (resp. $X^s(L_\chi) \neq \emptyset$) if and only if $\chi^{-1} \in Conv(St(V))$ (resp. $\chi^{-1} \in Conv(St(V))^0$). In particular, we get that $C^G(X)$ is equal to the cone over the convex hull $Conv(St(V))$ of $St(V)$. Comparing this with 2.1.6, we obtain that $C^G(X)$ is equal to the cone over the image of the moment map for the Fubini-Study symplectic form on X. For any $x \in X$ the image of the orbit closure $\overline{G \cdot x}$ under the moment map is equal to the convex hull $\mathbf{P}(x)$ of the state set $st(x)$ of x. This is a subpolytope of $Conv(St(V))$ of codimension equal to $\dim G_x > 0$. A wall $H(x)$ is the cone over $\mathbf{P}(x)$ with $\dim G_x > 0$. The union of walls is equal to the cone over the set of critical points of the moment map.

3.3.24. *Example.* — Let $G = SL(n+1)$ act diagonally on $X = (\mathbf{P}^n)^m$. We have

$$Pic(X) \cong Pic^G(X) \cong \mathbf{Z}^m.$$

Each line bundle L over X is isomorphic to the line bundle

$$L_{\mathbf{k}} = \pi_1^*(\mathcal{O}_{\mathbf{P}^n}(k_1)) \otimes \ldots \otimes \pi^*_m(\mathcal{O}_{\mathbf{P}^n}(k_m))$$

for some $\mathbf{k} = (k_1, \ldots, k_m) \in \mathbf{Z}^m$. Here $\pi_i : X \to \mathbf{P}^n$ denotes the projection map to the i-th factor. It is easy to see that $L_{\mathbf{k}}$ is nef (resp. ample) if and only if all k_i are nonnegative (resp. positive) integers. Let $\mathscr{P} = (p_1, \ldots, p_m) \in X$. Using the numerical criterion of stability, one verifies (see [MFK], p. 73) that $\mathscr{P} \in X^{ss}(L_{\mathbf{k}})$ if and only if for any proper linear subspace W of $\mathbf{P}^n$

$$\sum_{i,\, p_i \in W} k_i \leqslant \frac{\dim W + 1}{n+1} \left(\sum_{i=1}^{m} k_i\right).$$

The strict inequality characterizes stable points. This easily implies that

$$X^{ss}(L_{\mathbf{k}}) \neq \emptyset \Leftrightarrow (n+1) \max_i \{ k_i \} \leqslant \sum_{i=1}^{m} k\,.$$

Let

$$\Delta_{n,m} = \{ x = (x_1, \ldots, x_m) \in \mathbf{R}^m : \sum_{i=1}^{m} x_i = n+1,\ 0 \leqslant x_i \leqslant 1,\ i = 1, \ldots, m \}.$$

This is the so-called $(m-1)$-dimensional *hypersimplex* of type n.

We can express the previous condition for the non-emptiness of $X^{ss}(L_{\mathbf{k}})$ as follows

$$X^{ss}(L_{\mathbf{k}}) \neq \emptyset \Leftrightarrow (n+1)\, \mathbf{k} \in \left(\sum_{i=1}^{m} k_i\right) \boldsymbol{\Delta}_{n,m}.$$

Consider the positive cone over $\Delta_{n,m}$ in $\mathbf{R}^m$

$$C\,\Delta_{n,m} = \{ x \in \mathbf{R}^m : (n+1)\, x \in \left(\sum_{i=1}^{m} x_i\right) \Delta_{n,m},\ x_i \geqslant 0,\ i = 1, \ldots, m \}.$$

We have the injective map

$$\mathrm{Pic}^G(X) \to \mathbf{R}^m, \qquad L_{\mathbf{k}} \mapsto (k_1, \ldots, k_m),$$

which allows us to identify $\mathrm{Pic}^G(X)$ with a subset of $\mathbf{R}^m$. We then have

$$\mathrm{Pic}^G(X) \cap C\,\Delta_{n,m} = \{ L \in \mathrm{Pic}^G(X) : L \text{ is nef, } X^{ss}(L) \neq \emptyset \}.$$

Observe that $\mathscr{P} \in X^{sss}(L_{\mathbf{k}})$ if and only if there exists a proper linear subspace W of $\mathbf{P}^n$ such that

$$(n+1) \sum_{i,\, p_i \in W} k_i = (\dim W + 1) \sum_{i=1}^{m} k_i.$$

This is equivalent to the condition that $L_{\mathbf{k}}$ belongs to a hyperplane

$$H_{I,d} := \{ (x_1, \ldots, x_m) \in \mathbf{R}^m : (n+1) \sum_{i \in I} x_i = \left(\sum_{i=1}^{m} x_i\right) d \},$$

where I is a proper subset of $\{1, \ldots, m\}$ and d is an integer satisfying $1 \leqslant d \leqslant n$. Thus a chamber is a connected component of $C\,\Delta_{n,m} \setminus \bigcup_{I,d} H_{I,d}$. A wall is defined by intersection of the interior of $C\,\Delta_{n,m}$ with a subspace

$$H_{I_1, \ldots, I_s, d_1, \ldots, d_s} := \{ (x_1, \ldots, x_m) \in \mathbf{R}^m : (n+1) \sum_{i \in I_j} x_i = \sum_{i=1}^{m} x_i\, d_j,\ j = 1, \ldots, s \},$$

where $\{1, \ldots, m\} = I_1 \amalg \ldots \amalg I_s$ $(s \geqslant 2)$ and $d_1 + \ldots + d_s = n+1$ is an integral partition of $n+1$ with $d_1, \ldots, d_s > 0$.

Note that the set $\Delta_{n,m}$ is the image of the moment map for the natural action of the torus $T = (\mathbf{C}^*)^m$ on $\mathbf{P}(\bigwedge^{n+1}\mathbf{C}^m)$. Comparing with Example 3.3.23, we obtain that the closure of $C^{SL(n+1)}((\mathbf{P}^n)^m)$ is equal to $C^T(\mathbf{P}(\bigwedge^{n+1}\mathbf{C}^m))$. There is a reason for this. For any $\mathscr{P} = (p_1, \ldots, p_m) \in (\mathbf{P}^n)^m$ one can consider the matrix A of size $(n+1) \times m$ whose i-th column is a vector in $\mathbf{C}^{n+1}$ representing the point p_i. Let E(A) be the point of the Grassmann variety $G(n+1, m) \subset \mathbf{P}(\bigwedge^{n+1}\mathbf{C}^m)$ defined by the matrix A. A different choice of coordinates of the points p_i replaces E(A) by the point $t \cdot E(A)$ for some $t \in T$. In this way we obtain a bijection between $SL(n+1)$-orbits of points $(p_1, \ldots, p_m) \in (\mathbf{P}^n)^m$ with $\langle p_1, \ldots, p_m \rangle = \mathbf{P}^n$ and orbits of T on $G(n+1, m)$. This is called the *Gelfand-MacPherson correspondence*. The T-ample cone $C^T(G(n+1, m))$ equals $C^T(\mathbf{P}(\bigwedge^{n+1}\mathbf{C}^m))$ and hence coincides with the closure of the $SL(n+1)$-ample cone of $(\mathbf{P}^n)^m$. The Gelfand-MacPherson correspondence defines a natural isomorphism between the two GIT-quotients corresponding to the same point in $C\,\Delta_{n,m}$.

3.4. GIT-equivalence classes

3.4.1. *Definition. — Two elements l and l' in* $C^G(X)$ *are called* GIT-equivalent *(resp.* weakly GIT-equivalent*) if* $X^{ss}(l) = X^{ss}(l')$ *(resp.* $X^s(l) = X^s(l')$*).*

Let $E \subset C^G(X)$ be a GIT-equivalence class. We denote by $X^{ss}(E)$ the subset of X equal to $X^{ss}(l)$ for any $l \in E$. Clearly for any $l \in E$ the subset $X^s(l)$ is equal to the subset of the points in $X^{ss}(E)$ whose orbit is closed in $X^{ss}(E)$ and whose stabilizer is finite. This shows that this set is independent of the choice of l, so we can denote it by $X^s(E)$. In particular, GIT-equivalence implies weak GIT-equivalence. Examples of GIT-equivalence classes are chambers (Theorem 3.3.2). For these equivalence classes $X^{ss}(E) = X^s(E)$.

3.4.2. *Theorem. — Any* GIT*-equivalence class* E *is either a chamber or a union of cells. If it is not a chamber, it is contained in a wall* $H(x)$ *where x is any pivotal point of a cell* F *contained in* E.

Proof. — By Theorem 3.3.2 (ii), a chamber is a GIT-equivalence class. By Lemma 3.3.10, any cell is contained in a GIT-equivalence class. This implies that any GIT-equivalence class E, if not a chamber, is a union of cells. If F is a cell contained in E with a pivotal point x, then $x \in X^{ss}(E)$ and so $E \subset H(x)$. □

3.4.3. *Lemma. — Let l and l' be two points in* $C^G(X)$*. If* $X^{ss}(l) \subset X^{ss}(l')$*, then* $X^s(l') \subset X^s(l)$*. Consequently, if* $X^{ss}(l) \subset X^{ss}(l')$ *and* $X^s(l) \subset X^s(l')$*, then l and l' are weakly* GIT*-equivalent.*

Proof. — If $X^s(l') = \emptyset$, then there is nothing to prove. Assume that $X^s(l') \neq \emptyset$. Let $x \in X^s(l')$. The inclusion $X^{ss}(l) \subset X^{ss}(l')$ is easily seen to induce a morphism $X^{ss}(l)//G \to X^{ss}(l')//G$ which is an isomorphism over the open subset $X^s(l')/G$. Since this is a dominant morphism of projective varieties it must be surjective. This implies that $x \in X^{ss}(l)$. Now $G \cdot x$ is closed in $X^{ss}(l')$ and hence in $X^{ss}(l) \subset X^{ss}(l')$. In addition G_x is finite, so $x \in X^{s}(l)$. This proves the inclusion $X^{s}(l') \subset X^{s}(l)$. □

3.4.4. *Theorem. — Let* W *be a weak* GIT*-equivalence class and* E *a* GIT*-equivalence class. If* $E \subset W$, *then* $\mathrm{Conv}(E) \subset W$.

Proof. — Let l and l' be two points in E joined by a segment S of a straight line and l'' be a point on this segment. By the lower convexity of the functions $M^{\bullet}(x)$ we have $X^{s}(l) \cap X^{s}(l') \subset X^{s}(l'')$ and $X^{ss}(l) \cap X^{ss}(l') \subset X^{ss}(l'')$. Since $X^{ss}(l) = X^{ss}(l')$ and hence $X^{s}(l) = X^{s}(l')$, we obtain $X^{s}(l) \subset X^{s}(l'')$, $X^{ss}(l) \subset X^{ss}(l'')$. By the previous lemma, l and l'' are weakly GIT-equivalent. □

3.4.5. *Remark.* — Two points l, l' in $C^{G}(X)$ are said to be strongly GIT-equivalent if they induce the same stratification $s \in \mathcal{S}$ (see Definition 3.3.14). It would be nice to know precise interrelations among the three notions of GIT-equivalence.

3.4.6. *Definition. — A pair of chambers* (C, C') *are called relevant to a cell* F *if* $F \subset \overline{C} \cap \overline{C}'$ *and there is a straight path* $l : [-1, 1] \to C^{G}(X)$ *such that* $l([-1, 0)) \subset C$, $l(0) \in F$ *and* $l((0, 1]) \subset C'$.

3.4.7. *Proposition. — Let* (C, C') *be a pair of chambers relevant to a cell* F. *Then,*

$$X^{s}(F) = X^{s}(C) \cap X^{s}(C'), \qquad X^{ss}(F) \supset X^{s}(C) \cup X^{s}(C').$$

In particular, two cells relevant to the same pair of chambers are contained in the same weak GIT*-equivalence class.*

Proof. — The fact that $X^{s}(C) \cap X^{s}(C') \subset X^{s}(F)$ follows from the lower convexity of the function $M^{\bullet}(x)$. The fact that $X^{s}(F) \subset X^{s}(C) \cap X^{s}(C')$ follows from the continuity of $M^{\bullet}(x)$. Also, that $X^{s}(C) \subset X^{ss}(F)$ and $X^{s}(C') \subset X^{ss}(F)$ follow from the continuity of $M^{\bullet}(x)$. □

4. VARIATION OF QUOTIENTS

4.1. Faithful walls

4.1.1. *Definition. — We say that* $\mathrm{Pic}^{G}(X)$ *(or the action) is abundant if for any pivotal point* x *of any cell the isotropy homomorphism* $\rho_x : \mathrm{Pic}^{G}(X) \to \mathcal{X}(G_x)$ *has finite cokernel.*

The abundance helps to control the codimensions of walls.

4.1.2. *Proposition. — Assume that* $\mathrm{Pic}^{G}(X)$ *is abundant. Let* F *be a cell with a pivotal point* x. *Assume that the wall* $H(x)$ *is of codimension* k. *Then the radical of the stabilizer* G_x *is of dimension* k *or less.*

Proof. — This follows immediately from Proposition 3.3.18. □

4.1.3. *Theorem. — Let* B *be a Borel subgroup of* G *and let* G *act on* $X \times G/B$ *diagonally. Then* $\mathrm{Pic}^G(X \times G/B)$ *is abundant. In particular, when* G *is a torus,* $\mathrm{Pic}^G(X)$ *is abundant.*

Proof. — Notice first that $\mathrm{Pic}^G(G/B) \cong \mathscr{X}(T)$, where T is a maximal torus contained in B (see [KKV], p. 65). We want to show that for any point $(x, g[B])$ with reductive stabilizer the isotropy representation homomorphism

$$\rho_{(x,\,g[B])} : \mathrm{Pic}^G(X \times G/B) \to \mathscr{X}(G_{(x,\,g[B])})$$

is surjective. Obviously $G_{(x,\,g[B])} = G_x \cap G_{g[B]}$. By conjugation, we may assume that $R_{(x,\,g[B])} \subset R_x \subset T$, where R_x denotes the radical of the stabilizer G_x. Pick any character $\chi \in \mathscr{X}(G_{(x,\,g[B])})$. Extend it to a character $\tilde{\chi} \in \mathscr{X}(T)$. Let $L_{\tilde{\chi}} \in \mathrm{Pic}^G(G/B)$ be the line bundle associated to the character $\tilde{\chi}$, and let L be its inverse image in $\mathrm{Pic}^G(X \times G/B)$ under the projection map $X \times G/B \to G/B$. Then

$$\rho_{(x,\,g[B])}(L) = \tilde{\chi} \mid R_{(x,\,g[B])} = \chi \mid R_{(x,\,g[B])}.$$

Thus $\rho_{(x,\,g[B])}$ is surjective. Hence $\mathrm{Pic}^G(X \times G/B)$ is abundant.

The last statement follows immediately because $G = B$ when G is a torus. □

4.1.4. *Definition. — A cell* F *contained in the closure of a chamber with* $X^s(F) \neq \emptyset$ *is called* faithful *if for any pivotal point* x *of* F *the radical of* G_x *is one-dimensional, and* truly faithful *if the stabilizer* G_x *is a one-dimensional diagonalizable group. In this case, the identity component* G_x^0 *of* G_x *is a one-dimensional torus. A wall* H *is called* faithful (truly faithful) *if it contains a faithful (truly faithful) cell.*

Notice that in the absence of walls of codimension 0, any cell F is contained in the closure of some chamber. Also, if F is contained in the interior of $C^G(X)$ we have $X^s(F) \neq \emptyset$.

4.1.5. *Lemma. — Let* F *be a cell which has a non-empty intersection with the closure of another cell* $F' \neq F$. *Then*

(i) $X^{ss}(F') \subset X^{ss}(F)$, $X^{ss}(F') \neq X^{ss}(F)$;

(ii) $X^s(F) \subset X^s(F')$;

(iii) *the inclusion* $X^{ss}(F') \subset X^{ss}(F)$ *induces a morphism* $f: X^{ss}(F')//G \to X^{ss}(F)//G$ *which is an isomorphism over* $X^s(F)/G$;

(iv) *if* $X^s(F) \neq \emptyset$, *the morphism* f *is surjective and birational.*

Proof. — (i) Let $l' \in F'$ and $x \in X^{ss}(l') = X^{ss}(F')$. Then $M^{l'}(x) \leqslant 0$. So by continuity, $M^l(x) \leqslant 0$ for some and hence all $l \in F$. This proves that $X^{ss}(F') \subset X^{ss}(F)$. The assertion about the strict inclusion is obvious.

(ii) In the previous notation we have $M^l(x) < 0$ for any $x \in X^s(F)$. Thus, by continuity, $M^{l'}(x) < 0$ for some and hence all $l' \in F'$. Therefore $x \in X^s(F')$.

(iii) Follows from (i) and (ii).

(iv) Use the same argument as in the proof of Lemma 3.4.3. □

4.1.6. *Proposition. — Let* F *be a cell which has a non-empty intersection with the closure of another cell* F′. *Assume that* F *is faithful (resp. truly faithful). Then* F′ *is faithful (resp. truly faithful).*

Proof. — Let x be a pivotal point of F′. The closure of the orbit $G \cdot x$ in $X^{ss}(F)$ must coincide with $G \cdot x$, for otherwise there exists a pivotal point of F with stabilizer of dimension $\dim G_x \geqslant 1$. This shows that x is a pivotal point of F and hence it inherits all its properties. □

4.1.7. *Proposition. — Let* F *be a cell contained in the closure of a chamber. Assume that* F *has an empty intersection with the closure of any cell* F′ *with* $F' \neq F$. *Then for any pivotal point* x *of* F *the wall* $H(x)$ *is of codimension* 1 *and is not contained in a wall of codimension* 0.

Proof. — The assumption that $F \cap \overline{F'} = \emptyset$ for $F' \neq F$ implies that F is open in the relative topology of any wall H containing it. In fact, otherwise $\overline{H \backslash F} \cap F \neq \emptyset$, hence F have a non-empty intersection with the closure of one of the cells contained in $H \backslash F$. It follows that all walls containing F have the same dimension. By Proposition 3.3.20, $H(x)$ is contained in a wall H′ of codimension $\leqslant 1$. Since F is contained in the closure of a chamber, Proposition 3.3.18 implies that $\operatorname{codim} H(x) \geqslant 1$. Thus $H(x)$ must be of codimension 1. □

4.1.8. *Proposition. — Assume that* $\operatorname{Pic}^G(X)$ *is abundant. Let* F *be a cell intersecting the closure of a chamber in a non-empty set. If* $H(x)$ *is of codimension* 1 *for all pivotal points* x *of* F, *then* F *is faithful.*

Proof. — By Corollary 3.3.19 (i), $\dim R(G_x) \geqslant 1$. Proposition 4.1.2 implies that $\dim R(G_x) \leqslant 1$. □

4.1.9. *Corollary. — Let* G *act diagonally on* $X \times G/B$. *Then there are no walls of codimension* 0 *and all codimension* 1 *walls are truly faithful.*

Proof. — Let $(x, g[B]) \in X \times G/B$. Then $G_{(x, g[B])} = G_x \cap G_{g[B]} \subset G_{g[B]} = gBg^{-1}$. Now if $(x, g[B])$ is a pivotal point for a cell F, then $G_{(x, g[B])}$ is reductive. This implies that $G_{(x, g[B])}$ is a diagonalizable group. Assume that there is a wall H of codimension 0. Since each wall is a finite union of cells, there will be a cell $F \subset H$ which is not contained in any wall of positive codimension. Then for each pivotal point $\bar{x} = (x, g[B])$ of F, the wall $H(\bar{x})$ is of codimension 0. By Proposition 4.1.2 and Theorem 4.1.3, the stabilizer $G_{\bar{x}}$ is of dimension 0. This contradicts the definition of a pivotal point. Now, if H is a wall of codimension 1, then choosing a cell $F \subset H$ which is not contained in any wall of codimension 1, we repeat the argument to obtain that for any pivotal point $\bar{x}$ of F the stabilizer $G_{\bar{x}}$ is a one-dimensional diagonalizable group. □

Recall from 2.2.4 that GIT quotients of $X \times G/B$ by G can be identified with symplectic reductions of X by K. The previous corollary will assure that our main theorem on variation of quotients applies to symplectic reductions for general coadjoint orbits. This is one of the main motivations of this paper.

It is worthy to mention the following:

4.1.10. *Corollary. — Let* G *be a torus. Then there are no codimension* 0 *walls and all codimension* 1 *walls are truly faithful.*

As remarked in the introduction, the new feature in this corollary is that it takes into account the variation of moment maps as well as the characters of the torus.

4.2. Variation of quotients

In this section we assume that X is nonsingular and all walls in $C^G(X)$ are proper. The smoothness assumption on X only enters essentially when the bundle structures in a Bialynicki-Birula decomposition are used. Otherwise, e.g. the use of moment maps can be avoided by always dealing with rational points in $C^G(X)$. When irrational points are used, we are working in the category of Kähler quotients.

4.2.1. Let (C^+, C^-) be a pair of chambers relevant to a cell F. Let l_0 be a point in F. Lemma 3.3.16 implies that we can choose $l^+ \in C^+$ and $l^- \in C^-$ such that their induced stratifications of X can be arranged as follows:

$$X = X^{ss}(l_0) \cup X^{us}(l_0),$$

$$X = X^{s}(l^+) \cup S^{l^+}_{\alpha_1} \cup \ldots \cup S^{l^+}_{\alpha_p} \cup X^{us}(l_0),$$

and

$$X = X^{s}(l^-) \cup S^{l^-}_{\beta_1} \cup \ldots \cup S^{l^-}_{\beta_q} \cup X^{us}(l_0).$$

To simplify the notation we shall assume that each $Z^{\min}_{\alpha_i}$ or $Z^{\min}_{\beta_j}$ is connected.

Let

$$f_+ : X^s(C^+)/G \to X^{ss}(F)//G, \quad f_- : X^s(C^-)/G \to X^{ss}(F)//G$$

be the morphisms defined in Lemma 4.1.5. They are birational morphisms of projective varieties which are isomorphisms over the subset $X^s(F)/G$ of $X^{ss}(F)//G$. The goal of this section is to describe the fibers of the morphisms f_+ and f_-.

4.2.2. *Lemma. — Keep the previous notation and assume that the cell* F *is truly faithful; then all* G*-orbits of points from* $Z^{\min}_{\alpha_i}$ *are closed in* $X^{ss}(l_0)$*, and all non-stable closed orbits in* $X^{ss}(l_0)$ *meet some* $Z^{\min}_{\alpha_i}$*. In addition, up to conjugation, the* α_i *form the set of one-parameter subgroups (without parametrizations) of* G *that have nonempty fixed point set on* $X^{sss}(l_0)_c$*, where* $X^{sss}(l_0)_c$ *is the union of closed orbits in* $X^{sss}(l_0)$*. Similar statements are also true for* β_j *and* $Z^{\min}_{\beta_j}$*.*

Proof. — Since F is truly faithful, $X^{ss}(l_0)$ does not contain points with stabilizer of dimension 1. This implies that all G-orbits of points from $Z^{\min}_{\alpha_i}$ are closed in $X^{ss}(l_0)$. Now, by 4.2.1,

$$X^{ss}(l_0) = X^{s}(l^+) \cup S^{l^+}_{\alpha_1} \cup \ldots \cup S^{l^+}_{\alpha_p}.$$

Clearly points from $S^{l^+}_{\alpha_i}\setminus G\cdot Z^{\min}_{\alpha_i}$ are not in closed orbits of $X^{ss}(l_0)$. Hence all non-stable closed orbits in $X^{ss}(l_0)$ must meet some $Z^{\min}_{\alpha_i}$. This shows that

$$X^{sss}(l_0)_c = \bigcup_i G\cdot Z^{\min}_{\alpha_i}.$$

Since F is a truly faithful cell, the identity component of the isotropy subgroup of any point from $Z^{\min}_{\alpha_i}$ is exactly the one-parameter subgroup generated by α_i. Thus the last statement for α_i follows readily from the above observation. The assertions for β_j can be proved similarly. □

4.2.3. *Lemma.* — *Keep the previous notation and assume that the cell* F *is truly faithful*; *then we have*

(i) $p = q$;

(ii) *under a suitable arrangement (including choosing suitable Weyl chambers)*, $Z^{\min}_{\alpha_i} = Z^{\min}_{\beta_i}$, $1 \leqslant i \leqslant q$;

(iii) $\alpha_i = -c_i\,\beta_i$ $(1 \leqslant i \leqslant q)$ *for some positive number* c_i;

(iv) $S^{l^-}_{\beta_j} \cap S^{l^+}_{\alpha_i} = \emptyset$ *if* $j \neq i$.

Proof. — (i) and (ii) follow from the proof of Lemma 4.2.2 because

$$X^{sss}(l_0)_c = \bigcup_i G\cdot Z^{\min}_{\alpha_i} = \bigcup_i G\cdot Z^{\min}_{\beta_i}.$$

Now let us prove (iii). We have already seen that α_i and β_i generate the same subgroup of G, i.e. they differ by a constant multiple. Now by the constructions of the strata, we have $S^{l^+}_{\alpha_i} = G\cdot Y^{\min}_{\alpha_i}$ and $S^{l^-}_{\beta_i} = G\cdot Y^{\min}_{\beta_i}$, where $Y^{\min}_{\alpha_i}$ is the preimage over $Z^{\min}_{\alpha_i}$ under the Bialynicki-Birula contraction determined by α_i, and similarly $Y^{\min}_{\beta_i}$ is the preimage over $Z^{\min}_{\beta_i} = Z^{\min}_{\alpha_i}$ under the Bialynicki-Birula contraction determined by β_i. If α_i and β_i differ by a *positive* constant multiple, $S^{l^+}_{\alpha_i}$ and $S^{l^-}_{\beta_i}$ would coincide. Let us show that this is impossible. So, assume that $S^{l^+}_{\alpha_i}$ and $S^{l^-}_{\beta_i}$ coincide. Then pick a point $z \in Z^{\min}_{\alpha_i}$. Since the map

$$f_+ : X^s(l^+)//G \to X^{ss}(l_0)//G$$

is surjective, there is a point $x \in X^s(l^+)$ such that $G\cdot z \subset \overline{G\cdot x}$. Thus $x \in X^{sss}(l_0)$. Since $x \in X^s(l^+) \cap X^{sss}(l_0)$, we have that $x \notin X^s(l^-)$ because $X^s(l^+) \cap X^s(l^-) = X^s(l_0)$ (Proposition 3.4.7). Taking into account the stratification

$$X = X^s(l^-) \cup S^{l^-}_{\beta_1} \cup \ldots \cup S^{l^-}_{\beta_q} \cup X^{us}(l_0),$$

one sees that there must be $j \neq i$ such that $x \in S^{l^-}_{\beta_j}$. This is because we assume $S^{l^+}_{\alpha_i} = S^{l^-}_{\beta_i}$. This shows that there is a point $z' \in Z^{\min}_{\beta_j} = Z^{\min}_{\alpha_j}$ such that $G\cdot z' \subset \overline{G\cdot x}$. This contradicts the fact that $G\cdot z$ and $G\cdot z'$ are two distinct closed orbits in $X^{ss}(l_0)$. Therefore α_i and β_i differ by a *negative* constant multiple. This proves (ii) and (iii).

It remains to show (iv). Assume that there is a point $x \in S^{l^-}_{\beta_j} \cap S^{l^+}_{\alpha_i}$. Then there must be a point $z \in Z^{\min}_{\beta_j}$ and a point $z' \in Z^{\min}_{\alpha_i}$ such that $\overline{G \cdot x} \supset G \cdot z$ and $\overline{G \cdot x} \supset G \cdot z'$. Because $x \in X^{ss}(l_0)$, the above would imply that $G \cdot z$ and $G \cdot z'$ are mapped to the same point in the quotient $X^{ss}(l_0)//G$. But this can not happen because $G \cdot z$ and $G \cdot z'$ are different closed orbits in $X^{ss}(l_0)$. □

Let λ_i be the one-parameter subgroup generated by β_i and λ_i^{-1} be the one-parameter subgroup generated by α_i. Since F is a truly faithful cell, we obtain that $G^0_z = \lambda_i(\mathbf{C}^*)$ if $z \in Z^{\min}_{\beta_i}$. For convenience, we use the following notational convention. For any $\beta \in \{\beta_1, \ldots, \beta_p\}$, we shall use α to denote the corresponding element in $\{\alpha_1, \ldots, \alpha_p\}$ without specifying the sub-index (cf. Lemma 4.2.3). The convention also extends to λ and λ^{-1}, and so on.

4.2.4. *Proposition. — Keep the notation of* 4.2.1 *and the previous assumption. For any* $\beta \in \{\beta_1, \ldots, \beta_p\}$, *let* $p_+ : Y^{\min}_\beta \to Z^{\min}_\beta$, $p_- : Y^{\min}_\alpha \to Z^{\min}_\alpha$ *be the two natural projections. The subgroup of* G *that preserves the fiber of* $p_\pm$ *over* $z \in Z^{\min}_\beta$ *is* $G_z \cdot U(\lambda)$ (*resp.* $G_z \cdot U(\lambda^{-1})$).

Proof. — We shall consider only the map p_+. The other map is treated similarly. First, one checks easily that $G_z \cdot U(\lambda)$ preserves the fiber $p_+^{-1}(z)$. By 2.4.3 (i) any element of G which preserves the fiber must belong to $P(\lambda)$. Let $p \in P(\lambda)$ and $x \in p_+^{-1}(z)$ ($z \in Z^{\min}_\beta$) be such that $p \cdot x \in p_+^{-1}(z)$. Then we have

$$\lim_{t \to 0} \lambda(t) \cdot x = z, \qquad \lim_{t \to 0} \lambda(t) \cdot p \cdot x = z.$$

But

$$\lim_{t \to 0} \lambda(t) \cdot p \cdot x = \lim_{t \to 0} \lambda(t) \cdot p \cdot \lambda^{-1}(t)\, \lambda(t)\, x = p' \cdot z,$$

where $p' = \lim_{t \to 0} \lambda(t) \cdot p \cdot \lambda^{-1}(t)$. Hence $p' \cdot z = z$ which shows that $p' \in G_z$. Hence

$$\lim_{t \to 0} \lambda(t) \cdot (p')^{-1} p \cdot \lambda^{-1}(t) = \text{id}.$$

That is, $(p')^{-1} p \in U(\lambda)$, namely, $p \in p'\, U(\lambda) \subset G_z\, U(\lambda)$, which implies the claim. □

Following the above proposition, denote $p_\pm^{-1}(z)$ by $V^\pm$. In what follows, we will concentrate on V^+. The other set can be treated similarly.

Now, the group G_z acts on $U(\lambda)$ by conjugation. Thus for any $u \cdot z \in U(\lambda) \cdot z$ and $g \in G_z$ we have $g \cdot (u \cdot z) = gug^{-1} g \cdot z = gug^{-1} \cdot z$. This shows that the orbit $U(\lambda) \cdot z$ is G_z-invariant. Let us take a suitable identification of V^+ with the affine space $\mathbf{C}^n$ so that the point z is identified with the origin. By 1.3.3 the group G_z acts on V^+ linearly and has the point z in the closure of any orbit. So the action of G^0_z on V^+ is a good $\mathbf{C}^*$-action. It is well-known that it is equivalent to a positive grading on the ring of regular functions $\mathcal{O}(V^+) \cong \mathbf{C}[T_1, \ldots, T_n]$. We assume that each coordinate function T_i is homogeneous of some degree $q_i > 0$. Let $R = U(\lambda) \cdot z$. Being an orbit of a unipotent group acting on an affine variety, R is closed in V^+. Since it is G_z-invariant, it can be given by a system of equations $F_1 = \ldots = F_k = 0$ where the F_i are (weighted) homogeneous generators of the ideal of regular functions on V^+ vanishing on R. We can write $F_i = L_i(T_1, \ldots, T_n) + G_i(T_1, \ldots, T_n)$, where L_i is a linear function in $T_1, \ldots, T_n$ and G_i is a sum of (ordinary) homogeneous polynomials of degree 1. Since R is non-singular at the origin, we can replace the F_i by linear combinations enabling us to assume that there exist $1 \leqslant s_1 < \ldots < s_r \leqslant n$, $r = \operatorname{codim} R$, such that

$$L_i = T_{s_i} + \sum_{j \neq s_1, \ldots, s_r} a_{ij} T_j, \quad i = 1, \ldots, r, \quad L_i = 0, \quad i > r.$$

Let W be the linear subspace of V^+ defined by the equations $T_j = 0, j \neq s_1, \ldots, s_r$. It is obviously G_z-invariant. Consider the action map

$$a : U(\lambda) \times W \to V^+.$$

It is $U(\lambda) \cdot G_z$-equivariant.

4.2.5. *Lemma.* — *The map a is an isomorphism.*

Proof. — First we claim that $W \cap R = \{0\}$. In fact, since each polynomial F is weighted homogeneous, the variables T_j with $a_{ij} \neq 0$ and T_{s_i} do not enter in G_i. This shows that $R \cap W$ is given by the equations $T_i = 0$, $i = 1, \ldots, n$. This proves the claim. This also implies that the tangent space at the origin of V^+ is equal to the direct sum of the tangent spaces of R and W. Now the source $V := U(\lambda) \times W$ and the target V^+ are affine spaces of the same dimension. The restriction of the map to $U(\lambda) \times \{0\}$ and to $1 \times W$ are isomorphisms onto their images R and W, respectively. Thus the differential of the map a at the point $(1, 0)$ is bijective. Now

$$a^{-1}(a(1, 0)) = a^{-1}(z) = \{ (u, w) \in U(\lambda) \times W : w = u^{-1} \cdot z \}$$

consists of only one point $(1, z)$. We know that the map is G_z-equivariant and the action of $G_z^0 \cong \mathbf{C}^*$ on V and on V^+ is a good $\mathbf{C}^*$-action. This implies that the map a is defined by a homomorphism of positively graded rings $a^* : \mathcal{O}(V^+) \to \mathcal{O}(V)$. Let $\mathfrak{m}_{V^+}$ be the maximal ideal of the unique closed orbit $\{(0, 1)\}$ of V, and let $\mathfrak{m}_V$ be the maximal ideal of the unique closed orbit $\{z\}$ of V^+. The property that a is étale over z and the relation $a^{-1}(z) = (1, 0)$ imply that $a^*(\mathfrak{m}_{V^+})\, \mathcal{O}(V) = \mathfrak{m}_V$. By ([Bo], Chapter III, § 1, Proposition 1), the algebra $\mathcal{O}(V)$ is generated as a $\mathbf{C}$-algebra by $\mathfrak{m}_V$. This implies that the homomorphism a^* is surjective. Since both rings are integral domains of the same dimension, this shows that the map a^* is an isomorphism. Hence a is an isomorphism. □

4.2.6. *Corollary.* — *The quotient space of* $V^+ \backslash U(\lambda) \cdot z$ *by* $G_z\, U(\lambda)$ *can be identified with the quotient of* $W \backslash \{0\}$ *by* G_z *which is a quotient of a weighted projective space* $((W \backslash \{0\})/G_z^0)$ *by the finite group* $\pi_0(G_z) \cong G_z/G_z^0$. *The similar statement holds for* V^-.

Proof. — First note that from Lemma 4.2.5 the orbit space of $U(\lambda)$ on V^+ can be identified with W. In particular, the orbit space of $U(\lambda)$ on $V^+ \backslash U(\lambda) \cdot z$ can be identified with $W \backslash \{z\}$. This implies that the quotient of $V^+ \backslash U(\lambda) \cdot z$ by $G_z\, U(\lambda)$ can be identified with the quotient of $W\backslash\{0\}$ by G_z which is a quotient of a weighted projective space by $\pi_0(G_z)$ because the G_z^0-action on V^+ (and hence on W) is good. □

To prepare our last main theorem, we introduce the following. Let F be a truly faithful rational cell. Then $X^{ss}(F)\backslash X^s(F)$ has a *canonical* stratification by the so-called orbital type: two points x and y of $X^{ss}(F)\backslash X^s(F)$ are said to have the same orbital type if G_x and G_y are conjugate to each other. This stratification induces a stratification of $(X^{ss}(F)//G)\backslash(X^s(F)//G)$.

4.2.7. *Theorem.* — *Let* G *be a reductive algebraic group acting on a nonsingular projective variety* X. *Let* (C^+, C^-) *be a pair of chambers relevant to a truly faithful rational cell* F. *Then, there are two birational morphisms*

$$f_+ : X^s(C^+)//G \to X^{ss}(F)//G$$

and

$$f_- : X^s(C^-)//G \to X^{ss}(F)//G$$

so that by letting Σ_0 *be* $(X^{ss}(F)//G)\backslash(X^s(F)//G)$, *we have the following properties*:

(i) f_+ *and* f_- *are isomorphisms over the complement to* Σ_0;

(ii) *over each connected component* Σ_0' *of a stratum of* Σ_0, *each fiber of* f_+ (f_-) *is isomorphic to a quotient of a weighted projective space of dimension* d_+ (d_-) *by the finite group* [1] $\pi_0(G_z)$ *where* z *is some pivotal point of the cell* F;

(iii) $d_+ + d_- + 1 = \operatorname{codim} \Sigma_0'$.

Proof. — Statement (i) follows immediately from Lemma 4.1.5.

In showing (ii) and (iii), we apply the assertions and use the notation from 4.2.1 and 4.2.5.

To prove (ii), it suffices to consider any particular stratum S_β. Let λ be the corresponding one-parameter subgroup and $p_- : Y_\alpha^{min} \to Z_\alpha^{min}$, $p_+ : Y_\beta^{min} \to Z_\beta^{min}$ be the two natural projections. Note that for any point $z \in Z_\alpha^{min}(= Z_\beta^{min})$, $G_z^0 = \lambda(\mathbf{C}^*)$.

Let us denote the fiber $p_\pm^{-1}(z)$ by $V^\pm$. The group $U(\lambda^\pm)$ acts on $V^\pm$ (not linearly!), G_z^0 acts on V^+ (resp. on V^-) linearly with positive (resp. negative) weights. We need only to consider V^+; the other set can be treated similarly.

Now consider any non-stable closed orbit $G\cdot z \subset X^{ss}(F)$ where $z \in Z_\alpha^{min}$ for some α. We want to describe the fiber $f_-^{-1}([G\cdot z])$ (resp. $f_+^{-1}([G\cdot z])$) where $[G\cdot z] \in X^{ss}(F)//G$ is the induced point in the quotient. First we observe, by using Proposition 3.4.7, that

$$X^s(F) = X^s(C^+) \cap X^s(C^-).$$

By 4.2.1, we have

$$X = X^s(C^+) \cup S_{\alpha_1}^{l^+} \cup \ldots \cup S_{\alpha_p}^{l^+} \cup X^{us}(F).$$

(1) This finite group was overlooked in [Hu1] and also in [BP] and [Th2].

Thus by intersecting with $X^s(C^-)$ we obtain

$$X^s(C^-) = X^s(F) \cup (X^s(C^-) \cap S^{l^+}_{\alpha_1}) \cup \ldots \cup (X^s(C^-) \cap S^{l^+}_{\alpha_p}).$$

Now applying Lemma 4.2.2, we see that modulo the quotient relation in $X^{ss}(F)$, orbits from different strata $S^{l^+}_{\alpha_i}$ are identified with different closed orbits. Combining this with the above decomposition of $X^s(C^-)$, we conclude that the orbits of $X^s(C^-)$ whose induced points in the quotient $X^s(C^-)/G$ can be mapped to $[G\cdot z]$ by f_- are contained in

$$X^s(C^-) \cap S^{l^+}_{\alpha}.$$

Next, observe that $S^{l^+}_{\alpha} = GY^{\min}_{\alpha}$. So from Proposition 4.2.4, one sees that the fiber of $f_-^{-1}([G.z])$ can be identified with the orbit space of $X^s(C^-) \cap V^+$ by the group $G_z\cdot U(\lambda)$. The set $X^s(C^-) \cap V^+$ is not empty because f_- is surjective, and it is also open in V^+ because $X^s(C^-)$ is open in X. Furthermore, observe that

$$X^s(C^-) \cap V^+ \subset V^+ \backslash U(\lambda)\cdot z$$

because any point from $U(\lambda)\cdot z$ has an isotropy subgroup of positive dimension. Hence

$$(X^s(C^-) \cap V^+)/G_z\cdot U(\lambda)$$

is open in the quotient space of $V^+\backslash U(\lambda)\cdot z$ by $G_z\cdot U(\lambda)$ (the latter is a quotient of a weighted projective space by a finite group by Corollary 4.2.6). But as the fiber of f_-, $(X^s(C^-) \cap V^+)/G_z\cdot U(\lambda)$ is proper, we thus have that $X^s(C^-) \cap V^+/G_z\cdot U(\lambda)$ is equal to the quotient space $(V^+\backslash U(\lambda)\cdot z)/G_z\cdot U(\lambda)$. This completes the proof about the fiber of f_-.

Identical arguments can be applied to obtain a similar result for the map f_+. This finishes the proof of (ii).

To prove (iii), we first claim that, for any $z \in Z^{\min}_{\alpha}$,

$$\{ g \in G \mid g\cdot z \in Z^{\min}_{\alpha} \} = L(\lambda).$$

The inclusion

$$L(\lambda) \subset \{ g \in G \mid g\cdot z \in Z^{\min}_{\alpha} \}$$

follows from the fact that $Z^{\min}_{\alpha}$ is the set of semistable points in Z_α for the action of $L(\lambda)$ (Proposition 1.3.5).

To show the opposite inclusion, first notice that if $g\cdot z \in Z^{\min}_{\alpha}$ for some $z \in Z^{\min}_{\alpha}$ then the identity component of the isotropy group λ of z should be equal to the identity component of the isotropy group $g\lambda g^{-1}$ of $g\cdot z$ because all elements of $Z^{\min}_{\alpha}$ have the same stabilizer G_z. This implies that g belongs to the normalizer N_λ of λ. By Theorem 2.4.3 (i), we have

$$\{ g \in G \mid g\cdot z \in Z^{\min}_{\alpha} \} \subset \{ g \in G \mid g\cdot z \in Y^{\min}_{\alpha} \} = P(\lambda).$$

This shows that

$$\{ g \in G \mid g \cdot z \in Z_\alpha^{\min} \} \subset N_\lambda \cap P(\lambda).$$

It then suffices to prove that $N_\lambda \cap P(\lambda) = L(\lambda)$. Let $g \in N_\lambda \cap P(\lambda)$. Then $g^{-1} \cdot \lambda(t) \cdot g = \lambda(t)^k$ for some k and $\lim_{t \to 0} \lambda(t) \cdot g \cdot \lambda(t)^{-1}$ exists in G. This implies immediately that $\lim_{t \to 0} \lambda(t) \cdot g \cdot \lambda(t)^{-1} = \lim_{t \to 0} g \cdot \lambda(t)^{k-1}$ exists in G. By taking some linear representation of G, and diagonalizing $\lambda(t)$, we see that this is possible only if $k = 1$. This is equivalent to the fact that g centralizes $\lambda(t)$, and hence belongs to the subgroup $L(\lambda)$ of $P(\lambda)$. That is, $N_\lambda \cap P(\lambda) \subset L(\lambda)$.

The opposite inclusion is obvious. Thus, we have

$$N_\lambda \cap P(\lambda) = L(\lambda).$$

Now, by Proposition 1.3.5, $Z_\alpha^{\min}$ is the set of semistable points of $L(\lambda)$ in Z_α and we have $\Sigma_0' = GZ_\alpha^{\min}//G = Z_\alpha^{\min}//L(\lambda)$. Then, applying the Bialynicki-Birula decomposition theorem ([B-B], Theorem 4.1), we obtain

$$\dim X - \dim Z_\alpha^{\min} = \dim (\text{fiber of } p_+) + \dim (\text{fiber of } p_-).$$

The description of the Lie algebra of $P(\lambda)$ given in 1.2.1 shows that

$$\dim G = \dim L(\lambda) + \dim U(\lambda) + \dim U(\lambda^{-1}).$$

This gives

$$\begin{aligned}\operatorname{codim} \Sigma_0' &= (\dim X - \dim G) - (\dim Z_\alpha^{\min} - \dim L(\lambda) + 1)\\ &= \dim (\text{fiber of } p_+) - \dim U(\lambda) + \dim (\text{fiber of } p_-) - \dim U(\lambda^{-1}) - 1\\ &= (d_+ + 1) + (d_- + 1) - 1 = d_+ + d_- + 1. \quad \square\end{aligned}$$

4.2.8. *Remark.* — Theorem 4.2.7 can be generalized to the case of a faithful (but not necessary truly faithful) wall. In this case, we can still define the map $a : U(\lambda) \times W \to V^+$ which is equivariant with respect to the one-dimensional radical of G_z. We obtain that $W \backslash \{ z \}$ is contained in the union of $X^s(l^-)$ and some stratum S_{α_i}. Thus the orbit space $V^+ \cap X^s(l^-)/G_z \cdot U(\lambda)$ can be identified with the orbit space $(W \backslash \{ z \}) \cap X^s(l^-)/G_z$. This agrees with an example showed to us by C. Walters, where the fibers are isomorphic to Grassmannians and G_z is isomorphic to $GL(n)$.

We finish with the following result:

4.2.9. *Corollary.* — *Assume that all walls are of positive codimension. Let* F *be a truly faithful cell contained in the interior of* $C^G(X)$. *Assume that* $\Sigma_0(F) = (X^{ss}(F)//G) \backslash (X^s(F)//G)$ *is irreducible. Then* F *has empty intersection with the closure of any cell* $F' \neq F$. *In particular, any wall containing* F *is of codimension* 1.

Proof. — Assume the contrary. Let $x \in F \cap \overline{F'}$. Take a small open neighborhood $U \subset C^G(X)$ of x. Let $C_1, \ldots, C_n$ be the chambers which contain x in their closure.

Then $U\backslash(C_1 \cup \ldots \cup C_n)$ is equal to the intersection of U with a finite union of rational hyperplanes containing x. This easily implies that one can find three chambers C_1, C_2 and C_3, not necessarily distinct, such that C_1 contains x in its closure, (C_1, C_2) is relevant to F and (C_1, C_3) is relevant to F′. By Theorem 4.2.7, we have the maps $f_1 : X^s(C_1)/G \to X^{ss}(F')//G$, $f_2 : X^s(C_1)/G \to X^{ss}(F)//G$ with special fibers isomorphic to quotients of weighted projective spaces of positive dimensions by finite groups. Obviously $f_2 = \varphi \circ f_1$, where $\varphi : X^{ss}(F')//G \to X^{ss}(F)//G$ is a surjective birational map defined in Lemma 4.1.5. It is clear that φ maps the set $\Sigma_0(F')$ into $\Sigma_0(F)$. Take a point $y \in \varphi(\Sigma_0(F'))$. We claim that the fiber $\varphi^{-1}(y)$ is a point. First observe that the fiber $f_2^{-1}(y)$ is equal to $f_1^{-1}(\varphi^{-1}(y))$ and hence it is mapped to $\varphi^{-1}(y)$ with fibers over $\Sigma_0(F')$ isomorphic to the fibers of f_1. By the previous theorem, $f_2^{-1}(y)$ is finitely covered by an ordinary projective space. Thus we obtain a regular map of a projective space onto $\varphi^{-1}(y)$ with some fibers of positive dimension. However any non-constant regular map of a projective space is a finite map. Thus we have shown that the map φ has 0-dimensional fibers over the points from the closed subset $\varphi(\Sigma_0(F')) \subset \Sigma_0(F)$. On the other hand, since f_1 is an isomorphism over $X^s(F)/G$, the fibers of φ over the open subset $\Sigma_0(F)\backslash g(\Sigma(F'))$ must coincide with fibers of f_2 which are of positive dimension. This contradicts the well-known property of dimensions of fibers of a regular map of irreducible algebraic varieties.

The last assertion follows from Proposition 4.1.7.

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