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Appendix : An example of thick wall


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Among quotients associated to distinct $G$-linearized line bundles, those corresponding to chambers have a very good property: the fibers are orbits. Theorem 4.2.7 shows that between two relevant chambers the quotient is changed by a transformation similar to a Mori flip. Moreover, if $G$ is a torus, then two quotients corresponding to chambers are linked by a finite sequence of such transformations. In this appendix, we show by an example that this can fail for arbitrary reductive group $G$. For this, we produce a linear action of $G$ on a projective space, which admits a proper wall of codimension zero.

Let us fix some notation. We consider the connected reductive group $G = \mathbb{C}^* \times \text{SL}(2, \mathbb{C})$. Let $\chi_0$ be the character of $G$ defined by $\chi_0(t, g) = t$. Then $\chi_0$ generates the character group of $G$. If $T_1$ is the maximal torus of $\text{SL}(2, \mathbb{C})$ consisting in diagonal matrices, then $T = \mathbb{C}^* \times T_1$ is a maximal torus of $G$. Its character group is freely generated by $\chi_0$, and $\chi_1$ defined by the following formula:

$$\chi_1\left(t, \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}\right) = u, \quad t, u \in \mathbb{C}^*.$$

Let $W = \mathbb{C}^5$, $V = \mathbb{C}^8$. Let us choose an isomorphism $V \simeq \mathbb{C} \oplus \mathbb{C} \oplus W \oplus W \oplus W$. An element of $V$ is thus represented by a 5-tuple $(x_-, x_0, v_-, v_0, v_+)$ where $x_-, x_0 \in \mathbb{C}$ and $v_-, v_0, v_+ \in W$. We define an action of $G$ on $V$ by the following formula:

$$(t, g) \star (x_-, x_0, v_-, v_0, v_+) = (t^{-2} x_-, x_0, t^{-8} g \cdot v_-, g \cdot v_0, t^2 g \cdot v_+),$$

where $\cdot$ is the canonical action of $\text{SL}(2, \mathbb{C})$ on $W$. From now on, we use the notation of Section 1.1.5.

We represent the set of weights of the action of $T$ on $V$ by Figure 1. The coordinates in the basis $(\chi_0, \chi_1)$ of these weights are denoted by $(a, b)$ in the figure. In addition, the convex hulls of some parts of $\text{st}(V)$ are drawn with thick lines.
Formula (1) defines an action of \( G \) on \( X = \mathbb{P}(V) \) and a \( G \)-linearization on \( \mathcal{O}_X(1) \) as well; we denote by \( \mathcal{L} \) this \( G \)-linearized line bundle. According to Section 1.1.5, for \( \mathcal{L} \), a point \( x \in X \) is:

- semi-stable if and only if for all \( g \in G \), the origin belongs to the set \( \text{Conv}(\text{st}_V(g \cdot x)) \);
- stable if and only if for all \( g \in G \), the origin belongs to the interior of \( \text{Conv}(\text{st}_V(g \cdot x)) \);
- unstable if and only if there exists \( g \in G \) such that the origin does not belong to \( \text{Conv}(\text{st}_V(g \cdot x)) \).

Now we want to vary the ample \( G \)-linearized line bundle on \( X \). We also denote by \( \mathcal{L}_0 \) the trivial line bundle over \( X \) where \( G \) acts on the fibers by \( \mathcal{L}_0 \). Since the group \( \text{NS}(X) \) is isomorphic to \( \mathbb{Z} \), by \([\text{KKV}]\) each \( G \)-linearized line bundle on \( X \) is isomorphic to \( \mathcal{L}_0^\otimes n \otimes m \mathcal{L}_0 \) for some \( (m, n) \in \mathbb{Z}^2 \). It follows that the group \( \text{NS}^0(X) \) is isomorphic to \( \mathbb{Z}^2 \). From now on, we identify \( \text{NS}^0(X) \) with \( \mathbb{Z}^2 \), and so \( \text{NS}^0(X)_R \) with \( \mathbb{R}^2 \). Note that the line bundle corresponding to \( (m, n) \in \mathbb{Z}^2 \) is ample if and only if \( n \) is positive. Since two ample \( G \)-linearized line bundles on the same half-line from the origin are GIT-equivalent, we can restrict our study to the points of \( \text{NS}^0(X)_R \) of the form \((r, 1)\) with \( r \in \mathbb{R} \). We call the set of these points the horizontal line and \( r \) the abscissa of the point \((r, 1)\). We use these conventions in Figure 2.

Let \( r \in \mathbb{Q} \). There exists a power, say \( \mathcal{L}^\otimes n \otimes m \mathcal{L}_0 \) (with \( m = nr \in \mathbb{Z} \)), of \( \mathcal{L} \otimes r \mathcal{L}_0 \) which is the restriction (as a \( G \)-line bundle) of \( \mathcal{O}(1) \) for an embedding of \( X \) into a \( G \)-module. The sets \( \text{st}(x) \) corresponding to this embedding are obtained from \( \text{st}_V(x) \) by applying a dilation of factor \( n \) followed by a translation of vector \( (m, 0) \). So to study the stability for \( \mathcal{L} \otimes r \mathcal{L}_0 \), we can move the origin along the horizontal line in Figure 1 by \(-r\) and keep the weights of the action of \( V \). Finally the stability for \( \mathcal{L} \otimes r \mathcal{L}_0 \) of a point \( x \in X \) depends on the relative position of the point \((-r, 0)\) and the convex hulls in \( \mathcal{L}(T) \otimes \mathbb{R} \) of the sets \( \text{st}_V(g \cdot x) \) with \( g \in G \).

From now on, we denote by \( (e_1, e_2) \) the canonical basis of \( W \). Let \( x \in X \) and let \( \tilde{x} = (x_-, x_0, v_-, v_0, v_+) \) be a representative of \( x \) in \( V \). There exists \( g \in \text{SL}(2, \mathbb{C}) \) such that \( g \cdot v_- \) is proportional to \( e_1 \). But now, if \( r > 4 \) the point \((-r, 0)\) does not belong to the convex hull of \( \text{st}(g \cdot x) \) and \( x \) is not semi-stable for \( \mathcal{L} \otimes r \mathcal{L}_0 \). So if \( r > 4 \), \( X^w(\mathcal{L} \otimes r \mathcal{L}_0) \) is empty. Analogously, we prove that if \( r < -1 \) then \( \mathcal{L} \otimes r \mathcal{L}_0 \) is not effective.
Moreover, the "origins" of the form \((- r, 0)\) in Figure 1 which correspond to the intersection of the horizontal line and a wall belong to the boundary of a set \(\text{conv}(\text{st}(x))\) for some \(x \in X\). So the abscissa of the intersection of a wall and the horizontal line is \(r = 4, r = 3, r = 2, r = 0, r = -1\) or the segment \(0 \leq r \leq 2\).

Let \(x = [0:0:0:0:e_1:e_2]\). There are seven distinct sets of the form \(\text{st}(g \cdot x)\): two segments, four triangles and one rectangle. The point \((-4, 0)\) is either on the boundary or in the interior of these convex sets. So, \(r = 4\) is the abscissa of the wall \(H(x)\). In the same way, we show that \(r = 3\) is the wall \(H([0:0:0:0:e_1])\) and \(r = -1\) is the wall \(H([0:0:0:e_1:e_2])\).

Obviously, the walls \(H([1:0:0:0:0])\) and \(H([0:1:0:0:0])\) have \(r = 2\) and \(r = 0\) as their abscissa. Moreover, the intersection of the horizontal line and the wall \(H([1:1:0:0:0])\) is the interval \(0 \leq r \leq 2\).

So we obtain six walls, three chambers and six cells in the G-ample cone (see Figure 2). The cone \(\mathcal{G}^d(X)\) is partitioned into nine GIT-classes.

\[
\begin{array}{cccc}
1 & -1 & 2 & 3 \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{array}
\]

**FIG. 2.** — The G-ample cone

Theorem 4.2.7 compares quotients corresponding to two chambers \(C^+\) and \(C^-\) relevant to a cell \(F\). The starting point is that the set \(X^w(F)\) contains both \(X^w(C^+)\) and \(X^w(C^-)\), and so defines two morphisms:

\[
X^w(C^+)/G \xrightarrow{\mathcal{L}} X^w(F)/G \xrightarrow{\mathcal{L}^{-1}} X^w(C^-)/G.
\]

In the G-ample cone, the property \(X^w(F) \supset X^w(C)\) means that \(F\) intersects the closure of \(C\). Moreover, if we want to have \(X^w(F) = X^w(C^+) \cap X^w(C^-)\) it is natural to assume that \(C^+\) and \(C^-\) are relevant to \(F\). This explains why Theorem 4.2.7 concerns two relevant chambers to a face.

On the other hand, if there is no codimension zero wall, then any two chambers can be joined by a chain of relevant chambers. So quotients corresponding to two
arbitrary chambers are related by a sequence of birational transformations corresponding to relevant chambers.

Back to the example, if we want to relate the quotients associated to \( C^1 \) and \( C^2 \), we must look at the sequence of transformations

\[
\begin{array}{c}
X^\ast(C^1)//G \\
\downarrow \\
X^\ast(\mathcal{L}^\chi_0)//G \\
\downarrow \\
X^\ast(\mathcal{L})//G \\
\downarrow \\
X^\ast(\mathcal{L}^2\chi_0)//G \\
\downarrow \\
X^\ast(C^2)//G
\end{array}
\]

and so, we obtain \( X^\ast(\mathcal{L}^\chi_0)//G \) as a natural intervening quotient between \( X^\ast(C^1)//G \) and \( X^\ast(C^2)//G \).

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