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FINITE GROUP ACTIONS AND ÉTALE COHOMOLOGY

by JEREMY RICKARD

Abstract. — If a finite group G acts on a quasi-projective variety X, then $H_c^*(X, \mathbf{Z}/n)$, the étale cohomology with compact support of X with coefficients in $\mathbf{Z}/n$, has a $\mathbf{Z}/n[G]$-module structure. It is well known that there is a finer invariant, an object $R\Gamma_c(X, \mathbf{Z}/n)$ of the derived category of $\mathbf{Z}/n[G]$-modules, whose cohomology is $H_c^*(X, \mathbf{Z}/n)$. We show that there is a finer invariant still, a bounded complex $\Lambda_c(X, \mathbf{Z}/n)$ of direct summands of permutation $\mathbf{Z}/n[G]$-modules, well-defined up to chain homotopy equivalence, which is isomorphic to $R\Gamma_c(X, \mathbf{Z}/n)$ in the derived category. This complex has many properties analogous to those of the simplicial chain complex of a simplicial complex with a group action. There are similar results for l-adic cohomology.

1. Introduction

Recall that if G is a finite group and R is a commutative ring, then a "permutation" RG-module is a free R-module M on which G acts in such a way that it fixes setwise an R-basis of M. Of course, this R-basis is not usually determined uniquely, even up to isomorphism of G-sets.

If a finite group G acts simplicially on a finite simplicial complex X, and if we assume that the stabilizer of each simplex fixes the simplex pointwise (which will always be true after taking the barycentric subdivision of X), then it is clear that $C_*(X, R)$, the simplicial chain complex of X with coefficients in a ring R, is a bounded chain complex of finitely-generated permutation RG-modules, where the degree n term is the module with permutation basis given by the set of n-simplices of X. Thus the simplicial homology $H_*(X, R)$ (or, similarly, the cohomology $H^*(X, R)$) of X is the homology of a natural bounded chain complex of permutation modules. If, moreover, G acts freely on X, then $C_*(X, R)$ is a complex of free RG-modules.

More generally, if G acts on a topological space Y that is G-homotopy equivalent to the geometric realization of a finite simplicial complex X with a G-action, then the singular homology (or cohomology) of Y with coefficients in R is the homology of the singular chain (or cochain) complex of Y, which is a complex of RG-modules that is chain homotopy equivalent to a bounded complex of finitely-generated RG-modules, namely the simplicial chain (or cochain) complex of X. Thus in this situation as well there is a bounded chain complex of permutation RG-modules, natural up to chain homotopy, underlying the homology or cohomology of Y.

For algebraic varieties, étale cohomology and l-adic cohomology have many properties that are similar to those of singular cohomology for topological spaces. Indeed, if X is a quasi-projective variety defined over $\mathbf{C}$, then the étale cohomology $H^*_{ét}(X, \mathbf{Z}/n)$ is the same as the singular cohomology $H^*(X(\mathbf{C}), \mathbf{Z}/n)$, where $X(\mathbf{C})$ is the set of $\mathbf{C}$-rational points of X with the classical topology. Deligne and Lusztig prove in [5, Proposition 3.5] that if a finite group G acts freely on a variety X over a field k of characteristic different from the prime l, then the étale cohomology with compact support, $H^*_c(X, \mathbf{Z}/l^n)$, is the homology of a bounded chain complex of finitely-generated projective $\mathbf{Z}/l^n[G]$-modules. This result is similar to the case of simplicial complex, except that it is necessary to allow projective modules rather than just free modules (it is easy to see that this really is necessary by considering the case of a cyclic group of order p acting freely by translations on the affine line over a field of characteristic p).

In this paper we shall generalize this to the case of arbitrary (i.e. not necessarily free) actions. We prove that if the finite group G acts on a quasi-projective variety X over an algebraically closed field, and if R is a finite commutative coefficient ring, then there is a bounded chain complex $\Lambda_c(X, R)$ of finitely-generated direct summands of permutation R[G]-modules whose homology is the étale cohomology with compact support of X with coefficients in R. Moreover, this chain complex is natural up to chain homotopy. Note that, just as it was necessary to use projective modules rather than just free modules in the case of a free action, it is necessary to use summands of permutation modules rather than just permutation modules in the case of an arbitrary action.

In Section 4 we show how constructions that can be performed on the variety X, taking quotients and fixed points for the action of subgroups of G, correspond to constructions on the chain complex.

We hope that the results of this paper may be of independent interest, but we shall briefly outline our own reason for wanting these results.

This has to do with Broué's conjectures on equivalences between derived categories of blocks of finite group algebras [3]. Whenever such an equivalence occurs, we can always take it to be given by taking the tensor product with a bounded complex of bimodules [8], and so the problem becomes to find this complex of bimodules.

On the one hand, there are good reasons to think that, for general groups, the bimodules occurring in this complex may be taken to be summands of permutation modules for the direct product of the two groups concerned. If this were so, then it would give an explanation at the level of derived categories, at least for principal blocks, for the phenomenon of "isotypies" [3] (which are compatible families of perfect isometries), by allowing us to construct, using the "Brauer construction" (see Section 4), corresponding compatible families of equivalences of derived categories.

On the other hand, Broué has more specific conjectures about how this complex should arise in the case of finite reductive groups. In this case, where the groups concerned are some reductive group G and the normalizer $N_G(L)$ of some Levi subgroup of G, he conjectures that the restriction of the complex of bimodules to $G \times L$ should be isomorphic, in the derived category of $\mathbf{Z}_l[G \times L]$-modules, to $R\Gamma_c(X, \mathbf{Z}_l)$, where X is a Deligne-Lusztig variety. We refer to [3] for more details.

The results we prove in this paper show that these two aspects of Broué's conjectures are compatible.

Let us now set out some of the basic notation we shall use.

- If A is a ring then a module for A, or an A-module, will always mean a *left* module unless specified otherwise.
- A-*mod* will be the category of finitely generated left A-modules. Usually A will be (left) noetherian, so A-*mod* will be an abelian category.
- A-*proj* will be the category of finitely generated projective left A-modules.
- If M is an object of an additive category, then add-M will be the category of direct summands of finite direct sums of copies of M.

We shall use the language and basic machinery of derived categories [11, 7, 2]. In particular we shall use the following notation.

- $K^b(\mathscr{C})$ is the homotopy category of bounded chain complexes over an additive category $\mathscr{C}$.
- $D^b(\mathscr{A})$ is the bounded derived category of an abelian category $\mathscr{A}$.

For results on étale cohomology, the key reference is of course SGA4 [1]. A less encyclopaedic treatment can be found in the volume containing [11] (especially the first article in this volume), and [10, Chapter V] contains a brief introduction concentrating on the results most relevant to Deligne-Lusztig theory. We have attempted to give quite detailed references to [1] for the results we use.

2. Complexes representing étale cohomology

Throughout this section we adopt the following notation.

- k is an algebraically closed field.
- X is a separated scheme of finite type over k (for example, a variety).

If A is a ring and if $\mathscr{F}$ is a torsion sheaf of A-modules on X, then the étale cohomology with compact support of X with coefficients $\mathscr{F}$ is defined as the homology $H_c^*(X, \mathscr{F})$ of an object $R\Gamma_c(X, \mathscr{F})$ of the derived category $D^b(A\text{-}mod)$ of A-modules. Since $R\Gamma_c(X, \mathscr{F})$ is usually only defined up to isomorphism in the derived category, or in other words up to quasi-isomorphism, there are many different bounded complexes of A-modules that may be chosen to represent it. In this section we shall show that, with some reasonable restrictions on A and $\mathscr{F}$, there is a canonical (up to chain homotopy equivalence) choice of such a complex, and that this complex has nice properties.

As in the proof of [5, Proposition 3.5], where Deligne and Lusztig show that if a finite group G acts freely on X then $R\Gamma_c(X, \mathbf{Z}/l^n)$ may be represented by a bounded complex of projective $\mathbf{Z}/l^n[G]$-modules, we shall use the following result.

Proposition **2.1.** — *Let* A *be a right and left noetherian torsion ring, and let* $\mathscr{F}$ *be a constructible sheaf of* A*-modules on* X *such that the stalk* $\mathscr{F}_x$ *is a projective* A*-module for every* k*-rational point* x*. Then* $\mathrm{R}\Gamma_c(\mathrm{X}, \mathscr{F})$ *may be represented by a bounded complex of finitely generated projective* A*-modules.*

Proof. — For the convenience of the reader we shall sketch the proof from [5, Proposition 3.7], taking the opportunity to give references to some relevant facts about étale cohomology from [1].

By [1, XVII (4.2.8)], $\mathscr{F}$ has finite Tor-dimension (the set of k-rational points of X is a "conservative set of points" by [1, VIII (3.13)]). Therefore, by [1, XVII (5.2.10)], $\mathrm{R}\Gamma_c(\mathrm{X}, \mathscr{F})$ has finite Tor-dimension. By [1, XVII (5.3.6)], $\mathrm{R}\Gamma_c(\mathrm{X}, \mathscr{F})$ also has finitely generated homology. These two conditions on $\mathrm{R}\Gamma_c(\mathrm{X}, \mathscr{F})$ ensure that it is quasi-isomorphic to a bounded complex of finitely generated projective A-modules. □

The following technical lemma will be used several times in what follows. Note that the left derived functor $\mathrm{LJ}(\mathrm{R}\Gamma_c(\mathrm{X}, \mathscr{F}))$ may be calculated by applying J to the bounded complex of projective A-modules that, by Proposition 2.1, represents $\mathrm{R}\Gamma_c(\mathrm{X}, \mathscr{F})$.

Lemma **2.2.** — *Let* A *and* B *be two torsion rings, both left and right noetherian, and let* $\mathscr{F}$ *be a constructible sheaf of* A*-modules on* X *whose stalks* $\mathscr{F}_x$ *are projective* A*-modules for each* k*-rational point* x *of* X*. Let* J *be any additive functor from* A*-mod to* B*-mod. There is a natural isomorphism*

$$\mathrm{LJ}(\mathrm{R}\Gamma_c(\mathrm{X}, \mathscr{F})) \cong \mathrm{R}\Gamma_c(\mathrm{X}, \tilde{\mathrm{J}}(\mathscr{F}))$$

in $\mathrm{D}^b(\mathrm{B}\text{-}mod)$*, where* $\tilde{\mathrm{J}}(\mathscr{F})$ *is the sheafification of the presheaf* $\mathrm{J}(\mathscr{F})$.

Proof. — We may regard J(A), the image of the free A-module of rank one, as a B-A-bimodule. Then there is a natural morphism of functors

$$\theta : \mathrm{J}(\mathrm{A}) \otimes_{\mathrm{A}} - \to \mathrm{J}(-),$$

where $\theta(\mathrm{M})$ is the map corresponding with respect to the adjunction isomorphism

$$\mathrm{Hom}_{\mathrm{B}}(\mathrm{J}(\mathrm{A}) \otimes_{\mathrm{A}} \mathrm{M}, \mathrm{J}(\mathrm{M})) \cong \mathrm{Hom}_{\mathrm{A}}(\mathrm{M}, \mathrm{Hom}_{\mathrm{B}}(\mathrm{J}(\mathrm{A}), \mathrm{J}(\mathrm{M})))$$

to the map

$$\mathrm{M} \cong \mathrm{Hom}_{\mathrm{A}}(\mathrm{A}, \mathrm{M}) \to \mathrm{Hom}_{\mathrm{B}}(\mathrm{J}(\mathrm{A}), \mathrm{J}(\mathrm{M}))$$

induced by J. For a projective A-module M, $\theta(\mathrm{M})$ is an isomorphism.

By [1, XVII (5.2.9)] there is a natural isomorphism

$$\mathrm{J}(\mathrm{A}) \otimes^{\mathrm{L}}_{\mathrm{A}} \mathrm{R}\Gamma_c(\mathrm{X}, \mathscr{F}) \cong \mathrm{R}\Gamma_c(\mathrm{X}, \mathrm{J}(\mathrm{A}) \otimes^{\mathrm{L}}_{\mathrm{A}} \mathscr{F}).$$

Since the stalk of $\mathscr{F}$ at each k-rational point is projective, $\mathrm{J}(\mathrm{A}) \otimes^{\mathrm{L}}_{\mathrm{A}} \mathscr{F}$ is the same as $\mathrm{J}(\mathrm{A}) \otimes_{\mathrm{A}} \mathscr{F}$ [1, XVII (4.2.8)].

Recall that the tensor product of sheaves $\mathscr{G}_0 \otimes \mathscr{G}_1$ is the sheafification of the presheaf obtained by taking the tensor product of sections of $\mathscr{G}_0$ and $\mathscr{G}_1$; let us denote this presheaf by $\mathscr{G}_0 \otimes^0 \mathscr{G}_1$. There is a map of presheaves

$$J(A) \otimes^0_A \mathscr{F} \to J(\mathscr{F})$$

induced by θ, which induces an isomorphism on stalks at k-rational points, since the stalks of $\mathscr{F}$ at these points are projective. Therefore we get an isomorphism

$$J(A) \otimes_A \mathscr{F} \cong \tilde{J}(\mathscr{F})$$

when we sheafify.

Finally, θ induces an isomorphism

$$J(A) \otimes^L_A R\Gamma_c(X, \mathscr{F}) \cong LJ(R\Gamma_c(X, \mathscr{F})),$$

since $\theta(M)$ is an isomorphism for projective M. □

If, as above, $\mathscr{F}$ is a constructible sheaf of A-modules on X, and if M is an A-module, then we can apply the functor $\mathrm{Hom}_A(M, -)$ to $\mathscr{F}$ to get a presheaf of $\mathrm{End}_A(M)$-modules on X. Because $\mathrm{Hom}_A(M, -)$ is left exact, this presheaf is actually a sheaf. In general it will not be constructible, since it is not necessarily a sheaf of finitely generated modules. However, with suitable finiteness conditions on A and M it will be constructible. This is certainly the case if A is an Artin algebra (i.e., A is finitely generated as a module over an artinian centre) and if M is a finitely generated A-module. In all the examples we shall consider, A is actually a finite ring.

If, in addition to these conditions, M is such that the stalk $\mathscr{F}_x$ at each k-rational point of X is an object of add-M, then $\mathrm{Hom}_A(M, \mathscr{F})$ has $\mathrm{End}_A(M)$-projective stalks at k-rational points, since $\mathrm{Hom}_A(M, -)$ induces an equivalence of categories

$$\text{add-M} \to \mathrm{End}_A(M)\text{-}proj,$$

with a quasi-inverse equivalence given by $M \otimes_{\mathrm{End}(M)} -$. Also, the endomorphism ring $\mathrm{End}_A(M)$ of a finitely generated module for a torsion Artin algebra is itself a torsion Artin algebra — since it is finitely generated as a module for the centre of A — and so Proposition 2.1 applies to $\mathrm{Hom}_A(M, \mathscr{F})$. The following definition therefore makes sense.

Definition **2.3.** — *Let* A *be a torsion Artin algebra, and let* $\mathscr{F}$ *be a constructible sheaf of* A*-modules on* X. *If* M *is a finitely generated* A*-module such that* $\mathscr{F}_x$ *is in add-*M *for each k-rational point x of* X, *then define* $\Omega_c(X, \mathscr{F}, M)$ *to be the object* $M \otimes_{\mathrm{End}(M)} P^*$ *of* $K^b(A\text{-}mod)$, *where* P^* *is the bounded complex of projective* $\mathrm{End}_A(M)$*-modules which, by Proposition 2.1, represents* $R\Gamma_c(X, \mathrm{Hom}_A(M, \mathscr{F}))$.

Since the complex P^* of projectives is well-defined up to chain homotopy equivalence, the same is true of $\Omega_c(X, \mathscr{F}, M)$.

Let us also remark that there is always some module M with the properties required: since $\mathscr{F}$ is constructible, there are only finitely many isomorphism classes of stalks $\mathscr{F}_x$ at k-rational points, all of which are finitely generated A-modules, and so we may take M to be the direct sum of modules, one from each of these isomorphism classes.

The most important property of $\Omega_c(X, \mathscr{F}, M)$ is the following.

Proposition **2.4.** — *In the derived category* $D^b(A\text{-}mod)$, *there is an isomorphism between* $\Omega_c(X, \mathscr{F}, M)$ *and* $R\Gamma_c(X, \mathscr{F})$.

Proof. — By Lemma 2.2,

$$\begin{aligned}\Omega_c(X, \mathscr{F}, M) &\cong M \otimes^{L}_{\mathrm{End}(M)} R\Gamma_c(X, \mathrm{Hom}_A(M, \mathscr{F}))\\ &\cong R\Gamma_c(X, M \otimes_{\mathrm{End}(M)} \mathrm{Hom}_A(M, \mathscr{F}))\\ &\cong R\Gamma_c(X, \mathscr{F}),\end{aligned}$$

all isomorphisms being in the derived category. □

Proposition 2.4 tells us that, once we have chosen M, there is a canonical (up to chain homotopy equivalence) choice of a complex of objects of add-M that represents $R\Gamma_c(X, \mathscr{F})$. Now we shall see that this complex is even independent of M.

Proposition **2.5.** — *With the notation of Definition 2.3, if* N *is any finitely generated* A-*module then there is a natural isomorphism*

$$\Omega_c(X, \mathscr{F}, M) \cong \Omega_c(X, \mathscr{F}, M \oplus N)$$

of objects of $K^b(A\text{-}mod)$.

Proof. — Let $E = \mathrm{End}_A(M)$ and $\hat{E} = \mathrm{End}_A(M \oplus N)$, and let

$$J = \mathrm{Hom}_A(M \oplus N, M \otimes_E -),$$

considered as a functor from E-*mod* to $\hat{E}$-*mod*. Note that J takes projective E-modules to projective $\hat{E}$-modules and that there are the following two natural morphisms of functors.

First, there is a morphism

$$\alpha : J(\mathrm{Hom}_A(M, -)) \to \mathrm{Hom}_A(M \oplus N, -)$$

of functors from A-*mod* to $\hat{E}$-*mod*, where $\alpha(S)$ is an isomorphism if S is in add-M.

Second, there is a morphism

$$\beta : (M \oplus N) \otimes_{\hat{E}} J(-) \to M \otimes_E -$$

of functors from E-*mod* to A-*mod*, where $\beta(T)$ is an isomorphism if T is in E-*proj*.

There is a map of presheaves

$$J(\mathrm{Hom}_A(M, \mathscr{F})) \to \mathrm{Hom}_A(M \oplus N, \mathscr{F})$$

induced by α. The induced maps of stalks at k-rational points are all isomorphisms because the stalks of $\mathscr{F}$ at these points are all in add-M, and so after sheafifying we get an isomorphism

$$\tilde{\mathrm{J}}(\mathrm{Hom}_{\mathrm{A}}(\mathrm{M}, \mathscr{F})) \cong \mathrm{Hom}_{\mathrm{A}}(\mathrm{M} \oplus \mathrm{N}, \mathscr{F}).$$

Therefore, using Lemma 2.2,

$$\begin{aligned}\mathrm{R}\Gamma_c(\mathrm{X}, \mathrm{Hom}_{\mathrm{A}}(\mathrm{M} \oplus \mathrm{N}, \mathscr{F})) &\cong \mathrm{R}\Gamma_c(\mathrm{X}, \tilde{\mathrm{J}}(\mathrm{Hom}_{\mathrm{A}}(\mathrm{M}, \mathscr{F})))\\ &\cong \mathrm{LJ}(\mathrm{R}\Gamma_c(\mathrm{X}, \mathrm{Hom}_{\mathrm{A}}(\mathrm{M}, \mathscr{F}))).\end{aligned}$$

So if we let $\mathrm{P}^\bullet$ be the complex of projective E-modules that, by Proposition 2.1, represents $\mathrm{R}\Gamma_c(\mathrm{X}, \mathrm{Hom}_{\mathrm{A}}(\mathrm{M}, \mathscr{F}))$ and $\mathrm{Q}^\bullet$ be the complex of projective $\hat{\mathrm{E}}$-modules that represents $\mathrm{R}\Gamma_c(\mathrm{X}, \mathrm{Hom}_{\mathrm{A}}(\mathrm{M} \oplus \mathrm{N}, \mathscr{F}))$, then $\mathrm{Q}^\bullet \cong \mathrm{J}(\mathrm{P}^\bullet)$ in the homotopy category.

Therefore β induces an isomorphism

$$\begin{aligned}\Omega_c(\mathrm{X}, \mathscr{F}, \mathrm{M} \oplus \mathrm{N}) &\cong (\mathrm{M} \oplus \mathrm{N}) \otimes_{\hat{\mathrm{E}}} \mathrm{J}(\mathrm{P}^\bullet)\\ &\cong \mathrm{M} \otimes_{\mathrm{E}} \mathrm{P}^\bullet\\ &\cong \Omega_c(\mathrm{X}, \mathscr{F}, \mathrm{M})\end{aligned}$$

in $\mathrm{K}^b(\mathrm{A}\text{-}mod)$. □

In the light of the previous Proposition let us make the following definition.

Definition **2.6.** — *Let* A *be a torsion Artin algebra, and let* $\mathscr{F}$ *be a constructible sheaf of* A*-modules on* X*. Then* $\Omega_c(\mathrm{X}, \mathscr{F})$ *is defined to be* $\Omega_c(\mathrm{X}, \mathscr{F}, \mathrm{M})$*, where* M *is the direct sum of* A*-modules, one from each isomorphism class that is represented by* $\mathscr{F}_x$ *for some* k*-rational point* x *of* X.

The following theorem is mostly just a summary of what we already know about $\Omega_c(\mathrm{X}, \mathscr{F})$, together with some facts about it that follow easily from the corresponding facts about $\mathrm{R}\Gamma_c(\mathrm{X}, \mathscr{F})$.

Theorem **2.7.** — *a) If* A *is a torsion Artin algebra, then* $\Omega_c(\mathrm{X}, -)$ *is an additive functor from the category of constructible sheaves of* A*-modules on* X *to the homotopy category* $\mathrm{K}^b(\mathrm{A}\text{-}mod)$.

b) If all the stalks of $\mathscr{F}$ *at* k*-rational points of* X *are in add-*M *for some* A*-module* M*, then* $\Omega_c(\mathrm{X}, \mathscr{F})$ *is a complex of modules from add-*M.

c) The composition of $\Omega_c(\mathrm{X}, -)$ *with the usual quotient functor from* $\mathrm{K}^b(\mathrm{A}\text{-}mod)$ *to* $\mathrm{D}^b(\mathrm{A}\text{-}mod)$ *is isomorphic to* $\mathrm{R}\Gamma_c(\mathrm{X}, -)$.

d) If $f : \mathrm{Y} \to \mathrm{X}$ *is a finite morphism of separated schemes of finite type over* k*, then there is an induced map from* $\Omega_c(\mathrm{X}, \mathscr{F})$ *to* $\Omega_c(\mathrm{Y}, f^* \mathscr{F})$.

Proof. — *a)* If $\theta : \mathscr{F} \to \mathscr{G}$ is a map of constructible sheaves of A-modules on X then, if we define M to be a direct sum of A-modules, one from each isomorphism class which occurs as a stalk of either $\mathscr{F}$ or $\mathscr{G}$ at a k-rational point of X, then θ clearly induces a map

$$\Omega_c(\mathrm{X}, \mathscr{F}, \mathrm{M}) \to \Omega_c(\mathrm{X}, \mathscr{G}, \mathrm{M}).$$

But, by Proposition 2.5, $\Omega_c(X, \mathscr{F})$ and $\Omega_c(X, \mathscr{G})$ are naturally isomorphic to $\Omega_c(X, \mathscr{F}, M)$ and $\Omega_c(X, \mathscr{G}, M)$ respectively, so θ induces a map

$$\Omega_c(X, \theta) : \Omega_c(X, \mathscr{F}) \to \Omega_c(X, \mathscr{G}).$$

A similar argument shows that if θ and φ are composable maps between sheaves, then

$$\Omega_c(X, \theta \circ \varphi) = \Omega_c(X, \theta) \circ \Omega_c(X, \varphi).$$

b) By definition, $\Omega_c(X, \mathscr{F}, M)$ is a complex over add-M, and $\Omega_c(X, \mathscr{F})$ is isomorphic to $\Omega_c(X, \mathscr{F}, M)$ by Proposition 2.5.

c) This follows from Proposition 2.4.

d) For suitable M, f induces a map

$$\begin{aligned} R\Gamma_c(X, \mathrm{Hom}_A(M, \mathscr{F})) &\to R\Gamma_c(Y, f^* \mathrm{Hom}_A(M, \mathscr{F})) \\ &\cong R\Gamma_c(Y, \mathrm{Hom}_A(M, f^* \mathscr{F})). \quad \square \end{aligned}$$

Let us end this section with the following lemma, a corollary of Lemma 2.2, which will be useful later.

Lemma **2.8.** — *Let* A *and* B *be torsion Artin algebras, let* U *be a finitely generated* A-*module, and let* F *be an additive functor from* add-U *to* B-*mod. If* $\mathscr{F}$ *is a constructible sheaf on* X *of* A-*modules that are in* add-U, *then*

$$\Omega_c(X, \widetilde{F}(\mathscr{F})) \cong F(\Omega_c(X, \mathscr{F})),$$

where $\widetilde{F}(\mathscr{F})$ *is the sheafification of the presheaf* $F(\mathscr{F})$.

Proof. — Since End(U)-*proj* and End(F(U))-*proj* are equivalent to add-U and add-F(U) respectively, we get a functor

$$J : \mathrm{End}(U)\text{-}proj \to \mathrm{End}(F(U))\text{-}proj$$

by composing these equivalences with F. We can extend J to a functor on the whole of the module category End(U)-*mod*. For example, the functor

$$F(U) \otimes_{\mathrm{End}(U)} - : \mathrm{End}(U)\text{-}mod \to \mathrm{End}(F(U))\text{-}mod$$

restricts to a functor isomorphic to J. It should cause no confusion if we also call this extended functor J, and we shall do so.

Now, by definition of $\Omega_c(X, \mathscr{F})$,

$$\begin{aligned} F(\Omega_c(X, \mathscr{F})) &\cong F(U \otimes_{\mathrm{End}(U)} P^*) \\ &\cong F(U) \otimes_{\mathrm{End}(F(U))} J(P^*), \end{aligned}$$

where P^* is a bounded complex of projective End(U)-modules that is quasi-isomorphic to $R\Gamma_c(X, \mathrm{Hom}(U, \mathscr{F}))$. But, by Lemma 2.2,

$$\begin{aligned} J(P^*) &\cong R\Gamma_c(X, \tilde{J}(\mathrm{Hom}(U, \mathscr{F}))) \\ &\cong R\Gamma_c(X, \tilde{F}(\mathscr{F})) \\ &\cong \Omega_c(X, \tilde{F}(\mathscr{F})), \end{aligned}$$

and so the claimed isomorphism follows. □

3. Finite group actions

Throughout this section we adopt the following notation.

- G is a finite group.
- k is an algebraically closed field.
- R is a finite commutative ring.
- X is a quasi-projective variety defined over k with an action of G.
- Y is the quotient variety X/G.
- $\pi : X \to Y$ is the projection map.

Consider the constant sheaf R on X. Its direct image $\pi_* R$ is a sheaf of R[G]-modules on Y, and because π is a finite map there is a natural isomorphism

$$R\Gamma_c(Y, \pi_* R) \cong R\Gamma_c(X, R)$$

in the derived category $D^b(R\text{-}mod)$. We can therefore regard $R\Gamma_c(X, R)$ as an object of $D^b(R[G]\text{-}mod)$. Since $\pi_* R$ is a constructible sheaf, the results of Section 2 apply, and we can make the following definition.

Definition **3.1.** — *The object* $\Lambda_c(X, G, R)$ — *or just* $\Lambda_c(X, R)$ *if it is clear which group is involved* — *of* $K^b(R[G]\text{-}mod)$ *is* $\Omega_c(Y, \pi_* R)$.

Theorem **3.2.** — *In the derived category* $D^b(R[G]\text{-}mod)$, *the complex* $\Lambda_c(X, R)$ *is isomorphic to* $R\Gamma_c(X, R)$. *It is a complex of direct sums of direct summands of permutation modules of the form* R[G/H], *where* H *runs through the set of stabilizers of k-rational points of* X.

Proof. — This follows immediately by applying Theorem 2.7 to the sheaf $\pi_* R$ on Y, since the stalk of $\pi_* R$ at a point $\pi(x)$ of Y is just the permutation module R[G/H], where H is the stabilizer of x. □

If there is some restriction on which subgroups of G occur as point stabilizers then we get a restriction on which modules can occur in $\Lambda_c(X, R)$.

Corollary **3.3.** — *Let* K *be a subgroup of* G. *If the stabilizer of every k-rational point of* X *is conjugate to a subgroup of* K, *then* $\Lambda_c(X, R)$ *is, up to homotopy equivalence, a bounded complex of relatively* K*-projective summands of permutation modules.* □

If we consider an even more trivial special case, where all the point stabilizers are trivial, we recover Deligne and Lusztig's result for free actions.

Corollary **3.4.** — *If* G *acts freely on* X, *then* $\Lambda_c(X, R)$ *is, up to homotopy equivalence, a complex of projective modules. Thus in this case* $\Lambda_c(X, R)$ *coincides with the complex constructed by Deligne and Lusztig in* [5, *Proposition* 3.5]. □

As in [5, Proposition 3.5] we can get similar results for l-adic cohomology by taking an inverse limit.

Theorem **3.5.** — *Let* l *be a prime number. There is a canonical (up to chain homotopy equivalence) bounded complex of summands of permutation* $\mathbf{Z}_l[G]$*-modules, which we shall denote by* $\Lambda_c(X, G, \mathbf{Z}_l)$ *— or just* $\Lambda_c(X, \mathbf{Z}_l)$ *if it is clear what* G *is —, which is isomorphic to* $R\Gamma_c(X, \mathbf{Z}_l)$ *in* $D^b(\mathbf{Z}_l[G]\text{-}mod)$ *and such that*

$$\Lambda_c(X, \mathbf{Z}/l^n) \cong \Lambda_c(X, \mathbf{Z}_l) \otimes_{\mathbf{Z}_l} \mathbf{Z}/l^n.$$

Proof. — The proof works exactly as in [5], the only extra ingredient being the fact that, for a permutation $\mathbf{Z}_l[G]$-module M, the natural map

$$\mathrm{End}_{\mathbf{Z}_l[G]}(M) \otimes_{\mathbf{Z}_l} \mathbf{Z}/l^n \to \mathrm{End}_{\mathbf{Z}/l^n[G]}(M \otimes_{\mathbf{Z}_l} \mathbf{Z}/l^n)$$

is an isomorphism [9]. We have, for a suitable permutation $\mathbf{Z}_l[G]$-module M,

$$R\Gamma_c(Y, \mathrm{Hom}(M, \pi_* \mathbf{Z}/l^n)) \cong R\Gamma_c(Y, \mathrm{Hom}(M, \pi_* \mathbf{Z}/l^{n+1})) \otimes^{L}_{\mathbf{Z}/l^{n+1}} \mathbf{Z}/l^n,$$

and, by [6, XV, 3.3, Lemme 1], we can choose explicit bounded complexes K^*_n of projective modules, one for each positive integer n, such that K^*_n is isomorphic in the derived category $D^b(\mathrm{End}(M \otimes \mathbf{Z}/l^n)\text{-}mod)$ to $R\Gamma_c(Y, \mathrm{Hom}(M, \pi_* \mathbf{Z}/l^n))$, and such that K^*_n is the reduction mod l^n of K^*_{n+1}. Taking the inverse limit of the K^*_n, we get a complex K^*_∞ of finitely generated projective End(M)-modules. Using [6, XV, 3.3, Lemme 1] again, K^*_∞ is independent, up to homotopy equivalence, of the choice of K^*_n. So the theorem follows by taking

$$\Lambda_c(X, \mathbf{Z}_l) = M \otimes_{\mathrm{End}(M)} K^*_\infty. \quad \square$$

4. Quotients and fixed points

We shall keep the notation of Section 3. Also we shall be considering fixed points of group actions, and we shall use the standard notation X^G for the fixed points of a group G acting on an object X.

First we shall show that taking the quotient of X by some subgroup of G corresponds to taking the fixed points of the subgroup on the complex $\Lambda_c(X, R)$. Notice that, since the fixed point functor is not exact, it is not defined on the derived category. Therefore it is important that we consider the object $\Lambda_c(X, R)$ of the homotopy category rather than the coarser invariant $R\Gamma_c(X, R)$ in the derived category.

Theorem **4.1.** — *Let* H *be a subgroup of* G. *Then, with respect to the natural actions of* $N_G(H)$ *on the quotient variety* X/H *and on the* H*-fixed points of* R[G]*-modules, there is a natural isomorphism*

$$(\Lambda_c(X, G, R))^H \cong \Lambda_c(X/H, N_G(H), R)$$

in $K^b(R[N_G(H)]\text{-}mod)$.

Proof. — Let $N = N_G(H)$. Let M_G be the direct sum of all the permutation R[G]-modules $R[G/G_i]$, one for each subgroup G_i of G, and let $E_G = \mathrm{End}_{R[G]}(M_G)$. Similarly define M_N and E_N. Thus the categories of finitely generated summands of permutation modules for R[G] and R[N] are just add-M_G and add-M_N respectively.

If we let F be the H-fixed point functor from add-M_G to R[N]-*mod* then, by Lemma 2.8,

$$\Omega_c(X/G, \widetilde{F}(\pi_* R)) \cong F(\Omega_c(X/G, \pi_* R))$$

in $K^b(R[N]\text{-}mod)$, where $\widetilde{F}(\pi_* R)$ is the sheafification of $F(\pi_* R)$.

If we let $\varphi : X/H \to X/HN_G(H)$ and $\psi : X/HN_G(H) \to X/G$ be the projection maps, then $(\pi_* R)^H$ is isomorphic to the direct image $\psi_* \varphi_* R$ (where here R is the constant sheaf on X/H), which is the direct image of the sheaf $\varphi_* R$ of permutation R[N]-modules on $X/HN_G(H)$. In particular, $F(\pi_* R)$ is a sheaf, and so is isomorphic to $\widetilde{F}(\pi_* R)$. We have

$$\begin{aligned}\mathrm{Hom}_{R[N]}(M_N, (\pi_* R)^H) &\cong \mathrm{Hom}_{R[N]}(M_N, \psi_* \varphi_* R)\\ &\cong \psi_* \mathrm{Hom}_{R[N]}(M_N, \varphi_* R),\end{aligned}$$

and so, using the fact that ψ is a finite map,

$$\begin{aligned}&R\Gamma_c(X/G, \mathrm{Hom}_{R[N]}(M_N, (\pi_* R)^H))\\ &\qquad \cong R\Gamma_c(X/G, \psi_* \mathrm{Hom}_{R[N]}(M_N, \varphi_* R))\\ &\qquad \cong R\Gamma_c(X/HN_G(H), \mathrm{Hom}_{R[N]}(M_N, \varphi_* R)).\end{aligned}$$

Therefore, if we take P^* to be a bounded complex of projective E_N-modules representing

$$R\Gamma_c(X/HN_G(H), \mathrm{Hom}_{R[N]}(M_N, \varphi_* R)),$$

we have

$$\begin{aligned}\Lambda_c(X/H, N, R) &\cong M_N \otimes_{E_N} P^*\\ &\cong \Omega_c(X/G, (\pi_* R)^H)\\ &\cong (\Omega_c(X/G, \pi_* R))^H\\ &\cong (\Lambda_c(X, G, R))^H. \quad \square\end{aligned}$$

The analogue of this theorem for *l*-adic cohomology also holds, and follows easily by taking an inverse limit.

The previous theorem shows that the operation on the chain complex $\Lambda_c(X, R)$ that corresponds to taking the quotient of X by a subgroup of G is exactly the same as it is for the singular chain complex of a topological space; namely, taking the fixed points of the subgroup on the chain complex. We shall now see that this analogy holds for another operation on X: taking the fixed points of a subgroup of G. We shall first make some comments on this operation in the context of simplicial homology.

In general, if R is a commutative ring and X is a simplicial complex with a simplicial action of a finite group P, one cannot recover the simplicial chain complex $C_*(X^P, R)$ of the fixed point set X^P from the chain complex $C_*(X, R)$ of R[P]-modules. This can be seen even with 0-dimensional simplicial complexes: there are examples of P-sets S_1 and S_2 such that the permutation modules $R[S_1]$ and $R[S_2]$ are isomorphic, but the fixed point sets S_1^P and S_2^P have different cardinalities.

However, if R is a field of characteristic p and P is a p-group, then this problem does not arise. In this case, if P is a subgroup of a group G, there is a functor (the "Brauer construction", see [4]) from R[G]-*mod* to $R[N_G(P)]$-*mod* defined by

$$M \mapsto M(P) = M^P / \sum_{Q \lneqq P} \mathrm{Tr}_Q^P(M^Q),$$

where

$$\mathrm{Tr}_Q^P : M^Q \to M^P$$

is the relative trace map, defined by

$$\mathrm{Tr}_Q^P(x) = \sum_\pi \pi x,$$

the sum being over a set of representatives π of the cosets P/Q. The natural map

$$\mathrm{Br}_P : M^P \to M(P)$$

is called the Brauer morphism.

If M = R[S] is a permutation module, then M(P) is isomorphic to the permutation $R[N_G(P)]$-module $R[S^P]$. In fact there is a diagram of functors, commutative up to isomorphism,

$$\begin{array}{ccc} G\text{-sets} & \longrightarrow & N_G(P)\text{-sets} \\ \downarrow & & \downarrow \\ R[G]\text{-}mod & \longrightarrow & R[N_G(P)]\text{-}mod \end{array}$$

where the vertical arrows are the functors taking a permutation set to the permutation module with that set as basis, the first horizontal arrow is the P-fixed point functor, and the second horizontal arrow is the Brauer construction described above.

Thus, if X is a G-simplicial complex then the simplicial chain complex $C_*(X^P, R)$, considered as a complex of $R[N_G(P)]$-modules, is isomorphic to $C_*(X, R)$ (P), the

Brauer construction applied to the simplicial chain complex of X. We shall now see that there is an analogous result for étale cohomology.

Theorem **4.2**. — *Let* L *be an l-subgroup of* G. *Then* $\Lambda_c(X^L, N_G(L), \mathbf{Z}/l)$ *is isomorphic, in* $K^b(\mathbf{Z}/l[N_G(L)]\text{-}mod)$, *to* $\Lambda_c(X, G, \mathbf{Z}/l)$ (L).

Proof. — We have the following commutative diagram of natural maps of varieties.

$$\begin{array}{ccc} X^L & \xrightarrow{\alpha} & X \\ \downarrow{\beta} & & \downarrow{\pi} \\ X^L/N_G(L) & \xrightarrow{\gamma} & X/G \end{array}$$

First we shall show that $\pi_* \alpha_* \mathbf{Z}/l$ is isomorphic to the sheafification of $(\pi_* \mathbf{Z}/l)$ (L). There is a natural map

$$\tau : (\pi_* \mathbf{Z}/l)^L \to \pi_* \alpha_* \mathbf{Z}/l$$

of sheaves on X/G that is the composition of the inclusion of the L-fixed points into $\pi_* \mathbf{Z}/l$ with the map obtained by applying π_* to the adjunction map

$$\mathbf{Z}/l \to \alpha_* \alpha^* \mathbf{Z}/l = \alpha_* \mathbf{Z}/l.$$

If L′ is a proper subgroup of L, then we have the relative trace map

$$\mathrm{Tr}_{L'}^{L} : (\pi_* \mathbf{Z}/l)^{L'} \to (\pi_* \mathbf{Z}/l)^L.$$

If y is a k-rational point of X/G, then the maps on stalks at y induced by $\mathrm{Tr}_{L'}^{L}$ and τ are just

$$\mathbf{Z}/l[\pi^{-1}(y)]^{L'} \xrightarrow{\mathrm{Tr}_{L'}^{L}} \mathbf{Z}/l[\pi^{-1}(y)]^{L} \xrightarrow{\mathrm{Br}_L} \mathbf{Z}/l[\pi^{-1}(y)^L],$$

and so the composition is zero. Therefore τ factors through the Brauer morphism to give a map

$$(\pi_* \mathbf{Z}/l)\ (L) \to \pi_* \alpha_* \mathbf{Z}/l$$

from the presheaf obtained by applying the Brauer construction to $\pi_* \mathbf{Z}/l$. This map is an isomorphism on stalks at k-rational points, so gives rise to an isomorphism of sheaves from the sheafification of $(\pi_* \mathbf{Z}/l)$ (L) to $\pi_* \alpha_* \mathbf{Z}/l$.

Let M_N be the direct sum of permutation $\mathbf{Z}/l[N_G(L)]$-modules $\mathbf{Z}/l[N_G(L)/N_i]$, one for each subgroup N_i of $N_G(L)$, and let E_N be the endomorphism ring of M_N. Then

$$\begin{aligned} & R\Gamma_c(X^L/N_G(L), \mathrm{Hom}_{\mathbf{Z}/l[N_G(L)]}(M_N, \beta_* \mathbf{Z}/l)) \\ & \quad \cong R\Gamma_c(X/G, \gamma_* \mathrm{Hom}_{\mathbf{Z}/l[N_G(L)]}(M_N, \beta_* \mathbf{Z}/l)) \\ & \quad \cong R\Gamma_c(X/G, \mathrm{Hom}_{\mathbf{Z}/l[N_G(L)]}(M_N, \gamma_* \beta_* \mathbf{Z}/l)) \\ & \quad \cong R\Gamma_c(X/G, \mathrm{Hom}_{\mathbf{Z}/l[N_G(L)]}(M_N, \pi_* \alpha_* \mathbf{Z}/l)) \\ & \quad \cong R\Gamma_c(X/G, \mathrm{Hom}_{\mathbf{Z}/l[N_G(L)]}(M_N, (\pi_* \mathbf{Z}/l)\ (L))). \end{aligned}$$

So if P^* is a bounded complex of projective E_N-modules isomorphic to these objects of $D^b(E_N\text{-}mod)$ then

$$\begin{aligned}\Lambda_c(X^L, N_G(L), \mathbf{Z}/l) &\cong M_N \otimes_{E_N} P^* \\ &\cong \Omega_c(X/G, (\pi_* \mathbf{Z}/l)(L)),\end{aligned}$$

which is isomorphic, in $K^b(\mathbf{Z}/l[N_G(L)]\text{-}mod)$, to

$$\Omega_c(X/G, \pi_* \mathbf{Z}/l)(L) \cong \Lambda_c(X, G, \mathbf{Z}/l)(L)$$

by Lemma 2.8. □

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