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FIRST COHOMOLOGY OF ANOSOV ACTIONS OF HIGHER RANK ABELIAN GROUPS AND APPLICATIONS TO RIGIDITY

by ANATOLE KATOK* *and* RALF J. SPATZIER**

Abstract. — This is the first in a series of papers exploring rigidity properties of hyperbolic actions of $\mathbf{Z}^k$ or $\mathbf{R}^k$ for $k \geqslant 2$. We show that for all known irreducible examples, the cohomology of smooth cocycles over these actions is trivial. We also obtain similar Hölder and C^1 results via a generalization of the Livshitz theorem for Anosov flows. As a consequence, there are only trivial smooth or Hölder time changes for these actions (up to an automorphism). Furthermore, small perturbations of these actions are Hölder conjugate and preserve a smooth volume.

1. Introduction

The first untwisted smooth or Hölder cohomology for a smooth dynamical system plays a central role in the structure theory of such systems. For hyperbolic actions of $\mathbf{Z}$ or $\mathbf{R}$, the Hölder cohomology has been described by A. Livshitz [14]. It involves an infinite-dimensional moduli space, most conveniently described by periodic data. In the smooth case, similar results have been achieved by R. de La Llave, J. Marco and R. Moriyon [15].

While there is an abundance of Anosov flows and diffeomorphisms, one knows very few examples of Anosov actions of $\mathbf{Z}^k$ and $\mathbf{R}^k$ which do not arise from products and other obvious constructions. These examples exhibit a remarkable array of rigidity properties, markedly different from the rank one situation. At the heart of these phenomena lies a drastically different behaviour of the first cohomology. This is the central issue of this paper. In particular, we show that in all known examples satisfying suitable irreducibility assumptions, the first smooth or Hölder cohomology with coffiecients in $\mathbf{R}^n$ trivializes, i.e. every smooth or Hölder cocycle is cohomologous to a constant cocycle by a smooth or Hölder coboundary. We call these actions *standard*. As immediate applications, we see that all time changes of standard actions come from automorphisms of $\mathbf{Z}^k$ or $\mathbf{R}^k$. Furthermore, we show that any C^1-perturbation of a standard action preserves a smooth measure, and is Hölder-conjugate to the original action, up to an auto-

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morphism of $\mathbf{R}^k$. In fact, in most cases, we can show that the conjugacy between the original and the perturbed action is in fact smooth. Thus the actions are locally differentiably rigid. This phenomenon never appears for diffeomorphisms or flows. However, the derivation of differentiable rigidity requires a careful study of the transverse smooth structure for the action. This is complementary to our investigations of cocycles in this paper, and appears in [11]. In fact, the present paper is the first in a series of papers addressing rigidity phenomena of hyperbolic and partially hyperbolic actions of higher rank abelian groups. In [10], we extend the cohomology trivialization results of this paper to certain partially hyperbolic actions. In [12], we establish the smooth local rigidity of the orbit foliation of certain Anosov actions, and apply it to prove the smooth local rigidity of projective actions of irreducible lattices in higher rank connected semisimple groups of the noncompact type. Finally, we show in [13] that invariant measures for higher rank abelian actions are scarce, provided that some element has positive entropy.

The structure of this paper is as follows. In Section 2.1, we summarize the known general theory of Anosov actions. In Section 2.2, we introduce the standard examples. They are all homogeneous actions. Then we formulate the main result, Theorem 2.9 and its corollaries in Section 2.3. In Section 3, we summarize results about the decay of matrix coefficients of representations of semisimple groups. They are crucial for the proof of the main result in the semisimple case. In Section 4, we prove the main result. The proof follows a general scheme, which we describe in Section 4.1. It depends on specific estimates for various cases. We present these in Sections 4.2, 4.3 and 4.4. Finally, we discuss the immediate applications of the main result in Section 5.

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2. The standard Anosov $\mathbf{R}^k$-actions

2.1. Anosov actions

Definition **2.1.** — *Let* A *be* $\mathbf{R}^k$ *or* $\mathbf{Z}^k$. *Suppose* A *acts* C^∞ *and locally freely on a manifold* M *with a Riemannian norm* $\|\ \ \|$. *Call an element* $g \in A$ regular *or* normally hyperbolic *if there exist real numbers* $\lambda > \mu > 0$, $C, C' > 0$ *and a continuous splitting of the tangent bundle*

$$TM = E_g^+ + E^0 + E_g^-$$

such that E^0 *is the tangent distribution of the* A*-orbits and for all* $p \in M$, *for all* $v \in E_g^+(p)$ $(v \in E_g^-(p)$ *respectively) and* $n > 0$ $(n < 0$ *respectively) we have for the differential* $g_* : TM \to TM$

$$\| g_*^n(v) \| \leqslant C e^{-\lambda(n)} \| v \|.$$

Call an A*-action* Anosov *or* normally hyperbolic *if it contains a normally hyperbolic element. We call* E_g^+ *and* E_g^- *the* stable *and* unstable distribution *of g respectively.*

If M is compact, these notions do not depend on the ambient Riemannian metric. Note that the splitting and the constants in the definition above depend on the normally hyperbolic element.

Hirsch, Pugh and Shub developed the basic theory of normally hyperbolic transformations in [4].

Theorem **2.2.** — *Suppose* $g \in A$ *acts normally hyperbolically on a manifold* M. *Then there are Hölder foliations* W_g^s *and* W_g^u *tangent to the distributions* E_g^+ *and* E_g^- *respectively. We call these foliations the* stable *and* unstable foliations *of g. The individual leaves of these foliations are* C^∞*-immersed submanifolds of* M.

Theorem **2.3.** — *Let* M *be a closed manifold, and* $\alpha : A \times M \to M$ *an action with a normally hyperbolic element g. If* $\alpha^* : A \times M \to M$ *is a second action of* A *sufficiently close to* α *in the* C^1*-topology then g is also normally hyperbolic for* α^*. *The stable and unstable manifolds of* α^* *tend to those of* α *in the* C^k*-topology as* α^* *tends to* α *in the* C^k*-topology. Furthermore, there is a Hölder homeomorphism* $\varphi : M \to M$ *close to* id_M *such that* φ *takes the leaves of the orbit foliation of* α^* *to those of* α.

Let us call an orbit $\mathbf{R}^k.x$ of a locally free $\mathbf{R}^k$-action *closed* if the stationary subgroup S of x (and hence of each point of that orbit) is a lattice in $\mathbf{R}^k$. Thus any closed orbit is naturally identified with the k-torus $\mathbf{R}^k/S$. In fact, any orbit of an Anosov $\mathbf{R}^k$-action whose stationary subgroup contains a regular element a is closed (indeed, a fixes any point y in the closure of the orbit; consider the canonical coordinates of y with respect to a nearby point on the orbit to see that the orbit is closed).

Another standard fact about Anosov $\mathbf{R}^k$-actions is an Anosov-type closing lemma which is a straightforward generalization of a similar statement for Anosov flows [4].

Theorem **2.4** *(Closing Lemma). — Let* $g \in \mathbf{R}^k$ *be a regular element of an Anosov* $\mathbf{R}^k$*-action* α *on a closed manifold* M. *There exist positive constants* δ_0, C *and* λ *depending continuously on* α *in the* C^1*-topology and g such that :*
if for some $x \in M$ *and* $t \in \mathbf{R}$:

$$\mathrm{dist}(\alpha(tg)\, x, x) < \delta_0$$

then there exists a closed α*-orbit* $\mathcal{O}$, *a point* $y \in \mathcal{O}$, *a differentiable map* $\gamma : [0, t] \to \mathbf{R}^k$ *such that for all* $s \in [0, t]$ *we have*

1. $\mathrm{dist}(\alpha(sg)\,(x), \alpha(\gamma(s))\, y) \leqslant C e^{-\lambda(\min(s,\, t-s))}\, \mathrm{dist}(\alpha(tg)\,(x), x)$,
2. $\alpha(\gamma(t))\,(y) = \alpha(\delta)\,(y)$ *where* $\|\delta\| \leqslant C\, \mathrm{dist}(\alpha(tg)\,(x), x)$,
3. *and* $\|\gamma' - g\| < C\, \mathrm{dist}(\alpha(tg)\, x, x)$.

2.2. The standard actions

There are four constructions of Anosov actions from known ones:

1. cartesian products of two Anosov actions,
2. quotients or covers of an Anosov action by a finite group of automorphisms,
3. restrictions of a $\mathbf{Z}^k$-Anosov action to a subgroup which contains at least one regular element,
4. suspensions of Anosov $\mathbf{Z}^k$-action: Suppose $\mathbf{Z}^k$ acts on N. Embed $\mathbf{Z}^k$ as a lattice in $\mathbf{R}^k$. Let $\mathbf{Z}^k$ act on $\mathbf{R}^k \times \mathrm{N}$ by $z(x, m) = (x - z, zm)$ and form the quotient

$$\mathrm{M} = \mathbf{R}^k \times \mathrm{N}/\mathbf{Z}^k.$$

Note that the action of $\mathbf{R}^k$ on $\mathbf{R}^k \times \mathrm{N}$ by $x(y, n) = (x + y, n)$ commutes with the $\mathbf{Z}^k$-action and therefore descends to M. This $\mathbf{R}^k$-action is called the *suspension* of the $\mathbf{Z}^k$-action.

Now suppose at least one element $g \in \mathbf{Z}^k$ acts by an Anosov diffeomorphism on N. Then the suspension is an $\mathbf{R}^k$-Anosov action. Indeed, g, thought of as an element of $\mathbf{R}^k$, is a regular element.

Starting with Anosov flows and diffeomorphisms and taking products, quotients and covers, and in the case of diffeomorphisms also restrictions and suspensions, we obtain a collection of Anosov actions. They are not rigid, and play a role similar to products of rank one manifolds of non-positive curvature. We will see however that the product structure displays certain rigidity properties.

There is another less obvious construction which leads to more non-rigid examples. Start with an Anosov flow φ_t on a compact manifold B. Consider a compact fiber bundle $\mathrm{M} \to \mathrm{B}$ with fiber F. One can sometimes find a lift ψ_t of φ_t on M and a commuting $\mathbf{R}^{k-1}$-action which is vertical, i.e. preserves the fibers such that the resultant $\mathbf{R}^k$-action is Anosov. We will present specific examples of this type later in Example 2.10. This last construction can be combined with the first four to produce more examples of Anosov actions.

Now we will describe a class of Anosov actions which cannot be obtained this way. These will be called the *standard* actions. None of them have a finite cover with a smooth factor on which the action is not faithful, not transitive and is generated by a rank one subgroup. All examples of $\mathbf{R}^k$-actions in this class come from the following unified algebraic construction.

Let G be a connected Lie group, $\mathrm{A} \subset \mathrm{G}$ a closed Abelian subgroup which is isomorphic with $\mathbf{R}^k$, S a compact subgroup of the centralizer Z(A) of A, and Γ a cocompact lattice in G. Then A acts by left translation on the compact space $\mathrm{M} \overset{\text{def}}{=} \mathrm{S}\backslash\mathrm{G}/\Gamma$. The three specific types of standard Anosov examples discussed below correspond to:

a) for suspensions of Anosov automorphisms of nilmanifolds take $\mathrm{G} = \mathbf{R}^k \ltimes \mathbf{R}^m$ or $\mathrm{G} = \mathbf{R}^k \ltimes \mathrm{N}$, the semi-direct product of $\mathbf{R}^k$ with $\mathbf{R}^m$ or a simply connected nilpotent Lie group N;

b) for the symmetric space examples take G to be a semisimple Lie group of the non-compact type;

c) for the twisted symmetric space examples take $G = H \ltimes \mathbf{R}^m$ or $G = H \ltimes N$, a semi-direct product of a reductive Lie group H with semisimple factor of the non-compact type with $\mathbf{R}^m$ or a simply connected nilpotent group N.

We will now discuss the standard Anosov examples in more detail.

Example **2.5** *(Automorphisms of tori and nilmanifolds and their suspensions).* — Consider a $\mathbf{Z}^k$-action by toral automorphisms. Such an action is called *irreducible* if no finite cover splits as a product. We call an action by toral automorphisms *standard* if it is Anosov and if it contains a $\mathbf{Z}^2$-action such that every non-trivial element of $\mathbf{Z}^2$ acts ergodically with respect to Haar measure. Note that any standard action is irreducible.

There are many examples of standard Anosov actions by toral automorphisms. For example, any faithful action of $\mathbf{Z}^{n-1}$ on T^n with Anosov generators is standard. In fact, in this case any non-trivial element of $\mathbf{Z}^{n-1}$ is Anosov. Hence any higher rank subgroup of $\mathbf{Z}^{n-1}$ also acts by a standard action.

These examples directly generalize to Anosov $\mathbf{Z}^k$-actions by automorphisms of nilmanifolds. Note that the nilmanifolds always fiber as a torus bundle over a nilmanifold of smaller dimension by factoring out by the commutator subgroup. Furthermore, actions by automorphisms act via bundle maps. In particular, the fiber over the coset class of the identity in the base is mapped to itself. We inductively define such an action to be *standard* if the induced actions on the base and the fiber are standard. Explicit constructions of such actions on the k-step free nilpotent group over $\mathbf{R}^n$ can be found in [19].

We will also call any suspension of a standard $\mathbf{Z}^k$-action by toral or more generally nilmanifold automorphisms a *standard suspension action.* Note that the suspensions factor over a transitive action of $\mathbf{R}^k$ on T^k, and hence over a transitive rank one action.

The most important class of examples comes from symmetric spaces [7].

Example **2.6** *(Symmetric space examples).* — Let G be a semisimple connected real Lie group of the noncompact type and of $\mathbf{R}$-rank at least 2. Let A be the connected component of a split Cartan subgroup of G. Suppose Γ is an irreducible torsion-free cocompact lattice in G. Recall that the centralizer Z(A) of A splits as a product Z(A) = MA where M is compact. Since A commutes with M, the action of A by left translations on G/Γ descends to an A-action on $N \stackrel{\text{def}}{=} M\backslash G/\Gamma$. We call this action the *Weyl chamber flow of* A. It is an Anosov action. We will call all Weyl chamber flows *standard.*

Indeed, let Σ denote the restricted root system of G. Then the Lie algebra $\mathfrak{g}$ of G decomposes

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{a} + \sum_{\alpha \in \Sigma} \mathfrak{g}^\alpha$$

where $\mathfrak{g}^\alpha$ is the root space of α and $\mathfrak{m}$ and $\mathfrak{a}$ are the Lie algebras of M and A. Fix an ordering of Σ. If X is any element of the positive Weyl chamber $\mathscr{C}_p \subset \mathfrak{a}$ then $\alpha(X)$ is nonzero and real for all $\alpha \in \Sigma$. Hence $\exp X$ acts normally hyperbolically on G with respect to the foliation given by the MA-orbits.

As a specific example, let $G = SL(n, \mathbf{R})$. Take A to be the diagonal subgroup. For any split group, we have $M = \{1\}$. Thus the action of A on $SL(n, \mathbf{R})/\Gamma$ is Anosov for any cocompact torsion-free lattice Γ.

Let us note that we only need Γ to be torsion-free to assure that G/Γ is a manifold. All of our arguments in this paper directly generalize to the orbifold case.

Example **2.7** *(Twisted symmetric space examples)*. — Assume the notations of the Example 2.6. Let $\rho : \Gamma \to SL(n, \mathbf{Z})$ be a representation of Γ which is irreducible over $\mathbf{Q}$. Then Γ acts on the n-torus T^n via ρ and hence on $(M\backslash G) \times T^n$ via

$$\gamma(x, t) = (x\gamma^{-1}, \rho(\gamma)(t)).$$

Let $N \stackrel{\text{def}}{=} M\backslash G \times_\Gamma T^n \stackrel{\text{def}}{=} (M\backslash G \times T^n)/\Gamma$ be the quotient of this action. As the action of A on $M\backslash G \times T^n$ given by $a(x, t) = (ax, t)$ commutes with the Γ-action, it induces an action of A on N.

Suppose that $\rho(\gamma)$ for some element $\gamma \in \Gamma$ is an Anosov diffeomorphism on T^n. Note first that the image under ρ of the center Z of Γ is finite by Schur's Lemma. Hence we may suppose that Γ is a lattice in a semisimple group with finite center. By Margulis' superrigidity theorem, semisimplicity of the algebraic hull H of $\rho(\Gamma)$ and existence of an Anosov element $\rho(\gamma)$ the representation ρ of Γ extends to a homomorphism $G \to H_{ad}$ where H_{ad} is the adjoint group of H [16, Theor. (2′) and (3′)]. Note that $\rho(\Gamma)$ has finite center Z as follows for example from Margulis' finiteness theorem [16, Theor. (4′)]. Thus T^m/Z is an orbifold. To see that the A-action on N is Anosov it suffices to see that the A-action on $(M\backslash G) \times_\Gamma (T^m/Z)$ is Anosov (with an appropriate notion of Anosov for orbifolds). For this first note that γ is a semisimple element in G since Γ is cocompact. Let $\gamma = k_\gamma s_\gamma$ be the decomposition into the compact and split semisimple parts. Then s_γ is conjugate to an element $a' \in A$. As ρ extends to G, it follows that $\rho(s_\gamma)$ and hence $\rho(a')$ have no eigenvalues of modulus 1. We can pick $a \in A$ such that $\rho(a)$ does not have eigenvalues of modulus 1, and such that $\log a$ lies in an open Weyl chamber of $\mathfrak{g}$. Then a acts normally hyperbolically on $M\backslash G/\Gamma$.

We will now show that a acts normally hyperbolically on N. Let $(x, t) \in N$. Since Γ is cocompact there is a uniformly bounded sequence of elements $u_n(x) \in G$ such that $x^{-1} a^n x = u_n(x)\, \gamma_n(x)$ for some $\gamma_n(x) \in \Gamma$. Since the $u_n(x)$ are uniformly bounded in x and n, the stable tangent vectors for $x^{-1} ax$ are exponentially contracted by $\rho(\gamma_n)$ with estimates uniform in x. The same applies to unstable vectors. Since

$$\begin{aligned} a^n(x, t) &= (x(x^{-1} a^n x), t) \\ &= (xu_n(x), \rho(\gamma_n)\, t) \end{aligned}$$

and since a acts normally hyperbolically on $M\backslash G/\Gamma$, it follows that a is normally hyperbolic with respect to the orbit foliation of A. Thus the A-action is Anosov.

The above construction can be generalized considering toral extensions of other higher rank actions for which one of the monodromy elements is Anosov. For example, using a twisted Weyl chamber flow as above as the base we obtain nilmanifold extensions of the Weyl chamber flow. As A. Starkov pointed out, one can also start with the product of a Weyl chamber flow with a transitive action of some $\mathbf{R}^l$ on a torus and produce a toral extension which is Anosov and no finite cover splits as a product. These two extension constructions can be combined and iterated. This is our last class of *standard examples*.

We do not know whether these examples yield all algebraic Anosov actions under the other natural constructions discussed above.

Let us emphasize that for all standard actions the splitting $TM = E_g^+ + E^0 + E_g^-$ is smooth.

Finally let us describe some non-rigid higher rank Anosov actions which are non-trivial skew products over rank one actions.

Example **2.8.** — *a)* Denote the standard generators of $\mathbf{Z}^2$ by a_1 and a_2. Let a_1 act on T^3 by a hyperbolic toral automorphism α, and diagonally on $T^3 \times T^3$ by $\alpha \times \alpha$. Let a_2 act on $T^3 \times T^3$ by

$$\begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$$

where I is the 3×3-identity matrix. Then the flow determined by ta_1 in the suspension is a product. Any time change of ta_1 in the second factor still commutes with the action of ta_2. Note that this action is not a product, and that only one of the two generators is regular.

b) Another more elaborate example of this type was constructed by A. Starkov. Let N be the 3-dimensional Heisenberg group. Then $N = \mathbf{R}^2 \rtimes \mathbf{R}$ is a semi-direct product of $\mathbf{R}^2$ and $\mathbf{R}$. The action of

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

on $\mathbf{R}^2$ extends to a flow by automorphisms of N that acts trivially on the center. Let $H = \mathbf{R} \ltimes N$ be the corresponding semi-direct product. Let $G = H \times SL(2, \mathbf{R})$.

We can find a cocompact irreducible lattice Γ of G as follows. Let $\Lambda = \mathbf{Z} \ltimes \Lambda'$ be the semidirect product of $\mathbf{Z}$ with the integer points Λ' in the Heisenberg group. Let Γ' be a cocompact lattice in $SL(2, \mathbf{R})$ and $h : \Gamma' \to \mathbf{R}$ a homomorphism whose image is not commensurable with the integers. Interpret the graph of h as a subgroup of the product of the center $\mathbf{R}$ of N with $SL(2, \mathbf{R})$. Then set $\Gamma = \Lambda . \text{graph } h$.

Finally, $\mathbf{R}^3$ acts on this manifold by an Anosov action as follows. The first generator acts via the suspension, the second via the center of N and the third via the action of the diagonal subgroup of $SL(2, \mathbf{R})$ on $SL(2, \mathbf{R})/\Gamma'$ which in this case coincides with the geodesic flow of a certain Riemann surface.

This action is not locally rigid as we can perturb the action of the diagonal subgroup of $SL(2, \mathbf{R})$ to any nearby geodesic flow. Not all elements of $\mathbf{R}^3$ exponentially contract or expand at all on this space, and in this sense this is the continuous time counterpart of the non-standard irreducible action on T^6 by toral automorphisms we described in part *a)*.

Clearly this construction generalizes to other nilpotent and semisimple groups with cocompact lattices with non-trivial first Betti number. None of these examples will be called standard.

2.3. The main results

Recall that given an action of a group G on a manifold M and another group H, a map $\beta : G \times M \to H$ is called a *cocycle* if it satisfies the cocycle identity $\beta(g_1 g_2, m) = \beta(g_1, g_2 m)\, \beta(g_2, m)$ [23]. The simplest cocycles are the constant cocycles, i.e. those constant in M. They correspond to homomorphisms $G \to H$. If H is a Lie group, call two cocycles β and β^* C^∞ (Hölder)-*cohomologous* if there exists a C^∞ (Hölder)-function $P : M \to H$, called a C^∞ (Hölder)-*coboundary*, such that $\beta^*(a, x) = P(ax)^{-1}\, \beta(a, x)\, P(x)$ for all $a \in \mathbf{R}^k$ and $x \in M$.

Theorem **2.9.** — *a) Consider a standard Anosov* A*-action on a manifold* M *where* A *is isomorphic to* $\mathbf{R}^k$ *or* $\mathbf{Z}^k$ *with* $k \geq 2$. *Then any* C^∞*-cocycle* $\beta : A \times M \to \mathbf{R}^l$ *is* C^∞*-cohomologous to a constant cocycle.*

b) Any Hölder cocycle into $\mathbf{R}^l$ *is Hölder cohomologous to a constant cocycle.*

The Hölder result is obtained from the C^∞-result using the following straightforward generalization of the Livshitz theorem for Anosov diffeomorphisms and flows [14].

Theorem **2.10.** *(Livshitz Theorem for* $\mathbf{R}^k$*-Anosov actions). — Let* α *be a volume-preserving Anosov action of* $\mathbf{R}^k$ *on a compact manifold* M *and let* β *be a Hölder* $\mathbf{R}^l$*-cocycle over* α *such that* $\beta(a, x) = 0$ *for all* x *on any closed orbit of* α *and* a *with* $ax = x$. *Then*

$$\beta(a, x) = P(ax) - P(x)$$

where P *is an* $\mathbf{R}^l$*-valued Hölder function on* M. *Furthermore, if* β *is* C^1 *or* C^∞, *then* P *is correspondingly* C^1 *or* C^∞.

The assumption that α is volume-preserving can be weakened. Since this is irrelevant for our purposes we will not go into this matter.

We will also use the Livshitz theorem to obtain the C^∞-cocycle results for semisimple groups with $SO(n, 1)$ or $SU(n, 1)$ factors for which we do not have a uniform control of the exponential decay of C^∞-vectors for all unitary representations.

Another consequence of our Livshitz theorem is the following rigidity result for cocycles over products of Anosov actions.

Corollary **2.11.** — *Let* α_1 *and* α_2 *be volume preserving Anosov actions of* $\mathbf{R}^k$ *and* $\mathbf{R}^l$ *on closed manifolds* M_1 *and* M_2. *Then any* $\mathbf{R}^m$*-valued Hölder* (C^∞)*-cocycle over the product action is Hölder* (C^∞)*-cohomologous to the sum of cocycles constant on the first and the second factor.*

In particular, a cocycle over a product of standard Anosov actions of $\mathbf{R}^k$ *and* $\mathbf{R}^l$ *with k an* $\ell \geqslant 2$ *is cohomologous to a constant cocycle.*

Let us now describe three applications of the main result. Recall that an action α' of $\mathbf{R}^k$ is called a *time-change* of an action α of $\mathbf{R}^k$ if all α'-orbits are α-orbits.

Theorem **2.12.** — *a) All* C^∞*-time changes of a standard Anosov* $\mathbf{R}^k$*-action with* $k \geqslant 2$ *are* C^∞*-conjugate to the original action up to an automorphism.*

b) All Hölder time changes of a standard $\mathbf{R}^k$*-Anosov action with* $k \geqslant 2$ *are Hölder conjugate to the original action up to an automorphism.*

Note that one easily obtains more actions by composing a given action with an automorphism ρ of the acting group. In particular, this gives C^1-small perturbations. When two actions only differ by composition with an automorphism we say that the actions *agree up to an automorphism.* Call an $\mathbf{R}^k$-action *locally* C^∞ *(Hölder)-rigid* if any perturbation of the action which is C^1-close on a generating set is C^∞ (Hölder)-conjugate up to an automorphism.

Theorem **2.13.** — *The standard* $\mathbf{R}^k$*-Anosov actions with* $k \geqslant 2$ *are locally Hölder-rigid up to an automorphism.*

Theorem **2.14.** — *Any* C^1*-small* C^∞*-perturbation* α^* *of a standard* $\mathbf{R}^k$*-Anosov action* α *with* $k \geqslant 2$ *preserves a* C^∞ *volume* ω^*.

Notice that the Livshitz theorem as well as the main result are used in the proof. In fact, we can show that any conjugacy with the original action is volume preserving [11].

3. Preliminaries on matrix coefficients

Estimates on the decay of matrix coefficients of semisimple Lie groups play an essential role in representation theory. These estimates already appear in the work of Harish-Chandra. They were recently refined by several people [1, 2, 5, 17, 21]. While most estimates concern themselves with the matrix coefficients of so-called K-finite vectors (cf. below), both Ratner and Moore prove exponential decay for Hölder vectors in the real rank one case. Ratner manages this for arbitrary Hölder vectors for representations of $SL(2, \mathbf{R})$ while Moore needs a Hölder exponent bigger than $\dim K/2$ where K is the maximal compact subgroup. Although their work is not directly applicable, Moore's arguments can be generalized to the higher rank case. However, we prefer to give a more standard (and probably folklore) treatment for C^∞-vectors based on the K-finite case. At present, there seem to be no results in the literature about the decay of Hölder vectors with arbitrary exponent for general semisimple groups[1].

Let G be a connected semisimple Lie group with finite center. We will consider irreducible unitary representations π of G on a Hilbert space $\mathcal{H}$. Define the *matrix coefficient* of v and $w \in \mathcal{H}$ as the function $\varphi_{v,w} : G \to \mathbf{R}$ given by

$$g \to \langle \pi(g)\, v, w \rangle.$$

Fix a maximal compact subgroup K of G. Call a vector $v \in \mathcal{H}$ *K-finite* if the K-orbit of v spans a finite dimensional vector space. Let $\hat{K}$ denote the unitary dual of K. One can then decompose

$$\mathcal{H} = \bigoplus_{\mu \in \hat{K}} \mathcal{H}_\mu$$

where $\mathcal{H}_\mu$ is $\pi(K)$-invariant and the action of K on $\mathcal{H}_\mu$ is equivalent to $n\mu$ where n is an integer or $+\infty$, called the *multiplicity of* μ in $\mathcal{H}$. The K-finite vectors form a dense subset of $\mathcal{H}$. One calls $\mathcal{H}_\mu$ the *μ-isotypic component* of π.

Call π *strongly* L^p if there is a dense subset of $\mathcal{H}$ such that for v, w in this subspace, $\varphi_{v,w} \in L^p(G)$. Let A be a maximal split Cartan subgroup of G, and $\mathfrak{a}$ its Lie algebra. Fix an order on the roots and let $\mathscr{C}$ be the positive Weyl chamber. Further let $\rho : \mathfrak{a} \to \mathbf{R}$ be half the sum of the positive roots on $\mathscr{C}$. Howe obtained the following estimate for the matrix coefficients of $v \in \mathcal{H}_\mu$ and $w \in \mathcal{H}_\nu$ of a strongly L^p-representation π of G:

$$|\varphi_{v,w}(\exp tA)| \leqslant D \,\|v\|\, \|w\| \dim \mu \dim \nu\, e^{-(t/2p)\rho(A)}$$

where $A \in \overline{\mathscr{C}}$ and $D > 0$ is a universal constant [5, Corollary 7.2 and § 7]. Cowling [2] shows that every irreducible unitary representation of G with discrete kernel is strongly L^p for some p. Furthermore, if $\mathfrak{g}$ does not have factors isomorphic to $\mathfrak{so}(n, 1)$ or $\mathfrak{su}(n, 1)$ then p can be chosen independently of π.

A vector $v \in \mathcal{H}$ is called C^∞ if the map $g \in G \to \pi(g)\, v$ is C^∞. We will now combine the results above with more classical estimates on the size of Fourier coefficients of C^∞-vectors.

Let $m = \dim K$ and $X_1, \dots, X_m$ be an orthonormal basis of $\mathfrak{k}$. Set $\Omega = 1 - \sum_{i=1}^m X_i^2$. Then Ω belongs to the center of the universal enveloping algebra of $\mathfrak{k}$, and acts on the K-finite vectors in $\mathcal{H}$ since K-finite vectors are smooth.

Theorem **3.1.** — *Let v and w be C^∞-vectors in an irreducible unitary representation π of* G *with discrete kernel. Then there is a universal constant* $E > 0$ *and an integer $p > 0$ such that for all* $A \in \overline{\mathscr{C}}$ *and large enough* m

$$|\langle \exp(tA)\, v, w \rangle| \leqslant E e^{-(t/2p)\rho(A)} \,\|\Omega^m(v)\|\, \|\Omega^m(w)\|.$$

In fact, p can be any number for which π is strongly L^p. *Furthermore, if $\mathfrak{g}$ does not have factors isomorphic to $\mathfrak{so}(n, 1)$ or $\mathfrak{su}(n, 1)$, p only depends on* G.

1. After this paper was written, G. A. Margulis outlined an argument for the exponential decay of Hölder vectors in a private communication with the first author.

Proof. — By Schur's lemma, Ω acts as a multiple $c(\mu)\ id_{\mathscr{H}_\mu}$ *on* $\mathscr{H}_\mu$. Let $v = \Sigma_{\mu \in \widehat{K}} v_\mu$. By [22, Lemma 4.4.2.2], we have for all integers $m > 0$

$$\| v_\mu \| \leqslant c(\mu)^{-m} \dim^2 \mu \, \| \Omega^m(v) \|.$$

As in [22, Lemma 4.4.2.3] one sees that for m large enough

$$\sum_{\mu \in \widehat{K}} c(\mu)^{-2m} \dim^6 \mu < \infty.$$

We have analogous estimates for $w = \Sigma_{\mu \in \widehat{K}} w_\mu$. Pick $p > 0$ such that π is L^p. Then for m large enough, one sees that

$$\begin{aligned} | \langle \exp(tA)\, v, w \rangle | &= | \langle \sum_{\mu \in \widehat{K}} \exp(tA)\, v_\mu, \sum_{\nu \in \widehat{K}} w_\nu \rangle | \\ &\leqslant De^{-(t/2p)\rho(A)} \sum_{\mu, \nu \in \widehat{K}} \| v_\mu \| \, \| w_\nu \| \dim \mu \dim \nu \\ &\leqslant De^{-(t/2p)\rho(A)} \Big(\sum_{\mu \in \widehat{K}} \| v_\mu \|^2 \dim^2 \mu \Big)^{1/2} \Big(\sum_{\nu \in \widehat{K}} \| w_\nu \|^2 \dim^2 \nu \Big)^{1/2} \\ &\leqslant De^{-(t/2p)\rho(A)} \| \Omega^m(v) \| \, \| \Omega^m(w) \| \sum_{\mu \in \widehat{K}} c(\mu)^{-2m} \dim^6(\mu), \end{aligned}$$

as desired. □

Note that v and w only need to be C^k with respect to K for some large k.

Corollary **3.2.** — *Let* G *be a semisimple connected Lie group with finite center. Let* Γ *be an irreducible cocompact lattice in* G. *Assume that* $\mathfrak{g}$ *does not have factors isomorphic to* $\mathfrak{so}(n, 1)$ *or* $\mathfrak{su}(n, 1)$. *Let* $f_1, f_2 \in L^2(G/\Gamma)$ *be* C^∞*-functions orthogonal to the constants. Let* $\mathscr{C}$ *be a positive Weyl chamber in a maximal split Cartan* $\mathfrak{a}$. *Then there is an integer* $p > 0$ *which only depends on* G *and a constant* $E > 0$ *such that for all* $A \in \overline{\mathscr{C}}$

$$\langle (\exp tA)_* (f_1), f_2 \rangle \leqslant Ee^{-(t/2p)\rho(A)} \| f_1 \|_m \| f_2 \|_m$$

where $\| f \|_m$ *is the Sobolev norm of* f.

Proof. — Since Γ is irreducible, there are no L^2-functions on G/Γ orthogonal to the constants which are invariant under any non-compact element in G by Moore's theorem [23]. Hence every non-trivial irreducible component of $L^2(G/\Gamma)$ has discrete kernel. By Theorem 3.1 it suffices to see that any non-trivial irreducible component is strongly L^p for a p that only depends on G. This is exactly Cowling's result as $\mathfrak{g}$ does not have factors isomorphic to $\mathfrak{so}(n, 1)$ or $\mathfrak{su}(n, 1)$ [2]. □

We do not know if the corollary holds for G with factors of $\mathfrak{g}$ isomorphic to $\mathfrak{so}(n, 1)$ or $\mathfrak{su}(n, 1)$ with a p that depends on the lattice.

4. Cocycles

4.1. Scheme of proof of Theorem 2.9

Let us start with part *a)* of Theorem 2.9. We may and will always assume that $l = 1$. Pick a normally hyperbolic element $a \in A$. We will show that β is cohomologous to $\rho(b) = \int_M \beta(b, x)\, dx$, or that $\beta - \rho$ is cohomologous to 0. Thus we may assume that β has 0 averages.

Define the function f by $f(x) = \beta(a, x)$ (if $A = \mathbf{R}^k$, we could instead consider $f(x) = \frac{d}{dt}\Big|_{t=0} \beta(ta, x)$; as it turns out this is not necessary). Now we can define formal solutions of the cohomology equation by

$$P_a^+ = \sum_{k=0}^{\infty} a^k f \quad \text{and} \quad P_a^- = -\sum_{k=-\infty}^{-1} a^k f.$$

The first step is to show that P_a^+ and P_a^- are distributions. For the suspension case, Example 2.5, this uses the superexponential decay of Fourier coefficients of smooth functions. For the symmetric space case, Examples 2.6, we use estimates on the exponential decay of matrix coefficients for smooth functions which come from representation theory. For the twisted symmetric space examples, Example 2.7, both techniques are combined.

Hyperbolicity implies that the distribution P_a^+ has continuous derivatives of any order along the stable manifolds (and by definition in the direction of a in the continuous case) while P_a^- has continuous derivatives of any order along the unstable manifolds. Note that for Anosov actions by $\mathbf{Z}$ and $\mathbf{R}$, P_a^+ and P_a^- in general do not coincide even if they are distributions. In the higher rank case however, they do coincide, and thus P_a^+ is differentiable along both the stable and unstable manifolds which is the basis for proving P_a^+ is a smooth function.

To show that $P_a^+ = P_a^-$, note that $P_a^+ - P_a^- = \sum_{k=-\infty}^{\infty} a^k f$ is an a-invariant distribution. The cohomology equation together with the decay of matrix (Fourier) coefficients implies that it is also invariant with respect to the whole group. Pick $b \in A$ independent of a. The exponential decay of matrix coefficients implies that $\sum_{k=-\infty}^{\infty} b^k(P_a^+ - P_a^-)$ is a distribution. That forces the vanishing of $P_a^+ - P_a^-$.

From the definition of P_a^+ and the cocycle identity, it follows that P_a^+ is a distribution solution for the coboundary equation for all of A. Since P_a^+ is a coboundary, it has continuous derivatives of any order in the orbit directions. This gives us differentiability in a full set of directions. Then it follows from standard elliptic operator theory that P_a^+ is a C^∞-function.

Theorem 2.9 *b)* for Hölder cocycles is established indirectly. First we prove Theorem 2.10 which says that a cocycle whose restriction to every closed orbit is a coboundary is in fact a coboundary globally. Thus seemingly there are infinitely many independent obstructions to the vanishing of a cocycle. For **R**-actions (Anosov flows) this is indeed the case. Those obstructions (values of the cocycle over the generators of the stationary subgroup on each closed orbit) are continuous in the C^0-topology on cocycles. Thus since for standard Anosov actions they vanish for every C^∞-cocycle with zero averages due to Theorem 2.9 *a)*, the same is true for any C^0-limit of such cocycles over such an action. The proof is completed by showing that every continuous cocycle can be approximated by C^∞-cocycles in the C^0-topology.

We will now discuss the various cases in more detail.

4.2. Toral automorphisms and suspensions

Let us first note a general fact.

Lemma **4.1.** — *Let* A *be an abelian group acting on a measure space* X, *and let* $\beta : A \times X \to \mathbf{R}^m$ *be a measurable cocycle of this action. Suppose that for some* $a \in A$ *there is a measurable function* $P : X \to \mathbf{R}^m$ *such that* $\beta(a, x) - P(ax) + P(x)$ *is constant a.e. on ergodic components of* a. *Then* $\beta^*(b, x) \stackrel{\text{def}}{=} \beta(b, x) - P(bx) + P(x)$ *is constant a.e. on ergodic components of* a *for every* $b \in A$.

Proof. — Indeed we have

$$\begin{aligned}\beta^*(b, x) &= \beta^*(a + b, x) - \beta^*(a, bx)\\ &= \beta^*(b, ax) + \beta^*(a, x) - \beta^*(a, bx) = \beta^*(b, ax). \quad \square\end{aligned}$$

We will now prove rigidity of cocycles for standard suspensions and standard actions by toral automorphisms simultaneously.

Let α be the suspension of a standard action by $\mathbf{Z}^k$ on T^m on $M = \mathbf{R}^k \times_{\mathbf{Z}^k} T^m$. Let $\beta : \mathbf{R}^k \times M \to \mathbf{R}^l$ be a C^∞-cocycle over α. The coboundary P is found in two steps. We first straighten the cocycle on the fibers of the natural fibration $\pi : M \to T^k$.

The rigidity of C^∞-cocycles for standard actions by toral automorphisms is a particular case of the following proposition which is also the main part of the proof for the suspension case.

Proposition **4.2.** — *There is a continuous function* $Q : M \to \mathbf{R}^l$ *which is* C^∞ *along every fiber such that the cocycle* $\beta^* : \mathbf{R}^k \times M \to \mathbf{R}^l$ *defined by*

$$\beta^*(a, x) \stackrel{\text{def}}{=} \beta(a, x) - Q(ax) + Q(x)$$

is constant along the fibers of π *for all* $a \in \mathbf{R}^k$. *In particular,* β^* *restricted to any fiber defines a homomorphism from* $\mathbf{Z}^k$ *to* $\mathbf{R}^l$.

Proof. — As always we will assume that $l = 1$.

Pick a_1 and a_2 in $\mathbf{Z}^k$ such that every element $a_1^k a_2^l$ acts ergodically except when $k = l = 0$. Let $\Lambda \subset \mathbf{Z}^k$ be the subgroup generated by a_1 and a_2.

For $x \in M$ set $f_i(x) = \beta(a_i, x)$. We need to find a C^∞-function $Q: M \to \mathbf{R}$ that solves the difference equations

$$f_i(x) - \int_{\pi^{-1}(x)} f_i(y)\, dy = (\Delta_{a_i} Q)(x) \stackrel{\text{def}}{=} Q(a_i x) - Q(x).$$

Then

$$\beta^*(a, x) \stackrel{\text{def}}{=} \beta(a, x) - Q(ax) + Q(x)$$

is constant along the fiber $\pi^{-1}(\pi(x))$ for all $a \in \Lambda$, and β^* restricted to any fiber defines a homomorphism from Λ to $\mathbf{R}$.

To solve the difference equations above we will use Fourier series on the cover $\mathbf{R}^k \times T^m$ of M. Let $\mathbf{Z}^k$ act on $\mathbf{R}^k \times T^m$ by $\tilde{a}(r, t) \stackrel{\text{def}}{=} (r, at)$, and denote by $\tilde{\Delta}_a$ the associated difference operators. Let $\tilde{f}_i$ denote the lifts of the functions f_i to the covering space $\mathbf{R}^k \times T^m$. We may assume that $\int_{\pi^{-1}(x)} f_i(y)\, dy = 0$ and thus $\int_{\tilde{\pi}^{-1}(x)} \tilde{f}_i(y)\, dy = 0$ where $\tilde{\pi}$ is the projection onto the first factor of $\mathbf{R}^k \times T^m$. Also note that

$$\tilde{\Delta}_{a_1^{-1}} \tilde{f}_2 = \tilde{\Delta}_{a_2^{-1}} \tilde{f}_1.$$

Indeed, we have

$$f_2(a_1 x) + f_1(x) = \beta(a_1 + a_2, x) = f_1(a_2 x) + f_2(x).$$

Let

$$\tilde{f}_i(r, t) = \sum_I c_I^i(r)\, t^I$$

be the Fourier expansion of f_i where the coefficients c_I^i are functions on $\mathbf{R}^k$ and the $I = (i_1, \ldots, i_m)$ are multi-indices. Recall that $a \in Sl(m, \mathbf{Z})$ acts on the multiindices I via the contragredient representation $a^\vee$ of the natural representation of $Sl(m, \mathbf{Z})$ on $\mathbf{R}^m$. Since a_1 is ergodic with respect to the Lebesgue measure, the distance of $(a_1^k)^\vee(I)$ from 0 eventually increases exponentially in k for $I \neq 0$. Note that $c_0 = 0$. Also recall that if f_i is C^∞-then its Fourier coefficients c_I are C^∞-functions in r and decrease faster than any negative power of $|i_1| + \ldots + |i_m|$. Hence for any C^∞-function $f = \Sigma_I f_I t^I$, the sums of Fourier coefficients are absolutely convergent. In particular, the sum $O(I)(f) \stackrel{\text{def}}{=} \Sigma_{k=-\infty}^{\infty} f_{a_1^{k\vee}(I)}$ defines an a_1-invariant distributions $O(I)$ on every fiber $\pi^{-1}(r)$.

Lemma **4.3.** — *For every index set* I *we have* $O(I)(f_1) = 0$ *for every base point* r.

Proof. — Since $\tilde{\Delta}_{a_2^{-1}} \tilde{f}_1 = \tilde{\Delta}_{a_1^{-1}} \tilde{f}_2$ we get

$$\sum_{k=-l}^{l} c^1_{a_1^{k\vee} a_2^\vee(I)} - \sum_{k=-l}^{l} c^1_{a_1^{k\vee}(I)} = \sum_{k=-l}^{l} c^2_{a_1^{k\vee}(I)} - c^2_{a_1^{k-1\vee}(I)} = c^2_{a_1^{l\vee}(I)} - c^2_{a_1^{-l+1\vee}(I)}.$$

In the limit we get that $O(I)(f_1) = O(a_2^\vee(I))(f_1)$.

Since all nontrivial $a_1^k a_2^l$ are ergodic, no such element fixes an index set $I \neq 0$. Hence $\Sigma_{k=0}^{\infty} O(I)(f_1) = \Sigma_{k=0}^{\infty} O((a_2^k)^{\vee}(I))(f_1) \leqslant \Sigma_I |f_I| < \infty$ converges. Thus we see that $O(I)(f_1) = 0$. □

Now we will finish with the proof of Proposition 4.2. Let us show that $P^+ = \Sigma_{k=0}^{\infty} a^k f$ is given by a Fourier series whose coefficients decrease faster than any negative power of $|i_1| + \ldots + |i_m|$. This can be seen as follows: decompose $\mathbf{R}^m$ into stable and unstable subspaces E^- and E^+ for $a_1^{\vee}$. Let C^+ be the set of all elements in $\mathbf{Z}^m$ whose projection to E^+ has norm greater than or equal to the projection to E^-. Let C^- be the complement of C^+ in $\mathbf{Z}^m$. For $I \in C^+$ we estimate the Fourier coefficient of the solution given by P^+, and for $I \in C^-$ using the solution given by P^-. Therefore P^+ is a C^∞-function along the fibers. One sees that the sum of absolute values of Fourier coefficients for Q depends continuously on r. Hence Q is continuous.

Now the proposition follows from Lemma 4.1. □

Now Theorem 2.9 *a)* for suspensions follows from the following simple lemma.

Lemma **4.4.** — *Let* γ *be an* $\mathbf{R}$*-valued* C^∞*-cocycle of the homogeneous action of* $\mathbf{R}^k$ *on* T^k. *Then* γ *is* C^∞*-cohomologous to a homomorphism* $\rho : \mathbf{R}^k \to \mathbf{R}$.

Proof. — By the cocycle equation, the restriction of γ to $\mathbf{Z}^k \times \{t\}$ defines a homomorphism $\rho_t : \mathbf{Z}^k \to \mathbf{R}^l$ for any $t \in T^k$. Since for all $a \in \mathbf{Z}^k$ and $s \in \mathbf{R}^k$

$$\gamma(a, st) = \gamma(s + a, t) - \gamma(s, t) = \gamma(a, t),$$

$\rho = \rho_t$ is independent of $t \in T^k$. Extend ρ to a homomorphism from $\mathbf{R}^k$ to $\mathbf{R}$. Then $\gamma - \rho$ factors through a cocycle $\gamma^* : T^k \times T^k \to \mathbf{R}$. Set $P(t) = \gamma(t, 1)$. Then $\gamma(a, t) = P(t) - P(at, 1)$, and hence $\gamma - \rho$ is a C^∞ coboundary. Finally note that the coboundary is C^∞ along the orbits of $\mathbf{R}^k$ since in fact it is differentiable along these orbits. □

4.3. Symmetric space examples

Here we will discuss Theorem 2.9 *a)* for the standard examples of symmetric space type (Example 2.6) under the extra assumption that the Lie algebra $\mathfrak{g}$ of G does not have any factors isomorphic to $\mathfrak{so}(n, 1)$ or $\mathfrak{su}(n, 1)$. This will allows us to use Corollary 3.2. Cocycle rigidity of Weyl chamber flows for which $\mathfrak{g}$ has $\mathfrak{so}(n, 1)$- or $\mathfrak{su}(n, 1)$-factors (as well as the rigidity for Hölder cocycles over standard Anosov actions) is based on the generalized Livshitz theorem. We will discuss this in Section 4.5.

We will first consider the actions of a split Cartan subgroup A by left translations on G/Γ. As we noticed in Example 2.6, this action is Anosov if G is split. Otherwise, it is always *normally hyperbolic* to the orbit foliation $\mathcal{O}$ of the centralizer MA of A. This means that there is an element $g \in A$ and a continuous splitting of the tangent bundle

$$TM = E_g^+ + E_g^0 + E_g^-$$

such that E^0_g is tangent to $\mathcal{O}$ and for all $p \in M$, for all $v \in E^+_g(p)$ ($v \in E^-_g(p)$ respectively) and $n > 0$ ($n < 0$ respectively) we have for the differential $g_* : TM \to TM$

$$\| g^n_*(v) \| \leqslant C e^{-\lambda |n|} \| v \|$$

and for all $n \in \mathbf{Z}$ and $v \in E^0$ we have

$$\| g^n_*(v) \| \geqslant C' e^{-\mu |n|} \| v \|.$$

We call such an element *regular.*

The sum of the distributions $E^+_g \oplus E^-_g$ is completely non-integrable, i.e. Lie brackets tangent to this distribution span the whole tangent space. This property is crucial for proving that the first C^∞-cohomology is trivial [10]. For our current purposes, it suffices to show existence of a distribution solution to the cohomology equation.

Theorem **4.5.** — *Let* $\beta : A \times G/\Gamma \to \mathbf{R}^l$ *be a* C^∞*-cocycle where the Lie algebra* $\mathfrak{g}$ *of* G *does not have any factor isomorphic to* $\mathfrak{so}(n, 1)$ *or* $\mathfrak{su}(n, 1)$. *Assume that* $k \geqslant 2$ *and that* Γ *is irreducible. Then there is a constant cocycle* $\rho : A \times G/\Gamma \to \mathbf{R}^l$ *and an* $\mathbf{R}^l$*-valued distribution* P *on* G/Γ *such that*

$$\beta(a, x) - \rho(a) = aP - P.$$

Furthermore, the distribution derivatives of all orders of P *along both stable and unstable manifolds of some regular element and along the orbit foliation of* A *are continuous functions. Finally, if* β *is* M*-invariant, so is* P.

Proof. — Define

$$\rho(b) = \int_{G/\Gamma} \beta(b, x)\, dx.$$

It clearly suffices to show that $\beta(a, x) - \rho(a)$ is a coboundary. Thus we will assume that the averages $\int_{G/\Gamma} \beta(b, x)$ are all 0.

Again we will assume that $l = 1$. Let a_1 and a_2 be $\mathbf{R}$-linearly independent, and let Λ denote their span. We may assume that a_1 is regular. Set $f_1(x) = \beta(a_1, x)$ for $x \in G/\Gamma$. By Lemma 4.1 it suffices to find a C^∞-function P that satisfies the difference equations $\Delta_{a_1} P = f_1$.

To find the coboundary P let us first show that the formal solutions $P_+ = \Sigma_{k=0}^{\infty} a_1^k f$ and $P_- = -\Sigma_{k=-1}^{-\infty} a_1^k f$ define distributions on G/Γ. Let $g \in C^\infty(G/\Gamma)$. By Corollary 3.2 there is a positive integer m and constant $E > 0$ such that $| \langle a_1^k f, g \rangle | \leqslant E e^{-k\rho} \|f\|_m \| g \|_m$ where $\| \, \|_m$ is the Sobolev norm. Hence $\Sigma_{k=0}^{\infty} \langle a_1^k f, g \rangle$ converges absolutely, and there is a constant $A > 0$ such that $| \Sigma_{k=0}^{\infty} \langle a_1^k f, g \rangle | \leqslant A \| g \|_m$. Thus P_+ and similarly P_- are distributions. In fact, they are elements of the Sobolev space H^{-m}.

Lemma **4.6.** — *The distributions* P_+ *and* P_- *coincide.*

Proof. — As in the proof of Proposition 4.2 we have the difference equations

$$\Delta_{a_j^{-1}} f_i = \Delta_{a_i^{-1}} f_j.$$

Hence we get

$$\sum_{k=-l}^{l} a_1^k a_2 f_1 - \sum_{k=-l}^{l} a_1^k f_1 = \sum_{k=-l}^{l} a_1^{k+1} f_2 - a_1^k f_2 = a_1^{l+1} f_2 - a_1^{-l} f_2.$$

Since Γ is an irreducible lattice the matrix coefficients of elements in $L^2(G/\Gamma)$ orthogonal to the constants vanish [23, ch. 2]. Hence we see that for $g \in C^\infty(G/\Gamma)$

$$\sum_{k=-\infty}^{\infty} \langle a_1^k f_1, a_2^{-1} g \rangle - \sum_{k=-\infty}^{\infty} \langle a_1^k f_1, g \rangle = \lim_{l\to\infty} \langle a_1^{l+1} f_2 - a_1^{-l} f_2, g \rangle = 0.$$

Since $a_1^k a_2^m \to \infty$ as $(k, m) \to \infty$ and the matrix coefficients decay exponentially, the sum $\Sigma_{m=-\infty}^{\infty} \Sigma_{k=-\infty}^{\infty} \langle a_1^k f_1, a_2^m g \rangle = \lim_{m\to\infty} 2m \Sigma_{k=-\infty}^{\infty} \langle a_1^k f_1, g \rangle$ converges absolutely. Thus we get $\Sigma_{k=-\infty}^{\infty} \langle a_1^k f_1, g \rangle = 0$. □

Henceforth we will denote $P_+ = P_-$ by P.

Next we will show differentiability of P along the strong stable as well as the strong unstable manifold of a_1. Fix an ordering of the set of roots $\mathcal{R}$ of G with respect to the split Cartan $\mathbf{R}^k$ such that a_1 is in the positive Weyl chamber. Let X_α be an element in the root space E^α for a negative root α. Let $\frac{\partial}{\partial\alpha}$ denote the Lie derivative by X_α. Note that

$$\sum_{k=0}^{\infty} \frac{\partial}{\partial\alpha} (f_1 \circ a_1^k) = \sum_{k=0}^{\infty} e^{k\alpha(\log a_1)} \left(\frac{\partial}{\partial\alpha} f_1\right) \circ a_1^k$$

converges uniformly on G/Γ. Thus the derivative of P along stable directions X_α is a continuous function. Similarly all the higher derivatives along stable directions are continuous functions. Since $P = P_-$ by Lemma 4.6, a similar argument shows that the derivatives of P along unstable directions are also continuous functions. (As the stable and unstable directions X_α for $\alpha \in \mathcal{R}$ generate the Lie algebra $\mathfrak{g}$ of G as a Lie algebra, Theorem 2.1 of [10] shows that P is C^∞ on G/Γ.)

Finally, let us prove that derivatives of all orders of P along the orbit foliation of A are continuous functions. This follows immediately once we see that P is also a coboundary for β for all of A. Let $b \in A$. Then we see from the cocycle identity that

$$bP - P = \sum_{k=0}^{\infty} ba^n \beta(a, \ ) - a^n \beta(a, \ )$$

$$= \sum_{k=0}^{\infty} a^{n+1} b\beta(b, \ ) - a^n \beta(b, \ ) = \beta(b, \ ).$$

Thus P is a coboundary for β for all of A. That P is M-invariant is clear if we know that β is invariant. □

In fact, the distribution P is a C^∞-function. This follows from the complete non-integrability of the sum of the stable and unstable foliations and certain subelliptic estimates [10].

We are now ready to prove Theorem 2.9 *a)* for our class of symmetric space examples. Recall that the Weyl chamber flow is the action on $M\backslash G/\Gamma$ induced from the action of A on G/Γ by left translations.

Given a C^∞-cocycle $\bar\beta$ on X, lift it to a cocycle β on G/Γ. Since $\beta(a, x)$ is M-invariant so is P. Thus P projects to a distribution $\bar{P}$ on $M\backslash G/\Gamma$ which solves the coboundary equation $\bar\beta(a, x) = \bar{P}(x) - \bar{P}(ax)$ for all $a \in A$. Furthermore, derivatives of all orders of $\bar{P}$ along stable and unstable manifolds of some regular element and along the orbit foliation are continuous functions. In particular, let Δ be the sum of the Laplacians raised to the power m of the stable, unstable and orbit foliations raised to the power m. Then Δ is an elliptic operator, and $\Delta(\bar{P})$ is a function. As m is arbitrary large, P belongs to all Sobolev spaces by standard results on elliptic operators, and thus is C^∞.

4.4. Fiber bundle extensions of standard actions

As discussed in the description of Example 2.7 and of the actions on nilmanifolds in Example 2.5, the actions of this class are obtained by successive toral extensions of products of actions of the same kind with transitive actions on tori. For simplicity, we will just describe the case of a toral extension of a symmetric space example. The arguments in general are entirely analogous and are left to the reader.

Suppose $\mathbf{R}^k$ is a split Cartan subgroup in a real semisimple Lie group G without $SO(n, 1)$ and $SU(n, 1)$ local factors. Let Γ be an irreducible cocompact lattice in G and let $\rho : \Gamma \to SL(m, \mathbf{Z})$ be a representation of Γ by automorphisms of the m-torus T^m. Then let α be the $\mathbf{R}^k$-action on $G \times_\Gamma T^m$ given by $a([(g, t)]) = [(ag, t)]$ where $[(x, y)]$ denotes the equivalence class of $(x, y) \in G \times T^m$ in $G \times_\Gamma T^m$. Then the twisted Weyl chamber flow is the $\mathbf{R}^k$-action induced on $M\backslash G \times_\Gamma T^m$ where M again is the compact part of the stabilizer of A.

We will view $M\backslash G \times_\Gamma T^m$ as a torus bundle over $M\backslash G/\Gamma$. Every cocycle β can be decomposed into the orthogonal sum of a cocycle constant along the fibers and one with 0 averages over all fibers. The first component is a cocycle on the base, and is cohomologous to a constant by Theorem 2.9 for Weyl chamber flows. Thus it suffices to consider the second component which we will treat similarly to the suspension case.

Fix a relatively compact fundamental domain $\mathscr{B}$ for Γ acting on G. For $p \in G$ and $a \in \mathbf{R}^k$ write $p^{-1}ap = b_{a,p}\gamma_{a,p}$ with $\gamma_{a,p} \in \Gamma$ and $b_{a,p} \in \mathscr{B}$. Denote the restrictions of a function F on $G \times T^m$ to the tori T_p by F^p. We have a natural trivialization of the torus bundle over $\mathscr{B}$. Using this trivialization expand F^p as a Fourier series $F^p = \Sigma_I F^p_I x^I$.

Let β be a C^∞-cocycle. Pick a regular element $a \in \mathbf{R}^k$, and for $(p, x) \in G \times T^m$ set $f(p, x) = \beta(a, [(p, x)])$ where $[(p, x)]$ denotes the projection of (p, x) to $G \times_\Gamma T^m$.

Similar to the suspension case we will calculate obstructions to the solvability of the cocycle equation through Fourier coefficients. A straightforward calculation shows that for $I \in \mathbf{Z}^m$ and $p \in \mathscr{B}$ the expression

$$O_I(p) = \sum_{-\infty}^{\infty} f^{pb_{a^k,p}}_{\rho(\gamma_{a^k,p})\, I}$$

provides an obstruction to solving the cohomology equation on the fiber T_p. Note that

$$\rho(\gamma_{a^k,p}) = \rho(b_{a^k,p})^{-1}\, \rho(p^{-1} a^k p).$$

This and the fact that $\rho(\gamma_{a^k,p})\, I \in \mathbf{Z}^m$ implies that for $I \in \mathbf{Z}^m - \{0\}$, $\rho(p)\, I$ cannot belong to the stable subspace of $\rho(a)$. Hence $\| \rho(\gamma_{a^k,p})\, I \|$ grows exponentially uniformly in $p \in \mathscr{B}$ when $k \to \infty$. In particular, the obstruction $O_I(p)$ is finite and changes continuously in p as β is C^∞.

Lemma 4.7. — *For all* I *and every* $p \in \mathscr{B}$, *we have* $O_I(p) = 0$.

Proof. — As the obstructions change continuously in p, it suffices to show that $O_I(p) = 0$ for almost every p. Let $a' \in \mathbf{R}^k$. Arguing as in the proof of Lemma 4.3 we see that for every $l \in \mathbf{Z}$

$$O_I(p) = \sum_{k=-\infty}^{\infty} f^{pb_{a^k(a')^l,p}}_{\rho(\gamma_{a^k(a')^l,p})\, I}.$$

In order to show that $O_I(p) = 0$ it suffices to see that $\sum_{k,l=-\infty}^{\infty} f^{pb_{a^k(a')^l,p}}_{\rho(\gamma_{a^k(a')^l,p})\, I}$ is finite. Now assume that $\rho(a^k(a')^l)$ is ergodic. Note that

$$\rho(\gamma_{a^k(a')^l,p}) = \rho(b_{a^k(a')^l,p})^{-1}\, \rho(p^{-1} a^k (a')^l p).$$

Hence for $I \in \mathbf{Z}^m - \{0\}$, $\rho(\gamma_{a^k(a')^l,p})\, I$ grows exponentially if $\rho(p)\, I$ does not lie in one of finitely many proper subspaces of $\mathbf{R}^m$, namely the weak stable subspaces of non-hyperbolic elements of the representation ρ restricted to the plane generated by a and a'. Rational irreducibility of ρ implies that the set of such p is a proper subvariety of $\mathscr{B}$ and hence has measure 0. □

Now we construct a coboundary $P^+(p, x) = P^-(p, x) = P(p, x)$ whose Fourier coefficients at $p \in \mathscr{B}$ are given by $\sum_0^\infty f^{pb_{a^k,p}}_{\rho(\gamma_{a^k,p})\, I}$. We obtain exponential estimates uniform in p as in the final part of the proof of Proposition 4.2. In particular, P^+ is a distribution. Then using elliptic estimates as in the Weyl chamber flow case, one sees that P is C^∞.

4.5. Livshitz Theorem and Hölder rigidity

We now prove Theorem 2.10. Since the set of ergodic elements of any volume preserving Anosov action is dense and the set of regular elements is open, there exists an ergodic and hence topologically transitive regular element $a \in \mathbf{R}^k$. Let $x \in M$ be a point whose a-orbit is dense. We define

$$P((na)\,x) = \beta(na, x)$$

for all $n \in \mathbf{Z}$. Thus the function P is defined on a dense subset of M. The heart of the proof of the Livshitz theorem is the following

Lemma **4.8.** — *The function* P *extends uniquely to a Hölder continuous function on* M.

Proof. — It is sufficient to show that for some $\delta_0 > 0$ there exist positive numbers C and τ such that for any two points y and z for which P is defined with dist $(y, z) < \delta_0$ one has

$$\| P(z) - P(y) \| \leqslant C\,\mathrm{dist}(y, z)^{\tau}.$$

Let $y = (na)\,x$, $z = (ma)\,x$ and assume that $m \geqslant n$. Then the cocycle equation implies that

$$P(z) - P(y) = \beta(ma, x) - \beta(na, x) = \beta((m-n)\,a, y).$$

Choose the number δ_0 according to the Closing Lemma (Theorem 2.4), and apply that theorem to the orbit segment $(ta)\,y_{t \in [0, m-n]}$. In particular, there is a point y' whose α-orbit is closed and a sequence of vectors $\gamma_0 = 0, y_1, \ldots, \gamma_{m-n} \in \mathbf{R}^k$ where $\gamma_k = \gamma(k)$ is such that:

1. $\mathrm{dist}((ka)\,y, \gamma_k y') < Ce^{-\lambda(\min(k, m-n-k)}\,\mathrm{dist}(y, z)$,
2. $\| \gamma_{k+1} - \gamma_k - a \| < C \| a \|\,\mathrm{dist}(y, z)$
3. $\gamma_{m-n} y' = \delta y'$ where
4. $\| \delta \| < C\,\mathrm{dist}(y, z)$.

Now one has

$$\begin{aligned}
\beta((m-n)\,a, y) &= \sum_{k=0}^{m-n-1} \beta(a, (ka)\,y) \\
&= \sum_{k=0}^{m-n-1} \beta(\gamma_{k+1} - \gamma_k, \gamma_k y') \\
&\qquad + \sum_{k=0}^{m-n-1} [\beta(a, (ka)\,y) - \beta(\gamma_{k+1} - \gamma_k, \gamma_k y')] \\
&= \beta(\gamma_{m-n}, y') + \sum_{k=0}^{m-n-1} [\beta(a, (ka)\,y) - \beta(a, \gamma_k y')] \\
&\qquad + \sum_{k=0}^{m-n-1} \beta(\gamma_{k+1} - \gamma_k - a, (\gamma_k + a)\,y').
\end{aligned}$$

We will estimate each of the three components in the last expression. First, 3. implies that $(\gamma_{m-n} - \delta)y' = y'$, and hence $\beta(\gamma_{m-n} - \delta, y') = 0$ by the assumption on β. Therefore we get

$$\beta(\gamma_{m-n}, y') = \beta(\gamma_{m-n} - \delta, y') + \beta(\delta, (\gamma_{m-n} - \delta)y') = \beta(\delta, (\gamma_{m-n} - \delta)y').$$

By the Hölder property of the cocycle β and 4. above we have

$$\| \beta(\gamma_{m-n}, y') \| = \| \beta(\delta, (\gamma_{m-n} - \delta)y') \| \leqslant K \| \delta \|^{\kappa} \leqslant C^{\kappa} K(\mathrm{dist}(y, z))^{\kappa},$$

where κ is a Hölder exponent of β and K is the corresponding constant.

The k-th term in the second summand is estimated by 1. and again by the Hölder property of β by

$$K'(Ce^{-\lambda(\min(k, m-n-k))}\, \mathrm{dist}^{(y,z)})^{\kappa},$$

where K′ depends on a. Summing these estimates for $k = 0, \ldots, m - n - 1$ we obtain an estimate for the norm of the second sum by $D(\mathrm{dist}(y, z))^{\kappa}$, where D depends on C, λ and K′.

To estimate the terms in the third sum we will show that $\| \gamma_{k+1} - \gamma_k - a \|$ decreases exponentially with respect to $\min(k, m - n - k)$.

Since α is a locally free action, we have $\mathrm{dist}(x, bx) \geqslant C_1 \| b \|$ or equivalently

$$(*) \qquad \| b \| \leqslant C_1^{-1}\, \mathrm{dist}(x, bx)$$

for all small enough $b \in \mathbf{R}^k$. We can assume by 2. that the vector $\gamma_{k+1} - \gamma_k - a$ is small enough whenever $\mathrm{dist}(y, z)$ is small. Set $w = (\gamma_k + a)y'$. We are going to estimate $\mathrm{dist}(w, (\gamma_k + a)w)$ and hence, by 2., $\| \gamma_k + a \|$ from above. We have

$$\mathrm{dist}(w, (\gamma_k + a)w) = \mathrm{dist}((\gamma_k + a)y', \gamma_{k+1}y') \leqslant \mathrm{dist}(a(\gamma_k y'), a(ka)y) + \mathrm{dist}((k+1)ay, \gamma_{k+1}y').$$

The second term is estimated from 1. directly, the first from 1. and from the fact that $\alpha(a)$ expands distances by at most a bounded amount. Combining these estimates with $(*)$ we obtain

$$\| \gamma_{k+1} - \gamma_k - a \| \leqslant C_2\, e^{-\lambda \min(k, n-k)}\, \mathrm{dist}(y, z)$$

and using again the Hölder property of β

$$\sum_{k=0}^{m-n-1} \| \beta(\gamma_{k+1} - \gamma_k - a, (\gamma_k + a)y') \| \leqslant D'(\mathrm{dist}(y, z))^{\kappa}$$

for another constant D′. This finishes the proof of the lemma. □

Thus P can be extended to a Hölder function on M. Hence $P(ax) - P(x)$ is also a Hölder function. Since $P(ax) - P(x) = \beta(a, x)$ holds on a dense set, the latter equality holds everywhere. By Lemma 4.1 this proves the assertion of the theorem for Hölder cocycles.

The claims for C^1- and C^∞-cocycles follow as in [6] and similarly to Section 4.2. One first shows that P is C^1 (respectively C^∞) along the strong stable and the strong unstable directions of a regular element as well as the α-orbits. Since these directions span the tangent bundle linearly, P is C^1 (respectively C^∞) on M. Since the stable and unstable foliations are not C^∞, this last argument uses more than the standard elliptic regularity properties, cf. [6, 15, 8, 9]. □

We are now able to complete the proof of Theorem 2.9 *a)* for standard Weyl chamber flows where the Lie algebra $\mathfrak{g}$ of G has factors isomorphic to $\mathfrak{so}(n, 1)$ or $\mathfrak{su}(n, 1)$ as well as for fiber bundle extensions of such actions.

Proof of Theorem **2.9** *a).* — Suppose that α is a Weyl chamber flow of $\mathbf{R}^k$ on $M\backslash G/\Gamma$. Let $\beta : \mathbf{R}^k \times M\backslash G/\Gamma \to \mathbf{R}$ be a C^∞-cocycle over α and set $\widetilde{\beta} = \beta \circ \pi$ where $\pi : G/\Gamma \to M\backslash G/\Gamma$ is the canonical projection. Decompose $L(G/\Gamma) = \bigoplus_i V_i$ into irreducible representations $\rho_i : G \to V_i$ of G. Let $\widetilde{\beta}_i$ denote the projection of $\widetilde{\beta}$ to V_i. Then $\widetilde{\beta}_i$ is an M-invariant cocycle. Hence $\widetilde{\beta}_i$ descends to a cocycle β_i over α. Note that the argument in Section 4.2 applies verbatim to the $\widetilde{\beta}_i$ as we have uniform exponential decay on each non-trivial irreducible component. In particular, all the obstructions from the Livshitz theorem vanish for the β_i for ρ_i non-trivial. As $\widetilde{\beta}$ is the limit in the smooth (and hence uniform) topology to the finite sums of the $\widetilde{\beta}_i$ all the obstructions from the Livshitz theorem vanish for β. By the Livshitz theorem β is cohomologous to a constant. □

Next we will show how to smooth continuous cocycles. This together with the Livshitz theorem yield the cohomology vanishing result for Hölder cocycles from that for smooth cocycles.

Proposition **4.9.** — *Let α be a locally free $\mathbf{R}^k$-action on a compact differentiable manifold* M. *Let β be a continuous cocycle over α. Then β can be arbitrarily well approximated in the C^0-topology by C^∞-cocycles.*

Proof. — Let $\mathcal{O}$ be the orbit foliation of α. First notice that β can be approximated by a cocycle which is C^∞ along the orbits. For that observe that a shift of a cocycle $\beta_b(a, x) \stackrel{\text{def}}{=} \beta(a, bx)$ for any fixed $b \in \mathbf{R}^k$ is also a cocycle since $\mathbf{R}^k$ is abelian. Thus one can choose a C^∞-density ρ on $\mathbf{R}^k$ concentrated near the origin and define a new cocycle by

$$\beta_b(a, x) \stackrel{\text{def}}{=} \int_{\mathbf{R}^k} \rho(b)\, \beta_b(a, x)\, db$$

which is C^∞ along the orbits of α.

Now let us assume that α is C^∞ along the orbits. For any $v \in \mathbf{R}^k$, set

$$\omega(v, x) \stackrel{\text{def}}{=} \frac{d}{dt}\bigg|_{t=0} \beta(tv, x).$$

This defines the infinitesimal generator for β. The cocycle equation implies that ω is a closed form on the orbit foliation. Conversely, given a continuous field ω of closed forms on the orbits we can define a continuous cocycle by

$$\beta(a, x) = \int_0^1 \omega(a, (\lambda a)\, x)\, d\lambda.$$

Thus in order to prove the proposition, we need to approximate a continuous field ω of closed 1-forms on the orbit foliation by a C^∞-field of such forms.

First cover M by coordinate charts ("flow boxes") U_i such that the orbit foliation on each chart U_i is the foliation $x_{k+1} =$ constant, ..., $x_n =$ constant in local coordinates. Let $V_i \subset U_i$ be another cover of M. By the Poincaré Lemma we can write $\omega = d_{\mathcal{O}} F_U$ inside a chart $U = U_i$ where $d_{\mathcal{O}}$ is exterior differentiation along $\mathcal{O}$ and the function F_U is determined up to an arbitrary function constant on the local leaves. Furthermore, F_U can be chosen continuous on U. We will approximate F_U by a function F_U^* such that F_U^* is C^∞ on $V = V_i$, $d_{\mathcal{O}} F_U^*$ is close to $d_{\mathcal{O}} F_U$ on U and $F_U^* = F_U$ in a neighborhood of the boundary ∂U.

First choose a C^∞-function ρ such that $0 \leqslant \rho \leqslant 1$, $\rho = 1$ on V and $\rho = 0$ in a neighborhood of ∂U.

Next we can approximate F_U arbitrarily well by a C^∞-function G_U such that $d_{\mathcal{O}} G_U$ is also C^0-close to F_U. This can be done by approximating F_U in local coordinates with an appropriate smooth kernel. Now put $F_U^* = \rho G_U + (1 - \rho) F_U$. Then

$$\begin{aligned} d_{\mathcal{O}} F_U^* &= \rho\, d_{\mathcal{O}} G_U + (1 - \rho)\, d_{\mathcal{O}} F_U + (d_{\mathcal{O}} \rho)\,(G_U - F_U) \\ &= d_{\mathcal{O}} F_U + \rho\, d_{\mathcal{O}}(G_U - F_U) + (d_{\mathcal{O}} \rho)\,(G_U - F_U). \end{aligned}$$

Since $d_{\mathcal{O}} \rho$ is a fixed function and both $G_U - F_U$ and $d_{\mathcal{O}}(G_U - F_U)$ can be made arbitrarily small, $d_{\mathcal{O}} F_U^*$ can be made arbitrarily close to $d_{\mathcal{O}} F_U$. Furthermore in V, $d_{\mathcal{O}} F_U^* = d_{\mathcal{O}} G_U$ is C^∞ and near ∂U, $d_{\mathcal{O}} F_U^* = d_{\mathcal{O}} F_U = \omega$. Now put

$$\omega^* = \begin{cases} \omega & \text{outside } U, \\ d_{\mathcal{O}} F_U^* & \text{in } U. \end{cases}$$

The 1-form ω^* is C^∞ in V and is uniformly close to ω. Apply this approximation process inductively to the different flow boxes from the given finite cover. Note that the process keeps a 1-form smooth where it is already smooth. Thus we finally obtain a 1-form smooth on all the V_i, and thus on M, which is C^0-close to ω. □

Proof of Theorem **2.9** *b).* — Let β be a Hölder cocycle over α. Then $\beta = \lim \beta_n$ is the limit of smooth cocycles β_n. By Theorem 2.9 *a)*, the β_n are cohomologous to a constant cocycle β_n^* which is just the average of β_n over M. Hence $\beta^* = \lim \beta_n^*$ exists and is a constant cocycle. As $\beta_n - \beta_n^*$ are coboundaries, all the obstructions from the Livshitz theorem vanish for $\beta_n - \beta_n^*$, and hence for $\beta - \beta^*$. By the Livshitz theorem, $\beta - \beta^*$ is a Hölder coboundary. □

Proof of Corollary **2.11.** — Given a cocycle β over the product of two Anosov actions, decompose it as $\beta = \beta_1 + \beta_2 + \gamma$ where β_1 and β_2 are constant along the first and second factor respectively, and γ is orthogonal to both factors. It suffices to show that γ is cohomologous to 0.

Suppose $x \in M_1$ and $a_1 x = x$ for some $a_1 \in \mathbf{R}^k$. Pick $y \in M_2$ such that the a_2-orbit of y is uniformly distributed in M_2. Since β is a cocycle we get

$$\gamma((a_1, 0), (x, y)) = \frac{1}{n} \sum_{k=1}^{n} \gamma((a_1, 0), (x, a_2^k y)).$$

Since γ is orthogonal to the second factor and y is uniformly distributed the last sum tends to 0 as $n \to \infty$. Since such y are dense in M_2, we have $\gamma((a_1, 0), (x, y)) = 0$ for all y.

The same argument applies to the second factor. As any closed orbit for the product action is the product of closed orbits on the factors the Livshitz' theorem applies. □

5. Time changes, invariant volumes and local Hölder rigidity

Proof of Theorem **2.12.** — For part *a)*, suppose that α is a standard $\mathbf{R}^k$-action on a closed manifold M with $k \geqslant 2$. Let another action α^* of $\mathbf{R}^k$ be a C^∞-time change of α. For $a \in \mathbf{R}^k$ and $m \in M$ set $am = \alpha(a)(m)$ and $a^* m = \alpha^*(a)(m)$ respectively. Note that α^* has no isotropy at x if α has none. Then there is a unique continuous cocycle $\beta : \mathbf{R}^k \times M \to \mathbf{R}^k$ that satisfies the equation

$$ax = \beta(a, x)^* x.$$

Note that $\beta : \mathbf{R}^k \times M \to \mathbf{R}^k$ is C^∞.

By Theorem 2.9 *a)*, β is C^∞-cohomologous to a homomorphism $\rho : \mathbf{R}^k \to \mathbf{R}^k$ by a C^∞-coboundary $P : M \to \mathbf{R}^k$. Let us show next that ρ is an embedding. Pick $x \in M$ such that the isotropy of x w.r.t. α is trivial. Since P has compact range, we see that the image of ρ cannot be contained in a hyperplane of dimension less than k. As ρ is linear, $\rho : \mathbf{R}^k \to \mathbf{R}^k$ is an isomorphism.

Now set $\psi(x) = P(x)^* x$. Then $\psi(ax) = \rho(a)^* \psi(x)$. Therefore ψ is surjective. Suppose $\psi(x) = \psi(y)$. Then $y = bx$ for some $b \in \mathbf{R}^k$. Hence $\rho(b)^* \psi(x) = \psi(x)$. As ψ is homotopic to the identity and the identity is an orbit equivalence, ρ induces an isomorphism between the isotropy groups of x and $\psi(x)$. Hence $y = bx = x$, and ψ

is injective. The equivariance property implies that ψ is a diffeomorphism, establishing Theorem 2.12 *a)*.

The proof of part *b)* is essentially the same, replacing C^∞ by Hölder everywhere. □

Proof of Theorem **2.13.** — Consider a standard Anosov action α of $\mathbf{R}^k$ on a manifold M and a perturbation α^* sufficiently close to α in the C^1-topology. By Theorem 2.3 there is a Hölder homeomorphism $\varphi : M \to M$ that sends orbits to orbits. The pullback of the perturbed action under φ determines a Hölder time change. Theorem 2.12 *b)* allows to straighten φ into an isomorphism ψ (up to an automorphism of $\mathbf{R}^k$). Note that the resulting conjugacy is automatically smooth along the $\mathbf{R}^k$-orbits. □

Proof of Theorem **2.14.** — Without loss of generality we can assume by Theorem 2.13 that α^* and α are conjugate by a Hölder homeomorphism ψ (without an automorphism). To find an invariant volume for the perturbed action α^*, consider the Jacobian of the original volume ω for α^*. The logarithm of the Jacobian is a smooth cocycle over α^*. Hence the pullback of the logarithm of the Jacobian is a Hölder cocycle of the original action. By Theorem 2.9 *b)* this cocycle is Hölder cohomologous to a constant cocycle. Hence the original cocycle over α^* is Hölder cohomologous to a constant. Thus the obstructions in Theorem 2.10 vanish. As the original cocycle is C^∞, the coboundary is also C^∞ by Theorem 2.10. This implies that there is a positive C^∞-function $\rho : M \to \mathbf{R}$ and a constant C such that the Jacobian multiplies the volume $\rho\omega$ by C. Since the total volume is preserved we see that $C = 1$. □

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