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VOLUMES OF S-ARITHMETIC QUOTIENTS OF SEMI-SIMPLE GROUPS

by Gopal PRASAD*

With an appendix by Moshe Jarden and Gopal Prasad

Dedicated to the memory of Harish-Chandra.

Introduction

The purpose of this paper is twofold: The first is to give a computable formula for the volumes of the S-arithmetic quotients of $G_S := \prod_{v \in S} G(k_v)$, in terms of a natural Haar measure on G_S, where G is an arbitrary absolutely quasi-simple, simply connected algebraic group defined over a global field k (i.e. a number field or the function field of a curve over a finite field) and S is a finite set of places of k containing all the archimedean ones; see § 3. The second is to use the results involved in the volume computation to provide a " good " lower (and also upper) bound for the class number of G; this is done in § 4 of the paper.

Besides the results of C. L. Siegel for certain special classical groups, the only *general* results about the volumes of S-arithmetic quotients which were known until now were concerned with Chevalley groups (i.e. groups which split over k); see Harder [12]. There is quite a bit of literature on the class number problem for classical groups. We would like to mention here the work of C. L. Siegel, T. Tamagawa, M. Kneser and his school. The bounds for general absolutely quasi-simple, simply connected groups given in § 4 include the bounds for the special classical groups obtained by earlier authors.

In this work we have made use of a considerable amount of Bruhat-Tits theory of reductive groups over local fields. This theory is needed here even in the case S consists only of archimedean places i.e. when G_S is a connected real semi-simple Lie group.

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In [4], the volume formula of Theorem 3.7 and Theorem 4.3 have been used to prove the following finiteness assertions:

(1) *Given a positive real number c, there are only finitely many triples* $(k, \mathrm{G}, \mathrm{S})$ *consisting of a number field k, an absolutely quasi-simple group* G *defined over k and of absolute rank* > 1, *and a finite set* S *of places of k containing all the archimedean places, such that* $\mathrm{G}_\mathrm{S}(=\prod_{v\in \mathrm{S}} \mathrm{G}(k_v))$ *contains an* S*-arithmetic subgroup of covolume* $< c$.

(2) *Given a positive integer n, there are only finitely many pairs* (k, G) *consisting of a number field k and an absolutely quasi-simple, simply connected algebraic group* G *defined over k such that* $\mathrm{G}_\infty := \prod_{v\in \mathrm{V}_\infty} \mathrm{G}(k_v)$ *is compact and the class number of* G *(with respect to some coherent collection of parahoric subgroups* P_v *of* $\mathrm{G}(k_v)$*) is* $\leqslant n$; *here* V_∞ *is the set of all archimedean places of k.*

The first of the above results answers a question of Jacques Tits in the affirmative.

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0. Notation, conventions and preliminaries

In this section, we fix a number of notation and conventions to be used later, often without further reference.

0.0. As usual, **Q**, **R** and **C** will denote respectively, the fields of rational, real and complex numbers. **Z** will denote the ring of rational integers.

For a finite set S, $\#\mathrm{S}$ will denote its cardinality.

For a linear algebraic group H, $\mathrm{R}_u(\mathrm{H})$ will denote its unipotent radical, i.e. its maximal connected normal unipotent subgroup.

0.1. In the sequel, k is a global field and A the k-algebra of adèles of k endowed with the usual locally compact topology. Let V be the set of places of k, and V_∞ (resp. V_f) the subset of archimedean (resp. nonarchimedean) places. For $v \in \mathrm{V}$, k_v denotes the completion of k at v and $|\ |_v$ the normalized absolute value on k_v. The absolute value $|\ |_v$ has a unique extension to any algebraic extension of k_v, to be denoted in the same way. For $v \in \mathrm{V}_f$, $\mathfrak{o}_v$ denotes the ring of integers of k_v, $v(x)$ the additive valuation of $x \in k_v^\times$, $\mathfrak{f}_v$ the (finite) residue field and q_v the order of $\mathfrak{f}_v$. We recall that for $x \in k_v^\times$,

$$| x |_v = [\mathfrak{o}_v : x\mathfrak{o}_v]^{-1} = q_v^{-v(x)} \quad \text{if } x \in \mathfrak{o}_v,$$
$$| x |_v = [x\mathfrak{o}_v : \mathfrak{o}_v] = q_v^{-v(x)} \quad \text{if } x \notin \mathfrak{o}_v.$$

For $v \in \mathrm{V}_\infty$, $|x|_v = |x|$ if v is real, i.e. $k_v = \mathbf{R}$, and $|x|_v = |x|^2$ if v is complex, i.e. $k_v = \mathbf{C}$.

We have the *product formula*: For all $x \in k^\times$, $\prod_{v\in \mathrm{V}} |x|_v = 1$.

For $v \in V$, k_v is assumed to carry the Haar measure with respect to which the measure of $\mathfrak{o}_v$ is 1 if v is nonarchimedean, the measure of the unit interval $[0, 1]$ is 1 if v is real, and the measure of any square in $k_v (\cong \mathbf{C})$, with sides of length 1, is 2 if v is complex.

0.2. We shall denote by $\mathscr{G}$ an absolutely quasi-simple, simply connected algebraic group defined and quasi-split over k. Let $n = \dim \mathscr{G}$ and r be the absolute rank of $\mathscr{G}$.

If $\mathscr{G}/k$ is not a triality form of type ${}^6\mathsf{D}_4$, let ℓ be the smallest extension of k over which $\mathscr{G}$ splits; then $[\ell : k] \leqq 3$. If $\mathscr{G}/k$ is a triality form of type ${}^6\mathsf{D}_4$, let ℓ be a fixed extension of k of degree 3 contained in the Galois extension of k, of degree 6, over which $\mathscr{G}$ splits; there are three such extensions, all isomorphic to each other over k.

If k is a number field, let D_k (resp. D_ℓ) be the absolute value of the discriminant of $k/\mathbf{Q}$ (resp. $\ell/\mathbf{Q}$). Let $\mathfrak{d}(\ell/k)$ denote the *relative discriminant* of ℓ over k; it is an ideal in the ring of integers of k. It is well known that $| N_{k/\mathbf{Q}}(\mathfrak{d}(\ell/k)) | \cdot D_k^{[\ell:k]} = D_\ell$.

If k is the function field of a curve over a finite field, let q_k (resp. q_ℓ) be the cardinality of the finite field of the constant functions in k (resp. ℓ) and g_k (resp. g_ℓ) be the genus of k (resp. ℓ). Let $D_k = q_k^{2g_k - 2}$, $D_\ell = q_\ell^{2g_\ell - 2}$.

0.3. Let v be a nonarchimedean place of k such that $\ell_v := \ell \otimes_k k_v$ is a *ramified* field extension of k_v of degree 2. Let

$$\nu_v = \inf\{ |y|_v \mid y \in \ell_v, y + \bar{y} + 1 = 0 \},$$

where here, as well as in the sequel, for $y \in \ell_v$, $\bar{y}$ denotes its conjugate over k_v. Then $\nu_v \geqslant 1$ and $\nu_v = 1$ if and only if the characteristic of the residue field of k_v is odd. For later use, we fix a $\lambda_v \in \ell_v$, and a uniformizing element π_v of ℓ_v such that $\lambda_v + \bar{\lambda}_v + 1 = 0$, $|\lambda_v|_v = \nu_v$ and $\lambda_v \pi_v + \bar{\lambda}_v \bar{\pi}_v = 0$ (cf. Tits [33: 1.15]). Then

$$| \mathfrak{d}(\ell_v/k_v) |_v = | (\pi_v - \bar{\pi}_v)^2 |_v = | \pi_v^2 (1 + \lambda_v \bar{\lambda}_v^{-1})^2 |_v = q_v^{-1} \nu_v^{-2},$$

where $\mathfrak{d}(\ell_v/k_v)$ is the relative discriminant of ℓ_v/k_v.

0.4. *The integer* $\mathfrak{s}(\mathscr{G})$. If $\mathscr{G}$ splits over k, let $\mathfrak{s}(\mathscr{G}) = 0$. Now assume $\mathscr{G}$ *does not* split over k. On the relative root system ${}_k\Phi$ of $\mathscr{G}$, with respect to a maximal k-split torus $\mathscr{T}$, consider the ordering associated with a Borel k-subgroup containing $\mathscr{T}$. The integer $\mathfrak{s}(\mathscr{G})$ is then defined as follows. If ${}_k\Phi$ is reduced (which is the case if, and only if, $\mathscr{G}$ is *not* a k-form of type ${}^2\mathsf{A}_r$ with r even), then $\mathfrak{s}(\mathscr{G})$ is equal to the sum of the number of short roots and of short simple roots. If $\mathscr{G}$ is a k-form of type ${}^2\mathsf{A}_r$ with r even, then ${}_k\Phi$ is the non-reduced root system $\mathsf{BC}_{r/2}$ and $\mathfrak{s}(\mathscr{G}) = \frac{1}{2} r(r + 3)$, which is equal to the number of *all* roots in ${}_k\Phi$ plus the number of simple roots.

Note that if $\mathscr{G}$ is a k-form of type ${}^2\mathsf{A}_r$ (r odd), ${}^2\mathsf{D}_r$ (r arbitrary) or ${}^2\mathsf{E}_6$, then the root system ${}_k\Phi$ is the reduced root system of type $\mathsf{C}_{(r+1)/2}$, B_{r-1}, F_4 respectively and

$\mathfrak{s}(\mathscr{G})$ is $\frac{1}{2}(r-1)(r+2)$, $2r-1$, 26 respectively. If $\mathscr{G}$ is a triality form of type ${}^3\mathsf{D}_4$ or ${}^6\mathsf{D}_4$, then ${}_k\Phi$ is of type G_2 and $\mathfrak{s}(\mathscr{G}) = 7$.

0.5. In this paper, we assume familiarity with the Bruhat-Tits theory [6] and recall just some notation and facts. All we need is stated in [33], and in most cases, proofs can be found in one of these references.

Let K be a nonarchimedean local field. In the sequel, K will always be a finite extension of k_v, for a nonarchimedean place v. Let G be an absolutely quasi-simple, simply connected group defined over K. Let $\mathscr{B} = \mathscr{B}(G/K)$ be the associated Bruhat-Tits building. It is a contractible simplicial complex on which G(K) acts by simplicial automorphisms which are *special* (in particular, if $g \in G(K)$ leaves a simplex of $\mathscr{B}$ stable, then it fixes the simplex pointwise).

We recall that an *Iwahori subgroup* of G(K) can be defined as either the normalizer of a maximal pro-p subgroup of G(K), where p is the characteristic of the residue field of K, or as the subgroup of G(K) fixing a chamber (i.e. a maximal simplex) in $\mathscr{B}$. All Iwahori subgroups are conjugate in G(K). A *parahoric subgroup* P of G(K) is the stabilizer of a simplex of $\mathscr{B}$. Every parahoric subgroup is compact, open and contains an Iwahori subgroup. The maximal ones are the maximal compact subgroups of G(K) and are the stabilizers of the vertices of $\mathscr{B}$. A (maximal) parahoric subgroup P is *special* if it fixes a *special vertex* of $\mathscr{B}$. A vertex x of $\mathscr{B}$ is special if the *affine* Weyl group W is a semidirect product of the translation subgroup by the isotropy group W_x of x in W. If so, then W_x is canonically isomorphic to the Weyl group of the K-root system of G.

0.6. Let $\hat{K}$ be the maximal unramified extension of K and $\hat{\mathfrak{o}}$ be its ring of integers. Let $\hat{\mathscr{B}}$ be the building of $G(\hat{K})$ and $\hat{A} \subset \hat{\mathscr{B}}$ be the apartment of a maximal $\hat{K}$-split torus of G which is defined over K and which contains a maximal K-split torus. There is an action of the Galois group of $\hat{K}/K$ on $\hat{\mathscr{B}}$ and $\hat{A}$ is stable under this action; the fixed set in $\hat{\mathscr{B}}$ may be identified with $\mathscr{B}$ and the fixed set in $\hat{A}$ with an apartment A of $\mathscr{B}$ [33: 1.10]. A vertex of $\hat{\mathscr{B}}$ lying in A which is special for $\hat{\mathscr{B}}$ is also special for $\mathscr{B}$ [33: 1.10.2]. If G splits over $\hat{K}$, such a point, viewed as a vertex of $\mathscr{B}$ is called *hyperspecial* and its isotropy group in G(K) is a *hyperspecial* parahoric subgroup. If G is quasi-split over K and splits over an unramified extension of K, hyperspecial parahoric subgroups exist [33: 1.10.2]; these groups are the parahoric subgroups of G(K) of maximal volume [33: 3.8.2].

0.7. To any parahoric subgroup P of G(K), the Bruhat-Tits theory associates a smooth affine group scheme defined over the ring $\mathfrak{o}$ of integers of K, whose generic fiber is isomorphic to G/K and whose group of $\mathfrak{o}$-rational points is isomorphic to P (see [6: II] or [33: 3.4]). The coordinate ring of this group scheme is the $\mathfrak{o}$-algebra of those K-regular functions on G which on $\hat{P}$ take values in $\hat{\mathfrak{o}}$, where $\hat{P}$ is the parahoric subgroup of $G(\hat{K})$ associated with P.

1. Tamagawa forms on quasi-split groups

1.1. We fix a non-zero left-invariant exterior from ω on $\mathscr{G}$ of maximal degree and which is defined over k; such a form is unique up to multiplication by an element of $k^\times$ and is called a *Tamagawa form* on $\mathscr{G}/k$. As $\mathscr{G}$ is a semi-simple group, ω is bi-invariant.

1.2. For each $v \in V_f$ we fix, once and for all, a maximal parahoric subgroup $\mathscr{P}_v$ of $\mathscr{G}(k_v)$ with the following properties.

(i) If $\mathscr{G}$ splits over an unramified extension of k_v, then $\mathscr{P}_v$ is a *hyperspecial* parahoric subgroup.

(ii) If $\mathscr{G}$ does not split over any unramified extension of k_v (then $\mathscr{G} \times_k k_v$ is a residually split group over k_v), $\mathscr{P}_v$ is *special*. In case $\mathscr{G}$ is an outer form of type A_r, with r even, we assume moreover that the gradient (i.e. the vector part) of the affine simple root corresponding to this special parahoric subgroup is a *divisible* root.

(iii) $\Pi_{v \in V_\infty} \mathscr{G}(k_v) . \Pi_{v \in V_f} \mathscr{P}_v$ is an open subgroup of the adèle group $\mathscr{G}(\mathrm{A})$.

1.3. Let $\mathscr{G}_v$ be the smooth affine $\mathfrak{o}_v$-group scheme associated with the parahoric subgroup $\mathscr{P}_v$, whose generic fiber $(= \mathscr{G}_v \times_{\mathfrak{o}_v} k_v)$ is isomorphic to $\mathscr{G} \times_k k_v$ and whose group of $\mathfrak{o}_v$-rational points is isomorphic to $\mathscr{P}_v$ (see 0.7).

Let $c_v \in k_v^\times$ be such that $c_v\,\omega$ induces an invariant exterior form on the $\mathfrak{o}_v$-group scheme $\mathscr{G}_v$, of maximal degree, which is defined over $\mathfrak{o}_v$ and whose reduction to the group $\mathscr{G}_v \times_{\mathfrak{o}_v} \mathfrak{f}_v$ over the residue field $\mathfrak{f}_v$ is not zero. It is obvious that such a c_v exists and is unique up to multiplication by a unit. In particular, $\gamma_v := |\, c_v\,|_v$ is a well-defined positive real number; it is equal to 1 for all but finitely many v's.

1.4. If k is a number field, for an archimedean place v of k, let c_v be the positive real number such that with respect to the Haar measure determined by the form $c_v\,\omega$, the volume of any maximal compact subgroup of $\mathrm{R}_{k_v/\mathbf{R}}(\mathscr{G})\,(\mathbf{C})$ is 1, and let $\gamma_v = |\, c_v\,|_v$. We recall here that if v is real, then any maximal compact subgroup of $\mathrm{R}_{k_v/\mathbf{R}}(\mathscr{G})\,(\mathbf{C})$ is isomorphic to the unique (up to isomorphism) compact, simple, simply connected real-analytic Lie group of the same type as $\mathscr{G}$ and if v is complex, then any maximal compact subgroup of $\mathrm{R}_{k_v/\mathbf{R}}(\mathscr{G})\,(\mathbf{C})$ is the direct product of two copies of this group.

1.5. Let r be the absolute rank of $\mathscr{G}$ and let $m_1, \ldots, m_r$ $(m_1 \leqq \ldots \leqq m_r)$ be the *exponents* of the simple, simply connected, compact real-analytic Lie group of the same type as $\mathscr{G}$. Note that $\dim \mathscr{G} = r + 2\Sigma_{i=1}^r m_i$.

We list below the exponents (see Bourbaki [5]).

Type	Exponents
A_r	$1, 2, \ldots, r$.
B_r	$1, 3, 5, \ldots, 2r-1$.
C_r	$1, 3, 5, \ldots, 2r-1$.
D_r	$1, 3, 5, \ldots, 2r-5, 2r-3, r-1$ ($r-1$ has multiplicity 2 when r is even).
E_6	1, 4, 5, 7, 8, 11.
E_7	1, 5, 7, 9, 11, 13, 17.
E_8	1, 7, 11, 13, 17, 19, 23, 29.
F_4	1, 5, 7, 11.
G_2	1, 5.

1.6. *Theorem. — We have*

$$\prod_{v \in V} \gamma_v = (D_\ell/D_k^{[\ell:k]})^{\frac{1}{2}s(\mathscr{G})} \prod_{v \in V_\infty} \left| \prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v .$$

Proof. — Let $\mathfrak{L}$ be any (not necessarily finite) Galois extension of k containing ℓ, where ℓ is as in 0.2. Then $\mathscr{G}$ splits over $\mathfrak{L}$. Let $L(\mathscr{G})$ be the Lie algebra of left-invariant vector fields on $\mathscr{G}/k$, and $\mathfrak{g} = L(\mathscr{G}) \otimes_k \mathfrak{L}$. Let $\mathscr{C}$ be a maximal k-split torus of $\mathscr{G}$ and $\mathscr{T}$ be its centralizer. Then $\mathscr{T}$ is defined over k and it is a torus since $\mathscr{G}$ is quasi-split over k. Moreover, it splits over $\mathfrak{L}$ since $\mathscr{G}$ does. Let Φ be the root system of $\mathscr{G}$ with respect to $\mathscr{T}$, and $\Pi(\subset \Phi)$ be the set of simple roots with respect to the ordering on Φ obtained by fixing a Borel k-subgroup containing $\mathscr{T}$. Let $\{ H_a \mid a \in \Pi \} \cup \{ X_b \mid b \in \Phi \}$ be a Chevalley basis of $\mathfrak{g}$, where the H_a's constitute a basis of the Lie algebra $L(\mathscr{T}) \otimes_k \mathfrak{L}$ of $\mathscr{T}/\mathfrak{L}$ and for each $b \in \Phi$, X_b is an element of the root space $\mathfrak{g}_b$. We fix an enumeration of this Chevalley basis, and for $1 \leq i \leq n$ $(= \dim \mathscr{G})$, let $\mathfrak{X}_i$ be its i-th element. Let $\mathfrak{X}^i$ be the dual basis of the dual $\mathfrak{g}^*$ and let $\omega^{\mathrm{Ch}} = \mathfrak{X}^1 \wedge \ldots \wedge \mathfrak{X}^n$; ω^{Ch} is a $\mathscr{G}$-invariant exterior form on $\mathscr{G}$ of maximal degree. The form ω^{Ch} is defined over $\mathfrak{L}$ and any other choice of Chevalley basis or its enumeration gives only ω^{Ch} or $-\omega^{\mathrm{Ch}}$.

Since the space of $\mathscr{G}$-invariant exterior forms on $\mathscr{G}$ of maximal degree is 1-dimensional, there is an $\alpha \in \mathfrak{L}^\times$ such that $\omega = \alpha^{-1}\omega^{\mathrm{Ch}}$. As ω is defined over k, for every $\gamma \in \mathrm{Gal}(\mathfrak{L}/k)$, $\gamma(\omega) = \omega$. Now since $\gamma(\omega^{\mathrm{Ch}}) = \pm\,\omega^{\mathrm{Ch}}$, we conclude that $\gamma(\alpha)^2 = \alpha^2$ for all $\gamma \in \mathrm{Gal}(\mathfrak{L}/k)$ and hence $\alpha^2 \in k^\times$.

If k is a number field, $\det(\langle \mathfrak{X}_i, \mathfrak{X}_j \rangle)$, where $\langle \mathfrak{X}_i, \mathfrak{X}_j \rangle = \mathrm{Tr}(\mathrm{ad}\,\mathfrak{X}_i\,\mathrm{ad}\,\mathfrak{X}_j)$ is the inner product of $\mathfrak{X}_i$ with $\mathfrak{X}_j$ with respect to the Killing form on $\mathfrak{g}$, is an integer. Let m be its absolute value. Then m is uniquely determined by the absolute root system of $\mathscr{G}$; it does not depend on the choice of the Chevalley basis of $\mathfrak{g}$.

We fix a k-basis $X_1, \ldots, X_n$ of the Lie algebra $L(\mathscr{G})$ so that if $X^1, \ldots, X^n$ is the dual basis, $\omega = X^1 \wedge \ldots \wedge X^n$. If k is a number field, for every archimedean place v

of k, we fix a basis $Y_1^v, \ldots, Y_n^v$ of $L(\mathscr{G}) \otimes_k k_v$ such that with respect to the Killing form $\langle\ ,\ \rangle_v$ on $L(\mathscr{G}) \otimes_k k_v$, Y_i^v is orthogonal to Y_j^v for all $1 \leqq i \neq j \leqq n$, and moreover, if v is real, then $|\langle Y_i^v, Y_i^v \rangle_v|_v = 1$, whereas, if v is complex, then $\langle Y_i^v, Y_i^v \rangle_v = 1$ for all $i \leqq n$. Now let $Y_v^1, \ldots, Y_v^n$ be the dual basis and $\omega_v^K = Y_v^1 \wedge \ldots \wedge Y_v^n$. Then ω_v^K is an invariant exterior form on $\mathscr{G} \times_k k_v$, of maximal degree, defined over k_v; it determines a Haar measure on $\mathscr{G}(k_v)$ as well as on every maximal compact subgroup of $R_{k_v/\mathbf{R}}(\mathscr{G})(\mathbf{C})$. The volume of each of the latter subgroups is equal to

$$m^{\frac{1}{2}} \prod_{i=1}^{r} \frac{(2\pi)^{m_i+1}}{m_i!} \qquad \text{if } v \text{ is real,}$$

and

$$\left(m^{\frac{1}{2}} \prod_{i=1}^{r} \frac{(2\pi)^{m_i+1}}{m_i!}\right)^2 = \left| m^{\frac{1}{2}} \prod_{i=1}^{r} \frac{(2\pi)^{m_i+1}}{m_i!} \right|_v \qquad \text{if } v \text{ is complex;}$$

see, for example, [25: § 3] or [20].

Let $\mathfrak{d} = \det(\langle X_i, X_j \rangle)$, where $\langle X_i, X_j \rangle = \mathrm{Tr}(\mathrm{ad}\, X_i\, \mathrm{ad}\, X_j)$ is the inner product of X_i with X_j under the Killing form on $L(\mathscr{G})$. Then it is obvious that if v is a complex place, $\omega_v^K \otimes \omega_v^K$ equals $\mathfrak{d}\omega \otimes \omega$, and if v is real, then $\omega_v^K \otimes \omega_v^K$ equals either $\mathfrak{d}\omega \otimes \omega$ or $-\mathfrak{d}\omega \otimes \omega$. Now as the volume of any maximal compact subgroup of $R_{k_v/\mathbf{R}}(\mathscr{G})(\mathbf{C})$ with respect to the Haar measure determined by ω_v^K is

$$\left| m^{\frac{1}{2}} \prod_{i=1}^{r} \frac{(2\pi)^{m_i+1}}{m_i!} \right|_v,$$

we conclude that, for all archimedean v,

$$\gamma_v = |\mathfrak{d}|_v^{\frac{1}{2}} \left| m^{-\frac{1}{2}} \prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v.$$

But since $\omega^{\mathrm{Ch}} = \alpha\omega$, we find that $\alpha^4 m^2 = \mathfrak{d}^2$, which implies that $|\mathfrak{d}m^{-1}|_v = |\alpha^2|_v$, and hence for all archimedean v,

$$\gamma_v = |\alpha^2|_v^{\frac{1}{2}} \left| \prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v.$$

Therefore,

$$\begin{aligned}
\prod_{v \in V} \gamma_v^2 &= \prod_{v \in V_f} \gamma_v^2 \cdot \prod_{v \in V_\infty} \gamma_v^2 \\
&= \prod_{v \in V_f} \gamma_v^2 \cdot \prod_{v \in V_\infty} \left(|\alpha^2|_v \left| \prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v^2 \right) \\
&= \prod_{v \in V_f} \gamma_v^2 \cdot \prod_{v \in V_f} |\alpha^2|_v^{-1} \cdot \prod_{v \in V_\infty} \left| \prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v^2
\end{aligned}$$

(by the product formula (0.1); recall that $\alpha^2 \in k^\times$)

$$= \prod_{v \in V_f} (|\alpha^2|_v^{-1} \gamma_v^2) \cdot \prod_{v \in V_\infty} \left| \prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v^2.$$

Next we shall prove that

$$\prod_{v \in V_f} (|\,\alpha^2\,|_v^{-1}\,\gamma_v^2) = (D_\ell/D_k^{[\ell:k]})^{\mathfrak{s}(\mathscr{G})}.$$

This will establish the theorem.

Let $\mathscr{G}_v$ be the smooth affine $\mathfrak{o}_v$-group scheme associated with the parahoric subgroup $\mathscr{P}_v$ of $\mathscr{G}(k_v)$ (see 1.3). Let $L(\mathscr{G}_v)$ be the Lie algebra of $\mathscr{G}_v$; it is an $\mathfrak{o}_v$-Lie algebra. Since the generic fiber $\mathscr{G}_v \times_{\mathfrak{o}_v} k_v$ of $\mathscr{G}_v$ is $\mathscr{G} \times_k k_v$, it follows that

$$L(\mathscr{G}_v) \otimes_{\mathfrak{o}_v} k_v \cong L(\mathscr{G}) \otimes_k k_v.$$

We use this isomorphism to identify $L(\mathscr{G}_v)$ with an $\mathfrak{o}_v$-subalgebra of the k_v-Lie algebra $\mathfrak{g}_v := L(\mathscr{G}) \otimes_k k_v$. The k_v-span of $L(\mathscr{G}_v)$ is clearly all of $\mathfrak{g}_v$. Let $\{ Y_i^v \}$ be an $\mathfrak{o}_v$-basis of $L(\mathscr{G}_v)$ ($\subset \mathfrak{g}_v$), and let $a_v \in k_v^\times$ be such that $X_1 \wedge \ldots \wedge X_n = a_v\, Y_1^v \wedge \ldots \wedge Y_n^v$. Then it is obvious that $a_v\,\omega$ induces an invariant exterior form on the $\mathfrak{o}_v$-group scheme $\mathscr{G}_v$, which is defined over $\mathfrak{o}_v$ and whose reduction to the group $\mathscr{G}_v \times_{\mathfrak{o}_v} \mathfrak{f}_v$ is not zero. Hence, $|\,a_v\,|_v = \gamma_v$ (see 1.3).

Now let v be a nonarchimedean place of k such that $\mathscr{G}$ splits over the maximal unramified extension $\hat{k}_v$ of k_v. (Then $\ell \otimes_k k_v$ is a direct sum of certain *unramified* extensions of k_v; we note here, for future use, that for any unramified extension K of k_v, $|\,\mathfrak{d}(K/k_v)\,|_v = 1$.) Let $\hat{\mathfrak{o}}_v$ be the ring of integers of $\hat{k}_v$. Then it is clear that if $\{ Z_i^v \}$ is any $\hat{\mathfrak{o}}_v$-basis of $L(\mathscr{G}_v) \otimes_{\mathfrak{o}_v} \hat{\mathfrak{o}}_v$ and $b_v (\in \hat{k}_v^\times)$ is such that $X_1 \wedge \ldots \wedge X_n = b_v\, Z_1^v \wedge \ldots \wedge Z_n^v$, then $b_v\, a_v^{-1}$ is a unit and hence, $|\,b_v\,|_v = |\,a_v\,|_v = \gamma_v$. We observe now that since $\mathscr{P}_v$ is a hyperspecial parahoric subgroup of $\mathscr{G}(k_v)$, and $\mathscr{G}_v$ is the associated $\mathfrak{o}_v$-group scheme, $\mathscr{G}_v(\hat{\mathfrak{o}}_v)$ is a hyperspecial parahoric subgroup of $\mathscr{G}(\hat{k}_v)$ ([33: 2.6.1 and 3.4.1]). But as $\mathscr{G}$ splits over $\hat{k}_v$, this implies that there is an $\hat{\mathfrak{o}}_v$-basis $Z_1^v, \ldots, Z_n^v$ of $L(\mathscr{G}_v) \otimes_{\mathfrak{o}_v} \hat{\mathfrak{o}}_v$ which is a Chevalley basis of the split Lie algebra $\mathfrak{g}_v \otimes_{k_v} \hat{k}_v$ ([33: 3.4.2 and 3.4.3]). Now as $\omega^{\mathrm{Ch}} = \alpha\omega$, and, up to sign, ω^{Ch} is independent of the choice of the Chevalley basis and its enumeration, we conclude from this that $|\,\alpha^2\,|_v^{-1}\,\gamma_v^2 = 1$ *for every nonarchimedean place v such that $\mathscr{G}$ splits over the maximal unramified extension $\hat{k}_v$ of k_v.*

Let now $\mathscr{R}$ be the set of all nonarchimedean places v of k such that $\mathscr{G}$ does not split over any unramified extension of k_v, or equivalently, $\ell \otimes_k k_v$ contains a nontrivial ramified field extension of k_v. Then $\mathscr{R}$ is finite. Let $v \in \mathscr{R}$; then there are two possibilities:

1. $\ell \otimes_k k_v$ is a field, we shall denote it by ℓ_v, it is a ramified extension and $[\ell_v : k_v] = [\ell : k]$.
2. $\ell \otimes_k k_v$ is a direct sum of k_v and a ramified field extension ℓ_v of k_v of degree 2. This is the case if $\mathscr{G}/k$ is a form of type ${}^6\mathsf{D}_4$ of k-rank 2 and $\mathscr{G}/k_v$ is a form of type ${}^2\mathsf{D}_4$ of k_v-rank 3. In this case, the k-root system of $\mathscr{G}$ is of type G_2 and $\mathfrak{s}(\mathscr{G}) = 6 + 1 = 7$; its k_v-root system is of type B_3 which has 6 short roots and one short simple root.

To compute $|\,\alpha^2\,|_v^{-1}\,\gamma_v^2$, we shall construct a suitable $\mathfrak{o}_v$-basis of the Lie algebra $L(\mathscr{G}_v)$. For this purpose, we fix a maximal k_v-split torus $\mathscr{C}_v$ such that in the Bruhat-Tits building of $\mathscr{G}/k_v$ the vertex fixed by $\mathscr{P}_v$ lies on the apartment determined by $\mathscr{S}_v$. Let $\mathscr{T}_v$ be the centralizer of $\mathscr{S}_v$ in $\mathscr{G}$. Then as $\mathscr{G}$ is quasi-split over k (and so also over k_v), $\mathscr{T}_v$ is a torus, and it is clearly defined over k_v. Let $\Phi(\mathscr{T}_v)$ (resp. $\Phi(\mathscr{S}_v)$) be the root system of $\mathscr{G}$ with respect to $\mathscr{T}_v$ (resp. $\mathscr{S}_v$). We fix a Borel subgroup of $\mathscr{G}$ which contains $\mathscr{T}_v$ and is defined over k_v. This gives compatible orderings on $\Phi(\mathscr{S}_v)$ and $\Phi(\mathscr{T}_v)$. Let $\Phi(\mathscr{T}_v)^+$ (resp. $\Phi(\mathscr{S}_v)^+$) be the set of roots in $\Phi(\mathscr{T}_v)$ (resp. $\Phi(\mathscr{S}_v)$) positive with respect to this ordering and let $\Pi(\mathscr{T}_v)$ (resp. $\Pi(\mathscr{S}_v)$) be the set of simple roots.

We fix a minimal Galois extension L_v of k_v containing ℓ_v and denote by Γ the Galois group of L_v/k_v. Then $\mathscr{G}$, and so also the torus $\mathscr{T}_v$, splits over L_v. This implies that Γ operates on the character group $X^*(\mathscr{T}_v)$ of $\mathscr{T}_v$; under this action of Γ, $\Phi(\mathscr{T}_v)$, $\Phi(\mathscr{T}_v)^+$ and $\Pi(\mathscr{T}_v)$ are stable.

The restriction of roots in $\Phi(\mathscr{T}_v)$ to $\mathscr{S}_v$ gives a bijective correspondence between the set of Γ-orbits in $\Phi(\mathscr{T}_v)$ and the set $\Phi(\mathscr{S}_v)$; under this correspondence, the orbits in $\Pi(\mathscr{T}_v)$ correspond to the roots in $\Pi(\mathscr{S}_v)$, [32: 2.5]. Also, it is easy to see that the restriction to $\mathscr{S}_v$ of a root $\mathfrak{b}$ in $\Phi(\mathscr{T}_v)$ is a long root of the root system $\Phi(\mathscr{S}_v)$ if and only if $\mathfrak{b}$ is Γ-invariant.

For $\mathfrak{b} \in \Phi(\mathscr{T}_v)$, let $\Gamma_\mathfrak{b}$ be the isotropy group at $\mathfrak{b}$ in Γ, and let $\ell_v^\mathfrak{b}$ be the subfield of L_v fixed by $\Gamma_\mathfrak{b}$. Then for all $\mathfrak{b} \in \Phi(\mathscr{T}_V)$, $\ell_v^\mathfrak{b}$ is a ramified extension of k_v of degree $\leqslant 3$.

For every $b \in \Phi(\mathscr{S}_v)$, we fix a root $\mathfrak{b}$ in $\Phi(\mathscr{T}_v)$ such that (1) the restriction of $\mathfrak{b}$ to $\mathscr{S}_v$ is b, (2) if b is short, $\ell_v^\mathfrak{b} = \ell_v$, and (3) the root associated with $-b$ is the negative of the root associated with b.

Let π_v be a uniformizing element of ℓ_v. In case $\mathscr{G}/k_v$ is an outer form of type A_r with r even, we let λ_v be as in 0.3 and assume π_v so chosen that $\lambda_v \pi_v + \bar{\lambda}_v \bar{\pi}_v = 0$. The ring of integers of ℓ_v equals the direct sum of the $\pi_v^i\, \mathfrak{o}_v$, $0 \leqq i < [\ell_v : k_v]$.

The Lie algebra $\mathfrak{g}_v = L(\mathscr{G}) \otimes_k k_v$ splits over L_v and the action of the Galois group Γ on L_v induces an action on $\mathfrak{g}_v \otimes_{k_v} L_v$.

The following assertion can be proved using the considerations in §§ 1, 2 and 7.1 of [28] (see also [6: II, §§ 4.3, 4.4]).

There exists a Chevalley basis $\{ X_\mathfrak{b} \mid \mathfrak{b} \in \Phi(\mathscr{T}_v)\} \cup \{ H_\mathfrak{a} \mid \mathfrak{a} \in \Pi(\mathscr{T}_v)\}$ of the Lie algebra $\mathfrak{g}_v \otimes_{k_v} L_v$ such that:

(i) $\gamma(X_\mathfrak{b}) = X_{\gamma(\mathfrak{b})}$ for all $\gamma \in \Gamma$ and $\mathfrak{b} \in \Phi(\mathscr{T}_v)$ whose restriction to $\mathscr{S}_v$ is a non-divisible root in $\Phi(\mathscr{S}_v)$.

(ii) $\gamma(X_\mathfrak{b}) = -X_\mathfrak{b}$ for $\gamma \in \Gamma$, $\gamma \neq 1$, and any $\mathfrak{b} \in \Phi(\mathscr{T}_v)$ whose restriction to $\mathscr{S}_v$ is a divisible root.

(iii) If $\mathscr{G}/k_v$ is *not* an *outer* form of type A_r, with r even, then the union of the following sets is an $\mathfrak{o}_v$-basis of the Lie algebra $L(\mathscr{G}_v)$ ($\subset \mathfrak{g}_v$):

$$\{ H_a \mid a(\in \Pi(\mathscr{S}_v)) \text{ long} \}, \quad \{ \sum_{\gamma \in \Gamma/\Gamma_\mathfrak{a}} \gamma(\pi^i)\, H_{\gamma(\mathfrak{a})} \mid a(\in \Pi(\mathscr{S}_v)) \text{ short}, 0 \leqq i < [\ell_v : k_v]\},$$

$$\{ X_\mathfrak{b} \mid b(\in \Phi(\mathscr{S}_v)) \text{ long} \}, \quad \{ \sum_{\gamma \in \Gamma/\Gamma_\mathfrak{b}} \gamma(\pi^i)\, X_{\gamma(\mathfrak{b})} \mid b(\in \Phi(\mathscr{S}_v)) \text{ short}, 0 \leqq i < [\ell_v : k_v]\}.$$

(iv) If $\mathscr{G}/k_v$ is an outer form of type A_r with r even, let $\mathfrak{v}$ be the $\mathbf{R}$-valued homomorphism of the character group $X^*(\mathscr{Z}_v)$ which takes the value $+1$ at the unique root in $\Pi(\mathscr{Z}_v)$ whose restriction to $\mathscr{C}_v$ is a multipliable root and takes the value zero at all the other elements of $\Pi(\mathscr{Z}_v)$; note that $\mathfrak{v}(\mathfrak{b}) = 0$ or ± 2 if $\mathfrak{b}$ is a root whose restriction to $\mathscr{C}_v$ is neither multipliable nor divisible. Then the union of the following sets is an $\mathfrak{o}_v$-basis of the Lie algebra $L(\mathscr{G}_v)$:

$$\{ \pi_v^i H_{\mathfrak{a}} + \bar{\pi}_v^i H_{\bar{\mathfrak{a}}} \mid a \in \Pi(\mathscr{C}_v), i = 0, 1 \}, \quad \{ \lambda_v \pi_v X_{\mathfrak{b}}, \lambda_v^{-1} \bar{\pi}_v X_{-\mathfrak{b}} \mid b (\in \Phi(\mathscr{C}_v)^+) \text{ divisible} \},$$

$$\{ \pi_v^i X_{\mathfrak{b}} + \bar{\pi}_v^i X_{\bar{\mathfrak{b}}}; \lambda_v^{-1} \pi_v^i X_{-\mathfrak{b}} + \bar{\lambda}_v^{-1} \bar{\pi}_v^i X_{-\bar{\mathfrak{b}}} \mid b (\in \Phi(\mathscr{C}_v)^+) \text{ multipliable}, i = 0, 1 \},$$

$$\{ \lambda_v^{\frac{1}{2}\mathfrak{v}(\mathfrak{b})} \pi_v^i X_{\mathfrak{b}} + \bar{\lambda}_v^{\frac{1}{2}\mathfrak{v}(\mathfrak{b})} \bar{\pi}_v^i X_{\bar{\mathfrak{b}}} \mid b \in \Phi(\mathscr{C}_v), b \text{ nonmultipliable and nondivisible}, i = 0, 1 \};$$

where for $\mathfrak{b} \in \Phi(\mathscr{Z}_v)$, $\bar{\mathfrak{b}}$ denotes its conjugate under the nontrivial element of the Galois group of ℓ_v/k_v.

Now using the above basis of $L(\mathscr{G}_v)$, and the fact that $\omega^{\mathrm{Ch}} = \alpha\omega$, it is not difficult to see that $|\alpha^2|_v^{-1} \gamma_v^2 = |\mathfrak{d}(\ell_v/k_v)|_v^{-\mathfrak{s}(\mathscr{G})}$. Therefore,

$$\prod_{v \in V_f} (|\alpha^2|_v^{-1} \gamma_v^2) = \prod_{v \in \mathscr{R}} |\mathfrak{d}(\ell_v/k_v)|_v^{-\mathfrak{s}(\mathscr{G})} = (D_\ell / D_k^{[\ell:k]})^{\mathfrak{s}(\mathscr{G})};$$

see the appendix at the end of this paper. (Recall that for $v \in V_f - \mathscr{R}$, $\ell \otimes_k k_v$ is a direct sum of certain unramified extensions of k_v, and for any unramified extension K of k_v, $|\mathfrak{d}(K/k_v)|_v = 1$.) This proves the theorem.

2. Volumes of parahoric subgroups

We begin with the following general lemma.

2.0. *Lemma.* — *Let F be an arbitrary field.* G *and* G′ *be connected semi-simple F-groups. Assume that* G *is an inner F-form of* G′ *and* G′ *is quasi-split over F. Let F′ be a separable extension of F such that* G *is quasi-split over F′. Then* G *and* G′ *are isomorphic over F′. Moreover, if F′ is a Galois extension of F, we can find an F′-isomorphism* $\varphi : G \to G'$ *such that for all* γ *in the Galois group of* F'/F, $\varphi^{-1} \cdot {}^{\gamma}\varphi \in \mathrm{Int}(G)$.

Proof. — By assumption, there exists an isomorphism $f : G \to G'$ defined over a (fixed) separable closure F'_s of F' such that for all γ in the Galois group $\Gamma(F'_s/F)$ of F'_s/F, we have $a_\gamma := f^{-1} \cdot {}^{\gamma}f \in \mathrm{Int}(G)$. Choose a Borel subgroup B of G (resp. B′ of G′) and a maximal torus T (resp. T′) of B (resp. B′), all defined over F'. Then we can arrange that f maps B and T onto B′ and T′ respectively. Then so does ${}^{\gamma}f$ for all γ in the Galois group $\Gamma(F'_s/F')$ $(\subset \Gamma(F'_s/F))$ of F'_s/F'. Hence for $\gamma \in \Gamma(F'_s/F')$, a_γ preserves B, T and so it is of the form $\mathrm{Int}\, t_\gamma$ $(t_\gamma \in T)$.

Assume now that G is adjoint. Then t_γ is uniquely determined and it follows that $\gamma \mapsto t_\gamma$ is a 1-cocycle on $\Gamma(F'_s/F')$ with values in T. The Galois group $\Gamma(F'_s/F')$ acts on $X^*(T)$ by permuting the simple roots. These form a basis of $X^*(T)$ since G is adjoint. Therefore $\Gamma(F'_s/F')$ acts as a permutation representation and this implies that T is a direct product of certain tori of the form $R_{L/F'}(GL_1)$. Therefore it is cohomologically trivial over F' and so (a_γ) is a coboundary: there exists a $t \in T$ such that $t_\gamma = t \cdot {}^\gamma t^{-1}$. Then for $\gamma \in \Gamma(F'_s/F')$, $(f \cdot \operatorname{Int} t)^{-1} \cdot {}^\gamma (f \cdot \operatorname{Int} t) = \operatorname{Int} t^{-1} \cdot a_\gamma \cdot \operatorname{Int} {}^\gamma t = \operatorname{Id}$, hence the isomorphism $\varphi := f . \operatorname{Int} t$ is defined over F'.

If G is not adjoint, let $G \to \operatorname{Ad} G$ be the canonical central isogeny. If $\hat{t}$ is in the inverse image of the previous t, then again $\varphi := f \cdot \operatorname{Int} \hat{t}$ is defined over F'. Moreover, since $f^{-1} \cdot {}^\gamma f \in \operatorname{Int}(G)$ for all $\gamma \in \Gamma(F'_s/F)$, it is obvious that if F' is a Galois extension of F, then for all γ in the Galois group of F'/F, $\varphi^{-1} \cdot {}^\gamma \varphi \in \operatorname{Int}(G)$.

2.1. Let G be an absolutely quasi-simple, simply connected algebraic k-group which is an inner form of $\mathscr{G}$. It is known that for all but finitely many places v, G is quasi-split over k_v ([30: 4.9 (ii)]) and so it is isomorphic to $\mathscr{G}$ over k_v (Lemma 2.0).

We shall use the notation introduced in the previous section. Thus ω is the $\mathscr{G}$-invariant exterior form on $\mathscr{G}$ of maximal degree (and defined over k) fixed in 1.1.

Let $\varphi : G \to \mathscr{G}$ be an isomorphism defined over a (not necessarily finite) Galois extension K of k such that for every γ in the Galois group $\Gamma(K/k)$ of K/k, $\varphi^{-1} \cdot {}^\gamma \varphi$ is an inner automorphism of G. Then $\omega^* := \varphi^*(\omega)$ is an invariant exterior form on G of maximal degree; moreover it is defined over k (see [15: pp. 475-476]). If $\psi : G \to \mathscr{G}$ is some other isomorphism defined over an extension of k, then as any commutative quotient of $\operatorname{Aut}(\mathscr{G})/\operatorname{Int}(\mathscr{G})$ is of order $\leqslant 3$, it is clear that $\psi^*(\omega) = u(\psi)\, \omega^*$, where $u(\psi)$ is a root of unity of order $\leqslant 3$.

For each $v \in V$, ω (resp. ω^*), together with the normalized absolute value $|\ |_v$ on k_v (see 0.1), determines a Haar measure on $\mathscr{G}(k_v)$ (resp. $G(k_v)$) which we shall denote by ω_v (resp. ω_v^*). The Haar measure ω_v^* on $G(k_v)$ is *independent* of the choice of the isomorphism $\varphi : G \to \mathscr{G}$.

2.2. A collection $P = (P_v)_{v \in V_f}$ of parahoric subgroups P_v of $G(k_v)$ is said to be *coherent* if $\prod_{v \in V_\infty} G(k_v) \prod_{v \in V_f} P_v$ is an open subgroup of the adèle group $G(A)$.

Let a coherent collection $P = (P_v)_{v \in V_f}$ of parahoric subgroups be given. For $v \in V_f$, let G_v be the smooth affine $\mathfrak{o}_v$-group scheme associated with the parahoric subgroup P_v of $G(k_v)$ (0.7). Its generic fiber ($= G_v \times_{\mathfrak{o}_v} k_v$) is isomorphic to $G \times_k k_v$ and its group of integral points is isomorphic to P_v.

Let the parahoric subgroups $\mathscr{P}_v$ and the smooth affine $\mathfrak{o}_v$-group scheme $\mathscr{G}_v$ associated with $\mathscr{P}_v$ be as in 1.2 and 1.3 respectively. We shall denote by $\overline{\mathscr{G}}_v$ (resp. $\overline{G}_v$) the group $\mathscr{G}_v \times_{\mathfrak{o}_v} \mathfrak{f}_v$ (resp. $G_v \times_{\mathfrak{o}_v} \mathfrak{f}_v$) over the (finite) residue field $\mathfrak{f}_v$ of k_v. It is known (see [33: 3.5.2]) that since $\mathscr{G}$ and G are simply connected, the groups $\overline{\mathscr{G}}_v$ and $\overline{G}_v$ are connected; also the "reduction mod $\mathfrak{p}_v$" homomorphisms $\mathscr{P}_v = \mathscr{G}_v(\mathfrak{o}_v) \to \overline{\mathscr{G}}_v(\mathfrak{f}_v)$ and $P_v = G_v(\mathfrak{o}_v) \to \overline{G}_v(\mathfrak{f}_v)$ are surjective [33: 3.4.4]. Both $\overline{\mathscr{G}}_v$ and $\overline{G}_v$ admit a Levi decomposition over $\mathfrak{f}_v$ [33: 3.5]. Let $\overline{\mathscr{M}}_v$ (resp. $\overline{M}_v$) be a fixed maximal connected reductive $\mathfrak{f}_v$-subgroup such that

$$\overline{\mathscr{G}}_v = \overline{\mathscr{M}}_v . R_u(\overline{\mathscr{G}}_v) \ \text{(resp. } \overline{G}_v = \overline{M}_v . R_u(\overline{G}_v)\text{)},$$

where $R_u(\overline{\mathscr{G}}_v)$ (resp. $R_u(\overline{G}_v)$) is the unipotent radical of $\overline{\mathscr{G}}_v$ (resp. $\overline{G}_v$). As $\mathfrak{f}_v$ is a finite field, both $\overline{\mathscr{M}}_v$ and $\overline{M}_v$ are quasi-split over $\mathfrak{f}_v$ (see, for example, [2: Proposition 16.6]). We fix a Borel $\mathfrak{f}_v$-subgroup $\overline{\mathscr{B}}_v$ of $\overline{\mathscr{M}}_v$, $\overline{B}_v$ of $\overline{M}_v$ and a maximal $\mathfrak{f}_v$-torus $\overline{\mathscr{C}}_v$ of $\overline{\mathscr{B}}_v$, $\overline{T}_v$ of $\overline{B}_v$. Let $\overline{\mathscr{U}}_v$ (resp. $\overline{U}_v$) be the unipotent radical of $\overline{\mathscr{B}}_v$ (resp. $\overline{B}_v$).

Let $\mathscr{I}_v$ (resp. I_v) be the inverse image in $\mathscr{P}_v$ (resp. P_v) of $\overline{\mathscr{B}}_v(\mathfrak{f}_v)\cdot R_u(\overline{\mathscr{G}}_v)(\mathfrak{f}_v)$ (resp. $\overline{B}_v(\mathfrak{f}_v)\cdot R_u(\overline{G}_v)(\mathfrak{f}_v)$) under the reduction map $\mathscr{P}_v \to \overline{\mathscr{G}}_v(\mathfrak{f}_v)$ (resp. $P_v \to \overline{G}_v(\mathfrak{f}_v)$). Then $\mathscr{I}_v$ (resp. I_v) is an Iwahori subgroup of $\mathscr{G}(k_v)$ (resp. $G(k_v)$). Obviously, $[\mathscr{P}_v : \mathscr{I}_v] = [\overline{\mathscr{M}}_v(\mathfrak{f}_v) : \overline{\mathscr{B}}_v(\mathfrak{f}_v)]$ and $[P_v : I_v] = [\overline{M}_v(\mathfrak{f}_v) : \overline{B}_v(\mathfrak{f}_v)]$.

For all but finitely many v, P_v is a hyperspecial parahoric subgroup of $G(k_v)$ and hence there exists an isomorphism of $G(k_v)$ onto $\mathscr{G}(k_v)$ which carries P_v onto $\mathscr{P}_v$ (see [33: 2.5]); this isomorphism induces an isomorphism of the $\mathfrak{o}_v$-group scheme G_v onto the $\mathfrak{o}_v$-group scheme $\mathscr{G}_v$. Therefore, for all but finitely many v, $\overline{M}_v$ is isomorphic to $\overline{\mathscr{M}}_v$ over $\mathfrak{f}_v$.

2.3. *Proposition.* — *For* $v \in V_f$,

$$\omega_v^*(I_v) = \frac{\#\overline{T}_v(\mathfrak{f}_v)}{\#\overline{\mathscr{C}}_v(\mathfrak{f}_v)}\,\omega_v(\mathscr{I}_v)$$

and

$$\omega_v^*(P_v) = \frac{[\overline{M}_v(\mathfrak{f}_v) : \overline{B}_v(\mathfrak{f}_v)]}{[\overline{\mathscr{M}}_v(\mathfrak{f}_v) : \overline{\mathscr{B}}_v(\mathfrak{f}_v)]}\cdot\frac{\#\overline{T}_v(\mathfrak{f}_v)}{\#\overline{\mathscr{C}}_v(\mathfrak{f}_v)}\cdot\omega_v(\mathscr{P}_v).$$

Proof. — According to the Bruhat-Tits theory, the Iwahori subgroup $\mathscr{I}_v$ (resp. I_v) determines a smooth affine $\mathfrak{o}_v$-group scheme $\mathscr{G}_{\mathscr{I}_v}$ (resp. G_{I_v}) whose generic fiber is $\mathscr{G} \times_k k_v$ (resp. $G \times_k k_v$) and $\mathscr{G}_{\mathscr{I}_v}(\mathfrak{o}_v) \cong \mathscr{I}_v$ (resp. $G_{I_v}(\mathfrak{o}_v) \cong I_v$); see 0.7.

It is a well known consequence of a theorem of Steinberg [31] that since the residue field $\hat{\mathfrak{f}}_v$ of the maximal unramified extension $\hat{k}_v$ of k_v is algebraically closed, G is quasi-split over $\hat{k}_v$. Now since $\mathscr{G}$ is a quasi-split inner k-form of G, we conclude that G is isomorphic to $\mathscr{G}$ over $\hat{k}_v$ and there is an isomorphism $\varphi_v : G \times_k \hat{k}_v \to \mathscr{G} \times_k \hat{k}_v$ such that $\varphi_v^{-1}\cdot{}^{\gamma}\varphi_v$ is an inner automorphism of G for all $\gamma \in \mathrm{Gal}(\hat{k}_v/k_v)$; see Lemma 2.0. The exterior form $\varphi_v^*(\omega)$ is then defined over k_v and the Haar measure on $G(k_v)$ determined by it (and the absolute value $|\ |_v$ on k_v) is ω_v^* (2.1). Now let $\hat{\mathfrak{o}}_v$ be the ring of integers of $\hat{k}_v$. Then $\hat{I}_v := G_{I_v}(\hat{\mathfrak{o}}_v)$ (resp. $\hat{\mathscr{I}}_v := \mathscr{G}_{\mathscr{I}_v}(\hat{\mathfrak{o}}_v)$) is an Iwahori subgroup of $G(\hat{k}_v)$ (resp. $\mathscr{G}(\hat{k}_v)$) and in view of the conjugacy of Iwahori subgroups, we may (and we will) assume that $\varphi_v(\hat{I}_v) = \hat{\mathscr{I}}_v$. Then the isomorphism φ_v is induced from a unique isomorphism

$$G_{I_v} \times_{\mathfrak{o}_v} \hat{\mathfrak{o}}_v \to \mathscr{G}_{\mathscr{I}_v} \times_{\mathfrak{o}_v} \hat{\mathfrak{o}}_v,$$

defined over $\hat{\mathfrak{o}}_v$, which we denote again by φ_v, by base change $\hat{\mathfrak{o}}_v \hookrightarrow \hat{k}_v$; this is seen at once using the description of the coordinate rings of the group schemes G_{I_v} and $\mathscr{G}_{\mathscr{I}_v}$ (given in 0.7).

It is obvious that there is an $a_v (\in k_v^\times)$ such that the exterior form $a_v\,\omega$ induces an invariant exterior form on the group scheme $\mathscr{G}_{\mathscr{I}_v}$, which is defined over $\mathfrak{o}_v$ and whose reduction to the group scheme $\overline{\mathscr{G}}_{\mathscr{I}_v} := \mathscr{G}_{\mathscr{I}_v} \times_{\mathfrak{o}_v} \mathfrak{f}_v$ is not zero. Then as φ_v is an isomorphism defined over $\hat{\mathfrak{o}}_v$, the exterior form $\varphi_v^*(a_v\,\omega) = a_v\,\varphi_v^*(\omega)$ on the $\mathfrak{o}_v$-group scheme $\mathrm{G}_{\mathrm{I}_v}$ is defined over $\mathfrak{o}_v$ and its reduction to $\mathrm{G}_{\mathrm{I}_v} \times_{\mathfrak{o}_v} \hat{\mathfrak{f}}_v$, and hence also to $\overline{\mathrm{G}}_{\mathrm{I}_v} := \mathrm{G}_{\mathrm{I}_v} \times_{\mathfrak{o}_v} \hat{\mathfrak{f}}_v$, is not zero.

The inclusion $\mathrm{I}_v \subset \mathrm{P}_v$ (resp. $\mathscr{I}_v \subset \mathscr{P}_v$) induces a homomorphism $\overline{\mathrm{G}}_{\mathrm{I}_v} \to \overline{\mathrm{G}}_v$ (resp. $\overline{\mathscr{G}}_{\mathrm{I}_v} \to \overline{\mathscr{G}}_v$), where $\overline{\mathrm{G}}_v$ (resp. $\overline{\mathscr{G}}_v$) is as in 2.2. Also, there is a (unique) maximal $\mathfrak{f}_v$-torus of $\overline{\mathrm{G}}_{\mathrm{I}_v}$ (resp. $\overline{\mathscr{G}}_{\mathscr{I}_v}$) which is mapped isomorphically onto $\overline{\mathrm{T}}_v$ (resp. $\overline{\mathscr{E}}_v$) under this homomorphism. Since no confusion is likely, we denote this torus of $\overline{\mathrm{G}}_{\mathrm{I}_v}$ (resp. $\overline{\mathscr{G}}_{\mathscr{I}_v}$) again by $\overline{\mathrm{T}}_v$ (resp. $\overline{\mathscr{E}}_v$). Then $\overline{\mathrm{G}}_{\mathrm{I}_v} = \overline{\mathrm{T}}_v \cdot \mathrm{R}_u(\overline{\mathrm{G}}_{\mathrm{I}_v})$ and $\overline{\mathscr{G}}_{\mathscr{I}_v} = \overline{\mathscr{E}}_v \cdot \mathrm{R}_u(\overline{\mathscr{G}}_{\mathscr{I}_v})$.

Now as the Haar measure on $\mathscr{G}(k_v)$ given by $a_v\,\omega$ is $|\,a_v\,|_v\,\omega_v$, we conclude (cf. [24: I, 2.5]) that

$$\begin{aligned} |\,a_v\,|_v\,\omega_v(\mathscr{I}_v) &= \#\overline{\mathscr{G}}_{\mathscr{I}_v}(\mathfrak{f}_v)\cdot q_v^{-\dim \mathscr{G}} \\ &= \#\overline{\mathscr{E}}_v(\mathfrak{f})\cdot q_v^{-\dim \overline{\mathscr{E}}_v}. \end{aligned}$$

Similarily, as the Haar measure on $\mathrm{G}(k_v)$ given by $\varphi_v^*(a_v\,\omega) = a_v\,\varphi_v^*(\omega)$ is $|\,a_v\,|_v\,\omega_v^*$, we have

$$|\,a_v\,|_v\,\omega_v^*(\mathrm{I}_v) = \#\overline{\mathrm{T}}_v(\mathfrak{f}_v)\cdot q_v^{-\dim \overline{\mathrm{T}}_v}.$$

Now note that

$$\dim \overline{\mathscr{E}}_v = \hat{k}_v\text{-rank } \mathscr{G} = \hat{k}_v\text{-rank } \mathrm{G} = \dim \mathrm{T}_v,$$

and so we deduce from the above that

$$\omega_v^*(\mathrm{I}_v) = \frac{\#\overline{\mathrm{T}}_v(\mathfrak{f}_v)}{\#\overline{\mathscr{E}}_v(\mathfrak{f}_v)}\cdot \omega_v(\mathscr{I}_v).$$

Then

$$\begin{aligned} \omega_v^*(\mathrm{P}_v) &= [\mathrm{P}_v : \mathrm{I}_v]\;\omega_v^*(\mathrm{I}_v) \\ &= \frac{[\mathrm{P}_v : \mathrm{I}_v]}{[\mathscr{P}_v : \mathscr{I}_v]}\cdot\frac{\omega_v^*(\mathrm{I}_v)}{\omega_v(\mathscr{I}_v)}\cdot\omega_v(\mathscr{P}_v) \\ &= \frac{[\overline{\mathrm{M}}_v(\mathfrak{f}_v) : \overline{\mathrm{B}}_v(\mathfrak{f}_v)]}{[\overline{\mathscr{M}}_v(\mathfrak{f}_v) : \overline{\mathscr{B}}_v(\mathfrak{f}_v)]}\cdot\frac{\#\overline{\mathrm{T}}_v(\mathfrak{f}_v)}{\#\overline{\mathscr{E}}_v(\mathfrak{f}_v)}\cdot\omega_v(\mathscr{P}_v). \end{aligned}$$

This proves the proposition.

2.4. Let Δ_v be the basis of the *absolute* affine root system of G at v (i.e. the affine root system of G over the maximal unramified extension $\hat{k}_v$ of k_v) determined by the Iwahori subgroup I_v.

Let Θ_v be the subset of Δ_v corresponding to the parahoric subgroup P_v. The Galois group of $\hat{k}_v/k_v$ operates on Δ_v, leaving Θ_v stable. The Tits index [32] of the reductive group $\overline{\mathrm{M}}_v/\mathfrak{f}_v$ is obtained from the Dynkin diagram of Δ_v together with the action of the Galois group of $\hat{k}_v/k_v$ (i.e. " the local index " of G/k_v) by *deleting* the vertices corresponding to the roots in Θ_v and all the edges containing such vertices (see [33: 3.5.2]). Note that there is a canonical identification of the Galois group of $\hat{k}_v/k_v$ with the Galois group of $\hat{\mathfrak{f}}_v/\mathfrak{f}_v$, where $\hat{\mathfrak{f}}_v$ is the residue field of $\hat{k}_v$—it is an algebraic closure of $\mathfrak{f}_v$.

2.5. Fixing a $\hat{k}_v$-isomorphism of G onto $\mathscr{G}$, we identify the root system as well as the affine root system of $\mathscr{G}/\hat{k}_v$ with those of $G/\hat{k}_v$.

Let $d_v \in \Delta_v$ be the affine simple root corresponding to the parahoric subgroup $\mathscr{P}_v$. Then $\Delta_v - \{d_v\}$ can be identified with a basis of the absolute root system Ψ_v of the reductive group $\overline{\mathscr{M}}_v$; see [33: 3.5.2]. Since $\mathscr{P}_v$ is a hyperspecial parahoric subgroup if $\mathscr{G}$ splits over an unramified extension of k_v, otherwise ($\mathscr{G}$ is residually split over k_v and) $\mathscr{P}_v$ is a special parahoric subgroup, d_v is " special " ([28: 7.1]). Ψ_v is then a reduced and irreducible root system of rank $r_v := \hat{k}_v$-rank $\mathscr{G} = \hat{k}_v$-rank G. Hence the reductive group $\overline{\mathscr{M}}_v$ is in fact absolutely quasi-simple and its absolute rank is r_v.

2.6. We note here, for future use, the following empirical fact about connected semi-simple groups defined over a finite field: *If* H *is a connected semi-simple group defined over a finite field* $\mathfrak{f}$ *of cardinality* q, *then* $\# H(\mathfrak{f}) < q^{\dim H}$. This assertion can be checked by looking at the table of orders of finite groups of Lie type given in [25: Table 1]. Note that connected isogeneous groups over a finite field have an equal number of rational points [2: Proposition 16.8]; note also that it suffices to check the assertion for absolutely simple groups since every nonabelian simple group over $\mathfrak{f}$ is obtained by restriction of scalars from an absolutely simple group defined over a finite extension of $\mathfrak{f}$.

The two lemmas that follow (2.7 and 2.8) are needed for the proof of Proposition 2.10.

Let Θ_v be as in 2.4 and $t_v = \#\Theta_v - 1$.

2.7. *Lemma.* — (i) $\dim \overline{\mathscr{M}}_v \geqslant r_v(r_v + 2)$.

(ii) *If either* G *is not quasi-split over* k_v, *or* P_v *is not special, or* G *splits over* $\hat{k}_v$ *but* P_v *is not hyperspecial, then*

$$\dim \overline{\mathscr{M}}_v - \dim \overline{M}_v \geqslant 2r_v.$$

If, moreover, $t_v \geqslant 1$,

$$\dim \overline{\mathscr{M}}_v - \dim \overline{M}_v \geqslant 2(r_v + t_v - 1).$$

Proof. — The absolute root system of $\overline{M}_v$ is the root system Φ_v with the basis $\Delta_v - \Theta_v$ (see [33: 3.5.2]). Hence, $\dim \overline{M}_v = r_v + \#\Phi_v$.

Let d_v and Ψ_v be as in 2.5. Since Ψ_v is the absolute root system of $\overline{\mathscr{M}}_v$ (2.5), $\dim \overline{\mathscr{M}}_v = r_v + \#\Psi_v$. Now to prove assertion (i), we just need to note that among the reduced and irreducible root systems of a given rank s, one with the smallest cardinality is of type A_s, which has $s(s+1)$ roots.

We shall now prove (ii). We begin by observing that the root system Ψ_v can be identified with the root system consisting of the non-divisible roots of the root system of $\mathscr{G}/\hat{k}_v$, and with this identification, the gradient of d_v is the negative of the dominant (i.e. the highest) root of the root system of $\mathscr{G}/\hat{k}_v$, with respect to the basis determined by $\Delta_v - \{ d_v \}$, if $\mathscr{G}$ splits over $\hat{k}_v$, and is the negative of the dominant short root if $\mathscr{G}$ does not split over $\hat{k}_v$ but its $\hat{k}_v$-root system is reduced ([28: 2.10]).

We now first take-up the case where the $\hat{k}_v$-root system of G is not reduced. G/k_v is then an outer form of type A_r, with r even, say $r = 2n$, which does not split over $\hat{k}_v$. In this case G is quasi-split over k_v, $\bar{\mathscr{M}}_v$ is an absolutely quasi-simple group of type B_n and $\overline{M}_v$ is isogeneous to a $\mathfrak{f}_v$-group which is the direct product of a t_v-dimensional torus and a semi-simple group whose absolute Dynkin diagram is obtained from the Dynkin diagram of Δ_v by deleting the vertices corresponding to the roots in Θ_v and all the edges containing such vertices (2.4). From this description of $\bar{\mathscr{M}}_v$ and $\overline{M}_v$, (ii) follows at once. Note that since P_v is not special, if Θ_v contains a special root, then it contains at least two roots (so $t_v \geqq 1$).

We assume now that the $\hat{k}_v$-root system of G is reduced. If G is not quasi-split over k_v, then the Galois group of $\hat{k}_v/k_v$ does not fix any special (affine) root ([28: 7.2]), so if Θ_v contains a special root, being stable under the Galois group, it contains at least two special roots. It is easy to see that whenever Θ_v contains two or more special roots, Φ_v can be realized as an integrally closed proper subroot system of Ψ_v. This is also the case if G splits over $\hat{k}_v$ and Θ_v contains a non-special root. On the other hand, if G does not split over $\hat{k}_v$, and Θ_v contains a non-special root, then the dual root system $\Phi_v^\vee$ can be realized as an integrally closed proper subroot system of the dual $\Psi_v^\vee$.

It is well-known that for any integrally closed proper subroot system Φ of an irreducible and reduced root system Ψ of rank s, $\#\Psi - \#\Phi \geqq 2s$; see [3: Corollaire on p. 210 and Théorème 4]. From this we conclude that

$$\dim \bar{\mathscr{M}}_v - \dim \overline{M}_v = (\#\Psi_v - \#\Phi_v) \geqq 2r_v,$$

and if $t_v \geqslant 1$, then in fact $\#\Psi_v - \#\Phi_v \geqq 2r_v + 2(t_v - 1)$.

2.8. *Lemma. — Let* $\mathfrak{f}$ *be a finite field with* q *elements and* T *an* s*-dimensional* $\mathfrak{f}$*-torus. Then* $\#\mathsf{T}(\mathfrak{f}) \leqq (q+1)^s$.

Proof. — Let $\mathfrak{f}'$ be the smallest extension of $\mathfrak{f}$ over which T splits; $\mathfrak{f}'$ is necessarily a finite cyclic extension of $\mathfrak{f}$. Then T is isogeneous to a direct product of tori T_i such that the canonical representation of the Galois group $\Gamma(\mathfrak{f}'/\mathfrak{f})$ on $X^*(\mathsf{T}_i) \otimes_{\mathbf{Z}} \mathbf{Q}$ is irreducible. Since connected isogeneous groups over a finite field have the same number of rational elements ([2: Proposition 16.8]), this reduces us to the case where the representation of $\Gamma(\mathfrak{f}'/\mathfrak{f})$ on $X^*(\mathsf{T}) \otimes_{\mathbf{Z}} \mathbf{Q}$ is irreducible. Now let $[\mathfrak{f}' : \mathfrak{f}] = n$. Then $(s =)\dim \mathsf{T}$ equals the value of Euler's φ-function at n. It is not difficult to see, using, for example, Möbius inversion in the multiplicative form, that $\#\mathsf{T}(\mathfrak{f}) = P_n(q)$, where $P_n(x) \in \mathbf{Z}[x]$ is the

n-th cyclotomic polynomial. We recall that $P_n(x)$ is a monic polynomial of degree s whose roots are precisely the primitive n-th roots of unity. From this it follows at once that $\#\mathsf{T}(\mathfrak{f}) = P_n(q) \leqq (q+1)^s$.

2.9. Let γ_v be as in 1.3. Then (cf. [24: I, 2.5])

$$\gamma_v\, \omega_v(\mathscr{P}_v) = \#\overline{\mathscr{G}}_v(\mathfrak{f}_v)\cdot q_v^{-\dim \mathscr{G}} = \#\overline{\mathscr{M}}_v(\mathfrak{f}_v)\cdot q_v^{-\dim \overline{\mathscr{M}}_v};$$

where $\overline{\mathscr{G}}_v$ and $\overline{\mathscr{M}}_v$ are as in 2.2.

2.10. *Proposition.* — *For* $v \in V_f$,

(i)
$$\gamma_v\, \omega_v^*(I_v) = \frac{\#\overline{T}_v(\mathfrak{f}_v)}{q_v^{(r_v + \dim \overline{\mathscr{M}}_v)/2}}.$$

(ii)
$$\frac{\#\overline{T}_v(\mathfrak{f}_v)}{q_v^{(r_v + \dim \overline{\mathscr{M}}_v)/2}} \leqq (q_v+1)^{r_v}\, q_v^{-r_v(r_v+3)/2}.$$

(iii)
$$\gamma_v\, \omega_v^*(P_v) = \frac{\#\overline{M}_v(\mathfrak{f}_v)}{q_v^{(\dim \overline{M}_v + \dim \overline{\mathscr{M}}_v)/2}}.$$

(iv) *For all* $v \in V_f$,

$$\frac{\#\overline{M}_v(\mathfrak{f}_v)}{q_v^{(\dim \overline{M}_v + \dim \overline{\mathscr{M}}_v)/2}} < 1.$$

Moreover if either G *is not quasi-split over* k_v, *or* P_v *is not special, or* G *splits over* $\hat{k}_v$ *but* P_v *is not hyperspecial, then*

$$\frac{\#\overline{M}_v(\mathfrak{f}_v)}{q_v^{(\dim \overline{M}_v + \dim \overline{\mathscr{M}}_v)/2}} \leqq (q_v+1)\, q_v^{-r_v-1}.$$

Proof. — (1) According to Proposition 2.3,

$$\omega_v^*(I_v) = \frac{\#\overline{T}_v(\mathfrak{f}_v)}{\#\overline{\mathscr{E}}_v(\mathfrak{f}_v)}\cdot \omega_v(\mathscr{I}_v),$$

and hence,

$$\begin{aligned}\gamma_v\, \omega_v^*(I_v) &= \frac{\#\overline{T}_v(\mathfrak{f}_v)}{\#\overline{\mathscr{E}}_v(\mathfrak{f}_v)}\cdot\frac{\gamma_v\, \omega_v(\mathscr{P}_v)}{[\mathscr{P}_v : \mathscr{I}_v]}\\ &= \frac{\#\overline{T}_v(\mathfrak{f}_v)}{\#\overline{\mathscr{E}}_v(\mathfrak{f}_v)}\cdot\frac{\#\overline{\mathscr{M}}_v(\mathfrak{f}_v)\cdot q_v^{-\dim \overline{\mathscr{M}}_v}}{[\overline{\mathscr{M}}_v(\mathfrak{f}_v) : \overline{\mathscr{B}}_v(\mathfrak{f}_v)]} \qquad \text{(see 2.2 and 2.9)}\\ &= \frac{\#\overline{T}_v(\mathfrak{f}_v)}{\#\overline{\mathscr{E}}_v(\mathfrak{f}_v)}\cdot\frac{\#\overline{\mathscr{B}}_v(\mathfrak{f}_v)}{q_v^{\dim \overline{\mathscr{M}}_v}}.\end{aligned}$$

Now recall that $\overline{\mathscr{U}}_v$ (resp. $\overline{\mathrm{U}}_v$) is the unipotent radical of $\overline{\mathscr{B}}_v$ (resp. $\overline{\mathrm{B}}_v$). So

$$\overline{\mathscr{B}}_v(\mathfrak{f}_v) = \overline{\mathscr{C}}_v(\mathfrak{f}_v)\cdot\overline{\mathscr{U}}_v(\mathfrak{f}_v), \qquad \overline{\mathrm{B}}_v(\mathfrak{f}_v) = \overline{\mathrm{T}}_v(\mathfrak{f}_v)\cdot\overline{\mathrm{U}}_v(\mathfrak{f}_v),$$

$$\#\overline{\mathscr{U}}_v(\mathfrak{f}_v) = q_v^{\dim \overline{\mathscr{U}}_v}, \qquad \#\overline{\mathrm{U}}_v(\mathfrak{f}_v) = q_v^{\dim \overline{\mathrm{U}}_v},$$

and
$$\dim \overline{\mathscr{M}}_v = \dim \overline{\mathscr{C}}_v + 2\dim \overline{\mathscr{U}}_v; \qquad \dim \overline{\mathrm{M}}_v = \dim \overline{\mathrm{T}}_v + 2\dim \overline{\mathrm{U}}_v.$$

Moreover, as we have observed before,

$$\dim \overline{\mathscr{C}}_v = r_v = \dim \overline{\mathrm{T}}_v.$$

So
$$\gamma_v\, \omega_v^*(\mathrm{I}_v) = \frac{\#\overline{\mathrm{T}}_v(\mathfrak{f}_v)}{\#\overline{\mathscr{C}}_v(\mathfrak{f}_v)}\cdot\frac{\#\overline{\mathscr{B}}_v(\mathfrak{f}_v)}{q_v^{\dim \overline{\mathscr{M}}_v}}$$

$$= \frac{\#\overline{\mathrm{T}}_v(\mathfrak{f}_v)}{q_v^{r_v+\dim \overline{\mathscr{U}}_v}} = \frac{\#\overline{\mathrm{T}}_v(\mathfrak{f}_v)}{q_v^{(r_v+\dim \overline{\mathscr{M}}_v)/2}}.$$

Since $\#\overline{\mathrm{T}}_v(\mathfrak{f}_v) \leq (q_v+1)^{r_v}$ by 2.8, the second assertion of the proposition follows from Lemma 2.7 (i).

(2)
$$\gamma_v\, \omega_v^*(\mathrm{P}_v) = \frac{[\overline{\mathrm{M}}_v(\mathfrak{f}_v) : \overline{\mathrm{B}}_v(\mathfrak{f}_v)]}{[\overline{\mathscr{M}}_v(\mathfrak{f}_v) : \overline{\mathscr{B}}_v(\mathfrak{f}_v)]}\cdot\frac{\#\overline{\mathrm{T}}_v(\mathfrak{f}_v)}{\#\overline{\mathscr{C}}_v(\mathfrak{f}_v)}\cdot\gamma_v\, \omega_v(\mathscr{P}_v) \quad \text{(by 2.3)}$$

$$= \frac{[\overline{\mathrm{M}}_v(\mathfrak{f}_v) : \overline{\mathrm{B}}_v(\mathfrak{f}_v)]}{[\overline{\mathscr{M}}_v(\mathfrak{f}_v) : \overline{\mathscr{B}}_v(\mathfrak{f}_v)]}\cdot\frac{\#\overline{\mathrm{T}}_v(\mathfrak{f}_v)}{\#\overline{\mathscr{C}}_v(\mathfrak{f}_v)}\cdot\frac{\#\overline{\mathscr{M}}_v(\mathfrak{f}_v)}{q_v^{\dim \overline{\mathscr{M}}_v}} \quad \text{(by 2.9)}$$

$$= \frac{\#\overline{\mathrm{M}}_v(\mathfrak{f}_v)}{q_v^{\dim \overline{\mathscr{M}}_v}}\cdot\frac{\#\overline{\mathscr{B}}_v(\mathfrak{f}_v)}{\#\overline{\mathrm{B}}_v(\mathfrak{f}_v)}\cdot\frac{\#\overline{\mathrm{T}}_v(\mathfrak{f}_v)}{\#\overline{\mathscr{C}}_v(\mathfrak{f}_v)}$$

$$= \frac{\#\overline{\mathrm{M}}_v(\mathfrak{f}_v)}{q_v^{\dim \overline{\mathscr{M}}_v}}\cdot\frac{\#\overline{\mathscr{U}}_v(\mathfrak{f}_v)}{\#\overline{\mathrm{U}}_v(\mathfrak{f}_v)}$$

$$= \frac{\#\overline{\mathrm{M}}_v(\mathfrak{f}_v)}{q_v^{(\dim \overline{\mathrm{M}}_v+\dim \overline{\mathscr{M}}_v)/2}}.$$

(3) Let $\overline{\mathrm{Z}}_v$ be the connected component of the identity in the center of $\overline{\mathrm{M}}_v$ and $[\overline{\mathrm{M}}_v, \overline{\mathrm{M}}_v]$ be the derived group. Then the product map $\overline{\mathrm{Z}}_v \times [\overline{\mathrm{M}}_v, \overline{\mathrm{M}}_v] \to \overline{\mathrm{M}}_v$ is an isogeny defined over $\mathfrak{f}_v$ and, hence, $\#\overline{\mathrm{M}}_v(\mathfrak{f}_v) = \#\overline{\mathrm{Z}}_v(\mathfrak{f}_v)\,.\,\#[\overline{\mathrm{M}}_v, \overline{\mathrm{M}}_v]\,(\mathfrak{f}_v)$. So

$$\frac{\#\overline{\mathrm{M}}_v(\mathfrak{f}_v)}{q_v^{(\dim \overline{\mathrm{M}}_v+\dim \overline{\mathscr{M}}_v)/2}} = \frac{\#\overline{\mathrm{Z}}_v(\mathfrak{f}_v)\,.\,\#[\overline{\mathrm{M}}_v, \overline{\mathrm{M}}_v]\,(\mathfrak{f}_v)}{q_v^{\dim \overline{\mathrm{Z}}_v}\,.\,q_v^{\dim[\overline{\mathrm{M}}_v, \overline{\mathrm{M}}_v]}\,.\,q_v^{(\dim \overline{\mathscr{M}}_v-\dim \overline{\mathrm{M}}_v)/2}}$$

$$< \#\overline{\mathrm{Z}}_v(\mathfrak{f}_v)\; q_v^{-(\dim \overline{\mathscr{M}}_v-\dim \overline{\mathrm{M}}_v)/2-\dim \overline{\mathrm{Z}}_v}$$

$$\leq (q_v+1)^{\dim \overline{\mathrm{Z}}_v}\; q_v^{-(\dim \overline{\mathscr{M}}_v-\dim \overline{\mathrm{M}}_v)/2-\dim \overline{\mathrm{Z}}_v}.$$

We have used here the facts that $\#[\overline{\mathrm{M}}_v, \overline{\mathrm{M}}_v](\mathfrak{f}_v) < q_v^{\dim[\overline{\mathrm{M}}_v, \overline{\mathrm{M}}_v]}$ and $\#\overline{\mathrm{Z}}_v(\mathfrak{f}_v) \leqq (q_v+1)^{\dim \overline{\mathrm{Z}}_v}$, which follow from the observation in 2.6 and Lemma 2.8. Finally we note that $\dim \overline{\mathrm{Z}}_v = \#\Theta_v - 1 = t_v$. Lemma 2.7 (ii) now implies assertion (iv) of the proposition if either G is not quasi-split over k_v, or P_v is not special, or G splits over $\hat{k}_v$ but P_v is not hyperspecial. Note that if $t_v \geqq 1$,

$$(q_v+1)^{t_v} q_v^{-(r_v+t_v-1)-t_v} \leqq (q_v+1)\, q_v^{-r_v-1}.$$

Let us assume now that G is quasi-split over k_v (hence is isomorphic to $\mathscr{G}$ over k_v, see Lemma 2.0). Then $\#\overline{\mathrm{M}}_v(\mathfrak{f}_v)\, q_v^{-(\dim \overline{\mathrm{M}}_v + \dim \overline{\mathscr{M}}_v)/2}$ $(= \gamma_v\, \omega_v^*(\mathrm{P}_v))$ is maximal when P_v is isomorphic to $\mathscr{P}_v$, in which case it is equal to $\#\overline{\mathscr{M}}_v(\mathfrak{f}_v)\, q_v^{-\dim \overline{\mathscr{M}}_v}$, and, according to the observation in 2.6, this number is less than 1; recall that $\overline{\mathscr{M}}_v$ is (absolutely) quasi-simple (2.5).

2.11. *Remark.* — For particular groups, one can give bounds which are better than those provided by Proposition 2.10 (ii), (iv). If, for example, G is an inner or outer form of type **A** and v is such that $\mathrm{G}(k_v)$ is isomorphic to $\mathrm{SL}_{n_v}(\mathfrak{D}_v)$, where $\mathfrak{D}_v$ is a central division algebra over k_v of degree d_v, then

$$\frac{\#\overline{\mathrm{T}}_v(\mathfrak{f}_v)}{q_v^{(r_v+\dim \overline{\mathscr{M}}_v)/2}} = (q_v-1)^{-1}(q_v^{d_v}-1)^{(r+1)/d_v}\, q_v^{-r(r+3)/2}$$
$$< (q_v-1)^{-1}\, q_v^{-r(r+1)/2+1}$$

and
$$\frac{\#\overline{\mathrm{M}}_v(\mathfrak{f}_v)}{q_v^{(\dim \overline{\mathrm{M}}_v + \dim \overline{\mathscr{M}}_v)/2}} < (q_v-1)^{-1}\, q_v^{-(r^2+2r-(r+1)^2 d_v^{-1}-1)/2}.$$

To establish the latter, it is sufficient to consider the case where P_v is a maximal parahoric subgroup of $\mathrm{G}(k_v)$; we assume now this to be the case. Let F_v be the unique extension of $\mathfrak{f}_v$ of degree d_v. Then the dimension of $\overline{\mathrm{M}}_v$ is $d_v n_v^2 - 1$ and $\overline{\mathrm{M}}_v(\mathfrak{f}_v)$ is isomorphic to the subgroup of $\mathrm{GL}_{n_v}(\mathrm{F}_v)$ consisting of matrices whose determinant is of norm 1 over $\mathfrak{f}_v$, so its order is less than $(q_v-1)^{-1} q_v^{d_v n_v^2}$; $\overline{\mathrm{T}}_v(\mathfrak{f}_v)$ is isomorphic to the diagonal subgroup of this group, its order is $(q_v-1)^{-1}(q_v^{d_v}-1)^{n_v}$. Now as $d_v n_v - 1 = r$ and $\dim \overline{\mathscr{M}}_v = r^2+2r$, the above bounds are obvious.

On the other hand, if r is odd, say $r = 2n+1$, and G/k_v is an outer form of type $\mathbf{A}_r$, of k_v-rank n, which does not split over $\hat{k}_v$, then, as can be seen,

$$\frac{\#\overline{\mathrm{M}}_v(\mathfrak{f}_v)}{q_v^{(\dim \overline{\mathrm{M}}_v + \dim \overline{\mathscr{M}}_v)/2}} \leqq \frac{(q_v^{n+1}+1)\prod_{i=1}^{n}(q_v^{2i}-1)}{q_v^{(n+1)(n+2)}} \leqq q_v^{-(n+1)}.$$

3. Covolumes of the principal S-arithmetic subgroups

As in § 2, G is an inner k-form of $\mathscr{G}$ and $\varphi : \mathrm{G} \to \mathscr{G}$ is an isomorphism defined over some Galois extension K of k such that for every γ in the Galois group of K/k, $\varphi^{-1}\cdot{}^{\gamma}\varphi$ is an inner automorphism of G.

Let ω be an invariant exterior form on $\mathscr{G}$ defined over k and of maximal degree. Then $\omega^* = \varphi^*(\omega)$ is an invariant exterior form on G of maximal degree; it is defined over k (2.1).

We shall use the notation introduced in the preceding sections. Thus, for $v \in V_f$, the parahoric subgroups P_v's are as in 2.2.

3.1. The natural embedding of k in the k-algebra A of adèles, gives an embedding of $G(k)$ in $G(A)$; we identify $G(k)$ with a subgroup of $G(A)$ in terms of this embedding. Then $G(k)$ is a discrete subgroup of the locally compact group $G(A)$, and it is well-known ([1], [11]) that in the measure on $G(A)/G(k)$ induced by any Haar measure on $G(A)$, the volume of $G(A)/G(k)$ is finite.

3.2. ω^* determines a Haar measure ω^*_A on $G(A)$, which coincides with the product measure $\prod_{v \in V_\infty} \omega^*_v \cdot \prod_{v \in V_f} (\omega^*_v |_{P_v})$ on the open subgroup $\prod_{v \in V_\infty} G(k_v) \cdot \prod_{v \in V_f} P_v$; note that since G is semi-simple, the product $\prod_{v \in V_f} \omega^*_v(P_v)$ is absolutely convergent and hence the product measure $\prod_{v \in V_\infty} \omega^*_v \cdot \prod_{v \in V_f} (\omega^*_v |_{P_v})$ is a Haar measure on $\prod_{v \in V_\infty} G(k_v) \cdot \prod_{v \in V_f} P_v$ (cf. [26: § 1] where this is proved over number fields; a similar proof applies in the case of global function fields).

In the sequel, we shall let ω^*_A also denote the finite invariant measure on $G(A)/G(k)$ induced by the Haar measure ω^*_A on $G(A)$.

3.3. Let D_k be as in 0.2. The *Tamagawa number* $\tau_k(G)$ of G/k is by definition the positive real number $D_k^{-\frac{1}{2}\dim G}\, \omega^*_A(G(A)/G(k))$; in view of the product formula (see 0.1), it depends only on G/k and not on the choice of the invariant exterior k-form ω.

It was conjectured by André Weil that $\tau_k(G) = 1$. He and T. Tamagawa proved this for all *inner* forms of type A, and in case k is of characteristic different from two, for all k-forms of type B, C and all forms of type D except the triality forms of type D_4; M. Demazure verified the conjecture for the forms of type G_2 (see [34]). J. G. M. Mars then proved the conjecture for *outer* forms of type A ([23]), all forms of type F_4 and certain *inner* forms of type E_6 ([22]) over number fields. For split groups over number fields, the conjecture was proved by R. P. Langlands ([19]). Using some of his ideas, G. Harder ([13]) proved the conjecture for all split groups over global function fields, and K. F. Lai proved it for quasi-split groups over number fields ([18]).

R. Kottwitz ([17]), following a proposal of Jacquet-Langlands [15], has recently proved the conjecture for groups over number fields, without any case-by-case considerations, modulo the Hasse principle for the Galois cohomology of simply connected semi-simple groups ([16]). The Hasse principle has been known to hold for all groups of type other than E_8. V. I. Chernousov has just announced its verification also for the groups of type E_8. Hence, the work of Kottwitz ([17]) implies that $\tau_k(G) = 1$ if k is a number field.

3.4. Let S be a finite set of places of k, containing all the archimedean places, such that for some $v \in S$, $G(k_v)$ is noncompact, or, equivalently, G is isotropic over k_v. Let $G_S = \Pi_{v \in S} G(k_v)$. Then the strong approximation property ([27], [21]) implies that

$$G_S \cdot \prod_{v \notin S} P_v \cdot G(k) = G(A).$$

Let Λ be the image of $G(k) \cap (G_S \cdot \Pi_{v \notin S} P_v)$ under the natural projection

$$G_S \cdot \prod_{v \notin S} P_v \rightarrow G_S.$$

Then Λ is a *lattice* in G_S i.e., it is a discrete subgroup of G_S of finite covolume; we will say that it is the *principal* S-*arithmetic subgroup determined by the parahoric subgroups* P_v ($v \notin S$). The object of this section is to compute the volume $\mu_S(G_S/\Lambda)$ with respect to a natural measure μ_S (see 3.6 below).

Let ω_S^* denote the measure on G_S/Λ induced by the product measure $\Pi_{v \in S} \omega_v^*$ on $G_S(= \Pi_{v \in S} G(k_v))$. As

$$G(A) = G_S \cdot \prod_{v \notin S} P_v \cdot G(k),$$

$G(A)/G(k)$ has a natural identification with $G_S \Pi_{v \notin S} P_v/(G(k) \cap G_S \Pi_{v \notin S} P_v)$, and so there is a (principal) fibration $G(A)/G(k) \rightarrow G_S/\Lambda$ with fiber $\Pi_{v \notin S} P_v$. Hence,

$$D_k^{\frac{1}{2}\dim G} \tau_k(G) = \omega_A^*(G(A)/G(k)) = \omega_S^*(G_S/\Lambda) \cdot \prod_{v \notin S} \omega_v^*(P_v).$$

Therefore,

$$\omega_S^*(G_S/\Lambda) = D_k^{\frac{1}{2}\dim G} \tau_k(G) \left(\prod_{v \notin S} \omega_v^*(P_v)\right)^{-1}.$$

3.5. Let v be an archimedean place of k. Let $c_v (\in \mathbf{R}^\times)$ be as in 1.4 and $\gamma_v = | c_v |_v$. We recall that c_v is such that under the Haar measure induced by the invariant exterior form $c_v\, \omega$, any maximal compact subgroup of $R_{k_v/\mathbf{R}}(\mathscr{G})$ ($\mathbf{C}$) has volume 1. We claim that the volume of any maximal compact subgroup of $R_{k_v/\mathbf{R}}(G)$ ($\mathbf{C}$) in the Haar measure induced by the invariant form $c_v\, \omega^*$ is also 1. To prove this, we fix a basis $\mathscr{Y}_1^v, \ldots, \mathscr{Y}_n^v$ (resp. $Y_1^v, \ldots, Y_n^v$) of the Lie algebra $L(\mathscr{G}) \otimes_k k_v$ (resp. $L(G) \otimes_k k_v$) such that with respect to the Killing form $\langle\ ,\ \rangle_v$ on $L(\mathscr{G}) \otimes_k k_v$ (resp. $L(G) \otimes_k k_v$) $\mathscr{Y}_i^v$ is orthogonal to $\mathscr{Y}_j^v$ (resp. Y_i^v is orthogonal to Y_j^v) for all $1 \leq i \neq j \leq n$, and moreover, if v is real,

$$| \langle \mathscr{Y}_i^v, \mathscr{Y}_i^v \rangle_v |_v = 1 = | \langle Y_i^v, Y_i^v \rangle_v |_v \quad \text{for all } i \leq n,$$

and if v is complex, then

$$\langle \mathscr{Y}_i^v, \mathscr{Y}_i^v \rangle_v = 1 = \langle Y_i^v, Y_i^v \rangle_v \quad \text{for all } i \leq n.$$

Let $\mathscr{Y}_v^1, \ldots, \mathscr{Y}_v^n$ (resp. $Y_v^1, \ldots, Y_v^n$) be the dual basis and $\omega_{v,\mathscr{G}}^K = \mathscr{Y}_v^1 \wedge \ldots \wedge \mathscr{Y}_v^n$ (resp. $\omega_{v,G}^K = Y_v^1 \wedge \ldots \wedge Y_v^n$). Let $\theta_v : G \rightarrow \mathscr{G}$ be an isomorphism defined over the algebraic closure $\bar{k}_v (\cong \mathbf{C})$ of k_v such that for all γ in the Galois group of $\bar{k}_v/k_v$, $\theta_v^{-1} \cdot {}^\gamma\theta_v$ is an inner automorphism of G (2.0). Then $\theta_v^*(\omega_{v,\mathscr{G}}^K)$ is defined over k_v ([15: pp. 475-476]),

θ_v induces a Lie algebra isomorphism $L(G) \otimes_k \bar{k}_v \to L(\mathscr{G}) \otimes_k \bar{k}_v$, and as any isomorphism of Lie algebras preserves the Killing form, it follows that $\theta_v^*(\omega_{v,\mathscr{G}}^{\mathsf{K}}) = \pm\, \omega_{v,G}^{\mathsf{K}}$. Now we note that since G is a form of $\mathscr{G}$, the maximal compact subgroup of $R_{k_v/\mathbf{R}}(\mathscr{G})(\mathbf{C})$ and $R_{k_v/\mathbf{R}}(G)(\mathbf{C})$ are isomorphic and hence have equal volume ($= |\,a\,|_v$; where $a = m^{\frac{1}{2}} \prod_{i=1}^{r} \frac{(2\pi)^{m_i+1}}{m_i!}$, m is as in the proof of Theorem 1.6 and $m_1 \leqq \ldots \leqq m_r$ are the exponents (1.5)) with respect to the Haar measures determined by $\omega_{v,\mathscr{G}}^{\mathsf{K}}$ and $\omega_{v,G}^{\mathsf{K}}$ respectively. Now since ω is a multiple of $\omega_{v,\mathscr{G}}^{\mathsf{K}}$, and $\theta_v^*(\omega) = \zeta_v\, \omega^*$, where ζ_v is a root of unity (cf. 2.1), our claim is obvious.

3.6. For any archimedean place v of k, let μ_v be the Haar measure on $G(k_v)$ determined by the invariant exterior form $c_v\, \omega^*$ (c_v as in 1.4), and for v nonarchimedean, let μ_v be the *Tits measure* on $G(k_v)$, i.e., the Haar measure with respect to which every Iwahori subgroup of $G(k_v)$ has volume 1. Of course, $\mu_v = \omega_v^*(I_v)^{-1}\, \omega_v^*$ for all $v \in V_f$; where I_v is an Iwahori subgroup of $G(k_v)$. Let $\mu_S = \prod_{v \in S} \mu_v$ be the product measure on $G_S (= \prod_{v \in S} G(k_v))$; we shall denote the G_S-invariant induced measure on G_S/Λ also by μ_S.

Let ℓ, D_ℓ and $\mathfrak{s}(\mathscr{G})$ be as in 0.2 and 0.4 respectively.

3.7. *Theorem. — We have the following*

$$\mu_S(G_S/\Lambda) = D_k^{\frac{1}{2}\dim G}(D_\ell/D_k^{[\ell:k]})^{\frac{1}{2}\mathfrak{s}(\mathscr{G})} \left(\prod_{v \in V_\infty} \left| \prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v\right) \tau_k(G)\, \mathscr{E};$$

where

$$\mathscr{E} = \prod_{v \in S_f} \frac{q_v^{(r_v + \dim \mathscr{M}_v)/2}}{\#\overline{T}_v(\mathfrak{f}_v)} \cdot \prod_{v \notin S} \frac{q_v^{(\dim \overline{M}_v + \dim \mathscr{M}_v)/2}}{\#\overline{M}_v(\mathfrak{f}_v)},$$

and $S_f = S \cap V_f$.

Proof. — Clearly

$$\mu_S(G_S/\Lambda) = \Big(\prod_{v \in V_\infty} |\,c_v\,|_v\Big)\Big(\prod_{v \in S_f} \omega_v^*(I_v)\Big)^{-1} \omega_S^*(G_S/\Lambda)$$

$$= \Big(\prod_{v \in V_\infty} \gamma_v\Big)\Big(\prod_{v \in S_f} \omega_v^*(I_v)\Big)^{-1} D_k^{\frac{1}{2}\dim G}\, \tau_k(G) \Big(\prod_{v \notin S} \omega_v^*(P_v)\Big)^{-1}$$

(cf. 3.4)

$$= D_k^{\frac{1}{2}\dim G} \prod_{v \in V} \gamma_v \Big(\prod_{v \in S_f} \gamma_v\, \omega_v^*(I_v) \prod_{v \notin S} \gamma_v\, \omega_v^*(P_v)\Big)^{-1} \tau_k(G)$$

$$= D_k^{\frac{1}{2}\dim G}(D_\ell/D_k^{[\ell:k]})^{\frac{1}{2}\mathfrak{s}(\mathscr{G})} \left(\prod_{v \in V_\infty} \left| \prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v\right) \tau_k(G)\, \mathscr{E}$$

(by Theorem 1.6);

where, $$\mathscr{E} = (\prod_{v \in S_f} \gamma_v\, \omega_v^*(I_v) \cdot \prod_{v \notin S} \gamma_v\, \omega_v^*(P_v))^{-1}$$

$$= \prod_{v \in S_f} \frac{q_v^{(r_v + \dim \overline{\mathscr{M}}_v)/2}}{\# \overline{T}_v(\mathfrak{f}_v)} \cdot \prod_{v \notin S} \frac{q_v^{(\dim \overline{M}_v + \dim \overline{\mathscr{M}}_v)/2}}{\# \overline{M}_v(\mathfrak{f}_v)} \quad \text{(by Proposition 2.10)}.$$

This proves the theorem.

3.8. *Remark.* — If $V_\infty \neq \emptyset$, i.e. if k is a number field, then

$$\prod_{v \in V_\infty} \left| \prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v = \left(\prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} \right)^{[k:\mathbf{Q}]}.$$

3.9. *Remark.* — The reductive groups $\overline{\mathscr{M}}_v$, $\overline{M}_v$ and the tori $\overline{T}_v(\subset \overline{M}_v)$ can be described in terms of the local index of $\mathscr{G}/k_v$, of G/k_v and the subset Θ_v of 2.4 (see [33: 3.5]). Thus, in principle, $\mu_S(G_S/\Lambda)$ can be computed, using the formula given by the above theorem, when $\tau_k(G)$ is known, for example, if k is a number field (see 3.3).

3.10. *Remark.* — The factors $\dfrac{q_v^{(r_v + \dim \overline{\mathscr{M}}_v)/2}}{\# \overline{T}_v(\mathfrak{f}_v)}$ $(v \in S_f)$ and $\dfrac{q_v^{(\dim \overline{M}_v + \dim \overline{\mathscr{M}}_v)/2}}{\# \overline{M}_v(\mathfrak{f}_v)}$ $(v \notin S)$, of the Euler product $\mathscr{E}$ in the statement of Theorem 3.7, are all greater than 1. Moreover the former is at least $(q_v + 1)^{-r_v}\, q_v^{\frac{1}{2} r_v(r_v+3)}$. For $v \notin S$, if G is not quasi-split over k_v, then the later factor is at least $(q_v + 1)^{-1}\, q_v^{r_v+1}$ (and if G is anisotropic over k_v, then this factor is $\geqslant (q_v + 1)^{-r_v}\, q_v^{\frac{1}{2} r_v(r_v+3)}$; see Proposition 2.10 (ii), (iv). These observations are crucial for the proof of the finiteness assertions in [4].

3.11. *Remark.* — Let k be a number field and G be such that for some archimedean place v of k, $G(k_v)$ is noncompact and for every nonarchimedean v, G is quasi-split over k_v. We assume that for every nonarchimedean v, P_v is special and whenever G splits over the maximal unramified extension of k_v, it is hyperspecial. Then for all nonarchimedean v, $\overline{M}_v$ is isomorphic to $\overline{\mathscr{M}}_v$ over $\mathfrak{f}_v$, and hence,

$$\frac{q_v^{(\dim \overline{M}_v + \dim \overline{\mathscr{M}}_v)/2}}{\# \overline{M}_v(\mathfrak{f}_v)} = \frac{q_v^{\dim \overline{M}_v}}{\# \overline{M}_v(\mathfrak{f}_v)}.$$

Now let Λ_∞ be the projection of $G(k) \cap (\prod_{v \in V_\infty} G(k_v) \cdot \prod_{v \in V_f} P_v)$ into

$$G_\infty := \prod_{v \in V_\infty} G(k_v).$$

Then

$$\mu_\infty(G_\infty/\Lambda_\infty) = D_k^{\frac{1}{2}\dim G} (D_\ell / D_k^{[\ell:k]})^{\frac{1}{2} s(\mathscr{G})} \left(\prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} \right)^{[k:\mathbf{Q}]} \tau_k(G)\, \mathscr{E};$$

where $\mu_\infty = \mu_{V_\infty}$, and $\mathscr{E} = \prod_{v \in V_f} \dfrac{q_v^{\dim \overline{M}_v}}{\# \overline{M}_v(\mathfrak{f}_v)}$. Using the orders of finite groups of Lie type, given, for example, in [25: Table 1], we easily see that $\mathscr{E}$ is a product of the values of the Dedekind zeta function of k and certain Dirichlet L-functions at the integers $m_i + 1$, $i \leqq r$. If, moreover, the absolute rank of G_∞ equals that of any maximal compact subgroup, then (k is totally real and) using the functional equations of the Dedekind zeta function and Dirichlet L-functions, we get a very concise formula for the volume of G_∞/Λ_∞.

4. Class numbers of absolutely quasi-simple, simply connected groups

4.1. We shall assume in this section that G is *anisotropic* over k. If k is a number field, we assume moreover that $\prod_{v \in V_\infty} G(k_v)$ is *compact*; k is then totally real.

As in 2.2, let $P = (P_v)_{v \in V_f}$ be a coherent collection of parahoric subgroups. It is known that the set $(\prod_{v \in V_\infty} G(k_v) \cdot \prod_{v \in V_f} P_v) \backslash G(A)/G(k)$ of double cosets is finite ([1: Theorem 5.1], [11: 2.2.7 (iii)]); the cardinality of this set is called the *class number of* G *relative to* P and will be denoted by $\mathfrak{c}(P)$.

We shall denote the compact-open subgroup $\prod_{v \in V_\infty} G(k_v) \cdot \prod_{v \in V_f} P_v$, of G(A), by C. We shall use the notation introduced in the preceeding sections. In particular, ω_A^* is the Haar measure on G(A) defined in 3.2.

4.2. We fix representatives $g_i (\in G(A))$, $1 \leqq i \leqq \mathfrak{c}(P)$, of the double cosets in $C\backslash G(A)/G(k)$. Then

$$G(A) = \bigcup_{i=1}^{\mathfrak{c}(P)} C g_i\, G(k)$$

and so

$$\omega_A^*(G(A)/G(k)) = \sum_{i=1}^{\mathfrak{c}(P)} \frac{\omega_A^*(C)}{\# F_i}; \qquad (*)$$

where $F_i = g_i^{-1} C g_i \cap G(k)$ is a finite subgroup of $G(k)$ since $G(k)$ is a discrete subgroup of G(A) and C, and hence also $g_i^{-1} C g_i$, is a compact subgroup. If there is a finite upper bound for the orders of finite subgroups of $G(k)$ (which is the case if k is a number field—this follows, for example, from [29: LG, Chapter IV, Appendix 3, Theorem 1]), let $f = f(G/k)$ be the smallest integer such that the order of any finite subgroup of $G(k)$ is at most f, otherwise let $f = \infty$. Then as

$$\omega_A^*(G(A)/G(k)) = D_k^{\frac{1}{2}\dim G}\, \tau_k(G) \quad \text{(see 3.3)},$$

we conclude from $(*)$ that

$$\mathfrak{c}(P)\, \omega_A^*(C) \geqq D_k^{\frac{1}{2}\dim G}\, \tau_k(G) \geqq f^{-1}\, \mathfrak{c}(P)\, \omega_A^*(C).$$

So,

$$f D_k^{\frac{1}{2}\dim G}\, \tau_k(G)\, (\omega_A^*(C))^{-1} \geqq \mathfrak{c}(P) \geqq D_k^{\frac{1}{2}\dim G}\, \tau_k(G)\, (\omega_A^*(C))^{-1}.$$

Now we shall determine $(\omega^*_A(C))^{-1}$ using the results proved in §§ 1-3. Obviously,

$$\begin{aligned}(\omega^*_A(C))^{-1} &= (\prod_{v\in V_\infty}\omega^*_v(G(k_v))\cdot\prod_{v\in V_f}\omega^*_v(P_v))^{-1}\\ &= \prod_{v\in V}\gamma_v\cdot(\prod_{v\in V_\infty}\gamma_v\,\omega^*_v(G(k_v))\cdot\prod_{v\in V_f}\gamma_v\,\omega^*_v(P_v))^{-1}\\ &= (D_\ell/D_k^{[\ell:k]})^{\frac{1}{2}s(\mathscr{G})}\left(\prod_{v\in V_\infty}\left|\prod_{i=1}^{r}\frac{m_i!}{(2\pi)^{m_i+1}}\right|_v\right)(\prod_{v\in V_f}\gamma_v\,\omega^*_v(P_v))^{-1}\end{aligned}$$

since according to Theorem 1.6,

$$\prod_{v\in V}\gamma_v = (D_\ell/D_k^{[\ell:k]})^{\frac{1}{2}s(\mathscr{G})}\left(\prod_{v\in V_\infty}\left|\prod_{i=1}^{r}\frac{m_i!}{(2\pi)^{m_i+1}}\right|_v\right),$$

and it is clear from the definition of γ_v (see 3.5) that as $G(k_v)$ is compact for all $v\in V_\infty$, $\gamma_v\,\omega^*_v(G(k_v)) = 1$ for $\forall\, v\in V_\infty$.

Let

$$\begin{aligned}\zeta(P) &= (\prod_{v\in V_f}\gamma_v\,\omega^*_v(P_v))^{-1}\\ &= \prod_{v\in V_f}\frac{q_v^{(\dim\overline{M}_v+\dim\mathscr{M}_v)/2}}{\#\overline{M}_v(\mathfrak{f}_v)}\quad\text{(by Proposition 2.10 (ii)).}\end{aligned}$$

Then

$$\omega^*_A(C)^{-1} = (D_\ell/D_k^{[\ell:k]})^{\frac{1}{2}s(\mathscr{G})}\left(\prod_{v\in V_\infty}\left|\prod_{i=1}^{r}\frac{m_i!}{(2\pi)^{m_i+1}}\right|_v\right)\zeta(P).$$

Therefore, we conclude the following:

4.3. *Theorem.* — *Let*

$$c(P) = \#((\prod_{v\in V_\infty}G(k_v)\cdot\prod_{v\in V_f}P_v)\backslash G(A)/G(k)).$$

Then

$$c(P) \geqq D_k^{\frac{1}{2}\dim G}(D_\ell/D_k^{[\ell:k]})^{\frac{1}{2}s(\mathscr{G})}\left(\prod_{v\in V_\infty}\left|\prod_{i=1}^{r}\frac{m_i!}{(2\pi)^{m_i+1}}\right|_v\right)\tau_k(G)\,\zeta(P),$$

and

$$c(P) \leqq f D_k^{\frac{1}{2}\dim G}(D_\ell/D_k^{[\ell:k]})^{\frac{1}{2}s(\mathscr{G})}\left(\prod_{v\in V_\infty}\left|\prod_{i=1}^{r}\frac{m_i!}{(2\pi)^{m_i+1}}\right|_v\right)\tau_k(G)\,\zeta(P);$$

where

$$\zeta(P) = \prod_{v\in V_f}\frac{q_v^{(\dim\overline{M}_v+\dim\mathscr{M}_v)/2}}{\#\overline{M}_v(\mathfrak{f}_v)}.$$