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HYPERBOLIC 4-MANIFOLDS AND TESSELATIONS

by NICOLAAS H. KUIPER (1)

Dedicated to René Thom.

1. Introduction and survey

The remarkable discovery of complete hyperbolic manifold structures on non trivial plane bundles over oriented closed surfaces and of Moebius structures (that is conformally flat structures) on the corresponding non-trivial circle bundles by M. Gromov, B. Lawson and W. Thurston in the preceding paper [GLT], can be generalized by putting emphasis on *tesselations and discrete actions of groups* $\Gamma_{\nu, n}$. This makes the constructions more transparent and yields moduli, in particular rigidity *a)* for certain tesselated hyperbolic d-manifolds, *b)* for certain tesselated Moebius $d-1$-manifolds (called tesselated $\mathbf{CP}^1$-structures for $d-1=2$) (see [Go]), and therefore moduli and rigidity for discrete actions of groups $\Gamma_{\nu, n}$ on the hyperbolic space $\mathbf{H}^d$ and certain hyperbolic d-manifolds Σ^d, as well as on S^{d-1} and certain Moebius $d-1$-manifolds M^{d-1}. The groups $\Gamma_{\nu, n}$ are generated by involutions $g : g^2 = e$.

Our main interest is in dimension 4. Higher dimensions are simpler. To set the stage we describe in § 2 the *classical tesselations* of $\mathbf{H}^2$ [Cox] by ν-gons congruent to one of them, call it P, and $n \geqslant \nu$ meeting at each vertex. The *group* $\Gamma_{\nu, n}$ is generated by the ν involutions (half turns) about the middles of the sides of P as fixed points. We will see that necessarily the sum of the angles of P is $\Delta = 2\pi\nu/n$. Sufficient conditions are obtained by adding symmetry conditions on P in case the greatest common divisor $\gcd(\nu, n)$ of ν and n is $< \nu$. Tesselation and action are *rigid* if and only if $\gcd(\nu, n) = 1$. Then the tiles (ν-gons) must be *regular*. Then also the " orbifold " $\mathbf{H}^2/\Gamma_{\nu, n}$ is rigid.

If $\Gamma \subset \Gamma_{\nu, n}$ is of finite index and " acts " freely on $\mathbf{H}^2$, then $\Sigma^2 = \mathbf{H}^2/\Gamma$ is a *Riemann surface* tesselated by, say, $V\nu$-gons, with $\Gamma_{\nu, n}$ acting as group of symmetries which are all orientation preserving (!). A very simple example is for $n = 2\nu$, with ν odd (resp.

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$n = \nu$ even). The tiling then has a map colouring with two colours and we can choose for $\Gamma \subset \Gamma_{\nu,n}$ the freely acting subgroup of index 2 of the colour preserving elements of $\Gamma_{\nu,n}$.

In § 3 *we carry all of this over to higher dimensions*, to tesselations by " ν-Gons " P, n at each " Vertex ", of the hyperbolic space $\mathbf{H}^d$ (resp. $\Sigma^d = \mathbf{H}^d/\Gamma$) for $d = 4$ and 3 with the same abstract groups $\Gamma_{\nu,n}$ acting. There is one more necessary and sufficient condition for tiling expressed in terms of a global invariant for P called *torsion*. There also are included Moebius tesselations in

$$S^{d-1} = \partial_\infty \mathbf{H}^d \quad \text{and} \quad M^{d-1} = \partial_\infty \Sigma^d.$$

For $d = 3$ we get special $\mathbf{C}P^1$-tesselated surfaces with injective development maps into $\mathbf{C}P^1$ and Julia-curve limit sets. For dimension $d = 4$ we reproduce in particular the examples of [GLT].

By comparing deformation space dimensions in table 1 we conclude ([GLT]) that limit sets $\mathcal{J} \subset S^{d-1}$, self similar embedded circles called *Julia knots*, are in general not round circles and may be everywhere knotted hence nowhere tame. Any manifold Σ^4 we construct is homotopy equivalent to its *polyhedral core-surface* Σ, and if $\mathcal{J}$ is unknotted, then Σ^4 is a smooth 2-plane bundle over a smooth surface Σ^2. The tesselations and the actions of $\Gamma_{\nu,n}$ are rigid if and only if $\gcd(\nu, n) = 1$. Also the Julia knot $\mathcal{J}$ is then rigid with respect to the group of symmetries $\Gamma_{\nu,d}$ and the tiles then *must* be " regular " (homogeneous).

Our main observations are summarized in Theorems 1 to 6.

With their " template method " the authors of [GLT] discover the construction of polyhedral surfaces in $\mathbf{H}^4$ which in our approach appear as core-surfaces Σ_P. They study in detail the case where the template is a " regular " (homogeneous) standard ν-gonal (p, q)-torus-knot in a metric unit 3-sphere S^3. They calculate the non zero *normal* Euler number $\chi^\perp$ of the plane bundle Σ^4 when the regular torus-knot is unknotted $q = 1$. In § 4 we explain this relation of our paper with [GLT], in particular in Theorem 4. The formulas (4.2), (4.4) and (4.7) for the normal Euler number $\chi^\perp$ survive as formulas for the *self intersection number* of the polyhedral surface Σ in Σ^4, in case Σ is locally knotted.

In § 5 we use the formulas of [GLT] and obtain simple explicit examples for all plane-bundles for which $|\chi| \geqslant 3|\chi^\perp|$, where χ is the even Euler number of a closed surface and $\chi^\perp$ the Euler number of a 2-plane bundle over Σ. See Theorem 6.

In § 6 we elaborate the case of a complete hyperbolic 4-manifold tesselated by two regular 13-Gons, that has a locally knotted core surface Σ with $\chi^\perp = -7$, $\chi = -10$.

2. Tesselations of $\mathbf{H}^2$ and of Riemann surfaces Σ^2; actions of $\Gamma_{\nu,n}$

There are two models of the *hyperbolic d-space* $\mathbf{H}^d$ onto the interior D^d of the unit $d-1$-sphere

$$S^{d-1}_{(1)} = \{ x \in \mathbf{R}^4 : \| x \|^2 = \Sigma x_i^2 = 1 \} \subset \mathbf{R}^d$$

in the *euclidean space* $\mathbf{R}^d : \mathbf{H}^d_{\text{conf}}$ and $\mathbf{H}^d_{\text{proj}}$. For $d = 2$ they are related by stereographic and orthogonal projection of a lower half-sphere into the horizontal plane containing its boundary as illustrated in fig. 1 *a)*. In $\mathbf{H}^d_{\text{proj}}$ (fig. 1 *b)*) the convex sets and the straight lines are the same as in $\mathbf{R}^d$. In $\mathbf{H}^d_{\text{conf}}$ (fig. 1 *c)*) the straight lines are the intersections with those circles in $\mathbf{R}^d$ that meet $S^{d-1}_{(1)}$ orthogonally. Angles (between curves) are the same as in $\mathbf{R}^d$.

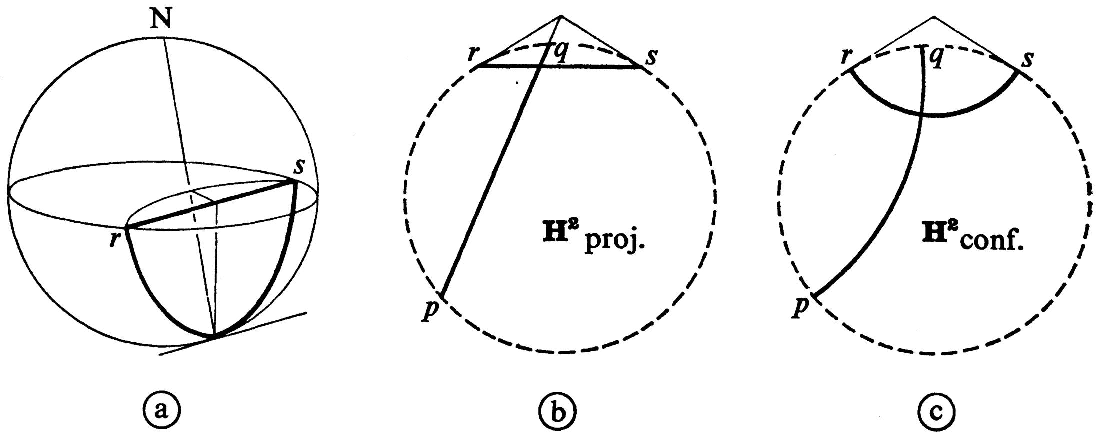


FIG. 1. — *a)* Stereographic and orthogonal projection
b) c) Orthogonal lines *pq* and *rs*

The groups of motions of $\mathbf{H}^d$ are the projective transformations of $D \subset \mathbf{R}^d \subset \mathbf{RP}^d$ for $\mathbf{H}^d_{\text{proj}}$, and they are the conformal transformations of $D \subset \mathbf{R}^d \subset S^d$ for $\mathbf{H}^d_{\text{conf}}$. In this case we can compactify $\mathbf{R}^d$ by one point to get a Moebius sphere S^d, and we can consider the group of isometries of $\mathbf{H}^d_{\text{conf}}$ as a subgroup of the Moebius group of S^d. These groups induce *Moebius groups* in the boundary, the $d - 1$-sphere and *Moebius space* $S^{d-1} = \partial\mathbf{H}^d = \partial_\infty \mathbf{H}^d$, also denoted $\mathbf{RP}^1$ for $d - 1 = 1$ and $\mathbf{CP}^1$ for $d - 1 = 2$. The groups are then the rational transformations of $\mathbf{RP}^1$ and $\mathbf{CP}^1$. A common notation for the group is $SO(d, 1)$.

By a *tesselation* of a locally homogeneous space in the sense of Ehresmann [Ehr], like a hyperbolic space or a Moebius space, we mean a covering by mutually equivalent (isometric resp. Moebius equivalent) connected pieces with boundary, called tiles, whose non-void interiors are disjoint. By way of introduction recall that the Euclidean plane has a tesselation obtained from *any* triangle or *any* quadrangle by the group of isometries generated by half turns around the middles of the edges as fixed points!

A general *tesselation* $T_{\nu, n}$ of the *hyperbolic plane* by mutually congruent convex ν-gons with $n \geqslant \nu$ meeting at each vertex is obtained as follows: Start from one convex ν-gon P with vertices v_i, edges $[v_i, v_{i+1}]$ and angles δ_i, $i \bmod \nu$. Then fit a congruent ν-gon by an orientation preserving involution g_i°, $(g_i^\circ)^2 = e =$ identity, that is a rotation over π, *around the centre of the side* $[v_i, v_{i+1}]$ as fixed point. If P has sides of mutually different length then this is the only way to start a tesselation. We can fit more copies around the vertex $v = v_1$ by analogous involutions denoted g_2, g_3, etc. (See fig. 2 *a), b), c)*). Note that the involutions around the centres of the sides of **P** are respectively

$$(2.1) \qquad g_1^\circ = g_1, \qquad g_2^\circ = g_1 g_2 g_1, \qquad g_3^\circ = g_1 g_2 g_3 g_2 g_1, \ldots, \qquad g_\nu^\circ = g_1 g_2 \ldots g_\nu \ldots g_2 g_1.$$

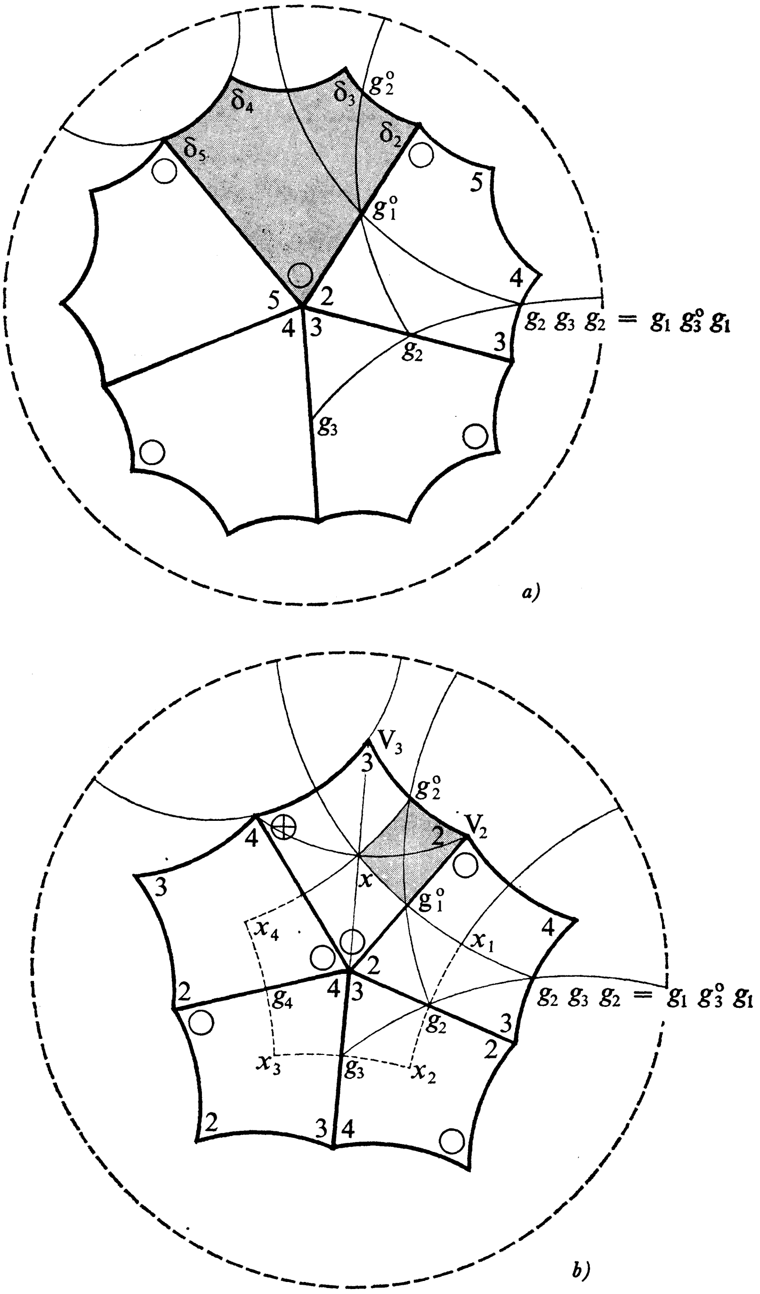


FIG. 2. — *a)* $(\nu, n) = (5. 5)$; *b)* $(\nu, n) = (4, 5)$, $x_i = g_i \ldots, g_2 g_1 x$ rigid

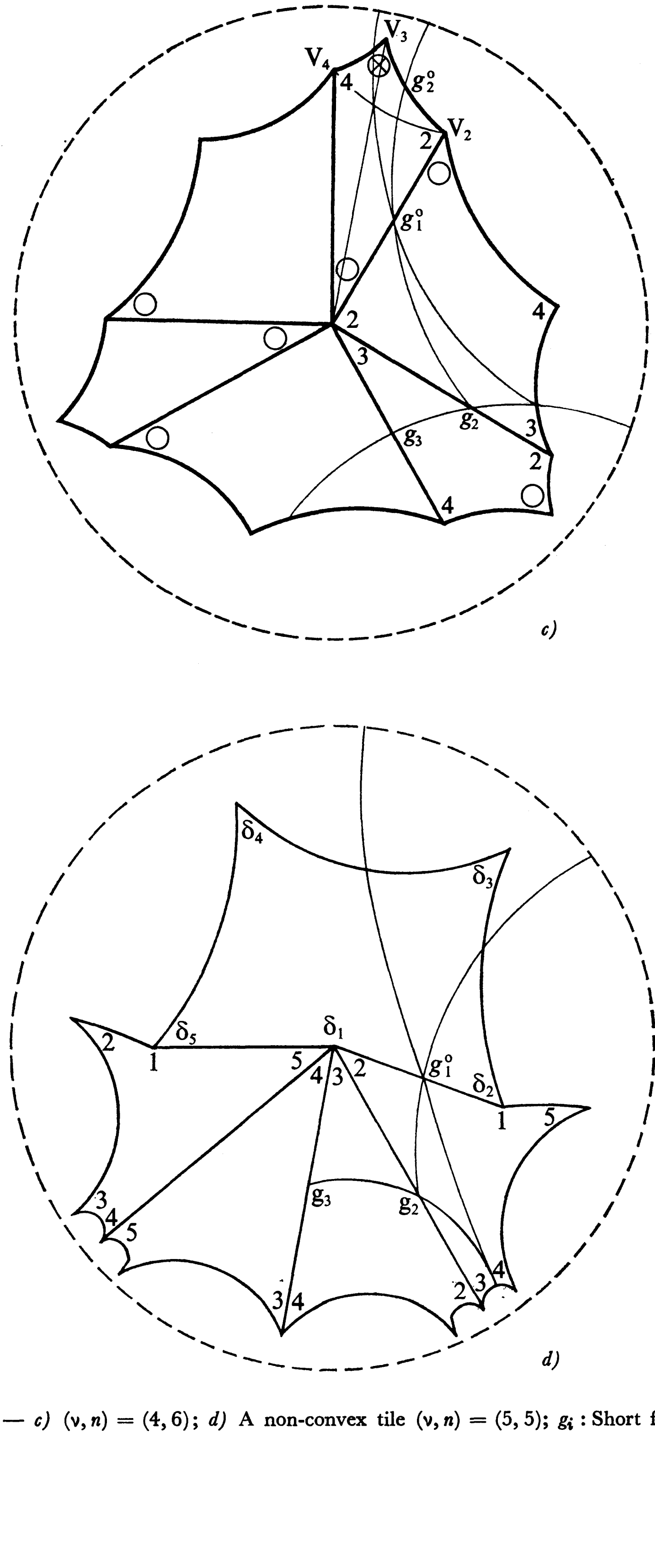


FIG. 2. — *c)* $(\nu, n) = (4, 6)$; *d)* A non-convex tile $(\nu, n) = (5, 5)$; g_i : Short for $[g_i]^{\mathrm{FIX}}$

Note that $g^\circ_{\nu+i} = g^\circ_i$. If our tesselation succeeds, then this set of involutions, or equivalently the set $g_1, \ldots, g_\nu$, generate a *group* $\Gamma_{\nu,n}$ *of isometries* of $\mathbf{H}^2$, which contains all involutions about all side centers of the tesselation.

As the Euler characteristic of a tile is $\chi(\text{tile}) = 1$, and the Gauss-curvature is $K = -1$, the theorem of Gauss-Bonnet yields:

$$(2.2) \qquad 2\pi\chi(\text{tile}) = \int K\,d\,\text{area} + \Sigma^\nu_{i=1}(\pi - \delta_i),$$
$$\text{hence} \quad (\nu - 2)\,\pi - \Sigma^\nu_i\,\delta_i = \text{area(tile)} > 0.$$

Therefore the *angle sum* $\Delta = \Sigma^\nu_i\,\delta_i$ is bounded between 0 and $(\nu - 2)\,\pi$. It is obvious (push in at one vertex) that Δ is an analytic function with no critical points on the 2ν-dimensional manifold of convex ν-gons in $\mathbf{H}^2$.

After ν steps around the vertex $v = v_1$, this point v has the same position with respect to the new polygon as with respect to P. Therefore $g_\nu \ldots g_2 g_1$ is a rotation around v, and we have more generally:

Lemma **1.** — *The product*

$$(2.3) \qquad g_{i\nu} \ldots g_2 g_1 = (g_\nu \ldots g_2 g_1)^i, \quad i \geqslant 0,$$

is a rotation in $\mathbf{H}^2$ *around* $v = v_1$.

After n steps we must have the first selfoverlap of interiors of tiles with *complete incidence with the original tile* $(g_n \ldots g_2 g_1)\,(\mathrm{P}) = \mathrm{P}$, and more generally for any integer $j \geqslant 0$:

Lemma **2.** — *We have*

$$(2.4) \qquad (g_{jn} \ldots g_2 g_1)\,(\mathrm{P}) = (g_n \ldots g_2 g_1)^j\,(\mathrm{P}) = \mathrm{P} \quad \textit{for } j \geqslant 0,$$

and also

$$(2.5) \qquad \delta_1 + \delta_2 + \ldots + \delta_n = 2\pi, \qquad \delta_i = \delta_j \quad \textit{for } j = i \bmod \nu.$$

From the geometry expressed in (2.2, 3, 4, 5) follows

$$(2.6) \qquad n \geqslant \nu \geqslant 5, \quad \text{or} \quad n > \nu \geqslant 4, \quad \text{or} \quad n \geqslant 7 > 3 = \nu.$$

Let $k = \gcd(\nu, n)$ be the greatest common divisor of ν and n and put

$$(2.7) \qquad \nu = \ell k, \qquad n = mk, \qquad \gcd(\ell, m) = 1.$$

Then, for $i = m \geqslant 1$, $j = \ell \geqslant 1$, Lemmas 1 and 2 yield the identity

$$(2.8) \qquad e = (g_\nu \ldots g_2 g_1)^m = (g_n \ldots g_1)^\ell = e,$$

as this isometry leaves fixed $v = v_1$ as well as P. We conclude:

Lemma **3.** — *The rotation* $g_\nu \ldots g_2 g_1$ *has order* m, *the rotation* $g_n \ldots g_2 g_1$ *has order* ℓ. *The polygon* P *has rotation symmetry of order* ℓ. *In particular, it has no imposed symmetry for* $\ell = 1$, i.e. $\gcd(\nu, n) = \nu$.

We now distinguish various cases mainly by the value of $\gcd(\nu, n) = k$.

I. *The case* $n = \nu = k \geqslant 5$, $\ell = 1$. — See fig. 2 *a)* for $\nu = n = 5$. Then $\Delta = 2\pi$, $g_\nu \dots g_2 g_1 = e$ (2.4, 5 I) and there is no obstruction to continue tiling at each new vertex and filling in the plane. The tesselations so obtained are isotopic and can be parametrised modulo isometries of $\mathbf{H}^2$, by the initial tiles modulo isometry, that is by a family which is an open subset of a real algebraic variety of dimension $2(\nu - 2)$. To get this dimension, fix one vertex 0 of a tile $P \subset \mathbf{H}^2_{\text{proj}} \subset \mathbf{R}^2$ and a half-line with origin 0 containing a side of P, move the other vertices preserving convexity, then multiply $\mathbf{R}^2$ by the unique scalar which restores the condition $\Delta = 2\pi$. The dimension of the family $\mathscr{M}(\Gamma_{\nu,\nu}, \mathbf{H}^2)$ of all discrete representations of the group $\Gamma_{\nu,\nu}$ in $\mathbf{H}^2$ found in this way, is obtained by substracting 2, as only the centres of the sides count ($g_\nu \dots g_2 g_1 = e$!):

(2.9) $$\dim \mathscr{M}(\Gamma_{\nu,\nu}, \mathbf{H}^2) = 2\nu - 6.$$

Note that $\Gamma_{\nu,\nu}$ acts simply transitively on the tiles of the tesselation.

Observe also that $\mathscr{M}(\Gamma_{\nu,\nu}, \mathbf{H}^2)$ has a stratification so that any two points in any open top-dimensional stratum represent non isometric $\Gamma_{\nu,\nu}$-representations.

Next, let in general $\Gamma \subset \Gamma_{\nu,n}$ be a subgroup of finite index acting freely on $\mathbf{H}^2$. Then $\Sigma^2 = \mathbf{H}^2/\Gamma$ is a Riemann surface $(K = -1)$ *tesselated by* V *congruent tiles* P. The family of such tesselated structures on the smooth surface Σ^2 has dimension $2\nu - 4$ for a given $\Gamma \subset \Gamma_{\nu,\nu}$, and the family of actions of $\Gamma_{\nu,\nu}$ on Σ^2 has dimension (as for $\mathbf{H}^2$)

(2.9) $$\dim \mathscr{M}(\Gamma_{\nu,\nu}, \Sigma) = 2\nu - 6.$$

II. *The case* $n = m\nu = mk, \ell = 1, m \geqslant 2, n \geqslant 8$. Then

$$\Delta = 2\pi/m = 2\pi\nu/n, \qquad g_n \dots g_2 g_1 = e \quad \text{and} \quad g_\nu \dots g_2 g_1$$

is a rotation of angle $2\pi/m$. Given the value of Δ and no other condition, there is again no obstruction to continue tiling at each new vertex and filling in the plane. The family of tesselations so obtained has dimension $2\nu - 4$. As the vertices such as v are defined in terms of the generators for $i = 1$ (see 2.3)), the family of tesselations has the same dimension as the family of representations, in $\mathbf{H}^2$ as well as in Σ, namely

(2.9) $$\dim \mathscr{M}(\Gamma_{\nu,m\nu}, \mathbf{H}^2) = \dim \mathscr{M}(\Gamma_{\nu,m\nu}, \Sigma) = 2\nu - 4.$$

III. *The case* $\gcd(\nu, n) = k = 1$, $\ell = \nu$. See fig. 2 *b)* for $(\nu, n) - (4, 5)$. Here $g_n \dots g_2 g_1 \neq e$ is a rotation sending P to P, and ν is the smallest for which

$$(g_n, \dots, g_1)^\nu = e.$$

The fixed point of this rotation, denoted x, is the centre of P which is (up to isometry) the *unique regular* ν*-gon with* angles $\delta_1 = \dots = \delta_\nu = \delta = 2\pi/n$,

$$\Delta = \Sigma_{i=1}^{\nu} \delta_i = 2\pi/m = 2\pi\nu/n.$$

The action is *rigid*:

$$\dim \mathcal{M}(\Gamma_{\nu,n}, \mathbf{H}^2) = \dim \mathcal{M}(\Gamma_{\nu,n}, \Sigma^2) = 0. \tag{2.9}$$

The convex polygon P^* with vertices $x, x_1 = g_1 x, x_2 = g_2 x_1, x_3 = g_3 x_2, \ldots, x_n = x$, see fig. 2 *b)*, is also regular, has centre v and can be used for a dual tesselation with νn-gons at every vertex. The angles are $2\pi/\nu$.

IV. *The case* $1 < \gcd(\nu, n) = k < \nu < n$. — See fig. 2 *c)* for $(\nu, n) = (4, 6)$, $k = 2$. Here $g_n \ldots g_2 g_1 \neq e$ is a rotation around x carrying P to P, and ℓ is the smallest integer for which $(g_n \ldots g_2 g_1)^\ell = e$. P has rotational symmetry of order ℓ, $\Delta = 2\pi\nu/n = 2\pi\ell/m$. One easily finds

$$\dim \mathcal{M}(\Gamma_{\nu,n}, \mathbf{H}^2) = \dim \mathcal{M}(\Gamma_{\nu,n}, \Sigma^2) = 2k - 2. \tag{2.9}$$

We summarize and complete in

Theorem **1.** — *Let* $\mathrm{T}_{\nu,n}$ *be a tesselation of* $\mathbf{H}^2$, *or of a closed orientable surface* $\Sigma = \mathbf{H}^2/\Gamma$, *by congruent* ν*-gons called tiles, with angles* $\delta_1, \ldots, \delta_\nu$, n *tiles meeting at each vertex, invariant under the group* $\Gamma_{\nu,n} \supset \Gamma$ *as defined above, that is generated by involutions about the side centres of one tile* P. *Let* $k = (\nu, n)$ *be the* gcd *of* ν *and* n, $\nu = \ell k$, $n = mk$. *Then*:

a) The total angle of a tile is

$$\Delta = \delta_1 + \ldots + \delta_\nu = 2\pi\nu/n, \tag{2.5}$$

the area of a tile is $(\nu - 2)\,\pi - \dfrac{\nu}{n}\,2\pi$.

b) If $\gcd(\nu, n) = \nu = k$ *then* P *is arbitrary except for* (2.5). *If* $\gcd(\nu, n) = k < \nu$ *then* P *has a center of symmetry for rotations of order* $\ell = \nu/k$. *If* $\gcd(\nu, n) = k = 1$ *then* P *is unique and regular.*

c) $\dim \mathcal{M}(\Gamma_{\nu,n}, \mathbf{H}^2) \quad \dim \mathcal{M}(\Gamma_{\nu,n}, \Sigma)$
$= 2k - 6$ *for* $n = \nu = k \geqslant 5$,
$= 2k - 4$ *for* $n = m\nu$, $m \geqslant 2$, $n \geqslant 8$,
$= 2k - 2$ *for* $1 \leqslant k = \gcd(\nu, n) < \nu$.

In particular.

d) The action of $\Gamma_{\nu,n}$ *is rigid for* $k = \gcd(\nu, n) = 1$.

e) Each family or isotopy class of tesselations $\mathrm{T}_{\nu,n}$ *is connected and contains one tesselation by regular* ν*-gons with angles* $\delta = 2\pi/n$.

Non-convex tiles. — We have not really used the convexity of the polygon P in our arguments. The condition $\Delta = 2\pi\nu/n$, together with the rotation symmetry of order ℓ for $k = \gcd(\nu, n) < \nu = k\ell$, permits one and at most one reentrant angle $\delta_1 : \pi \leqslant \delta_1 < 2\pi$, and this only if $n = \nu \geqslant 5$. *The families* $\mathcal{M}(\Gamma_{\nu,n}, \mathbf{H}^2)$ *and* $\mathcal{M}(\Gamma_{\nu,n}, \Sigma^2)$ *should be enlarged correspondingly*. See fig. 2 *d)* for an example.

Deformation and $\Gamma_{\nu,n}$*-module dimension.* — Let $\Gamma_{\nu,n}$ denote a specific action for a specific tesselation $T_{\nu,n}$, obtained from a specific polygon P. If $\gcd(\nu, n) = \nu$ then $F_{\nu,n} = P$ is a *fundamental domain* in $\mathbf{H}^2$ for the action of $\Gamma_{\nu,n}$. That means it covers the quotient space $\mathbf{H}^2/\Gamma_{\nu,n}$ completely and every interior point of $F_{\nu,n}$ is met exactly once. If $\gcd(\nu, n) = k < \nu = k\ell$, then P has a centre x, and as *fundamental domain* we can take the polygon $F_{\nu,n}$ with successive vertices $x, g_1^\circ, v_2, g_2^\circ, \ldots, g_{1+k}^\circ$. See fig. 2 *b)*. Let $F_{\nu,n}^\nabla, T_{\nu,n}^\nabla, \Gamma_{\nu,n}^\nabla, P^\nabla$ be a second set of data for a tesselation in the same family. Consider a diffeomorphism $h : F_{\nu,n} \to F_{\nu,n}^\nabla$ of fundamental domains which respects the correspondance of vertices and sides and the identification of the isomorphic *abstract* groups $\Gamma_{\nu,n}$ and $\Gamma_{\nu,n}^\nabla$. Then h can be extended in a unique way to a homeomorphism $h : \mathbf{H}^2 \to \mathbf{H}^2$ for which

$$\Gamma_{\nu,n}^\nabla = h\Gamma_{\nu,n}h^{-1}.$$

Note that h is *quasi-conformal.*

Vice versa, let $h : \mathbf{H}^2 \to \mathbf{H}^2$ be a homeomorphism. Given $\Gamma_{\nu,n}$ we obtain an action by homeomorphisms $\Gamma_{\nu,n}^\dagger = h\Gamma_{\nu,n}h^{-1}$, generated by involutory homeomorphisms $hg_i^\circ h^{-1}$, $i = 1, \ldots, \nu$, and with all the consequences we had for $\Gamma_{\nu,n}$. Suppose $\Gamma_{\nu,n}^\dagger$ is an action by isometries of $\mathbf{H}^2$. Then $\Gamma_{\nu,n}^\dagger$ is called *module equivalent* to $\Gamma_{\nu,n}$. Clearly h sends the points (defined by involutions) $g_1^\circ, \ldots, g_\nu^\circ$ to analogous points for $\Gamma_{\nu,n}^\dagger$. The "consequences" are now in terms of isometries, e.g. involutions and periodic rotations like (2.3).

Now suppose the point $hg_i^\circ h^{-1}$ is very near g_i° for $i = 1, \ldots, \nu$. Then these points form a configuration in $\mathbf{H}^2$ belonging to some tesselation in the family, say $T_{\nu,n}^\nabla$. But by continuation by reflections in side centres we then see that all vertices of suitable fundamental domains $F_{\nu,n}^\dagger$ and $F_{\nu,n}^\nabla$ coincide, and so do all their images under $\Gamma_{\nu,n}^\dagger$ and $\Gamma_{\nu,n}^\nabla$. So the actions of $\Gamma_{\nu,n}^\dagger$ and $\Gamma_{\nu,n}^\nabla$ coincide and the family which we denoted by $\mathscr{M}(\Gamma_{\nu,n}, \mathbf{H}^2)$ contains with any $\Gamma_{\nu,n}^\nabla$ all nearby actions $\Gamma_{\nu,n}^\dagger$.

Let us now consider a one-parameter family of actions $\Gamma_{\nu,n}(t)$ beginning with a tesselation action for $t = 0$ and such that, for some value of t, the action does not belong to a tesselation. Then there is a smallest such value t_0. The tesselations and the ν-gon for $0 \leqslant t < t_0$ would converge to a situation where at least one of the angles of the ν-gon is $\delta_i = 0$, and the action $\Gamma_{\nu,n}(t)$ degenerates for $t \to t_0$. Therefore

Theorem **1 A.** — *Deformation components of our tesselations give full deformation components of discrete actions of* $\Gamma_{\nu,n}$ *on* $\mathbf{H}^2$.

Example 1 (see Figure 3). For odd $\nu = 2g + 1 \geqslant 5$, $n = 2\nu$, there is a tesselated surface $\Sigma^2 = \mathbf{H}^2/\Gamma$, where $\Gamma \subset \Gamma_{\nu,2\nu}$ is of index 2 and consists of the colour preserving elements of the map colouring with two colours of the tesselation in $\mathbf{H}^2$. The surface Σ^2 is the double covering of $S^2 = \mathbf{H}/\Gamma_{\nu,n}$ branched at "the middles of the sides of P" and at one more point, represented by each of the vertices of P. A fundamental domain is $P \cup g_1 P$ and the corresponding tiling is seen for $\nu = 5$ in Figure 3. The involution g_1 acts on Σ^2, it interchanges the two tiles and has $\nu + 1$ fixed points. We use this example generalized to dimension 4 in § 5 and § 6.

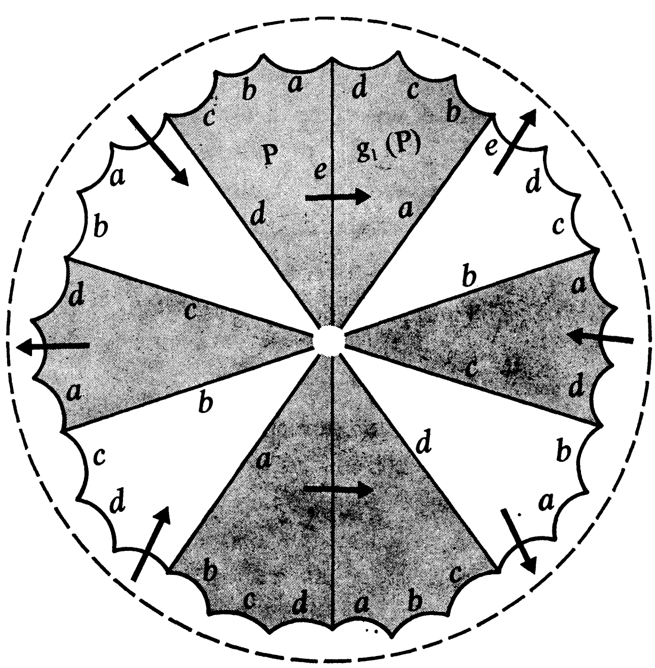


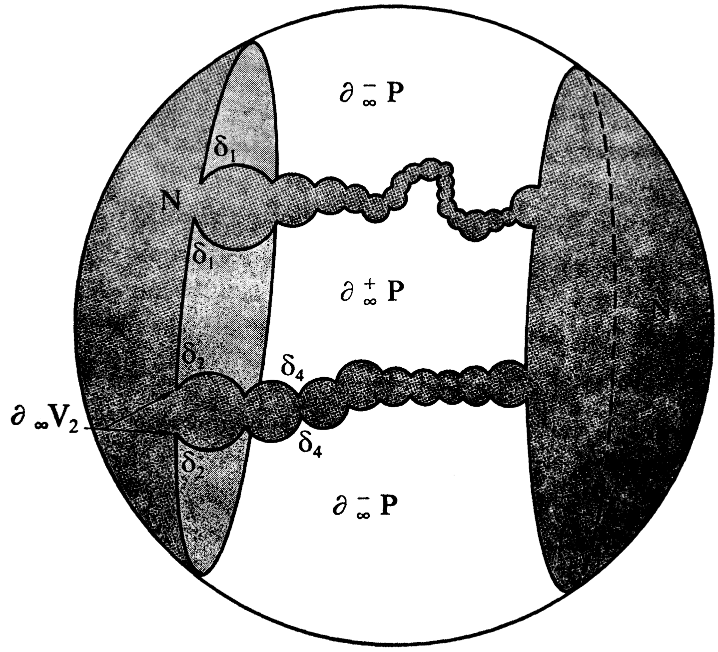
FIG. 3. — Fundamental domain $P \cup g_1(P)$ for Σ^2, a convex $4g$-gon for $g \geqslant 2$

3. Tesselations and actions of $\Gamma_{\nu, n}$ in $\mathbf{H}^d$

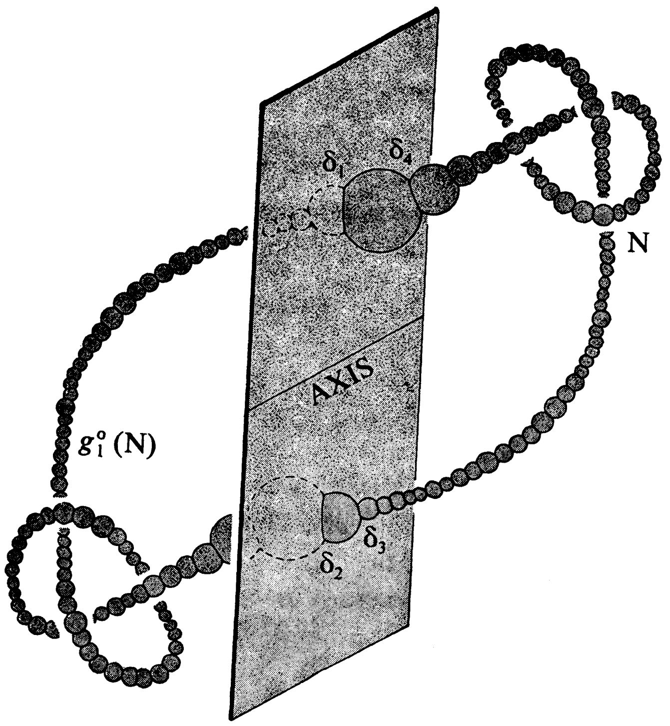
3.1. ν-Gons as tiles. — In this section we formulate *the case $d = 4$ in detail*, but cover the *case $d = 3$* as well *by side remarks*. A *convex ν-Gon* $P \subset \mathbf{H}^4_{\text{proj}} \subset \mathbf{R}^4$ is by definition the intersection of ν successive half-spaces bounded by hyperbolic 3-planes, such that the boundary ∂P is the union of ν successive slices called *Sides* and denoted by $[v_i, v_{i+1}]$, $i \bmod \nu$. The Side $[v_i, v_{i+1}]$ is bounded by two complete 2-planes v_i and v_{i+1}, called *Vertices*. The closures of these Vertices in $\mathbf{R}^4$ are assumed disjoint. The interior angle of P along v_i is δ_i, and the *total angle* of the ν-Gon is

$$(3.1) \qquad \Delta = \delta_1 + \ldots + \partial_\nu.$$

We want to tesselate $\mathbf{H}^4$ with ν-Gons as tiles. For a special example we can start from a tesselation ν-gon in $\mathbf{H}^2 \subset \mathbf{H}^4$, as in § 2, figure 2, and define the ν-Gon in $\mathbf{H}^4$ by 3-planes orthogonal to $\mathbf{H}^2$ through the sides of the given ν-gon. Figure 2 illustrates the relations (the same as before) between the generators g_1°, g_1, of $\Gamma_{\nu, n}$ which we will define now in the higher dimensional context. Given a ν-Gon P, the intersection $\mathbf{H} = \partial_\infty P$ of its closure $\overline{P}$ in $\mathbf{R}^4$ with $S^3 = \partial_\infty \mathbf{H}^4$ is called a *Moebius ν-Gon* in S^3. Its complement in S^3 is a *collar* N *of successive round balls (" beads ") in* S^3, whose successive boundaries meet


$\partial_\infty^- P$
δ_1
N
δ_1
$\partial_\infty^+ P$
δ_3
δ_4
$\partial_\infty V_2$
δ_4
δ_2
$\partial_\infty^- P$
N

a)


δ_1
δ_4
N
AXIS
g_1^o (N)
δ_3
δ_2

b)

Fig. 4

in ν successive circles $\partial_\infty v_i$, and no other two beads meet. The case of dimension $d = 3$ is illustrated in figure 4 *a)*, where

$$\partial_\infty \mathrm{P} = \partial_\infty^+ \mathrm{P} \cup \partial_\infty^- \mathrm{P} = \mathrm{Q} = \mathrm{Q}^+ \cup \mathrm{Q}^-$$

is a pair of *collared Moebius* ν*-gons* in $\mathrm{S}^2 = \mathbf{CP}^1$ with circle arcs as sides and each collared by the collar $\mathrm{N} \subset \mathrm{S}^2$.

For $d = 4$ the collar " necklace " N is a topological solid torus with a well defined *isotopy knot class* [N] possibly unknotted (and a unique isotopy class $[\gamma]$ for any " core curve " $\gamma \subset \mathrm{N}$ (see figure 4 *b)*). *Vice versa*, a collar $\mathrm{N} \subset \mathrm{S}^3$ has as closed complement a Moebius ν-Gon Q and it determines a tile $\mathrm{P} \subset \mathbf{H}^4$ by the definition

$$\mathrm{P} = [\text{convex hull } \mathrm{Q}] \cap \mathbf{H}^4, \qquad \mathrm{H}^4 = \mathbf{H}^4_{\mathrm{proj}} \subset \mathbf{R}^4.$$

The 2-planes v_i and v_{i+1} have as unique shortest connecting arc, an orthogonal line segment $[d_i, a_{i+1}]$ from $d_i \in v_i$ to $a_{i+1} \in v_{i+1}$ (" d " for departure, " a " for arrival). See figure 5 *c)*. In figure 5 *a)*, *b)*, *d)* and *e)*, the case $d = 3$ is illustrated. The 2-plane in the Side $[v_i, v_{i+1}]$, orthogonal to $[d_i, a_{i+1}]$ and meeting it in the middle, is the 2-plane of symmetry of this Side. Next we introduce the involution g_i°, an isometry of $\mathbf{H}^4$ with $(g_i^\circ)^2 = e$, which has this *symmetry plane as fixed point set* $(g_i^\circ)^{\mathrm{F}}$. It carries P to a congruent copy $g_i^\circ(\mathrm{P})$, that fits precisely along the common Side $[v_i, v_{i+1}]$. If no two Sides of P are congruent, then this is the only way to start a tesselation.

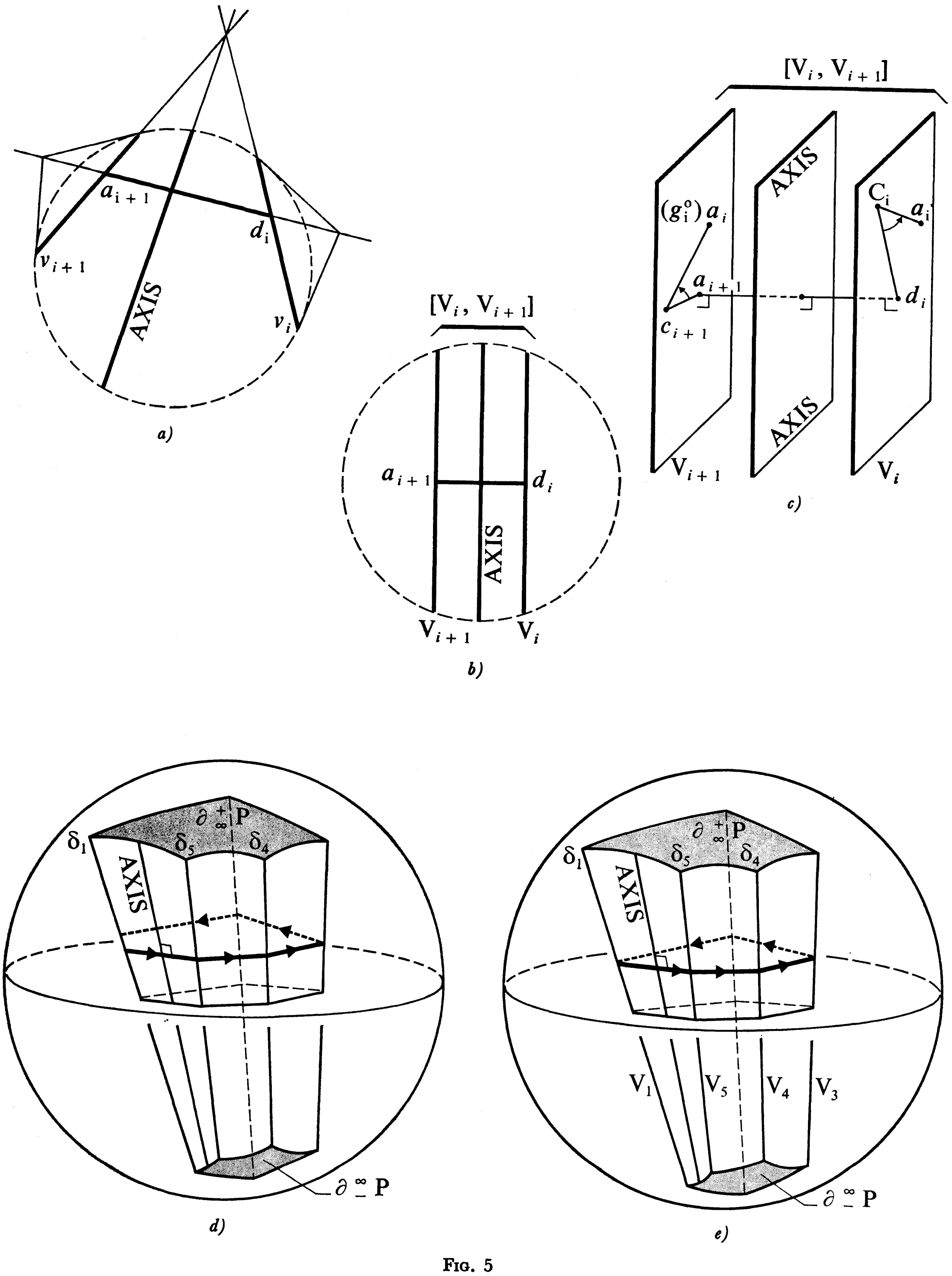
In figure 4 *b)* we use a special model for S^3, where $\partial_\infty(g_1^\circ)^{\mathrm{F}}$ is a straight line in euclidian space $\mathbf{R}^3$, and $\mathrm{S}^3 = \mathbf{R}^3 \cup \{\infty\}$ is the one-point compactification. Then we see that the union of the Moebius ν-Gons ∂_∞ P and $g_i^\circ(\partial_\infty \mathrm{P})$ has as closed complement in S^3 a *new collar*, the connected sum

$$\mathrm{N}_2 = \mathrm{N} \# g_i^\circ(\mathrm{N}) \subset (\mathrm{N} \cup g_i^\circ(\mathrm{N})). \tag{3.2}$$

Its isotopy knot class is seen to be the connected sum (symbol $\#$) of the isotopy knot classes [N] and $[g_1^\circ(\mathrm{N})] = [\mathrm{N}]$. Similarly, the isotopy knot classes of the core-curve N_2 is, with obvious notation

$$[\gamma] \# [\gamma] = [\gamma]^{\#2}.$$

Adding more ν-Gons while keeping the union connected gives a nested sequence of collars $\mathrm{N} \supset \mathrm{N}_2 \supset \mathrm{N}_3 \ldots$, where $[\mathrm{N}_j] = [\mathrm{N}]^{\# j}$, which converges to a compact set $\mathcal{J} = \bigcap_j \mathrm{N}_j$. The analogous nested sequence also exists for a collared Moebius ν-gon in S^2 for $d = 3$. Given the tile $\mathrm{P} \subset \mathbf{H}^4$ we can fit copies around the Vertex $v = v_1$ by involutions $g_1 = g_1^\circ, g_2, g_3$, etc., as in § 2 and illustrated in figure 2. If the tesselation succeeds then $g_1^\circ, \ldots, g_\nu$ or equivalently $g_1^\circ, \ldots, g_\nu^\circ$ generate a group action of $\Gamma_{\nu, n}$ by isometries of $\mathbf{H}^4$, a group which contains involutions about symmetry-planes of all Sides of the tesselation. After ν steps around Vertex v, that Vertex has the same position with respect to the tile $g_\nu \ldots g_1(\mathrm{P})$ as with respect to P. Therefore the product $g_\nu \ldots g_1 \in \Gamma_{\nu, n}$ is the product of a rotation about v over the angle Δ, and the normal extension to $\mathbf{H}^4$ of an isometry T of the hyperbolic 2-plane v. This T is called the *torsion* of P; up to motions it is an *invariant of the Moebius* ν-Gon $\partial_\infty \mathrm{P} = \mathrm{Q}$ in S^3,



a_{i+1}
d_i
v_{i+1}
AXIS
v_i
a)
$[V_i, V_{i+1}]$
a_{i+1}
d_i
AXIS
V_{i+1}
V_i
b)
$[V_i, V_{i+1}]$
AXIS
$(g_i^o)\,a_i$
C_i
a_i
a_{i+1}
d_i
c_{i+1}
AXIS
V_{i+1}
V_i
c)
δ_1
$\partial_\infty^+ P$
δ_5
δ_4
AXIS
$\partial_-^\infty P$
d)
δ_1
$\partial_\infty^+ P$
δ_5
δ_4
AXIS
V_1
V_5
V_4
V_3
$\partial_-^\infty P$
e)

FIG. 5

and of the *Moebius collar* N in S^3. The equations (2.4, 5, 7, 8) hold now as before. In particular we deduce the *following necessary and sufficient conditions for a* ν*-Gon* in $\mathbf{H}^4$ *to* produce a *tesselation of type* $\Gamma_{\nu, n}$, $\gcd(\nu, n) = k$, $\nu = \ell k$, $n = mk$:

(3.3) The *total angle* is $\Delta = 2\pi\nu/n$.

(3.4) *The total torsion is periodic of order* m: $T^m = \mathrm{id} \mid v$

In particular T has a fixed point

(3.5) $c_1 = c \in v_1 = v,$

and

(3.6) T is an *elliptic* 2*-plane* isometry.

The *torsion* angle is $\theta = 2\pi j/m$ for $j = 0, 1, \ldots$, or $m - 1$, and θ is an isotopy invariant of the tesselation and a deformation invariant of the $\Gamma_{\nu, n}$-action on $\mathbf{H}^4$ *as well as on* S^3. The above conditions are *sufficient in case* $\gcd(\nu, n) = \nu$, as we can just continue adding tiles and fill $\mathbf{H}^4$ like in § 2.

For $\gcd(\nu, n) = k < \nu$ we have one more condition for sufficiency, namely P *must admit a symmetry* of order ℓ

(3.7) $(g_n \ldots g_1)^\ell = e$

(cf. (2.8)). *For dimension* $d = 3$ the torsion T is a *translation* in a line in $\mathbf{H}^3$. In $S^2 = \partial\mathbf{H}^3$ it is expressed by a *dilatation* with two fixed points. Then (3.4, 5, 6) are replaced by the condition that the torsion (a *distance* in $\mathbf{H}^3$, a dilation invariant in S^2) is

(3.8) $\theta = 0 \in \mathbf{R}.$

See figure 5 *d)* and *e)*. The other conditions are unchanged.

3.2. The core surfaces of P and $\mathbf{H}^4$, and the Julia knot J. — With $c_1^\circ \in v = v_1$ we find $c_i^\circ \in v_i$, $i = 1, \ldots, \nu$ inductively by $c_{i+1}^\circ = g_i^\circ(c_i^\circ)$ and a unique $c_g \in gv$ for any $g \in \Gamma_{\nu, n}$. The points $c_1^\circ, \ldots, c_\nu^\circ$ form a ν-gon in ∂P. Take a point x in the interior of the convex set P, but choose it in the *fixed point* or *fixed plane* $(g_n \ldots g_1)^F$ of the isometry $g_n \ldots g_1$ in case $\gcd(\nu, n) < \nu$ and $\ell > 1$ in (3.7). The cone from x on the polygon in ∂P is called a *core cone* cc(P) of P. Of course P can be retracted *radially* into $x \in P \subset \mathbf{H}^d_{\mathrm{proj}} \subset \mathbf{R}^d$. By an isometry of $\mathbf{H}^d$ we can assume $x = 0 \in P \subset \mathbf{H}^d_{\mathrm{proj}} \subset \mathbf{R}^d$. We can easily modify the retraction, keep the core cone pointwise fixed and let the Sides of P move over themselves, in order to obtain a *retraction* of P as well as $\overline{P}$ *onto* $cc(P) \subset P$. We can assume invariance of the retraction under the isotopy subgroup of P in $\Gamma_{\nu, n}$.

Let $U(cc(P))$ be a tubular neighborhood of $cc(P) \subset P$ which is the union of an ε^2-neighborhood of cc(P) in P and an ε-ball around $x \in P$, where $\varepsilon > 0$ is small and refers to hyperbolic distance. Then we see that there is also an *isotopy* of (P, ∂P) inside (P, ∂P), moving each Side and Vertex of P inside itself, carrying

(3.9) (P, ∂P) onto $(U(cc(P)),\ \partial U(cc(P)))$

and keeping the core cone cc(P) pointwise fixed.

The unions of the transforms under all $g \in \Gamma_{\nu,n}$ of the spaces $cc(\mathrm{P}) \subset \mathrm{U}(cc(\mathrm{P})) \subset \mathrm{P}$ yield respectively

a) a complete *polyhedral core surface* $\Sigma_{\mathrm{P}} \subset \mathbf{H}^4$ tesselated by core cones congruent to $cc(\mathrm{P})$,

b) a tubular neighborhood U, and *c)* the whole of $\mathbf{H}^4$.

The sequence $\Sigma_{\mathrm{P}} \subset \mathrm{U} \subset \mathbf{H}^4$ is invariant under $\Gamma_{\nu,n}$. Set

$$\Omega = \mathrm{S}^3 \backslash \mathscr{J} = \bigcup\nolimits_g g\mathrm{Q}, \qquad \mathrm{Q} = \partial_\infty \mathrm{P}.$$

The invariant isotopy extends to an invariant isotopy of Ω, carrying Ω onto $\partial\mathrm{U} \subset \mathbf{H}^4$, invariant under the action of $\Gamma_{\nu,n}$ on $\overline{\mathrm{H}^4}$.

The set $\mathscr{J}$ is the limit set of the action of $\Gamma_{\nu,n}$ on $\mathbf{H}^4$ and on S^3. That is, it is the limit set of the set

$$\{gy, g \in \Gamma_{\nu,n}\}$$

for any $y \in \mathrm{P}$, or $y \in \partial_\infty \mathrm{P} = \mathrm{Q}$. It is compact, connected, and invariant under any $g \in \Gamma_{\nu,n}$. It is therefore self-similar under Moebius transformations of S^3. Below, we prove that $\mathscr{J}$ is an embedded circle. We can call it the *Julia-curve* or *Julia knot* of the tesselation and of the action of $\Gamma_{\nu,n}$ in $\mathbf{H}^4$, S^3, resp. (for $d = 3$) in $\mathbf{H}^3$, S^2.

Consider the tesselation of $\mathbf{H}^2$ by regular ν-gons with $\delta_i = 2\pi/n$ and group $\Gamma_{\nu,n}$. There is a piecewise projective homeomorphism

$$\kappa : \mathbf{H}^2_{\mathrm{proj}} \to \Sigma_{\mathrm{P}}$$

which carries the regular tesselation of $\mathbf{H}^2$ onto the tesselation of Σ_{P} and which commutes with the action of $\Gamma_{\nu,n}$. It is a quasi-conformal homeomorphism which extends to a continuous one-to-one map

$$\partial_\infty \kappa : \mathrm{S}^1 = \partial\mathbf{H}^2 \to \partial_\infty \Sigma_p = \mathscr{J}.$$

Therefore $\partial_\infty \kappa$ is an embedding and $\mathscr{J}$ *is a self-similar knot.* It may be unknotted, possibly lying in a round $\mathrm{S}^2 \subset \mathrm{S}^3$, or even a circle. If $\mathrm{J} = \mathrm{S}^1$ is a circle, then the action is called *Fuchsian.* This happens if and only if the action is the natural extension of a tesselation action in the 2-plane $\mathbf{H}^2 \subset \mathbf{H}^3$ with $\partial\mathbf{H}^2 = \mathrm{S}^1$. A non-Fuchsian action has a limit set $\mathscr{J}$ in a round $\mathrm{S}^2 \subset \mathrm{S}^3$ if and only if it is the natural extension of a non-Fuchsian tesselation action in a 3-plane $\mathbf{H}^3 \subset \mathbf{H}^4$. If [N] *is properly knotted* (not unknotted), then $\mathscr{J}$ is knotted in any neighborhood of any of its points and $\mathscr{J}$ is *nowhere* tame. See [Ma], p. 202, example F.4, and p. 212.

3.3. The tesselated manifolds Σ^4 and M^3 and the deformation dimensions. — Suppose the group $\Gamma_{\nu,n}$ acts on a regular tesselation in $\mathbf{H}^2$ and a sub-group $\Gamma \subset \Gamma_{\nu,n}$ of finite index and acting freely is fixed once and for all as abstract subgroup in $\Gamma_{\nu,n}$. If $\Gamma_{\nu,n}$ acts as before on $\mathbf{H}^4$ and $\Omega \subset \mathrm{S}^3$, then Γ is seen to act freely also on $\mathbf{H}^4$ and on Ω.

Therefore $\Sigma^4 = \mathbf{H}^4/\Gamma$ is a tesselated smooth complete 4-manifold with hyperbolic structure and $M^3 = \Omega/\Gamma$ is a compact tesselated Moebius-3-manifold (i.e. a conformally flat 3-manifold) with Ω as a covering space. The number of tiles is V.

The surface $\Sigma = \Sigma_p/\Gamma$ is a *polyhedral core surface* to which Σ^4 retracts tile by tile. So Σ^4 is *homotopy equivalent to the surface* Σ. The smooth 4-manifold with boundary $(\mathbf{H}^4 \cup \Omega)/\Gamma$ is isotopic *to the* 4-manifold with boundary $\overline{U}_p = \overline{U}_p/\Gamma$ by a tile by tile isotopy. Then M is PL-equivalent to the boundary ∂U of the tubular neighborhood U of the core surface Σ of Σ^4. The PL manifolds U, $\overline{U}$ and ∂U, can be made *smooth* and the smooth structures are unique up to equivalence.

Clearly the polyhedral manifold Σ^4 *is obtained from the tesselated surface* Σ by taking a disc bundle over the complement of the union of small discs around the centers of the polyhedral tiles of Σ, and sticking in balls, shaped like (i.e. homeomorphic to) the tiles P, and attached along their boundaries ∂P. The 3-manifold M bounds this manifold. Some *topological invariants* of Σ^4 are the *homotopy type of* Σ, the *self-intersection* $\mathbf{H}_0(\Sigma \cap \Sigma)$ of the core surface Σ, and the possibly very complicated *homotopy type of the end* M^3 of Σ^4. All these invariants are also invariants of the (deformation family of the) tile P.

The isotopy families of tesselations or $\Gamma_{\nu,n}$-actions on $\mathbf{H}^d$ are modulo hyperbolic motions parametrized by their tiles, the ν-gons modulo motions. It is easy to calculate their *deformation or module dimensions* as in § 2 for $d = 2$. The deformation components of tesselation actions of $\Gamma_{\nu,n}$ on $\mathbf{H}^d$ consist of tesselation actions only. See Theorem 2.*g*. and Remark 1 below.

Let DIM(d) be the common deformation dimension of the families $\mathcal{M}(\Gamma_{\nu,n}, \mathbf{H}^d)$, $\mathcal{M}(\Gamma_{\nu,n}, \Sigma^d)$, $\mathcal{M}(\Gamma_{\nu,n}, S^{d-1})$, and $\mathcal{M}(\Gamma_{\nu,n}, M^{d-1})$. As $\mathcal{M}(\Gamma_{\nu,n}, \mathbf{H}^{d-1})$ has a natural embedding (by extension of the action on $\mathbf{H}^{d-1} \subset \mathbf{H}^d$ to $\mathbf{H}^d$) into $\mathcal{M}(\Gamma_{\nu,n}, \mathbf{H}^d)$ we have the inequalities

$$\mathrm{DIM}(2) \leqslant \mathrm{DIM}(3) \leqslant \mathrm{DIM}(4).$$

Here is the table of dimensions (see the relations (2.9) and Theorem 1 *c)*); "diff" denotes the common value of DIM(4) − DIM(3) and DIM(3) − DIM(2):

TABLE 1

DIM(d)		$d = 2$	3	4	Diff.
$n = \nu \geqslant 5$		$2\nu - 6$	$3\nu - 10$	$4\nu - 14$	$\nu - 4$
$n = m\nu, m \geqslant 2$	$\nu \geqslant 4$	$2\nu - 4$	$3\nu - 8$	$4\nu - 12$	$\nu - 4$
	$\nu = 3$	2	2	2	0
$\gcd(\nu, n) = k < \nu$ and $\nu \geqslant 3$		$2k - 2$	$3k - 3$	$4k - 4$	$k - 1$
$\gcd(\nu, n) = 1$		0	0	0	0

We summarize our earlier observations, as well as conclusions from table I, in theorems 2 and 3.

Theorem **2.** — *The case of dimension $d = 3$. A ν-Gon* P *in* $\mathbf{H}^3$ *or* Σ^3, *or a collar bounded Moebius ν-gon* Q *in* S^2 *or* M^2, *can be a tile of a tesselation with* $\Gamma_{\nu, n}$ *acting on it if and only if*

a) The angle sum is $\Delta = 2\pi\nu/n$ *and the torsion is* $\theta = 0$, *and*

b) P *as well as* Q *admits a* symmetry, *that is a rotation of order* $\ell = \nu/k$, $k = \gcd(\nu, n)$.

c) The action is rigid if and only if $\gcd(\nu, n) = 1$. *Then it is regular, Fuchsian, and it is the natural extension to* $\mathbf{H}^3$ *of the rigid regular action of* $\Gamma_{\nu, n}$ *in* $\mathbf{H}^2 \subset \mathbf{H}^3$.

d) The Fuchsian caracter is rigid but not the action if and only if $\nu = 3$ *or* 4, *and* $\gcd(\nu, n) = \nu$.

e) In all other cases, except for a proper algebraic subset of the deformation space, the limit set $\mathscr{J} \subset S^2 = \partial_\infty \mathbf{H}^3$ *is a proper self-similar Julia-Jordan curve.*

f) For $\Gamma_{\nu,\nu}$-*tesselations non-convex tiles with one angle* $\geqslant \pi$ *must be admitted and included in the deformation families.*

g) The $\Gamma_{\nu, n}$-*actions of our tesselations form an open set in the deformation space of discrete* $\Gamma_{\nu, n}$-*actions on* $\mathbf{H}^3$, Σ^3, *in* S^2 *and on* M^2.

h) All our tesselated 3-manifolds Σ^3 *are diffeomorphic to each other and to* $\Sigma^2 \times \mathbf{R}$.

Remark **1.** — For a *collar-bounded Moebius ν-gon* Q in S^2 we obtained the limit set $\mathscr{J} = \bigcap_i N_i$ as an intersection of an infinite sequence of collars in S^2. The union of the tiles is then embedded (by the " development map ") in S^2.

We can also start, *more generally*, from an immersed disc whose boundary is immersed as a ν-gon of circular arcs, with a collar, suitably immersed in S^2 as a union of embedded round discs (beads). Here, we seem to need $\nu \geqslant 9$ (two tiles). Such a " ν-gon " can be used as tile for a tesselation of an open disc $\mathbf{X}$ with $\mathbf{CP}^1$-structure (obtained by gluing tiles) under the conditions *a)* and *b)* as before. But the development map $\mathbf{X} \to S^2$ is in general not a covering of its image, and the action of $\Gamma_{\nu, n}$ is discrete on $\mathbf{X}$ but not in S^2. The deformation dimensions are the same as before. See [Go] for a beautiful deformation theory of $\mathbf{CP}^1$-structures on surfaces.

Remark **2.** — The condition *b)* is empty when $\gcd(\nu, n) = \nu$.

Remark **3.** — Question: Are all discrete $\Gamma_{\nu, n}$-actions in S^2 obtained from our tesselations?

Theorem **3.** — *The case of dimension $d = 4$. A ν-Gon* P *in* $\mathbf{H}^4$ *or* Σ^4, *or a Moebius ν-Gon* Q *with collar* N *in* S^3 *resp.* M^3, *can be a tile of a tesselation with* $\Gamma_{\nu, n}$ *as an invariant group action, if and only if*

a) The anglesum is $\Delta = 2\pi\nu/n$ *and the (elliptic) torsion has value* $\theta = 2\pi j/m \bmod 2\pi$, *for* $j = 0, 1, \ldots,$ *or* $m - 1$, *and*

b) P *as well as* Q *admits an isometry resp. Moebius transformation of order* $\ell = \nu/\kappa$, $k = \gcd(\nu, n)$, $n = m\nu$.

c) The action is rigid if and only if $\gcd(\nu, n) = 1$. *Then the tesselation is regular as described in Theorem* 4.

d) The Fuchsian caracter is rigid but not the action if and only if $\nu = 3$ *or* 4, *and* $\gcd(\nu, n) = \nu$.

e) In all other cases, except for a proper algebraic subset in the deformation space, the limit-set $J \subset S^3$ *is a self-similar proper Julia knot, everywhere knotted and nowhere tame or nowhere knotted. It will be seen to be tame in some cases in Theorem* 5.

f) For $\Gamma_{\nu,\nu}$*-tesselations non-convex tiles with one angle* $\geqslant \pi$ *must be admitted and included in the deformation families, also for knotted isotopy classes* [N].

g) The $\Gamma_{\nu,n}$*-actions of our tesselations form an open set in the deformation space of discrete* $\Gamma_{\nu,n}$*-actions on* $\mathbf{H}^4$, Σ^4, *in* S^3 *and on* M^3.

h) The set of diffeomorphism types or homeomorphism types of hyperbolic manifolds Σ^4 *we constructed is enormous. Apart from their homotopy type (that of the core surface* Σ*), they can be distinguished further by*

1. *the diffeomorphism type of* $M^3 \times \mathbf{R}$ *(by* [S]*), and*

2. *the self-intersection number of the core surface* $\Sigma \subset \Sigma^4$. *In particular for the unknotted case, this is the Euler number (see* (4.4)*) of the normal bundle of a smooth core surface* $\Sigma^2 \subset \Sigma^4$.

Observe that the diffeomorphism type of M^3 is a metric but not a diffeomorphism invariant of Σ^4.

Proofs.

c) As $\mathrm{DIM}(d) = 0$, no deformation is possible.

d) As $\mathrm{DIM}(d) - \mathrm{DIM}(2) = 0$, the extension of the actions in the deformation class on $\mathbf{H}^2 \subset \mathbf{H}^d$, exhausts the actions on $\mathbf{H}^d$ (with $S^1 = \partial_\infty \mathbf{H}^2 \subset S^{d-1} = \partial_\infty \mathbf{H}^d$ as invariant limit set).

e) As $\mathrm{DIM}(2) < \mathrm{DIM}(3) < \mathrm{DIM}(4)$, the extension of the actions on $\mathbf{H}^2$ to $\mathbf{H}^3$ and on $\mathbf{H}^3$ to $\mathbf{H}^4$ do not exhaust the deformation classes, So $\mathscr{J}$ is a proper Julia knot in S^{d-1} in general. This is precisely the case if the boundaries of the beads do not have a common orthogonal circle or two-sphere, an algebraic condition.

g) The proof is analogous to that for case $d = 2$.

Note that for dimension $d = 3$ (Theorem 2), the core surface $\Sigma \subset \Sigma^3$ is nowhere knotted; it can be smoothed (made C^∞) in a smoothed tubular neighborhood U to give $\Sigma^2 \subset U(\Sigma^2) \subset \overline{U}(\Sigma^2)$, where $\overline{U}(\Sigma^2)$ is diffeomorphic to Σ^3 and to the trivial segment bundle over the surface Σ^2. For dimension $d = 4$, see section 4.

4. The normal Euler number, regular tesselations. Connection with [GLT].

4.1. The normal Euler number $\chi^\perp(\Sigma)$ of the core surface $\Sigma \subset \Sigma^4$ in the unknotted case. — Suppose the collar $N = S^3 \backslash \partial_\infty P$ is unknotted in S^3. Then $\Sigma \subset \Sigma^4$ is unknotted at every vertex of the polyhedral core surface Σ. There are V vertices with congruent surface germs at " centers " gx of broken 2-tiles of Σ for $g \in \Gamma_{\nu,n}$. There is a dual tesselation of Σ by F dual broken 2-tiles with " centers " in gc, with ν of then meeting in each vertex $g'x$, $g' \in \Gamma_{\nu, n}$. They have mutually congruent surface germs at their centers gc. As in [GLT] one has

$$\text{(4.1)} \qquad V\nu = Fn.$$

Over the complement of the union of small round balls with radius ε around the vertices of Σ in the hyperbolic 4-manifold Σ^4, the surface Σ can be smoothed to have a locally parallel (flat connection) normal plane bundle. The remaining part of the surface can be smoothed by inserting discs, to give a smooth surface $\Sigma^2 \subset \Sigma^4$ whose normal plane bundle is diffeomorphic to Σ^4. The normal *Euler number* is obtained by an *integration* over Σ^2. This integral is equal to a sum of contributions (see [GLT], also [Ba2])

$$\chi^{\perp}(\Sigma) = \Sigma_y \chi^{\perp}(y)$$

over the vertices y of Σ. In our case this gives

$$\text{(4.2)} \qquad \chi^{\perp}(\Sigma) = V.\chi^{\perp}(x) + F.\chi^{\perp}(c).$$

It remains to find an expression for the normal Euler number $\chi^{\perp}(y)$ of a nowhere knotted polyhedral surface at a vertex. An expression was proposed by Banchoff [Ba2] for a polyhedral surface in $\mathbf{R}^4$. In [GLT] the authors present the beautiful formula (4.3) below. To describe this we cut the polyhedral cone induced by the surface Σ at a vertex y in the tangent space $\mathbf{R}^4$ of $y \in \Sigma^4$, by a unit 3-sphere, and get a polygonal geodesic unknot $\zeta(y)$ in S^3. Let $2\pi T(y) \in \mathbf{R}$ be the total torsion (see [GLT] for the definition of total torsion) and $SL(y) \in \mathbf{Z}$ be the self-linking number of $\zeta(y)$ ([2]). Following [GLT], the normal Euler number at y is given by

$$\text{(4.3)} \qquad \chi^{\perp}(y) = T(y) - SL(y).$$

Substitution in (4.2) yields the integer

$$\text{(4.4)} \qquad \chi^{\perp}(\Sigma) = V.(T(x) - SL(x)) + F.(T(c) - SL(c)).$$

Let Σ' be obtained from Σ by a small isotopy. For a generic Σ' the intersection number $\Sigma' \cap \Sigma$ is (always) equal to $\chi^{\perp}(\Sigma)$ and called the *self-intersection number* of Σ. The formulas (4.3), (4.4), (4.7) remain valid for the cases where the core surface Σ is knotted at some vertices. We use this in § 6. Note that, by ([GLT]),

$$-\chi(\Sigma) = V.((\nu/2) - 1 - (\nu/n)).$$

4.2. The template construction. — Following [GLT] we construct special so called *template* ν-*Gons* as tiles. Start from a collar of ν beads in S^3, any two consecutive ones being tangent but no other two meeting. Let h_i^- be the half space in $\mathbf{R}^4$ (!) which

([2]) The self-linking of a curve in 3-space was first defined by Călugăreanu [Cal] in 1959 and studied with interesting results by W. Pohl [Po] in 1968. Banchoff [Ba1] defined it in 1976 for polygons, and he also defined a normal Euler class for a polyhedral surface in $\mathbf{R}^4$ in 1984 [Ba2].

meets the i-th bead of the collar exactly in its boundary. The closure h_i^+ of its complement contains the whole bead. Multiply the convex set

$$\mathbf{K}_0 = \bigcap_{i=1}^{\nu} h_i^-$$

in the vector space $\mathbf{R}^4$ *by a scalar* $t < 1$ to obtain $\mathbf{K}_t = t\mathbf{K}_0$ and a convex set in the projective model $\mathbf{H}^4 = \mathbf{H}^4_{\mathrm{proj}}$

$$\mathrm{P}_t = \mathbf{K}_t \cap \mathbf{H}^4. \tag{4.6}$$

Clearly there exists $t_0 > 0$ such that P_t for $t > t_0$ is a ν-Gon in the sense of our definition. The collar in S^3 of the ν-Gon P_t has beads $th_i^+ \cap \mathrm{S}^3$, $i = 1, \ldots, \nu$.

Example. — Any collar of *congruent beads* in the *Euclidean unit sphere* S^3, whose centers are the vertices of a polygon in S^3 with equal sides, can be so obtained.

As $t > t_0$ increases, the angle sum Δ decreases, so there is n_0 such that any angle sum $\Delta = 2\pi\nu/n$ can be attained for $n > n_0$. The condition $m\theta = 0 \bmod 2\pi$ is harder to see.

The centers of the beads form a geodesic ν-gon γ in the *Euclidean unit sphere* $\mathrm{S}^3 \subset \mathbf{R}^4$ called a *template*. Let $\mathrm{C}(\gamma)$ be the cone from $x = 0$ over γ. The template surface $\mathrm{C}(\gamma) \cap \mathrm{P}_t$ is a cone in P_t with vertices $c_i^\circ \in v_i^\circ$ (v_i° a Vertex (2-plane) in P_t) and $m_i \in [v_i^\circ, v_{i+1}^\circ]$, where $[v_i^\circ, v_{i+1}^\circ]$ is a Side of P_t. Our *real surprise* is now *first* that g_i° carries c_i° to $g_i^\circ c_i^\circ = c_{i+1}^\circ$, so that $c = c_i^\circ \in v_i^\circ = v$ is just the invariant point of the elliptic torsion T in the 2-plane v, and *second* that the template surface and its images under $g \in \Gamma_{\nu,n}$ form the polyhedral core surface $\Sigma_{\mathrm{P}} \subset \mathbf{H}^4$, which happens to be *flat at every vertex* $gc \in gv$. The dual tiles of Σ are flat n-gons of which ν come together in every point gx, $g \in \Gamma_{\nu n}$. The formula (4.4) for the Normal Euler number $\chi^\perp(\Sigma)$ now reduces to

$$\chi^\perp(\Sigma) = \mathrm{V}.(\mathrm{T}(x) - \mathrm{SL}(x)) \tag{4.7}$$

for template tesselated 4-manifolds Σ^4.

For the regular tesselations (see 4.3) the core surface is the same as that in [GLT], where the full group of symmetries of this core surface leads to a different description.

4.3. Regular tesselations. — A ν-Gon P is called *regular* if there is an elliptic isometry of $\mathbf{H}^d$ sending P to P, and every Vertex and Side onto the next. For the case $\gcd(\nu, n) = 1$ there is such an isometry $(g_n \ldots g_1)^j$ for some j by (3.7).

In *dimension* $d = 3$ assign to the Vertex (a line) v_i of P the distance θ_i between a_i and d_i with factor $+1$ (resp. -1) if the vector from a_i to d_i points in the direction of $\partial_\infty^+ \mathrm{P}$ (resp. $\partial_\infty^- \mathrm{P}$). The sum of these numbers clearly is the torsion $\theta = \Sigma\theta_i = 0$. For a regular ν-Gon $\theta_1 = \theta_2 = \ldots = \theta_\nu$, and as $\theta = 0$, each θ_i must be zero and $a_i = d_i$ for $i = 1, \ldots, \nu$. Then we have a plane ν-gon with vertices $a_1, \ldots, a_\nu$, which must be regular, and the lines v_i meet this plane orthogonally. It follows easily that our regular $\Gamma_{\nu,n}$-tesselation or action is the natural extension of that in $\mathbf{H}^2 \subset \mathbf{H}^3$ to $\mathbf{H}^3$ as announced in theorem 2 *b)*.

In dimension $d = 4$ we assume for a regular ν-Gon P in $\mathbf{H}^4 = \mathbf{H}^4_{\mathrm{proj}}$ that $x = 0 \in \mathbf{R}^4$ is a fixed point of the elliptic isometry that carries v_i to v_{i+1}. The collar of ν beads is invariant, so that all beads in the Euclidean three-sphere $S^3 = \partial \mathbf{H}^4 \subset \mathbf{R}^4$ are congruent and successive centers have constant distances. So we get a template tile. These centers form a regular polygon in S^3. If it is not a plane regular ν-gon in $S^1 \subset S^3$, then in suitable complex coordinates z_1, z_2 for $\mathbf{R}^4$, the elliptic isometry is

$$(z_1, z_2) \to (z_1\, \omega^q, z_2\, \omega^p)$$

where $\omega = \exp 2\pi i/\nu$, and p and q are coprime. The regular ν-gon *(template)* is the *standard regular ν-gonal (p, q)-torus-knot* with vertices

$$(z_1, z_2) = (\cos \varepsilon \,.\, \omega^{jq}, \sin \varepsilon \,.\, \omega^{jp}) \tag{4.8}$$

for $j = 0, 1, \ldots \nu$.

These vertices are equidistant on the smooth (p, q)-torus-knot

$$(z_1, z_2) = (\cos \varepsilon \,.\, \omega^{qs}, \sin \varepsilon \,.\, \omega^{ps}), \quad s \in \mathbf{R}, \tag{4.9}$$

on the standard torus $|z_1| = \cos \varepsilon$, $|z_2| = \sin \varepsilon$, in S^3. A specific template tesselation is now obtained by choosing $\Delta = 2\pi\nu/n$ for $n > n_0$, and ε is restricted by a condition on the total torsion $2\pi T(y)$ *of the template*. (This condition should correspond to our condition on θ.) In [GLT] the normal Euler number is calculated in the unknotted case ($q = 1$!), where the self-linking number is $SL(x) = p$:

$$\chi^{\perp}(\Sigma) = V.(T - p).$$

With $\chi(\Sigma) = V.((\nu/2) - 1 - (\nu/n))$ it is found in [GLT] that

$$|\chi^{\perp}(\Sigma)/\chi(\Sigma)| < 1.$$

Then the plane bundle $\Sigma^4 \to \Sigma^2$ has a transversal foliation by Milnor [Mi] and Wood [W].

Question. — Is every *tesselation* of $\mathbf{H}^4$ with unknotted collar N in the deformation class of a regular tesselation? If true, this would imply the inequality (4.9) for all " unknotted " tesselations, and the existence of a transversal foliation whenever Σ^4 is a 2-plane bundle constructed with our tesselation method.

We summarize in

Theorem **4.** — *Rigid tesselations are regular. a) Regular tesselations in* $\mathbf{H}^4$ *are " template ". If not Fuchsian then the template is a regular standard ν-gonal (p, q)-torus-knot* (4.8). *b)* ([GLT]) *In case $\mathcal{J}$ is unknotted, then the* 4*-manifold Σ^4 is a plane bundle with normal Euler number* (4.4), *simplified to* (4.7) *for template tesselations. For the regular non Fuchsian case one has* $0 < |\chi^{\perp}(\Sigma)/\chi(\Sigma)| < 1$, *and the plane bundle has a transversal foliation.*

We also note

Theorem **5.** — *If the $\Gamma_{\nu,n}$ tesselation* T *of* $\mathbf{H}^d$ *can deform to a Fuchsian tesselation* T', *then its limit Julia curve $\mathcal{J}$ is tame unknotted.*

Proof. — There is a quasi conformal homeomorphism h from $\mathbf{H}^4 \cup \Omega$ onto $\mathbf{H}^4 \cup \Omega'$, which commutes with the action of $\Gamma_{\nu,n}$ and extends to $\mathbf{H}^4 \cup S^3$. It carries $\mathscr{J}$ to the circle $\mathscr{J}' = S^1$. Take $\mathbf{H}^4 = \mathbf{H}^4_{\text{conf}} \subset \mathbf{R}^4$, and define h first on a fundamental domain for the action of $\Gamma_{\nu,n}$ on $\mathbf{H}^4 \cup \Omega$. Then extend.

5. Calculations

In this last section we use the formulas in [GLT] to calculate simple examples. We find:

Theorem **6.** — *a)* Let $\chi = 2 - 2g$ *be the even Euler number of an oriented closed surface* Σ *of genus* g, *and* $\chi^{\perp}$ *the Euler number of a real 2-plane bundle over* Σ *with total space* Σ^4. *If*

$$|\chi| \geqslant 3\,|\chi^{\perp}|, \tag{5.1}$$

in particular if $g \geqslant 3$ *and* $|\chi^{\perp}| = 1$, *then* Σ^4 *has a complete hyperbolic metric and the total space* M^3 *of the corresponding circle bundle has a Moebius structure (conformally flat structure).*

*b) The (chosen) examples have a tesselation by two congruent (*resp. *Moebius equivalent) tiles. The tesselated manifolds* Σ^4 *and* M^3 *have deformation dimension* $4\nu - 14$, *where* $\nu = |\chi| + 3 \geqslant 7$ *is the number of " Sides " of the tile. The tile can be chosen " regular ". The fundamental group* Γ *of* Σ^4 *is of index 2 in the group* $\Gamma_{\nu,2\nu}$.

Note. — The simplest example is for $g = 3$, $\nu = 7$, $n = 14$, $\chi^{\perp} = -1$, $\chi = -4$, and Σ^4 is then covered by two regular " 7-Gons " as tiles, with beautiful symmetry. See Figure 3. The greatest value for $\chi^{\perp}/\chi$ we obtain is $7/11 = 0.428$. In [GLT] the upper bound, 1, is found for all *regular unknotted* template constructions: $|\chi^{\perp}/\chi| < 1$. The authors also recall that then, by Milnor-Wood [W], the groups of the bundles $\mathbf{R}^2 \to \Sigma^4 \to \Sigma^2$ and $S^1 \to M^3 \to \Sigma^2$ reduce to discrete subgroups of $\text{Diffeo}(\mathbf{R}^2)$ and $\text{Diffeo}(S^1)$ respectively. This implies the existence of foliations of Σ^4 and M^3 transversal to the leaves of the bundles. In fact all cases which we calculated obeyed the stronger inequality $|\chi^{\perp}/\chi| < \frac{1}{2}$. Then by Milnor [Mi] a reduction of the groups is possible to discrete subgroups in $GL(\mathbf{R}, 2) \subset \text{Diffeo}(\mathbf{R}^2)$, and in $GL(\mathbf{R}, 2) \subset \text{Diffeo}(S^1)$ respectively, in which $\pi_1(\Sigma)$ is represented by the holonomy. The corresponding transversal foliation in Σ^4 has one compact leaf in Σ^4, namely Σ^2, the leaf of $0 \subset \mathbf{R}^2$.

Proof. — We want to construct the regular examples with group $\Gamma_{\nu,n}$, where $n = 2\nu \geqslant 10$. We start from the unknotted case ($q = 1$) of the standard (q, p)-torus-knot

$$\{ (\cos \varepsilon . e^{ti}, \sin \varepsilon . e^{pti}) : 0 \leqslant t < 2\pi \} \subset S^3 \subset \mathbf{C}^2 = \mathbf{R}^4. \tag{5.2}$$

The points obtained for $t = j2\pi/\nu$, $j = 0, 1, \ldots, \nu - 1$, are the vertices of a regular template. We want to construct in S^3 a regular embedded collar of congruent " beads " with centers in these vertices. If such a collar shall exist then all diagonals of the template must be longer than the sides. To determine how much we need some preparation.

Let $\alpha(t)$ (resp. $a(t) = 2\sin(\alpha(t/2))$ be the distance in S^3 (resp. in $\mathbf{R}^4$), between the points $(\cos\varepsilon, \sin\varepsilon)$ and $(\cos\varepsilon . e^{ti}, \sin\varepsilon . e^{pti})$. Then

(5.3) $$a(t)^2 = 2[\cos^2\varepsilon(1 - \cos t) + \sin^2\varepsilon(1 - \cos pt],$$

and, if we let

(5.3′) $$u = tg^2\,\varepsilon \quad \text{and} \quad \mathrm{B}(t) = \frac{1}{4}\,a(t)^2$$

then

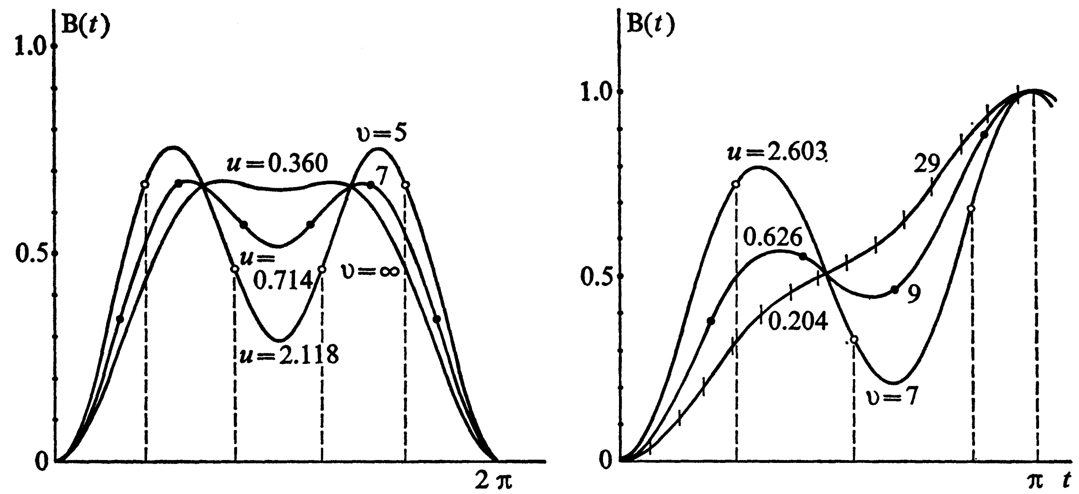
(5.4) $$\mathrm{B}(t) = \{\sin^2(t/2) + u\sin^2(pt/2)\}/(1 + u).$$

The distances in S^3 (resp. $\mathbf{R}^4$) between the vertices for $j = 0$ and $j = j$ of the regular template are denoted by α_j $\left(\text{resp. } a_j = 2\sin\frac{1}{2}\alpha_j\right)$ and, if $\mathrm{B}_j = a_j^2/4$, then

(5.5) $$\mathrm{B}_{\gamma - j} = \mathrm{B}_j = \mathrm{B}(2j\pi/\nu).$$

Now recall that the " outer angles " of the collar must be $2\pi/n = \pi/\nu$ (see § 2). By straightforward spherical trigonometry we find that the collar is embedded if and only if

(5.6) $$a_1^2/a_j^2 = \mathrm{B}_1/\mathrm{B}_j < \cos^2(\pi/2\nu) < 1 \quad \text{for } 2 \leqslant j \leqslant \nu - 2.$$



a) $p = 2$, $\nu = 5, 7, \infty$ *b)* $p = 3$, $\nu = 7, 9, 29$

FIG. 6

In figure 6 *a)* and Table II the values of B_j for $p = 2$ and some values of ν with " associated " values of u are illustrated in the graph of the function $\mathrm{B}(t)$. Idem in Figure 6 *b)* for $p = 3$. See also Table II. Observe that (5.5) is not satisfied for $p = 2$, $\nu = 5$ and for $p = 3$, $\nu = 7$. Note that, by (5.3), $\mathrm{B}(t)$ is the sum of two sinusoidal functions for $0 \leqslant t \leqslant 2\pi$ with minimal values zero at the ends, and zero, resp. $p - 1$, relative minima in between. Therefore if u is small or if ν is large for given p, then evidently (5.6) is satisfied.

TABLE II

p	v	u	B_1	B_2	B_3	B_4	$\cos^2(\pi/2v)$
2	5	2.118	**.725**	**.524**			.90
	7	.714	.364	.752	.633		.95
	9	.522	.219	.604	.750	.677	.97
	∞	.35					
3	7	2.604	**.739**	**.305**	.705		
	9	.626	.361	.542	.461	.885	.97
	11	.391	.216	.485	.493	.617	.98
	29	.204	.0270	.1004	.2006	.304	.997
4	9	2.879	**.750**	**.193**			
	11	.584	.355	.395	.390	.884	
	13	.330	.211	.379	.344	.618	
	15	.244	.143	.327	.345	.452	

There remains to discuss the values of u for given p and ν. The torsion $\tau \in \mathbf{R}^+$ of the template was calculated in [GLT]. The authors found for $1 \leqq q < p$, p, q coprime,

$$\text{(5.7)} \qquad \cos\tau = \frac{\cos^2\varepsilon.\sin^2(2q\pi/\nu).\cos(2\pi p/\nu) + \sin^2\varepsilon.\sin^2(2\pi p/\nu).\cos(2q\pi/\nu)}{\cos^2\varepsilon.\sin^2(2q\pi/\nu) + \sin^2\varepsilon.\sin^2(2\pi p/\nu)}$$

$$= \frac{\sin^2 2qw.\cos 2pw + u\sin^2 2pw.\cos 2qw}{\sin^2 2qw + u\sin^2 2pw},$$

where $w = \pi/\nu$, $u = tg^2\,\varepsilon$, $\nu \geqslant 2p+1$, and we need $q = 1$ (unknot). As $\cos\tau$ is a *mean* of $\cos 2pw$ and $\cos 2w$, we have

$$\text{(5.8)} \qquad 2w < \tau < 2pw, \qquad 1 < \mathrm{T} < p, \qquad 1\frac{1}{2} \leqslant \mathrm{T} \leqslant p - \frac{1}{2}$$

where $2\pi\mathrm{T} = \nu\tau$ is the total torsion of the template, p the self-linking number, and $2(p-\mathrm{T}) = -\chi^{\perp}$ must be an *integer*.

Solving (5.7) for u we obtain

$$u = \frac{\sin^2 2w}{\sin^2 2pw}\cdot\frac{\cos 2\mathrm{T}w - \cos 2pw}{\cos 2w - \cos 2\mathrm{T}w}$$

$$\text{(5.9)} \qquad u = \frac{\sin^2 2w}{\sin^2 2pw}\cdot\frac{\sin(p-\mathrm{T})\,w.\sin(p+\mathrm{T})\,w}{\sin(\mathrm{T}-1)\,w.\sin(\mathrm{T}+1)\,w}.$$

We *choose* $T = (p+1)/2$, *so that*

$$-\chi^{\perp} = 2(p - T) = p - 1.$$

Then (5.8) simplifies to

$$u = \frac{\sin^2 2w}{\sin^2 2pw} \cdot \frac{\sin((3p+1)\,w/2)}{\sin((p+3)\,w/2)}. \tag{5.10}$$

For $p = 2$, $-\chi^{\perp} = 1$, we find

$$u = \frac{\sin^2 2w}{\sin^2 4w} \cdot \frac{\sin(7w/2)}{\sin(5w/2)}. \tag{5.10.2}$$

The values of u for $\nu = 5, 7$ and ∞ are shown in Figure 6 *a)*. The value u decreases with increasing $\nu \geqslant 7$ so that the existence condition (5.5) is clearly always satisfied in these cases.

For $p = 3$ the values of u corresponding to $\nu = 7, 9$ and 29 are used in Figure 6 *b)*. The crucial feature of $B(t)$ is the smallest relative minimum greater than zero, which is the first minimum of $B(t)$ to come for increasing $t > 0$. See Table III for the calculated smallest useful values of $|\chi|$, given $-\chi^{\perp} = p - 1$, $\nu = |\chi| + 3 \geqslant 2p + 1$, for $2 \leqslant p \leqslant 10$. Note that Table III suggests that

$$|\chi| \geqslant 3\,|\chi^{\perp}| - 2,$$

instead of (5.1), already suffices for existence in case $|\chi^{\perp}| \geqslant 6$.

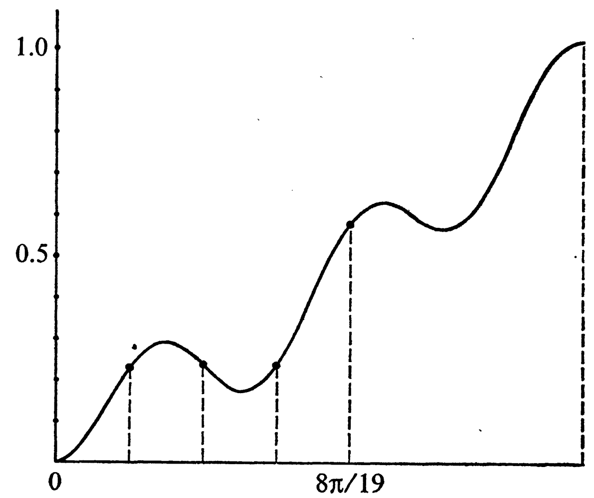


FIG. 6 *c)*. — A marginal cases: $p = 7$, $\nu = 19$, $|\chi^{\perp}/\chi| = 0.375$
$B_1 = 0.242319$, $B_2 = 0.244345$, $B_3 = 0.253583$, $B_4 = 0.632146$
$B_1/B_3 = 0.955579$, $B_1/B_2 = 0.991706 < 0.993181 = \cos^2(\pi/38)$

TABLE III. — m = minimum $(\chi/\chi^{\perp} = (v-3)/(p-1))$

m	p	$3p-3$	$3p-2$	$3p-1$	$3p$	$3p+1$
4	2			*		7
3	3		*		9	
3.67	4	*		11		9
3	5*		*		15	
2.80	6	*		17		19
2.67	7*		19		21	
2.86	8	*		23		25
2.75	9*		25		27	
2.89	10	*		29		31
2.80	11*		31		33	
2.99	221*		661		663	

We now prove that asymptotically for increasing p the relation (5.1), i.e.

$$|\chi| \geqslant 3|\chi^{\perp}|,$$

suffices for existence.

Substitute $\nu = 3p$ (that means $|\chi| = 3|\chi^{\perp}|$) and $w = \pi/\nu = \pi/3p$ in (5.10). This yields

$$u = \frac{(2\pi/3p)^2 \sin((3p+1)\,\pi/6p)}{\sin^2(2p\pi/3p)\,.\sin((p+3)\,\pi/6p)} + \mathcal{O}(p^{-4})$$

$$= \frac{(4\pi^2/9p^2)\sin(\pi/2)}{(3/4)\,(\sin(\pi/6) + (\pi/2p)\cos(\pi/6))} + \mathcal{O}(p^{-4}),$$

hence

$$u = \frac{32\pi^2}{27p^2}\left(1 - \frac{\pi\sqrt{3}}{2p}\right) + \mathcal{O}(p^{-4}). \tag{5.11}$$

Next we calculate with (5.4), (5.5) and (5.11):

$$\frac{B_1}{B_2} = \frac{\sin^2(\pi/3p) + u\sin^2(p\pi/3p)}{\sin^2(2\pi/3p) + u\sin^2(2p\pi/3p)}$$

$$= \frac{(\pi^2/9p^2) + u.(3/4)}{(4\pi^2/9p^2) + u.(3/4)} + \mathcal{O}(p^{-4})$$

$$= \frac{1 + 8\big(1 - (\pi\sqrt{3}/2p)\big)}{4 + 8\big(1 - \pi\sqrt{3}/2p\big)} + \mathcal{O}(p^{-4})$$

$$= (9/12) + 0(p^{-1}) = (3/4) + \mathcal{O}(p^{-1}).$$

$$(5.12)\qquad \frac{B_1}{B_3} = \frac{1 + 8\big(1 - (\pi\sqrt{3}/2p)\big)}{9 + 0} + \mathcal{O}(p^{-2}) = 1 - \frac{4\pi\sqrt{3}}{9p} + \mathcal{O}(p^{-2}).$$

As $\qquad \cos^2(\pi/2\nu) = \cos^2(\pi/6p) = 1 - (\pi^2/36p^2) + \mathcal{O}(p^{-3}),$

we conclude that (5.6) is satisfied for $j = 2, 3$, and clearly for other small values of j, and for all $4 \leqslant j \leqslant \nu - 4$ since the corresponding t-values $2j\pi/\nu$ are beyond that of the first relative minimum of the function $B(t)$ for $t > 0$. The asymptotic fulfilment of (5.6) for $p \to \infty$ can be extended to all values by checking for small values for which the term $\mathcal{O}(p^{-2})$ in (5.12) might be large. The existence in the interval $3.|\chi^{\perp}| \leqslant \chi| \leqslant 3.|\chi^{\perp}| + 4$ and for small values of p is seen in table III. The existence for $|\chi| > 3.|\chi^{\perp}| + 4$ is then a consequence, and Theorem 6 is proved.

Remark. — By taking $n = \nu$ instead of $n = 2\nu$ we can obtain slightly *higher values of* $|\chi^{\perp}/\chi|$, but we need $V > 2$ tiles to cover Σ^4. Then $|\chi| = \frac{1}{2}(\nu - 4).V$ is relatively large. Now the champions for small p are as follows:

p	$n = \nu$	$\lvert\chi^{\perp}/\chi\rvert = (p-1)/(\nu-4)$
2	7	0.333
3	9	0.400
4	11	$\boxed{0.428}$
5	15	0.364
6	17	0.385
7	19	0.400
8	23	0.368
9	25	0.381

6. Complete hyperbolic 4-manifolds with a knotted core surface covered by two regular tiles

6.1. An example. — We start from the regular trefoil knot

$$(6.1)\qquad \gamma = \{(\cos\varepsilon.e^{2it}, \sin\varepsilon.e^{3it}) : 0 \leqslant t < 2\pi\} \subset S^3.$$

We would expect to need many beads with centers in the ν vertices on γ of a regular template, in order to get an *embedded regular necklace* in S^3. This can be guessed from the graphs (for variable u) of the " self-distance functions ", defined as before, but now for (6.1):

$$(6.2)\qquad \frac{1}{4}a^2(t) = B(t) = (\sin^2 t + u\sin^2(3t/2))/(1 + u).$$

For the total torsion $2\pi T = \nu\tau$ we have

$$2 = q < T < p = 3 \quad \text{and} \quad 2T \in \mathbf{Z}$$

as the torsion condition also holds for this knotted case. Then $T = 1.5$. From the torsion formula (5.7) we obtain

$$u = \frac{\sin^2(4w)}{\sin^2(6w)} \cdot \frac{\sin(11w/2)}{\sin(9w/2)}, \quad w = \pi/\nu.$$

The smallest value for ν for which the collar embedding inequality (5.6) is satisfied is $\nu = 13$. The genus of the core surface is then 6. With the formulas, we find $u = 0.753642$ and the values of $B_j = B_{\nu-j} = B(2j\pi/13)$:

$$\begin{array}{lll} B_1 = 0.312, & B_2 = 0.810, & B_3 = 0.853, \\ B_4 = 0.523, & B_5 = 0.344, & B_6 = 0.408. \end{array}$$

The condition (5.6) is satisfied as is also illustrated in the graph of the self-distance function $B(t)$ in Figure 7.

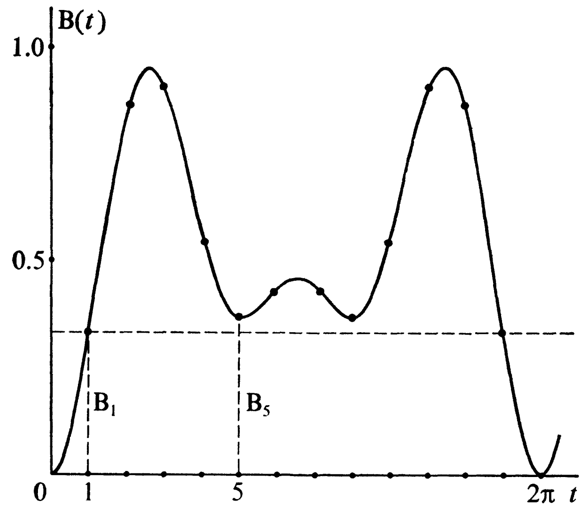


FIG. 7. — Self-distance trefoil knot

As the self-linking of a regular (p, q)-torus-knot (or, equivalently, a regular ν-gonal (p, q)-torus-knot) is $SL(\lambda p, q) = pq$ (see 6.2), Lawson's formula (4.7) gives the self-*intersection number* of $\Sigma \subset \Sigma^4$ for our example with knotted core surface:

$$\chi^{\perp} = 2[T(y) - SL(y)] = 2\left(2\frac{1}{2} - 6\right) = -7.$$

Note that $\chi = \chi(\Sigma) = -10$. The core surface Σ is topologically isotopic to a smooth surface with two branching points. Question: can that surface have minimal area?

We also find a simple compact conformally flat 3-manifold, tesselated by two congruent regular Moebius 13-Gons, whose interiors are homeomorphic to the complement in S^3 of a trefoil knot.

6.2. The self-linking (number) $SL(\gamma)$ **of the regular (homogeneous)** (p, q)**-torus-knot** $\gamma = \gamma_{p,q} \subset S^3 \subset \mathbf{C}^2$.

Lemma. — $SL(\gamma) = pq$ *for* $\gamma = \{(z_1, z_2)\} = \{\cos \varepsilon e^{qti}, \sin \varepsilon e^{pti}\}$.

Proof. — Note that SL is the same for the smooth regular (p, q)-torus-knot as for the regular ν-gonal (p, q)-torus-knot. For small $\varepsilon > 0$, γ approaches a q-fold covering of the circle $z_2 = 0$ in S^3, with the principal normal turning around p times. If we replace the principal normal vector field by the normal in the direction of the point $(0, 1) \subset S^3 \subset \mathbf{C}^2$, the new vector field defines a " quasi-self-intersection " which is p less. Now we move the torus, its knot and this new vector field over S^3 in the direction of $(0, 1)$ until the torus becomes the boundary of a very thin (δ^2)-tubular neighborhood of a round circle with small radius δ and center $(0, 1) \in S^3$. The new vector field χ has become the principle normal vector field. We turn it over $\pi/2$ and obtain roughly parallel vectors orthogonal to the " plane " of the circle, which we now consider as being in a euclidean 3-space. Orthogonal projection into that plane yields a knot diagram with $(q - 1)\, p$ crossings. If we exchange in all those crossings " up " and " under ", then the selflinking $SL(\chi)$ is replaced by $-SL(\chi)$ and so

$$SL(\chi) - (-SL(\chi)) = 2(q - 1)\, p.$$

Hence $SL(\chi) = (q - 1)\, p$ and $SL(\gamma) = p + (q - 1)\, p = qp$.

Open problems.

1. Is $T(\chi) \leqslant SL(\gamma)$ for any unknotted ν-gon in $S^3(\mathbf{R}^3)$ with all sides equal and each diagonal longer than a side?

2. Is the unknotted Julia knot $\mathcal{J}$ in our examples of Σ^4 always tame in S^3? What is the Hausdorff dimension of $\mathcal{J}$? (See [SY].)

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