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NON-ARITHMETIC GROUPS IN LOBACHEVSKY SPACES

by M. GROMOV and I. PIATETSKI-SHAPIRO

0. Introduction

In this paper we construct *non-arithmetic* lattices Γ (both cocompact and non-cocompact: see 1.3 for the definition) in the projective orthogonal group $\mathrm{PO}(n, 1) = \mathrm{O}(n, 1)/\{+1, -1\}$ for all $n = 2, 3, \dots$. We obtain our Γ by “interbreeding” two *arithmetic* subgroups Γ_1 and Γ_2 in $\mathrm{PO}(n, 1)$ as follows. Recall that $\mathrm{PO}(n, 1)$ is the isometry group of the *Lobachevsky space* L^n and assume the subgroups $\Gamma_i \subset \mathrm{PO}(n, 1)$, for $i = 1, 2$, have no torsion. Then the quotient spaces $V_i = \Gamma_i \backslash L^n$ are *hyperbolic manifolds* (i.e. complete Riemannian of constant curvature) and Γ_i is the fundamental group of V_i for $i = 1, 2$. Next, to make the interbreeding possible, we assume there exist connected submanifolds $V_1^+ \subset V_1$ and $V_2^+ \subset V_2$ of dimension n with boundaries $\partial V_1^+ \subset V_1$ and $\partial V_2^+ \subset V_2$, such that

a) The hypersurface $\partial V_i^+ \subset V_i$ for $i = 1, 2$ is totally geodesic in V_i . That is, the universal covering of ∂V_i^+ is a hyperplane in the universal covering L^n of V_i . In particular, ∂V_i^+ is an $(n - 1)$ -dimensional hyperbolic manifold.

b) The manifolds ∂V_1^+ and ∂V_2^+ are isometric.

Now we produce the *hybrid manifold* V by gluing together V_1^+ and V_2^+ according to an isometry between ∂V_1^+ and ∂V_2^+ . This V carries a natural metric of constant negative curvature coming from those on V_1^+ and V_2^+ and this metric is complete apart from a few irrelevant exceptional cases (see 2.10). Then the universal covering of V equals L^n and the fundamental group Γ of V is a lattice in $\mathrm{PO}(n, 1) = \mathrm{Is} L^n$. Note that if the subgroups Γ_1 and Γ_2 are *cocompact* (i.e. if V_1 and V_2 are compact) then also Γ is cocompact.

Also note that the fundamental group Γ_i^+ of V_i^+ *injects* into Γ_i for $i = 1, 2$ (see 2.10) and that in the relevant cases Γ_i^+ satisfies the following.

0.1. Density property (see 1.7). — *The subgroup $\Gamma_i^+ \subset \mathrm{PO}(n, 1)$ is Zariski dense in $\mathrm{PO}(n, 1)^0$ for $i = 1, 2$, where 0 stands for “the identity component of”.*

This density for $i = 1$ implies (see 1.2 and 1.6) the following

0.2. Commensurability property. — *If the group Γ (as well as Γ_1) is arithmetic then Γ and Γ_1 are commensurable. That is there exists a hyperbolic manifold admitting locally isometric finite covering maps onto V and onto V_1 .*

Similarly, arithmeticity of Γ implies commensurability between Γ and Γ_2 and hence, commensurability between Γ_1 and Γ_2 . Therefore, *one obtains a non-arithmetic Γ by taking Γ_1 and Γ_2 non-commensurable* (compare 2.6, 2.7 and 2.8).

0.3. Historical remarks. — *a)* Examples of non-arithmetic lattices Γ in L^3 (the existence of non-arithmetic lattices in L^2 is trivial) were first found by Makarov (see [M]) among *reflection groups* that are groups generated by reflections in some hyperplanes. Then non-arithmetic reflection lattices were constructed in L^4 and L^5 . It is yet unknown for which n there exists a non-arithmetic reflection lattice in L^n , but one does know this n cannot be too large. In fact, no reflection lattice exists in L^n for $n \geq 995$ (see [V], [N] and references therein).

b) A famous theorem by Margulis asserts that every lattice in a simple Lie group G with $\text{rank}_{\mathbb{R}} G \geq 2$ is arithmetic. The remaining non-compact groups (groups with $\text{rank}_{\mathbb{R}} = 1$) are (up to local isomorphism): $O(n, 1)$, $U(n, 1)$, and their quaternion and Cayley analogues. Apart from $O(n, 1)$ where our interbreeding provides non-arithmetic lattices for all n , the existence of non-arithmetic lattices is only known for $SU(2, 1)$ and $SU(3, 1)$. Non-arithmetic lattices in these two groups were constructed by Mostow (see [Mo]) by using reflections in complex hyperplanes.

0.4. Questions. — Call a discrete subgroup $\Gamma_0 \subset PO(n, 1)$ *subarithmetic* if Γ_0 is Zariski dense and if there exists an arithmetic subgroup $\Gamma_1 \subset PO(n, 1)$ such that $\Gamma_0 \cap \Gamma_1$ has finite index in Γ_0 . Does every lattice Γ in $PO(n, 1)$ (maybe for large n) contain a subarithmetic subgroup? Is Γ generated by (finitely many) such subgroups? If so, does $V = \Gamma \backslash L^n$ admit a “nice” partition into “subarithmetic pieces”?

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1. Rudiments of arithmetic groups

1.1. Integral points in linear reductive groups. — A connected Lie group G is called *reductive* if the center of G is compact and G/Center is semisimple. Such a G obviously contains a unique maximal compact normal subgroup $K \subset G$. The quotient group $G' = G/K$, clearly is of *adjoint type*. That is the adjoint representation $\text{ad} : G' \rightarrow \text{Aut } L'$ is injective, where L' denotes the Lie algebra of G' and Aut is the group of linear automorphisms of L' . Our basic example is $G' = PO(n, 1)^0$.

Sufficiently dense subgroups. — Call $\Gamma \subset G$ *sufficiently dense* if the image of Γ in $G' \subset \text{Aut } L'$ is Zariski dense in G' .

Let $G \subset \text{GL}_N \mathbf{R}$ be a reductive subgroup and let $\Gamma \subset G$ be the subgroup of integral matrices in G with $\det = \pm 1$. That is

$$\Gamma = G \cap \text{GL}_N \mathbf{Z}.$$

Property A. — We say that G satisfies A if Γ is sufficiently dense in G .

1.2. Basic Theorem. — *A reductive subgroup $G \subset \text{GL}_N \mathbf{R}$ satisfies A if and only if Γ is a lattice in G , that is, $\text{Vol } G/\Gamma < \infty$.*

Proof. — The implication

$$\text{Vol } G/\Gamma < \infty \Rightarrow \text{Zariski density of } \Gamma' \text{ in } G'$$

holds true for all discrete subgroups $\Gamma \subset G$ and is called *Borel density theorem*. A short proof of this can be found in [Z] and [G]₂.

Let us indicate the (well-known, see [B]₁) proof of the implication $\text{Vol } G/\Gamma < \infty \Leftarrow A$.

Step 1. — By elementary properties of reductive groups (see [B]₂), G equals the identity component of the Zariski closure $\bar{G} \subset \text{GL}_N \mathbf{R}$. Therefore, G contains the identity component $\bar{\Gamma}_0$ of the Zariski closure $\bar{\Gamma} \subset \text{GL}_N \mathbf{R}$.

Note that the inclusion $\bar{\Gamma}_0 \subset G$ is automatic in all our cases and so Step 1 can be omitted.

Step 2. — Property A immediately implies that the homomorphism $G \rightarrow G'$ maps $\bar{\Gamma}_0$ onto G' . It follows that $\bar{\Gamma}_0$ is reductive.

Step 3. — The Zariski density of integral points in $\bar{\Gamma}$ implies that $\bar{\Gamma}$ is defined over \mathbf{Q} . In fact one only needs Zariski density of *rational* points in $\bar{\Gamma}$. This easily follows from the very definition of the Zariski closure.

Step 4. — Since $\bar{\Gamma}$ is reductive, there exists a polynomial map $P : (\mathbf{R}^N)^k \rightarrow \mathbf{R}'$ for some k and ℓ , such that

a) The set of linear transformations of \mathbf{R}^N fixing P equals $\bar{\Gamma}$.

Furthermore, since $\bar{\Gamma}$ is defined over \mathbf{Q} one can choose the above P integral. That is

b) $P((\mathbf{Z}^N)^k) \subset \mathbf{Z}'$.

The existence of P is easy (see [B]₁) and follows directly from Step 2. (We included Step 3 only to bring our discussion nearer to the standard language.)

Step 5. — The orbit $\bar{\Gamma}(\mathbf{Z}^N)$ is *closed* in $\text{GL}_N \mathbf{R}/\text{GL}_N \mathbf{Z}$, where the quotient space $\text{GL}_N \mathbf{R}/\text{GL}_N \mathbf{Z}$ is identified in a natural way with the space of lattices in \mathbf{R}^N . (Note that this step brings us from algebra to geometry.)

Proof of step 5. — Observe that for each lattice $L \subset \mathbf{R}^N$ there exists a finite subset $F \subset L$, such that the values of P on F^k uniquely determine P among the polynomials of the same degree on $(\mathbf{R}^N)^k$. Thus the inequality $P \circ g = P$ on F^k implies $g \in \bar{\Gamma}$ for all $g \in \mathrm{GL}_N \mathbf{R}$ and the diagonal action of $\mathrm{GL}_N \mathbf{R}$ on $(\mathbf{R}^N)^k$.

If L lies in the closure of the orbit $\bar{\Gamma}(\mathbf{Z}^N)$, then there exists a sequence g_i converging to 1 in $\mathrm{GL}_N \mathbf{R}$ and a sequence γ_i in $\bar{\Gamma}$ such that $g_i L = \gamma_i \mathbf{Z}^N$ for all $i = 1, 2, \dots$. This follows from the very definition of the topology in the space of lattices, that is $\mathrm{GL}_N \mathbf{R} / \mathrm{GL}_N \mathbf{Z}$.

Since P is integer valued (i.e. \mathbf{Z}^l -valued) on $(\mathbf{Z}^N)^k$ and $\bar{\Gamma}$ -invariant, the equality $g_i L = \gamma_i \mathbf{Z}^N$ shows that $P \circ g_i$ is integer valued on F^k .

Since $P \circ g$ is continuous in g and F is finite, we have $P \circ g_i = P$ on F^k for almost all i . This implies $P \circ g_i = P$ on all of $(\mathbf{R}^N)^k$ by our choice of F . Therefore, $g_i \in \bar{\Gamma}$ and $L = g_i^{-1} \gamma_i(\mathbf{Z}^N) \in \bar{\Gamma}(\mathbf{Z}^N)$. Q.E.D.

Step 6. — If the orbit $G(\mathbf{Z}^N)$ is *precompact* in $\mathrm{GL}_N \mathbf{R} / \mathrm{GL}_N \mathbf{Z}$, then by the previous step $G/\Gamma = G(\mathbf{Z}^N)$ is compact. That is, Γ is a *cocompact* lattice in G . Note that this case is sufficient for our examples of *compact* hybrids V .

If $G(\mathbf{Z}^N)$ is not precompact the proof of the lattice property

$$\mathrm{Vol} G(\mathbf{Z}^N) < \infty$$

is more complicated (see § 16 in [B]₁ and § 10 in [R]). Yet, in the cases needed for our purpose the proof is relatively simple (see § 2).

1.3. Arithmetic groups. — A discrete subgroup Γ in a reductive group G is called *arithmetic* if there exists a reductive subgroup $\bar{G} \subset \mathrm{GL}_N \mathbf{R}$ for some $N = 1, 2, \dots$ satisfying A and a continuous surjective homomorphism $\rho: \bar{G} \rightarrow G$ such that

- (i) the kernel of ρ is a *compact* subgroup in \bar{G} ;
- (ii) the ρ -image of $\bar{G} \cap \mathrm{GL}_N \mathbf{Z}$ is *commensurable* with Γ . That is, the intersection

$$\Gamma \cap \rho(\bar{G} \cap \mathrm{GL}_N \mathbf{Z})$$

has finite index in Γ as well as in $\rho(\bar{G} \cap \mathrm{GL}_N \mathbf{Z})$.

Remarks. — a) Since G is reductive and $\mathrm{Ker} \rho$ is compact, the group \bar{G} is *necessarily* reductive.

b) Since $\bar{G} \cap \mathrm{GL}_N \mathbf{Z} \subset \bar{G}$ is a lattice by 1.2, the subgroup $\Gamma \cap \rho(\bar{G} \cap \mathrm{GL}_N \mathbf{Z})$ has finite index in Γ . Thus, it is enough to assume in (ii) that this subgroup has finite index in $\rho(\bar{G} \cap \mathrm{GL}_N \mathbf{Z})$.

c) For our applications, we only need $G = \mathrm{PO}(n, 1)$ and $\mathrm{PO}(n, 1) \times \mathrm{PO}(n, 1)$.

1.4. Criterion for non-arithmeticity. — *Let $H \subset G$ be a reductive subgroup. Then the intersection of H with an arithmetic subgroup $\Gamma \subset G$ is arithmetic in H if and only if this intersection $H \cap \Gamma$ is sufficiently dense in H .*

Proof. — Use $\bar{H} = \rho^{-1}(H) \subset \bar{G} \subset \mathrm{GL}_N \mathbf{R}$ and 1.2.

1.4.A. Corollary. — *If $\Gamma \subset G$ is arithmetic and $H \cap \Gamma$ is sufficiently dense in H then $H \cap \Gamma$ is a lattice in H . That is, $\text{Vol } H/H \cap \Gamma < \infty$.*

Proof. — Apply 1.2 again.

1.5. Remarks. — a) If Γ is cocompact in G , then 1.4.A obviously implies that $\Gamma \cap H$ is cocompact in H , provided Γ is arithmetic.

b) The above corollary can be used as a criterion of non-arithmeticity for Γ . For example, let H be isomorphic to $\text{SL}_2 \mathbf{R}$ or $\text{PSL}_2 \mathbf{R}$. Then an elementary argument shows that a discrete subgroup $\Gamma' \subset H$ is either sufficiently dense (here it is equivalent to Zariski dense) or virtually cyclic (i.e. contains a cyclic subgroup of finite index). Therefore, the intersection of an *arithmetic* subgroup $\Gamma \subset G$ with every H isomorphic to $\text{SL}_2 \mathbf{R}$ or $\text{PSL}_2 \mathbf{R}$ is either a lattice in H or a virtually cyclic group. (This observation is due to D. Toledo.)

1.6. Commensurability criterion. — *Let Γ and Γ_1 be arithmetic subgroups in G such that $\Gamma \cap \Gamma_1$ is sufficiently dense in G . Then $\Gamma \cap \Gamma_1$ has finite index in Γ as well as in Γ_1 .*

Proof. — Observe that $\Gamma \times \Gamma_1$ is an arithmetic subgroup in $G \times G$ and that $\Gamma \cap \Gamma_1 \subset G$ equals $G \cap (\Gamma \times \Gamma_1)$ for the diagonal embedding $G \subset G \times G$. Hence, $\Gamma \cap \Gamma_1$ is a lattice in G by 1.4.A which implies the desired commensurability.

1.6.A. Example : Commensurability of hyperbolic manifolds (compare 0.2). — Let V and V_1 be n -dimensional hyperbolic manifolds whose fundamental groups Γ and Γ_1 are *arithmetic* subgroups in $\text{PO}(n, 1)$. Let $V^+ \subset V$ and $V_1^+ \subset V_1$ be connected mutually isometric submanifolds with sufficiently dense fundamental groups Γ^+ and Γ_1^+ . That is, the images of Γ^+ and Γ_1^+ in Γ and Γ_1 respectively are Zariski dense in the ambient group $\text{PO}(n, 1)$. Then there exists a hyperbolic manifold V' which admits a finite locally isometric covering map onto V and onto V_1 .

Proof. — Since V^+ is isometric to V_1^+ the image of Γ^+ in $\text{PO}(n, 1)$ is conjugate to that of Γ_1^+ . Therefore, we may assume that the intersection $\Gamma' = \Gamma \cap \Gamma_1$ in $\text{PO}(n, 1)$ contains the image of Γ^+ . According to 1.6 this Γ' has finite index in Γ as well as in Γ_1 . Hence, the manifold $V' = \Gamma' \backslash L^n$ finitely covers V and V_1 .

1.7. Density criterion for hyperbolic manifolds with boundary. — Let V^+ be a connected n -dimensional manifold of constant negative curvature with non-empty totally geodesic boundary ∂V^+ having finitely many connected components. Assume V^+ is complete as a metric space and $\text{Vol } V^+ < \infty$.

1.7.A. Lemma. — *Let the (image of the) fundamental group of every component of ∂V^+ have finite index in the fundamental group of V^+ . Then $n = 2$ and V^+ is simply connected. It follows that V^+ is isometric to a k -gon in L^2 with vertices at infinity.*

Proof. — The finite index condition shows that the universal covering \tilde{V}^+ also has finitely many boundary components. Then one may assume without loss of generality that the deck transformation group Γ maps every component into itself. Let ∂_0 be one of the components of $\partial\tilde{V}^+$ and let $\bar{\partial}_i \subset \partial_0$ be the normal projections of the remaining components ∂_i , $i = 1, \dots, k$, to ∂_0 . The condition $\text{Vol } V^+ < \infty$ implies that $\bigcup_{i=1}^k \bar{\partial}_i \subset \partial_0$ is a subset of full measure. Hence, $n = 2$, and the action of deck transformations is trivial. Q.E.D.

1.7.B. Corollary (compare 0.1). — *If $\text{Vol } \partial V^+ < \infty$, then the fundamental group Γ^+ of V^+ is Zariski dense in $\text{PO}(n, 1)^0$.*

Proof. — Since $\text{Vol } \partial V^+ < \infty$ the Zariski closure $\bar{\Gamma}^+ \subset \text{PO}(n, 1)$ of Γ^+ contains $\text{PO}(n-1, 1)$ by Borel density theorem (see 1.2), where $\text{PO}(n-1, 1) \subset \text{PO}(n, 1)$ is identified with the isometry group of the space L^{n-1} serving as the universal covering of each component of ∂V^+ . By the above lemma, $\dim \bar{\Gamma}^+ > \dim \text{PO}(n, 1)$ because the (algebraic!) group $\bar{\Gamma}^+$ has at most finitely many connected components. It follows that $\bar{\Gamma} = \text{SO}(n, 1)$, since $\text{O}(n-1, 1)^0$ is a *maximal* connected subgroup in $\text{SO}(n, 1)$.

2. Arithmetic subgroups in $\text{O}(n, 1)$.

2.1. Orthogonal groups. — Let $K \subset \mathbf{R}$ be a number field and F be a non-singular quadratic form in $n+1$ variable with coefficient in K . Denote by $\Gamma(F) \subset \text{GL}_{n+1} \mathbf{R}$ the group of K -integral automorphisms of F . That is the group of F -orthogonal matrices with entries from the ring of integers in K . If the form F has real type (p, q) , then $\Gamma(F)$ is contained in (some conjugate of) the orthogonal group $\text{O}(p, q)$. We are mainly interested in the case $p = n$ and $q = 1$.

Suppose K is totally real of degree $d+1$ and let $I_i: K \subset \mathbf{R}$, $i = 0, \dots, d$ be the various embeddings where I_0 is the original embedding $K \subset \mathbf{R}$. For our applications we shall only need the fields \mathbf{Q} and $\mathbf{Q}(\sqrt{2})$. Note that the embedding $I_1: \mathbf{Q}(\sqrt{2}) \subset \mathbf{R}$ is obtained from I_0 by applying the automorphism $I: \alpha + \beta\sqrt{2} \mapsto \alpha - \beta\sqrt{2}$ to $\mathbf{Q}(\sqrt{2})$.

The following classical theorem (see [B]₁, for example) provides a variety of arithmetic subgroups in $\text{O}(n, 1)$.

2.2. Arithmeticity of $\Gamma(F)$. — *If the forms $I_i F$ are positive definite for $i = 1, \dots, d$, then the subgroup $\Gamma(F) \subset \text{O}(p, q)$ is arithmetic. In particular, $\Gamma(F)$ is discrete and $\text{Vol } \text{O}(p, q)/\Gamma(F) < \infty$.*

Proof. — The pertinent group \bar{G} here (compare 1.3) for $G = \text{O}(p, q)$ is the Cartesian product of the real orthogonal groups $\text{O}(I_i F)$, $i = 0, 1, \dots, d$ (where $\text{O}(I_0 F) = \text{O}(F) = \text{O}(p, q)$). Thus $\bar{G} \subset \text{GL}_N \mathbf{R}$ for $N = (n+1)(d+1)$, where \mathbf{R}^N is given a K -rational basis, that is, a basis of vectors whose projections to the copies of

\mathbf{R}^{d+1} lie in $K \subset \mathbf{R}^{d+1}$, where K embeds into \mathbf{R}^{d+1} by $x \mapsto (I_0(x), \dots, I_d(x))$ for all $x \in K$. Then the verification of the A-property of \bar{G} and arithmeticity of $\Gamma(F)$ is straightforward (see [B]₁).

2.3. Cocompactness of $\Gamma(F)$. — *The above arithmetic group $\Gamma(F)$ is cocompact in $O(p, q)$ if and only if F has no non-trivial zero in K .*

This is a simple corollary of Mahler compactness theorem for lattices in \mathbf{R}^N (see [B]₁, [R]).

2.3.A. *If $d + 1 \geq 2$, then $\Gamma(F)$ is cocompact.*

Proof. — If $F(x, x) = 0$, then also $I_i F(I_i(x), I_i(x)) = 0$ for $i > 0$, as I_i is an isomorphism. Since $I_i F$ is positive definite for $i > 0$, we have $I_i(x) = 0$ and thus $x = 0$.

2.4. Remark. — If $K = \mathbf{Q}$, then $\Gamma(F)$ may be both cocompact and non-cocompact for $n = 2, 3, 4$. But $\Gamma(F)$ is not cocompact for $n \geq 5$ as every indefinite rational quadratic form in five variables has a non-trivial rational zero by the Minkovski-Hasse theorem.

2.5. Action of $\Gamma(F)$ on L^n . — Let F be of signature $(n, 1)$ and consider the (pseudo)-sphere $S = S_F = \{x \in \mathbf{R}^{n+1} \mid F(x, x) = -1\} \subset \mathbf{R}^{n+1}$. This S has two connected components isometric to L^n for the metric induced from the pseudo-Euclidean metric F on \mathbf{R}^{n+1} . Thus $S/\{+1, -1\} = L^n$ and $PO(n, 1) = PO(F)$ acts isometrically on L^n . If $\Gamma \subset \Gamma(F)$ is a subgroup of finite index without torsion, then $\Gamma/\{+1, -1\}$ acts *freely* on L^n and the quotient space $\Gamma \backslash L^n$ is a hyperbolic manifold such that, according to 1.2,

$$\text{Vol}(\Gamma \backslash L^n) < \infty.$$

2.5.A. Congruence subgroups in $\Gamma(F)$. — Take a prime ideal \mathfrak{p} in the ring of integers of K and define the *congruence subgroup* $\Gamma_{\mathfrak{p}}(F) \subset \Gamma(F)$ by

$$\Gamma_{\mathfrak{p}}(F) = \{\gamma \in \Gamma(F) \mid \gamma \equiv \text{Id} \pmod{\mathfrak{p}}\}.$$

If $|\mathfrak{p}|$ is sufficiently large, then $\Gamma_{\mathfrak{p}}(F)$ has no torsion and the action of $\Gamma_{\mathfrak{p}}(F)$ on L^n is free (see [B]₁, [R]).

2.6. Commensurable manifolds. — *Let F_1 and F_2 be two forms over K of type $(n, 1)$ for $n \geq 2$, such that the corresponding groups $\Gamma(F_1)$ and $\Gamma(F_2)$ are commensurable (we stick to the assumptions in 2.2 so that these groups are arithmetic) in the following sense. There exists an isometry α of the (Lobachevsky) space $L_1 = S_{F_1}/\{+1, -1\}$ onto $L_2 = S_{F_2}/\{+1, -1\}$ which sends some subgroup of finite index $\Gamma_1 \subset \Gamma(F_1)/\{+1, -1\}$ (acting on L_1) into $\Gamma(F_2)/\{+1, -1\}$ (acting on L_2). Then the forms F_1 and F_2 are similar over K . That is there exists a linear K -isomorphism $\mathbf{R}^{n+1} \rightarrow \mathbf{R}^{n+1}$ sending F_1 to λF_2 for some $\lambda \in K$.*

Proof. — There obviously exists a unique (up to $\{+1, -1\}$) linear map $\bar{\alpha} : \mathbf{R}^{n+1} \rightarrow \mathbf{R}^{n+1}$ sending F_1 to F_2 such that the induced map $L_1 \rightarrow L_2$ is α . Denote by $\bar{\Gamma}_1 \subset \Gamma(F_1)$ the $\{+1, -1\}$ -extension of Γ_1 . Since $\bar{\Gamma}_1$ is Zariski dense in $O(n, 1)$ and the action of $O(n, 1)$ on \mathbf{R}^{n+1} is \mathbf{C} -irreducible for $n \geq 2$, the K -linear span of $\bar{\Gamma}_1$ in $\text{End } \mathbf{R}^{n+1}$ equals $\text{End } K^{n+1} \subset \text{End } \mathbf{R}^{n+1}$. Since $\bar{\alpha}$ sends $\bar{\Gamma}_1$ in $\Gamma(F_2)$, the K -span of $\bar{\Gamma}_1$ goes to that of $\Gamma(F_2)$ and then the equality $\text{Span}_K \bar{\Gamma}_1 = \text{End } K^{n+1}$ implies that $\bar{\alpha} = \mu \bar{\alpha}_0$, where $\bar{\alpha}_0$ is defined over K and $\mu \in \mathbf{R}^\times$. Now, α_0 sends F_1 to $\mu^{-2} F_2$ and since $F_1 \neq 0$, the factor μ^{-2} lies in K . Q.E.D.

2.7. Corollary. — *Let F_1 and F_2 be diagonal,*

$$F_1 = \sum_{i=1}^{n+1} a_i x_i^2 \quad \text{and} \quad F_2 = \sum_{i=1}^{n+1} b_i x_i^2$$

for a_i and b_i in K . Then for $n+1$ even the ratio of the discriminants

$$\prod_{i=1}^{n+1} a_i \mid \prod_{i=1}^{n+1} b_i \quad \text{lies in } (K^\times)^2.$$

Proof. — A linear transformation over K with determinant D multiplies discriminants by D^2 and similarity $F \mapsto \lambda F$ multiplies the discriminant of F by λ^{n+1} .

2.7.A. Example. — *a)* Let $K = \mathbf{Q}$ and

$$\begin{aligned} F_1 &= x_0^2 + x_1^2 + \dots + x_{n-1}^2 - x_n^2 \\ F_2 &= 2x_0^2 + x_1^2 + \dots + x_{n-1}^2 - x_n^2. \end{aligned}$$

Then for $n+1$ even the groups $\Gamma(F_1)$ and $\Gamma(F_2)$ are not commensurable as 2 is not a square in \mathbf{Q} . Also note that these groups are not cocompact as $F_i(x, x) = 0$ for $x = (0, 0, \dots, 0, 1, 1)$ and $i = 1, 2$ (compare 2.4).

b) Let $K = \mathbf{Q}(\sqrt{2})$ and

$$\begin{aligned} F_1 &= x_0^2 + x_1^2 + \dots + x_{n-1}^2 - \sqrt{2} x_n^2 \\ F_2 &= 3x_0^2 + x_1^2 + \dots + x_{n-1}^2 - \sqrt{2} x_n^2. \end{aligned}$$

Here again the corresponding groups are not commensurable for $n+1$ even, but now these groups are cocompact (see 2.3.A).

2.8. Totally geodesic submanifolds in hyperbolic manifolds. Take a $(k+1)$ -dimensional linear subspace $R_0 \subset \mathbf{R}^{n+1}$ which meets the sphere $S = S(F) \subset \mathbf{R}^{n+1}$. Then the intersection $S_0 = S \cap R_0$ is a totally geodesic submanifold in S of dimension k . For a subgroup $\Gamma \subset \Gamma(F)$ denote by $\Gamma_0 \subset \Gamma$ the subgroup stabilizing R_0 . If the subspace R_0 is K -rational and Γ_0 has finite index in Γ , then Γ_0 is arithmetic. That is, the image of Γ_0 in the full isometry group $\text{Is } S_0 = O(k, 1)$ gives a *proper immersion* $\Gamma_0 \backslash S_0 \rightarrow \Gamma \backslash S$ (by Step 5 in 1.2).

2.8.A. Embedding criterion. — Denote by $I_0 \in O(n, 1)$ the orthogonal reflection of \mathbf{R}^{n+1} in R_0 .

If I_0 normalizes Γ , then the canonical map $\Gamma_0 \backslash S_0 \rightarrow \Gamma \backslash S$ is a proper embedding, provided Γ has no torsion.

Proof. — Suppose two distinct points s and s' from S_0 go to the same point in $\Gamma \backslash S$. That is $s' = \gamma(s)$ for some $\gamma \in \Gamma$. Since s and s' are fixed by I_0 , the commutator $\delta = \gamma^{-1} I_0 \gamma I_0^{-1}$ fixes s . Since I_0 normalizes Γ this δ is contained in Γ and as Γ has no torsion and acts freely on S_0 , we obtain $\delta = \text{Id}$. Since S_0 equals the fixed point set of I_0 , the equality $[\gamma, I_0] = \text{Id}$ implies that $\gamma \in \Gamma_0$. Q.E.D.

2.8.B. Remark. — If $\Gamma_0 \backslash S_0 \rightarrow \Gamma \backslash S$ is an embedding, then, obviously, the corresponding map $\Gamma'_0 \backslash S_0 \rightarrow \Gamma' \backslash S$ also is an embedding for every subgroup $\Gamma' \subset \Gamma$.

Corollary. — If the group generated by Γ and $I_0 \Gamma I_0^{-1}$ is discrete without torsion, then the map $\Gamma_0 \backslash S_0 \rightarrow \Gamma \backslash S$ is an embedding.

2.8.C. Example. — Let F_0 be a quadratic form in variables x_1, \dots, x_n over $K \subset \mathbf{R}$ of type $(n-1, 1)$ and $F = ax_0^2 + F_0$ for $a > 0$ in K . Then the reflection I_0 in the hyperplane $R_0 = \{x_0 = 0\} \subset \mathbf{R}^{n+1}$,

$$I_0 : (x_0, x_1, \dots, x_n) \mapsto (-x_0, x_1, \dots, x_n)$$

lies in $\Gamma(F)$ and the previous discussion applies to the congruence subgroups $\Gamma_p(F) \subset \Gamma(F)$ with $|p|$ sufficiently large. Therefore the hyperbolic manifold

$$V(F_0, p) = \Gamma_p(F_0) \backslash L^{n-1}$$

(where we identify L^{n-1} with $S_0/\{+1, -1\}$) isometrically embeds into $V(F, p) = \Gamma_p(F) \backslash L^n$.

Note that for p prime to 2 both manifolds $V(F, p)$ and $V(F_0, p)$ are orientable. In fact, if $-1 \not\equiv 1 \pmod{p}$, then $\Gamma_p(F) \subset SO(n, 1)$ and $\Gamma_p(F_0) \subset SO(n-1, 1)$.

The hypersurface $V(F_0, p)$ does not necessarily bound in $V(F, p)$. (In fact for large $|p|$ it does not bound). However, there exists an obvious double covering $\tilde{V}(F, p)$ of $V(F, p)$, such that the lift of $V(F_0, p)$ to $\tilde{V}(F, p)$ consists of two disjoint copies of $V(F_0, p)$ which do bound some connected submanifold $V^+ \subset \tilde{V}(F, p)$. That is the boundary ∂V^+ is the union of two copies of $V(F_0, p)$.

2.9. Interbreeding hyperbolic manifolds. — Take the forms $F_i = a_i x_0^2 + F_0$ as in the previous example for $i = 1, 2$, and assume for the uniformity of notation that $V(F_0, p)$ does not bound in either of the two manifolds $V(F_i, p)$. (As we mentioned earlier, this is the case for large $|p|$.) Then we take the corresponding manifolds $V_i^+ \subset \tilde{V}(F_i, p)$ for $i = 1, 2$ and recall that V_1^+ and V_2^+ have isometric boundaries equal to $2V(F_0, p)$.

If $n+1$ is even and a_1/a_2 is not a square in K then the forms F_1 and F_2 are not similar over K (compare 2.7) and the groups $\Gamma(F_1)$ and $\Gamma(F_2)$ are not commensurable (see 2.6). In this case the manifold V obtained by gluing V_1^+ to V_2^+ along the boundary is non-arithmetic (i.e. the fundamental group is not arithmetic: compare 0.2, 1.6.A).

If $(n + 1)$ is odd, we consider a K -rational hyperplane $R' \subset \mathbf{R}^{n+1}$ normal to R_0 . For example, let $F_0 = \sum_{i=1}^n b_i x_i^2$, where $b_1 > 0$ and take

$$R' = \{x_1 = 0\} \subset \mathbf{R}^{n+1}.$$

Then the corresponding hypersurfaces $V'_i \subset V(F_i, p)$ are normal to $V(F_0, p)$. Therefore, their "halves" $V'_1 \cap V_1^+$ and $V'_2 \cap V_2^+$ glue together to a *totally geodesic* hypersurface $V' \subset V$. If V is arithmetic, then so is V' (see 1.4). But V' is non-arithmetic for $n - 1 = \dim V' \geq 2$ by the previous argument and thus the non-arithmeticity of V (i.e. of the fundamental group Γ of V) is established for all $n \geq 3$. We leave the (trivial) case where $n = 2$ to the reader.

2.10. Final hyperbolic remarks. — To complete our discussion we need two simple facts from hyperbolic geometry.

2.10.A. The fundamental group of V^+ injects into that of V .

Proof. — The submanifold $V^+ \subset V$ has convex (in fact, totally geodesic) boundary and so every class in $\pi_1(V^+)$ is represented by a *geodesic* loop in V^+ . Such a loop is not contractible in V , as V is complete of negative curvature. Q.E.D.

2.10.B. The manifold V obtained by gluing V_1^+ and V_2^+ (see § 0) is complete provided these manifolds as well as their (totally geodesic) boundaries have finite volumes.

Proof. — The claim is obvious if $V_1^+ = V_2^+$ is compact.

If V_1^+ is non-compact then the geometry at infinity is described with the following notion.

2.10.C. Cusps. — An n -dimensional *cusp with boundary* is a Riemannian manifold $C^+ = F^+ \times \mathbf{R}_+$, where F^+ is a compact flat manifold with totally geodesic boundary and where the metric in C^+ is $dt^2 + e^{-t}g$, where $t \in \mathbf{R}_+$ and g is the flat metric on F^+ .

Observe that a compact connected flat manifold F^+ with a non-empty boundary either is isometric to a product $F_0 \times [-a, a]$ for some compact flat manifold F_0 without boundary, or has a double covering isometric to $F_0 \times [-a, a]$. In both cases the connected components of the levels of the distance function $\text{dist}(x, \partial F^+)$ foliate F^+ into closed connected totally geodesic submanifolds F_θ for $\theta \in [0, a]$. It follows that a connected cusp with non-empty boundary is canonically foliated into leaves $C_\theta = F_\theta \times \mathbf{R}_+$. Note that this splitting of C_θ is unique. In fact, for each $x \in C_\theta$, there exists a unique closed connected $(n - 2)$ -dimensional hypersurface $F(x) \subset C_\theta$ passing through x , such that

- a) the induced metric in $F(x)$ is flat;
- b) also the induced metrics in the *parallel* hypersurfaces (which are defined as the level of the distance function to $F(x)$ in C_θ) are flat.

Since the hypersurfaces $F_\theta \times t \subset C_\theta$ have these properties, the hypersurface $F(x)$ for $x = (f, t)$ equals $F_\theta \times t$.

The $(n - 2)$ -dimensional volume of $F_0 \times t$ is obviously $\text{const exp}(n - 2) t$. Hence, if $n \geq 3$, the parameter $t = t(x)$ for $x = (f, t)$ can be recaptured (up to an additive constant) by taking $\log \text{Vol } F(x)$, for those x , for which the hypersurface is normally orientable and $\log 2 \text{Vol } F(x)$ for the others.

Now it is clear that a manifold C , obtained by gluing together two cusps $C_i^+ = F_i^+ \times \mathbf{R}_+$ by isometries along their boundary cusps $\partial F_i^+ \times \mathbf{R}_+$, is again a cusp. In fact, the foliations on C_i^+ define a geodesic foliation of C into $(n - 1)$ -dimensional cusps C_0 without boundary and the cusp structure in C is seen with $t = \log \text{Vol } F(x)$.

Finally, we conclude the proof of 2.10.B by invoking the following.

2.10.D. Proposition. — *Let V^+ be a complete hyperbolic manifold with totally geodesic boundary. If $\text{Vol } V^+ < \infty$, then the complement to a compact subset in V^+ is isometric to a (possibly disconnected) cusp.*

Proof. — If V^+ has no boundary, this is standard (see [B]₁, [R], [G]₁), and the case with boundary follows by taking the double of V^+ .

This proposition and the above discussion show that the glued manifold V is cuspidal at infinity. Since cusps are complete, V is complete. Q.E.D.

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