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ON THE C^1 Ω-STABILITY CONJECTURE

by J. PALIS*

In his remarkable proof of the C^1 *Stability Conjecture*, Mañé stated that from his result it became realistic to expect a proof of the similar conjecture for the nonwandering set, the C^1 Ω-*Stability Conjecture*. We show in this paper that this is indeed the case; actually, we just change the last argument in Section 1 of Mañé's paper. Besides some combinatorial reasoning, perhaps the novelty here is the following strategy: instead of showing directly that if f is Ω-stable then it satisfies Axiom A, we first show that it can be C^1 approximated by an Axiom A diffeomorphism. We then prove that f itself satisfies Axiom A. A key simple fact is that if f is Ω-stable then no cycle can exist among hyperbolic basic sets of f [7]. Using repeatedly one of Mañé's result we succeed in reaching cycles as above unless f satisfies Axiom A, thus proving that Ω-stability of a diffeomorphism implies that it satisfies Axiom A (see discussion below on the complete statement and converse).

We begin by briefly recalling some basic definitions and known facts concerning this conjecture.

Let M be a C^∞ compact manifold without boundary and let $\mathrm{Diff}^r(M)$ denote the set of C^r diffeomorphisms of M with the C^r topology for $r \geqslant 1$. For $f \in \mathrm{Diff}^r(M)$, we denote by $\Omega(f)$ its nonwandering set and by $P(f)$ its set of periodic points. We say that f is C^r structurally stable, resp. Ω-stable, if there exists a neighborhood $\mathscr{U}$ of f in $\mathrm{Diff}^r(M)$ such that for each $g \in \mathscr{U}$ there is a homeomorphism h of M, resp. $h : \Omega(f) \to \Omega(g)$, satisfying $hf(x) = gh(x)$ for all $x \in M$, resp. $x \in \Omega(f)$.

The C^r Stability Conjecture formulated in [6] states that $f \in \mathrm{Diff}^r(M)$ is structurally stable if and only if $\Omega(f)$ is hyperbolic and $\Omega(f) = \overline{P(f)}$ (Axiom A) and for every $x \in M$ its stable and unstable manifolds $W^s(x)$ and $W^u(x)$ are in general position (transversality condition).

We can formulate a similar conjecture for the Ω-stability as follows. If f satisfies Axiom A then by [8] we can write $\Omega(f)$ as a finite union of closed, transitive sets in which the periodic points are dense: $\Omega(f) = \Omega_1 \cup \ldots \cup \Omega_s$. Each Ω_i is called a basic

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set. A cycle on $\Omega(f)$ is a sequence $\Omega_{i_1}, \ldots, \Omega_{i_k}$ with points $x_1, y_1 \in \Omega_{i_1}, \ldots, x_k, y_k \in \Omega_{i_k}$ such that $W^s(x_1) \cap W^u(y_2) \neq \emptyset, \ldots, W^s(x_k) \cap W^u(y_1) \neq \emptyset$. The C^r Ω-Stability Conjecture states: f is Ω-stable if and only if it satisfies Axiom A and there are no cycles on $\Omega(f)$. The conjecture arises quite naturally from the following facts. On one hand, Smale's Ω-stability theorem [8] states that if f satisfies Axiom A and there are no cycles on $\Omega(f)$ then f is Ω-stable. On the other hand, if f satisfies Axiom A and there is a cycle on $\Omega(f)$ then by [7] f is not Ω-stable: it can be C^r approximated by g such that $P(g) \not\supseteq P(f)$.

Before showing the results we recall that Λ is a basic set for f if it is closed, hyperbolic and transitive, with a dense subset of periodic orbits. We also require it to be isolated, i.e. maximal invariant set for f in some neighborhood of it. If Λ is a basic set for f, and the maximal invariant set in some neighborhood $U(\Lambda)$, then Λ is persistent: if g is C^r close to f then there is a (unique) basic set $\Lambda(g)$ near Λ and f/Λ is conjugate to $g/\Lambda(g)$; $\Lambda(g)$ is called the continuation of Λ [2]. The stable manifold $W^s(\Lambda)$ consists of points whose ω-limit sets are in Λ; similarly for the unstable manifold $W^u(\Lambda)$. Also, as observed in [4], if f is Ω-stable then all periodic orbits of f are persistently hyperbolic: if g is near f then its periodic orbits are hyperbolic. That is, $f \in \mathcal{F}^r(M)$ where $\mathcal{F}^r(M)$ denotes the interior of the set of C^r diffeomorphisms whose periodic orbits are all hyperbolic. Finally, if $f \in \mathcal{F}^r(M)$ and Λ_i, $1 \leqslant i \leqslant s$, are basic sets for f then by [7] there can be no cycle on $\bigcup_i \Lambda_i$.

We can now present the main fact toward the proof of the Ω-Stability Conjecture.

Theorem A. — If $f \in \mathcal{F}^1(M)$ then f can be C^1 approximated by an Axiom A diffeomorphism.

This and our previous discussions have as immediate consequence the following result.

Corollary. — The elements of an open *and* dense *subset of $\mathcal{F}^1(M)$ satisfy Axiom A and the no-cycle condition.*

Proof of Theorem A. — Suppose first we want to show that f itself satisfies Axiom A. Since $f \in \mathcal{F}^1(M)$, following [4] we may suppose by induction that $\overline{P_k(f)}$ is hyperbolic for all $0 \leqslant k \leqslant j$, where $P_k(f)$ is the set of periodic points of index k (dimension of the stable manifold). We then have to prove that $\overline{P_{j+1}(f)}$ is hyperbolic. So let $\Lambda_1, \ldots, \Lambda_s$ be the decomposition of the union of the $\overline{P_k(f)}$ for $0 \leqslant k \leqslant j$ into (hyperbolic) basic sets. According to theorems I.4 through I.7 in [4], it is enough to show that $P_{j+1}(f)$ does not accumulate on $\bigcup_k \Lambda_k$. This is clearly the case for $P_1(f)$ since $P_0(f)$ is either empty or consists of sources. Thus we may consider only the basic sets corresponding to $\overline{P_k(f)}$ for $1 \leqslant k \leqslant j$. That f satisfies Axiom A is therefore reduced to the proof of the following statement:

(a) if $P_{j+1}(f)$ accumulates on $\bigcup_k \Lambda_k$ then we can create a cycle on $\bigcup_k \Lambda_k$ for some f^* C^1 close to f, $f^* = f$ on $\bigcup_k \Lambda_k$.

In fact, from the existence of a cycle, we can, as in [7], create new periodic points of the same index by small C^1 perturbations. We conclude that $f^* \notin \mathscr{F}^1(M)$ and so $f \notin \mathscr{F}^1(M)$ since $\mathscr{F}^1(M)$ is an open subset of $\mathrm{Diff}^1(M)$.

We now approximate f by g so that if $W^s(\Lambda_{j_1}(g))$ intersects $W^u(\Lambda_{j_2}(g))$ it does so persistently: the corresponding intersection for any $\tilde{g} \in C^1$ near g also occurs. After this we will prove that (a) above occurs for g; later in Theorem B we will show that f itself satisfies Axiom A. To see that we can take such a g close to f we consider a small neighborhood $\mathscr{U}$ of f on which the continuation of the basic sets Λ_k are defined. We claim that the set $g \in \mathscr{U}$ such that $W^s(\Lambda_{j_1}(g))$ intersects $W^u(\Lambda_{j_2}(g))$ persistently or not at all for $1 \leqslant j_1 \leqslant j_2 \leqslant s$ is of second category. This follows from the stable manifold theorems for basic sets [2], [3] including its continuous dependence on the map on compact parts and the existence of fundamental domains. For this last point, there is an easier proof in [1], [5] using a shadowing property.

So we consider such a g and introduce a partial ordering on the $\Lambda_k = \Lambda_k(g)$, $1 \leqslant k \leqslant s$, as follows: $\Lambda_{k_1} \leqslant \Lambda_{k_2}$ if $W^s(\Lambda_{k_1}) \cap W^u(\Lambda_{k_2}) \neq \emptyset$. The advantage of working with g is that such an intersection (and thus the ordering relation) is persistent. We now show how to choose a (finite) sequence of small perturbations of g so that, at each stage, either we have an Axiom A diffeomorphism or else, through the subsequent perturbation, a new pair of basic sets becomes directly ordered by the relation above and they do so in a persistent way. Clearly the statement implies the theorem: the number of basic sets is finite and we cannot reach a cycle among them since the diffeomorphisms are in $\mathscr{F}^1(M)$. To prove it, let us suppose $\overline{P_{j+1}(g)}$ to be nonhyperbolic and let Λ_{j_1} be such that $P_{j+1}(g)$ accumulates on it and, among the Λ_k with this property, Λ_{j_1} has maximum index. Then by theorem I.7 of [4], we can take Λ_{j_2} on which $P_{j+1}(g)$ accumulates and g_1 near g such that $\Lambda_{j_1}(g_1) < \Lambda_{j_2}(g_1)$ and we can make this relation persistent because of the index condition. We claim we may assume that $\Lambda_{j_1} \nless \Lambda_{j_2}$ which shows that we are indeed adding a new persistent relation $\Lambda_{j_1}(g_1) < \Lambda_{j_2}(g_1)$. In fact if $W^s(\Lambda_{j_1}) \cap W^u(\Lambda_{j_2}) \neq \emptyset$ we then choose $j_3 = j_2$ such that $P_{j+1}(g)$ accumulates on Λ_{j_3} and g_2 near g so that $\Lambda_{j_2}(g_2) < \Lambda_{j_3}(g_2)$. By persistence, we also have $\Lambda_{j_1}(g_2) < \Lambda_{j_2}(g_2)$. As in [7], using that $\Lambda_{j_2}(g_2)$ is a basic set, we can perturb g_2 to $\tilde{g}_2$ so that $\Lambda_{j_1}(\tilde{g}_2) < \Lambda_{j_3}(\tilde{g}_2)$ and, by the index condition, we may assume this relation to hold persistently. Notice that $j_3 \neq j_1$ for otherwise we reach a 2-cycle. Again, it may happen that already for g we have $\Lambda_{j_1} < \Lambda_{j_3}$. If so, as before, we do not perform any of the three previous perturbations; instead we take Λ_{j_4} in which $P_{j+1}(g)$ accumulates and g_3 near g so that $\Lambda_{j_3}(g_3) < \Lambda_{j_4}(g_3)$ and from that we get $\tilde{g}_3$ near g_3 such that $\Lambda_{j_1}(\tilde{g}_3) < \Lambda_{j_4}(\tilde{g}_3)$. In this way either we get an Axiom A diffeomorphism or else we achieve a new persistent relation $\Lambda_{j_1}(g^*) < \Lambda_{j_i}(g^*)$ for some g^* near g and some index j_i *except* in one of the following possibilities:

(b) $W^s(\Lambda_{j_1}(g))$ already intersects $W^u(\Lambda_i(g))$ for all Λ_i in which $P_{j+1}(g)$ accumulates,

(c) we reach a sequence $\Lambda_{i_1}, \ldots, \Lambda_{i_\ell}$ such that $P_{j+1}(g)$ accumulates on all of these basic sets and $\Lambda_{j_1} < \Lambda_{i_k}$ for all $1 \leqslant k \leqslant \ell$ and also there are maps $g_1, \ldots, g_\ell$ C^1 near g such that $\Lambda_{i_1}(g_1) < \Lambda_{i_2}(g_1), \ldots, \Lambda_{i_\ell}(g_\ell) < \Lambda_{i_1}(g_\ell)$.

In the second case we repeat our previous procedure choosing an element in this sequence with maximum index. Then, either we achieve an Axiom A diffeomorphism or a new persistent relation with this element as claimed, or else this sequence yields a 2-cycle for some g_k above, which is a contradiction. On the other hand, if (b) above holds, that is $\Lambda_{j_1} < \Lambda_i$ for all Λ_i on which $P_{j+1}(g)$ accumulates, then this property cannot be shared by any other basic set besides Λ_{j_1}, for otherwise we have a 2-cycle for g. So, in this last case we just put aside Λ_{j_1} and start anew: we choose some Λ_{j_2} on which $P_{j+1}(g)$ accumulates and having maximum index (except for that of Λ_{j_1}) among the basic sets which are accumulated upon by $P_{j+1}(g)$. Then, in this turn, we definitely either reach an Axiom A diffeomorphism or else we add a new persistent relation for a pair of basic sets. This follows from the fact that $\Lambda_{j_2} \nless \Lambda_{j_1}$ for, otherwise, we have a 2-cycle for g, which is not possible.

It is clear that we can now proceed inductively and the induction step is exactly the same as above. This proves statement (a) above for g which implies that $\overline{P_{j+1}(g)}$ is hyperbolic. Thus by induction $\overline{P_k(g)}$ is hyperbolic for all k and so g satisfies Axiom A. This finishes the proof of Theorem A.

We now reach the main goal of this paper proving the following theorem.

Theorem B. — *If* $f \in \mathrm{Diff}^1(M)$ *is* Ω*-stable then it satisfies Axiom* A.

Proof. — Using Theorem A we obtain an Axiom A diffeomorphism g C^1 close to f, and thus Ω-conjugate to f. As in Theorem A, by [4] we may assume that $P_k(f)$ is hyperbolic for $0 \leqslant k \leqslant j$ and to show that $\Omega(f)$ is hyperbolic it is enough to show that $P_{j+1}(f)$ does not accumulate on $\overline{\bigcup_k P_k(f)}$. This means $\overline{P_{j+1}(f)}$ to be homogeneous in the sense that all periodic points have the same index. But the basic sets in $\Omega(g)$ are homogeneous and any pair of (periodic) points have stable and unstable manifolds that mutually intersect each other at points that belong to $\Omega(g)$. Thus if $h : \Omega(g) \to \Omega(f)$ is a conjugacy between $g/\Omega(g)$ and $f/\Omega(f)$ then it is enough to show that the image by h of each basic set in $\Omega(g)$ is homogeneous.

Suppose that this is not so. We now make use of Thom's transversality theorem and the continuous dependence of stable and unstable manifolds on the map on compact parts. By slightly perturbing f if necessary, we may assume that only one pair of periodic points of f with different indices have their stable and unstable manifolds mutually intersecting each other. Moreover, we can suppose that one of these points of intersection is in $\Omega(f)$: for hyperbolic $p, q \in P(f)$, if $W^s(p)$ and $W^u(q)$ have a point of transversal intersection then any point in $W^u(p) \cap W^s(q)$ is in $\Omega(f)$. But this is clearly absurd because f is Ω-conjugate to g, and so at least one of these two periodic orbits must be accumulated by others with a different index and having with them the mutual intersection property of stable and unstable manifolds. This concludes the proof of the theorem.

REFERENCES

[1] R. BOWEN, On Axiom A diffeomorphisms, *Conference Board Math. Sciences*, **35**, *Amer. Math. Soc.*, 1977.

[2] M. HIRSCH and C. PUGH, Stable manifolds and hyperbolic sets, Global Analysis, *Proc. AMS Symp. in Pure Math.*, **14** (1970), 133-163.

[3] M. HIRSCH, J. PALIS, C. PUGH and M. SHUB, Neighborhoods of hyperbolic sets, *Invent. Math.*, **9** (1970), 121-134.

[4] R. MAÑÉ, A proof of the C^1 Stability Conjecture, *Publ. Math. I.H.E.S.*, **66** (1987), 161-210.

[5] S. NEWHOUSE, Lectures on dynamical systems, *Progress in Math.*, **8**, Birkhäuser, 1980.

[6] J. PALIS and S. SMALE, Structural stability theorems, *Proc. AMS Symp. in Pure Math.*, **14** (1970), 223-231.

[7] J. PALIS, A note on Ω-stability, same *Proceedings*, **14** (1970), 221-222.

[8] S. SMALE, The Ω-stability theorem, same *Proceedings*, **14** (1970), 289-297.

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