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POSITIVE SCALAR CURVATURE AND THE DIRAC OPERATOR ON COMPLETE RIEMANNIAN MANIFOLDS

by MIKHAEL GROMOV and H. BLAINE LAWSON, Jr. (*)

PREFACE

One of the principal aims of this study is to understand spaces which carry metrics of positive scalar curvature. In recent years this subject has been the focus of lively research, and we would like to begin with a brief discussion of the developments (1).

Certainly, one of the simplest invariants of a riemannian manifold is its scalar curvature function. In general dimensions, this function (whose value at a point is just the average sectional curvature) is a weak measure of the local geometry, and one might suspect it to be unrelated to the global topology of the manifold. Evidence for this suspicion can be found in the work of Kazdan-Warner and Aubin. For example, in [KW] it is proved that on a compact manifold of dimension ≥ 3, every smooth function which is negative somewhere, is the scalar curvature of some riemannian metric. However, the intriguing fact is that there are manifolds of high dimension which carry no metrics whose scalar curvature is everywhere ≥ 0.

The first examples of such manifolds were given in 1962 by A. Lichnerowicz [Li], who reasoned as follows. Over any riemannian spin manifold there exists a fundamental elliptic operator, called the *Dirac operator* (2). Using Bochner's method, Lichnerowicz showed that on compact manifolds of positive scalar curvature, this operator is invertible. In dimensions $4k$ he then concluded, via the Atiyah-Singer Index Theorem, that a certain basic topological invariant of the manifold, called the $\hat{A}$-genus, must vanish. For any oriented 4-manifold M, it is a fact that $8\hat{A}(M) = \text{signature}(M)$. Since there are many spin 4-manifolds of non-zero signature, and since $\hat{A}$ is a " multiplicative " invariant, the above argument produces many examples of manifolds which do not carry positive scalar curvature.

(*) Research partially supported by NSF Grant number MCS 830 1365.
(1) We also recommend to the reader the historical account of L. Berard-Bergery [B1].
(2) This operator was first constructed by Atiyah and Singer in their work on the Index Theorem.

Note that the spin assumption is essential here, since the complex projective plane has both positive curvature and non-zero $\hat{A}$-genus.

The next major contribution to the subject was made by N. Hitchin [H] who used the Dirac operator to conduct an extensive investigation of spaces of positive scalar curvature. One surprising and beautiful result of this study was a proof that half of the exotic spheres in dimensions 1 and 2 (mod 8) cannot carry metrics of positive scalar curvature. The major ingredient in this proof is a refined version of the index theorem, due also to Atiyah and Singer [ASV].

Although it is very impressive, all of this work still left open the question of Geroch: Can the torus T^n, $n \geq 3$, carry a metric of positive scalar curvature? This question was settled (for $n \leq 7$) in a beautiful series of papers by R. Schoen and S. T. Yau [SY_1], [SY_2]. Their principal result is the following " Splitting Theorem ". *Let* M^n ($n \leq 7$) *be a compact manifold of positive scalar curvature. Then any class* $\alpha \in H_{n-1}(M^n; \mathbf{Z})$ *can be represented by a manifold which carries positive scalar curvature.* This is also true for homology classes in any (possibly infinite) covering space of M^n. This result allows one to define inductively a class of manifolds *(Schoen-Yau manifolds)* which do not carry positive scalar curvature. The class contains all sufficiently large 3-manifolds, and hence it is quite rich.

The above Splitting Theorem is proved by choosing a manifold of least area in the homology class α, and then applying the formula for the second variation of area. The positivity of the second variation operator easily implies that the metric induced on this minimizing hypersurface is conformally equivalent to one of positive scalar curvature.

The major breakthrough made by Schoen and Yau was the discovery that the standard second variational formula could be rewritten in a particularly useful way. (See 11.11 below.) Although the computation is trivial, this important fact eluded geometers for many years. It now forms the keystone of a solid arch between geometric measure theory and riemannian geometry. This connection has led to an intriguing circle of ideas involving minimal hypersurfaces, scalar curvature and the Dirac operator.

It should be mentioned that the restriction on dimension in the Splitting Theorem results from the breakdown of interior regularity for solutions to the Plateau Problem in n-manifolds for $n \geq 8$. Although it appears that Schoen and Yau have recently succeeded in extending their techniques to all dimensions, the Splitting Theorem, as stated above, remains conjectural.

The next step in the development of the subject of positive scalar curvature was the introduction of the notion of *enlargeability* [GL_1]. With this concept it was possible to introduce the fundamental group into the Dirac operator methods used in the past. The key realization was that π_1 enters the problem through the geometry of the universal covering. A manifold was defined to be *enlargeable* if for every $\varepsilon > 0$ there exists a finite covering which admits an ε-contracting map of non-zero degree onto the unit sphere. These highly contracting maps can be used to produce non-trivial bundles which are almost flat by pulling back some fixed bundle from the sphere. Applying the Bochner Method to the Dirac operator with coefficients in these bundles proves directly that *enlargeable spin manifolds cannot carry positive scalar curvature.*

Enlargeability is a homotopy-theoretic property, and the category of enlargeable manifolds is a rich and interesting one. It contains, for example, all solvmanifolds, all hyperbolic manifolds and all sufficiently large 3-manifolds. In particular, it properly contains all Schoen-Yau manifolds.

The category has some nice properties. Products of enlargeable manifolds are enlargeable, and the connected sum of an enlargeable manifold with any other manifold is again enlargeable. In fact, any manifold which admits a map of non-zero degree onto an enlargeable manifold is enlargeable.

There is an important generalization of the notion of enlargeability which is given by simply replacing the word " degree " with " $\hat{A}$-degree " in the discussion above. (The $\hat{A}$-degree of a map f is defined to be $\hat{A}(f^{-1}$ (any regular value)).) With this more general definition it remains true that enlargeable spin manifolds cannot carry positive scalar curvature. As an example, the product manifold $X \times Y$, where X is hyperbolic and Y is spin with $\hat{A}(Y) \neq 0$, cannot carry positive scalar curvature.

This more general result has an interesting interpretation. In a sense, the basic enlargeable manifolds are manifolds of $K(\pi, 1)$-type, such as solvmanifolds, manifolds of non-positive curvature, or $K(\pi, 1)$ 3-manifolds. Suppose now that a manifold X is mapped onto such a $K(\pi, 1)$ space. Then the $\hat{A}$-degree of this map is one of the *higher $\hat{A}$-genera* of X. The higher $\hat{A}$-genera are defined in strict analogy with the Novikov higher signatures (or " higher L-genera "). In fact, the general arguments given in [GL_1] are related spiritually to the constructions of Lusztig in his proof of the homotopy invariance of certain higher signatures [Lu]. The results in [GL_1] can be interpreted as establishing certain cases of the general statement that for spin manifolds of positive scalar curvature, the higher $\hat{A}$-genera must vanish.

Geometric intuition strongly suggested that the conclusions reached in [GL_1] should be true of a much broader class of manifolds. One of the principal deficiencies of [GL_1] was that the methods applied only to compact manifolds, whereas the natural place to analyze largeness was on the universal covering space. This difficulty has now been overcome.

In this article we deal directly with *non-compact* riemannian manifolds, and analyze there the interplay between largeness and positive scalar curvature. This leads both to improved results in the compact case and to new results for non-compact spaces. Our methods follow two distinct lines of development. One involves the Dirac operator; the other uses minimal hypersurfaces.

The adaptation of Dirac operator methods to non-compact manifolds requires a replacement for the role of the Atiyah-Singer Theorem in the compact case. This has led to the formulation and proof of a " *Relative Index Theorem* " for operators (on L_2-sections) over open manifolds. The result seems particularly well adapted to the study of scalar curvature. It also has some independent interest, and has been subsequently greatly generalized by Jeff Cheeger.

Using the Relative Index Theorem with some additional techniques we are able to substantially widen the results in [GL_1]. For example, it is shown that a *compact manifold* X *which admits a metric with sectional curvature* $\leqq 0$, *cannot carry a metric of positive scalar curvature.* The same holds for any compact spin manifold admitting a map of non-zero $\hat{A}$-degree onto X. In dimension 3, these methods lead directly to the " correct " result that *a compact 3-manifold which has a* $K(\pi, 1)$*-factor in its prime decomposition cannot carry positive scalar curvature.* (One uses here only the simple trick of the " inflating balloon ", given in 7.36ff., and not the lengthier, more delicate results of that section.)

The operator techniques have also led to the establishment of topological conditions under which *a non-compact manifold admits no complete metric of positive scalar curvature.* For example, there is a non-compact version of the result above: *Any manifold which admits a complete hyperbolic metric of finite volume cannot carry a complete metric of positive scalar curvature.* Here " hyperbolic " means that the sectional curvature is bounded between two negative constants.

An important aspect of the question of positive scalar curvature in the open case is clearly presented by the following sequence of results. Let X be any (compact) enlargeable manifold. Then a complete metric on:

$X \times \mathbf{R}$ cannot have positive scalar curvature
$X \times \mathbf{R}^2$ cannot have uniformly positive scalar curvature
$X \times \mathbf{R}^3$ can have uniformly positive scalar curvature.

In dimensions $\geqq 4$ the second statement will be proved only under a mild restriction on the metric. On the other hand in dimension 3 the results are particularly strong. The second statement remains true with $S^1 \times \mathbf{R}^2$ replaced by any 3-manifold which is homotopy equivalent to S^1 or which is diffeomorphic to the interior of a compact manifold with boundary such that $H_1 \partial X \to H_1 X$ is not zero.

Concerning metrics of uniformly positive scalar curvature, we shall prove an interesting local non-existence theorem (where " local " here means local at infinity). We shall define what we call a " bad end " of a non-compact manifold, and prove that any complete metric on a manifold with a bad end *cannot have uniformly positive scalar curvature.*

Our second line of argument here involves minimal hypersurfaces. These techniques are most powerful in dimension 3. Via the second variational formula, one gains rather strong control over the geometry of even incomplete 3-manifolds of positive scalar curvature. For example, suppose X is a compact 3-manifold with boundary, and with scalar curvature $\geqq 1$. Then it is shown that *any closed curve* γ *such that* $[\gamma] = 0$ *in* $H_1(X, \partial X)$ *and* distance$(\gamma, \partial X) > 2\pi$, *already bounds in its* 2π*-neighborhood.* (In particular, $[\gamma] = 0$ in $H_1(X)$.) Notice, for example, that this immediately implies: There exists no complete metric of uniformly positive scalar curvature on $S^1 \times \mathbf{R}^2$; also, any 3-manifold admitting such a metric and with π_1 finitely generated is simply-connected at infinity. There are, of course, many other consequences. (See § 10.)

The key to the method here is to apply the second variational formula to *non-simply-connected* regions on a stable minimal surface.

Several mathematicians have made contributions to this area. Among the first were Burago and Toponogov [BT] whose were, at that time, not equipped with Osserman's interior regularity results or the Schoen-Yau formula. (Hence, they considered only positive Ricci curvature.) More recently there has been an important paper [SY_3] by Schoen and Yau, and a series of beautiful results by Fischer-Colbrie and Schoen in [FCS].

In this last paper, Fischer-Colbrie and Schoen introduce a " symmetrization " trick. Given a stable minimal surface Σ in a 3-manifold of positive scalar curvature, they construct a warped product metric of positive scalar curvature on $\Sigma \times S^1$. Realizing that this trick extends to all dimensions, we have combined it with our notions of largeness to give results for incomplete manifolds of dimensions $\geqq 7$. The basic result is this. (See § 12.) *Let* X *be a compact oriented riemannian n-manifold,* $n \leqq 7$, *with scalar curvature* $\geqq 1$. *Then there exists no* ε_n*-contracting map* $(X, \partial X) \to (S^n, *) =$ *unit sphere, of non-zero degree, where* $\varepsilon_n = \sqrt{\dfrac{n}{n-1}}\,\dfrac{1}{2^n\pi}$.

The extension of symmetrization to higher dimensions was also noticed by Schoen and Yau, who have recently announced similar results. They have also announced a technique for by-passing the singularities which can appear on minimizing hypersurfaces in dimensions $\geqq 8$. This means that a large portion of the results discussed here can be proved by either of the two complementary techniques (*i.e.*, by " Dirac operator " or " minimal surface " methods). It is interesting to compare the methods since ultimately each has its own strengths.

There are several categories of results which are at the moment provable only by Dirac operator techniques. One involves the higher $\hat{A}$-obstruction. For example, *a product* $X \times Y$, *where* X *can carry sectional curvature* $\leqq 0$ *and where* Y *is spin with* $\hat{A}(Y) \neq 0$, *cannot carry positive scalar curvature.* This is provable in dimensions $\geqq 8$ only by operator methods.

Another such category of results is that where, basically, one must invoke the index theorem for families. For example, *a compact manifold* X *which can represent a non-zero rational homology class in a manifold* Z *of non-positive sectional curvature* (i.e., *for which there exists a continuous map* $f\colon X \to Z$ *such that* $f_*[X] \neq 0$ *in* $H_n(Z; \mathbf{Q})$), *cannot carry positive scalar curvature.* This appears to be often provable only by Dirac operator methods.

Another example is the following. *Suppose* X *is a compact manifold such that* $X \times \mathbf{R}^m$ *carries a complete metric of non-positive curvature for some m. Then* X *cannot carry positive scalar curvature.* It is, incidentally, unknown whether every compact $K(\pi, 1)$-manifold has this property.

On the other hand, Dirac operator methods fail for any manifold which does not admit a spin covering space, e.g., $T^4 \# \mathbf{P}^2(\mathbf{C})$.

The relationship between the Dirac operator and minimal hypersurfaces is one of the intriguing mysteries of riemannian geometry. Their roles in the study of scalar curvature are analogous to the roles of the Hodge operator on 1-forms and geodesics in the study of Ricci curvature. Even in this latter case, no deep relationship is understood, although there is some interesting speculation.

It certainly seems to be the scalar curvature function which mediates between harmonic spinors and minimal hypersurfaces. In fact, many of the results proved here ultimately have applications to the topological structure of complete stable hypersurfaces in manifolds with scalar curvature $\geqq 0$. There are also applications to the topological " placement " of minimal hypersurfaces and to the structure of stable cones. (See § 11.)

Note that, thus far, we have only discussed the negative side of the scalar curvature question. Nevertheless, much is known about how to construct positive scalar curvature metrics, and using this knowledge we are able to formulate some delicate conjectures.

We begin by recalling Hitchin's Theorem. Let Ω_*^{spin} denote the spin cobordism ring. Then there is a graded ring homomorphism

$$\hat{\mathscr{A}} : \Omega_*^{\mathrm{spin}} \to \mathrm{KO}_*(\mathrm{pt.})$$

defined by Atiyah and Milnor [M_2], which strictly generalizes the classical $\hat{A}$-genus in dimensions $4k$. By means of the results in [AS], this invariant can be realized as the index of the Dirac operator (taken in a suitable sense). In particular, for compact spin manifolds X of positive scalar curvature, one concludes that $\hat{\mathscr{A}}(X) = 0$.

In the simply connected case, this result has almost been proved to be sharp. More explicitly, the following is proved in [GL_2]. Let X be a compact simply-connected manifold of dimension $\geqq 5$. *If* X *is not spin, then* X *carries a metric of positive scalar curvature.* If X is spin, the existence of a positive scalar curvature metric on X is completely determined by a finite number of spin cobordism invariants. In particular, there exists a surjective, graded ring homomorphism $\hat{\mathfrak{a}} : \Omega_*^{\mathrm{spin}} \to \mathfrak{ko}_*$ which factors $\hat{\mathscr{A}}$, *i.e.* for which there is a commutative diagram

$$\begin{tikzcd} & \mathfrak{ko}_* \arrow[d, "\pi"] \\ \Omega_*^{\mathrm{spin}} \arrow[ur, "\hat{\mathfrak{a}}"] \arrow[r, "\hat{\mathscr{A}}"'] & \mathrm{KO}_*(\mathrm{pt.}), \end{tikzcd}$$

and if X is spin, then *there exists positive scalar curvature on* X *if and only if* $\hat{\mathfrak{a}}(X) = 0$. One conjectures that π is an isomorphism. This has been shown to be true if one tensors with the rational numbers.

It should be mentioned that the key differential geometric step in proving the results above is the following codimension-3 surgery theorem. Suppose X carries a metric with positive scalar curvature. Then any manifold obtained from X by doing surgery on an embedded sphere of codimension ≥ 3, also carries positive scalar curvature. This theorem was independently proved by Schoen and Yau in [SY_2]. Both the statement of their theorem and the method of proof are different from those in [GL_2]. Nevertheless the results are completely equivalent.

It should be mentioned that the " equivariant " version of these questions is more delicate. L. Berard-Bergery has found examples of G-manifolds with positive scalar curvature which admit no invariant positive scalar curvature metric [B_2].

The general spaces of positive scalar curvature are generated from basic ones by surgery. The basic ones appear from S^3-actions in the spirit of [LY] or from fibre bundles having such spaces as fibres.

The results discussed above are quite suggestive. The homomorphism $\hat{\mathscr{A}}$ determines a transformation of generalized homology theories: $\hat{\mathscr{A}}: \Omega_*^{\mathrm{spin}}(\cdot) \to \mathrm{KO}_*(\cdot)$. In particular, for any group Π this gives us a homomorphism

$$\hat{\mathscr{A}}: \Omega_*^{\mathrm{spin}}(\Pi) \to \mathrm{KO}_*(\Pi)$$

where by $h_*(\Pi)$ we mean $h_*(\mathrm{K}(\Pi, 1))$. Note that any compact spin manifold X with fundamental group Π determines a class [X] in $\Omega_*^{\mathrm{spin}}(\Pi)$ via the classifying map $\mathrm{X} \to \mathrm{K}(\Pi, 1)$ (taking $\pi_1 \mathrm{X}$ isomorphicially to Π).

For " reasonable " finitely presented groups Π, the following conjecture seems plausible.

CONJECTURE. — *Let* X *be a compact spin manifold with* $\pi_1 \mathrm{X} = \Pi$. *Then there exists a metric of positive scalar curvature on* X *if and only if* $\hat{\mathscr{A}}([\mathrm{X}]) = 0$.

There is some significant evidence for this whenever Π is geometrically approachable. The reader will find that all the results in this paper concerning the nonexistence of positive scalar curvature metrics on compact spin manifolds, are subsumed in this statement (for various groups Π). Quite recently T. Miyazaki [Mi] has written a very nice paper in which he carries over the results of [GL_2] to manifolds X with $\pi_1 \mathrm{X} \cong \mathbf{Z}/k\mathbf{Z}$ or $\pi_1 \mathrm{X} \cong \mathbf{Z}^m$ for $m < \dim(\mathrm{X})$. In particular, he proves the existence half of the conjecture (modulo torsion) in these cases.

Further evidence comes from a parallel approach to the scalar curvature question which involves infinite dimensional bundles. The original point of enlargeability was to construct, over each covering space, a k-plane bundle with connection, with the property that the curvature went uniformly to zero as one passed up the tower of coverings. Each such bundle could be pushed forward and considered on the base manifold. Passing to the limit, one might expect to find an interesting infinite-dimensional, *flat* bundle E_0

over the original manifold, so that one could apply the Bochner Method directly to the Dirac operator with coefficients in E_0.

Jonathan Rosenberg has pointed out to us that the appropriate index theorem for such an approach was recently proved by Mischenko and Fomenko. The index takes its values in the K-Theory of the group C^*-algebra $C^*(\pi_1 X)$. Deep results on the generalized Novikov Conjecture imply the " only if " part of the conjecture above (again modulo torsion), for a wide range of groups Π. This includes fundamental groups of compact non-positively curved manifolds, and discrete subgroups of connected Lie groups. Rosenberg has proved the obstruction to hold for torsion classes which survive the " complexification " map $KO_*(\Pi) \to KU_*(\Pi)$. The details of this appear in Rosenberg's paper following this one.

The first section of this paper presents a detailed summary of our principal results. However, before setting to this task we want to express our great debt to Jeff Cheeger who patiently explained to us a number of fundamental points concerning operators on non-compact manifolds and whose comments and insights were invaluable during the development of this work. We would also like to thank Marc Culler and Robert Osserman for valuable conversations.

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o. A GLIMPSE AT THE MAIN RESULTS

The principal new analytic tool developed in this paper is the Relative Index Theorem for pairs of elliptic operators which agree outside a compact set. The theorem is stated and proved here for generalized Dirac operators. However the result holds in considerably greater generality and will be the subject of a forthcoming paper of Jeff Cheeger and the authors.

A generalized Dirac operator on a riemannian manifold X is a first order elliptic operator defined on a bundle of modules over the Clifford bundle of X by the formula

$$D = \sum_j e_j \cdot \nabla_{e_j}.$$

Here the dot denotes Clifford multiplication. Examples include the Clifford bundle itself, the spinor bundle (when it exists), and either of these tensored with an arbitrary " coefficient " bundle. We begin the paper with a detailed discussion of these operators, including a proof of their essential self-adjointness over complete manifolds. A central point in the discussion is the generalized Bochner-Lichnerowicz-Weitzenböck formula, which has the form:

$$\text{(BLW)} \qquad D^2 = \nabla^* \nabla + \mathscr{R}$$

where $\nabla^*\nabla$ is the " connection Laplacian " and where $\mathscr{R}$ is an explicit zero-order symmetric operator defined in terms of the curvature of the bundle. We say that the operator D is *strictly positive at infinity* if there is a constant $r_0 > 0$ so that $\mathscr{R} \geqq r_0 \operatorname{Id}$ outside a compact subset. Under this assumption the kernel of D on L^2-sections is finite dimensional.

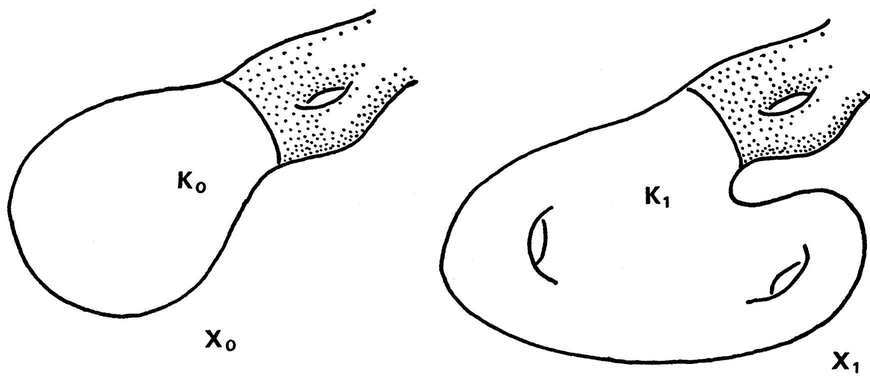
Over oriented manifolds of even dimension the module bundle S has a parallel orthogonal splitting $S = S^+ \oplus S^-$ with respect to which D can be written as

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}.$$

Thus, if D is positive at infinity, the operator $D^+ : \Gamma(S^+) \to \Gamma(S^-)$ has finite dimensional kernel and cokernel, and we can define

$$\begin{aligned} \operatorname{index}(D^+) &\equiv \dim(\ker D^+) - \dim(\operatorname{coker} D^+) \\ &= \dim(\ker D^+) - \dim(\ker D^-). \end{aligned}$$

Suppose now that we are given two generalized Dirac operators D_0 and D_1 over two complete riemannian manifolds X_0 and X_1 respectively. These operators are said to *coincide at infinity* if outside of compact subsets $K_0 \subset X_0$ and $K_1 \subset X_1$ there is an isometry of manifolds and bundles which carries D_0 onto D_1.



Under these assumptions there is a well defined *relative topological index* $\mathrm{ind}_t(D_0^+, D_1^+)$, obtained by compactifying the pair in the region of agreement and then taking the difference of the indices of the resulting operators. This index has a purely topological definition. It can also be expressed in the spirit of Chern-Weil Theory as the " difference " of integrals of certain characteristic polynomials in the curvature tensors.

Two basic examples are as follows. Suppose D_0 and D_1 are the Dirac operators on spinors over X_0 and X_1. In this case the relative index is given by $\mathrm{ind}_t(D_0^+, D_1^+) = \hat{A}(Y)$ where Y is any compact manifold obtained by taking the " difference " of X_0 and X_1. (Remove a neighborhood of infinity in the region of agreement, and attach the resulting compact manifolds along their " common " boundary.) For the second example let S be the bundle of spinors over a complete riemannian manifold X, and let E be a complex k-dimensional vector bundle over X which is *trivialized* at infinity. Assume E has a unitary connection compatible with the trivialization at infinity. (In particular, the connection is flat at infinity. Such connections always exist.) Consider the Dirac operators D_k and D_E on the bundles $S \otimes \mathbf{C}^k$ and $S \otimes E$ respectively. These operators agree at infinity and their relative topological index is given by the integral

$$\mathrm{ind}_t(D_E^+, D_k^+) = \int_X \hat{\mathrm{ch}}\, E \wedge \hat{\mathbf{A}}(X)$$

where $\hat{\mathrm{ch}}\, E = \mathrm{ch}\, E - \mathrm{ch}\, \mathbf{C}^k = \mathrm{ch}_1 E + \mathrm{ch}_2 E + \dots$ is the " reduced " Chern character of E, $\hat{\mathbf{A}}(X)$ is the total $\hat{A}$-class of X, and where all these characteristic differential forms are written canonically in terms of the curvature tensors of E and X respectively. Note that $\hat{\mathrm{ch}}\, E$ has compact support since E is flat outside a compact subset of X.

The fundamental result is the following.

THE RELATIVE INDEX THEOREM. — *Let* D_0 *and* D_1 *be two generalized Dirac operators over complete riemannian manifolds* X_0 *and* X_1. *Suppose these operators coincide at infinity and are positive there. Then*

$$\mathrm{index}(D_1^+) - \mathrm{index}(D_0^+) = \mathrm{ind}_t(D_0^+, D_1^+).$$

We also prove a refined version of this theorem for pairs of operators which coincide and are positive on some *family of ends* of the manifolds X_0 and X_1. (See § 4 for a statement.) Here the difference of the indices is a " semi-topological " invariant. It seems that this result should also be quite useful in the study of complete riemannian manifolds.

The Relative Index Theorem plays a major role in proving the main geometric results of this paper. However, it also enables us to define some interesting integer invariants for metrics of positive scalar curvature on a compact spin manifold X. Given two such metrics g_0 and g_1 on X, we introduce on $X \times \mathbf{R}$ a metric of the form $ds^2 = g_t + dt^2$ where

$$g_t = \begin{cases} g_0 \text{ for } t = 0 \\ g_1 \text{ for } t = 1 \\ \text{any smooth homotopy from } g_0 \text{ to } g_1 \text{ for } 0 \leqq t \leqq 1. \end{cases}$$

We then define

$$i(g_0, g_1) = \operatorname{index}(D^+)$$

where D is the canonical Dirac operator on $X \times \mathbf{R}$. The Relative Index Theorem proves that this integer is independent of the choice of homotopy g_t; and our refined index theorem shows that it satisfies a " cocycle condition "

$$i(g_0, g_1) + i(g_1, g_2) + i(g_2, g_0) = 0.$$

(Note that $i(g_0, g_1) = -i(g_1, g_0)$, by a trivial change of orientation.) The vanishing theorems can be applied to show that *if g_0 is homotopic to g_1 through metrics of positive scalar curvature, then* $i(g_0, g_1) = 0$. Thus, $i(g_0, g_1)$ is constant on connected components of the space of positive scalar curvature metrics.

This invariant is non-trivial. If g_0 is the constant curvature metric on S^7, then the function $g \to i(g, g_0)$ takes on *all* integer values. This function is, incidentally, invariant on the orbits of the group of diffeomorphisms acting on the space of metrics on S^7.

There is a family of similar non-trivial invariants which can be defined. A systematic study of these invariants will be made in a forthcoming paper of the authors.

The main geometric results of this paper concern the non-existence of metrics with positive scalar curvature. To state results we must introduce the notion of enlargeability. Let S^n denote the euclidean n-sphere of radius one. Given $\varepsilon > 0$, we say that a riemannian manifold X is ε-*hyperspherical* if there exists an ε-contracting map $f\colon X \to S^n$ which is constant outside a compact subset and of non-zero $\hat{A}$-degree. By $\hat{A}$-*degree* we mean $\hat{A}(f^{-1}$ (any regular value of f)). If $\dim X = n$, this is the usual notion of degree.

Definition A. — A compact manifold X is said to be *enlargeable* if for any $\varepsilon > 0$ and any riemannian metric on X, there exists a spin covering manifold $\tilde{X} \to X$ which is ε-hyperspherical (in the lifted metric).

This definition differs from the one given in [GL_1] in that here we no longer require the coverings to be finite. This broadens enormously the class of enlargeable manifolds. For example, *any compact manifold which admits a metric with sectional curvature* $\leqq 0$ *is enlargeable.* As shown in [GL_1] *any compact solvmanifold is enlargeable.* (A solvmanifold is a solvable Lie group modulo a cocompact discrete subgroup, e.g. a torus.)

The property of enlargeability depends only on the homotopy type of the manifold. In fact, considerably more is true. Any compact spin manifold which admits a map of non-zero $\hat{A}$-degree onto an enlargeable manifold is itself enlargeable. Thus, the connected sum of an enlargeable manifold with any spin manifold is again enlargeable. Furthermore, the product of an enlargeable manifold with any spin manifold if non-zero $\hat{A}$-genus, is again enlargeable.

THEOREM A. — *An enlargeable manifold cannot carry a metric of positive scalar curvature. In fact any metric of non-negative scalar curvature on an enlargeable manifold must be Ricci flat.*

COROLLARY A. — *A compact manifold* X *which carries a metric of non-positive sectional curvature cannot carry a metric of positive scalar curvature. In fact any metric with scalar curvature* $\geqq 0$ *on* X *must be flat.*

One can conclude that if X carries a metric with sectional curvature < 0, then there is no metric on X with scalar curvature $\geqq 0$.

We now take up the question of complete metrics on non-compact manifolds. Recall that a smooth map $f\colon X \to Y$ between riemannian manifolds is ε-contracting if $\|f_* V\| \leqq \varepsilon \|V\|$ for all tangent vectors V on X. We shall say that f is (ε, Λ^2)-*contracting* (or ε-*contracting on 2-forms*) if

$$\|f_* V \wedge f_* W\| \leqq \varepsilon \|V \wedge W\|$$

for all tangent 2-frames V, W on X. Clearly an ε-contracting map is ε^2-contracting on 2-forms. However, to be contracting on 2-forms, it is only necessary to be contracting in $(n-1)$-directions at each point (where $n = \dim X$). That is, if $f_*^t \circ f_*$ has eigenvalues $\lambda_1^2, \ldots, \lambda_n^2$, where at each point $\lambda_1^2 \leqq 1$ and $\lambda_j^2 \leqq \varepsilon^2$ for $j \geqq 2$, then f is (ε, Λ^2)-contracting.

In analogy with the above we say a riemannian manifold X is (ε, Λ^2)-*hyperspherical* if it admits an (ε, Λ^2)-contracting map $f\colon X \to S^n$ which is constant at infinity and of non-zero $\hat{A}$-degree.

DEFINITION B. — A (not necessarily compact) manifold X is said to be Λ^2-*enlargeable* if for any $\varepsilon > 0$ and any riemannian metric on X there exists a spin covering manifold which is (ε, Λ^2)-hyperspherical (in the lifted metric).

The property of Λ^2-enlargeability depends only on proper homotopy type of the manifold. The product of an enlargeable manifold with a Λ^2-enlargeable manifold is again Λ^2-enlargeable. So also is the connected sum of a compact spin manifold with a Λ^2-enlargeable manifold. This property has a further contagious aspect.

Proposition B. — *Let* X *be a connected manifold, and suppose* $U \subset X$ *is an open submanifold with* $\pi_1 U \to \pi_1 X$ *injective. Then if* U *is* Λ^2*-enlargeable, so is* X.

Any enlargeable manifold is clearly Λ^2-enlargeable. However, the interesting examples are the non-compact ones.

Examples. — *The following manifolds are* Λ^2*-enlargeable.*

(1) $X \times \mathbf{R}$ *where* X *is enlargeable.*

(2) *Any hyperbolic* (1) *manifold of finite volume.*

(3) $X - K$ *where* X *is compact with sectional curvature* ≤ 0 *and where* K *is any compact subset which misses a compact geodesic hypersurface.*

(4) *The product of any of the above examples with an enlargeable manifold (e.g. with a manifold of curvature* ≤ 0 *or with a spin manifold of non-zero* $\hat{A}$*-genus).*

(5) *The connected sum of any of the examples above with any compact spin manifold.*

Note that included in Example (1) above are manifolds of the form $X \times \mathbf{R}$ where X is a compact spin manifold of non-zero $\hat{A}$-genus. Included in Example (3) is any manifold of the form $T^n - K$ where K is a compact subset in the complement of a linear subtorus $T^{n-1} \subset T^n$. The main result is the following.

Theorem B. — *A* Λ^2*-enlargeable manifold cannot carry a complete metric of positive scalar curvature.*

Using recent results of J. Kazdan [K] we can also conclude that any complete metric with scalar curvature ≥ 0 on a Λ^2-enlargeable manifold must be Ricci flat.

Corollary B_1. — *A manifold* X *which carries a hyperbolic metric of finite volume cannot carry a complete metric of positive scalar curvature.*

Corollary B_2. — *There is no complete metric of positive scalar curvature on* $X \times \mathbf{R}$ *if* X *is either*

(1) *a compact manifold which admits a metric of non-positive sectional curvature*

or

(2) *a compact spin manifold of non-zero* $\hat{A}$*-genus.*

Results of this kind in dimension three were first obtained by R. Schoen and S. T. Yau [SY_3].

Any covering of a compact manifold of positive scalar curvature is complete and of *uniformly positive scalar curvature, i.e.* $\kappa \geq \kappa_0$ for some constant $\kappa_0 > 0$. The existence of such a metric has stronger implications than does the existence of a complete metric with only $\kappa > 0$. Suppose that X is a (compact) enlargeable manifold. Then, as

(1) Here « hyperbolic » means complete with sectional curvature bounded between two negative constants.

we have seen, $X \times \mathbf{R}$ cannot carry a complete metric with $\kappa > 0$. However, $X \times \mathbf{R}^2$ can carry such a metric. Consider, for example, the product of a flat torus with a paraboloid of revolution. Nevertheless, $X \times \mathbf{R}^2$ cannot carry a complete metric with $\kappa \geqq \kappa_0$, for some $\kappa_0 > 0$. (We shall prove this only under some mild restrictions on the metric.) Note that the process ends here. That is, $X \times \mathbf{R}^3$ always carries a complete metric with $\kappa \geqq \kappa_0 > 0$.

The central result is the following.

THEOREM C. — *Let* X *be a compact enlargeable manifold. Then there is no complete metric* g *of uniformly positive scalar curvature on* $X \times \mathbf{R}^2$ *which satisfies either :*

(1) $\text{Ricci} \geqq -c^2 g$ *for some constant* c,

or

(2) *there are no properly embedded surfaces, cohomologous to* $\{x\} \times \mathbf{R}^2$, *which are of finite area.*

Of course, the techniques of the proof apply to a much wider class of manifolds. For a general statement of results, see § 7.

It should be pointed out that the techniques introduced in § 7 of this paper are perhaps more important than the results. The techniques, once understood, can be applied in a variety of situations. A good example of this is the following " local " non-existence result.

DEFINITION D. — A connected manifold X is said to have a *bad end* if there exists a (compact, oriented) enlargeable hypersurface $Z \subset X$ and an unbounded component X_+ of $X - Z$ with a map $\overline{X}_+ \to Z$ whose restriction to $Z \subset \overline{X}_+$ has non-zero degree. (More generally, the map $\overline{X}_+ \to Z$ may be replaced by a map to any enlargeable manifold.) The set X_+ is, of course, the " bad end. " The result is basically that bad ends cannot carry uniformly positive scalar curvature.

THEOREM D. — *Let* X *be a connected manifold with a bad end* X_+. *Then there exists no complete metric* g *on* X *satisfying the conditions* $\kappa \geqq \kappa_0$ *and* $\text{Ricci} \geqq -c^2 g$ *(for constants* c, $\kappa_0 > 0$*) on the end* X_+.

The above results are particularly strong when applied to manifolds of dimensions 3 and 4.

Recall that any compact orientable 3-manifold X admits a unique decomposition $X = X_1 \# \dots \# X_N$ into a connected sum of prime 3-manifolds. A prime 3-manifold which is not diffeomorphic to $S^1 \times S^2$ is either a $K(\pi, 1)$-manifold or it is covered by a homotopy 3-sphere.

THEOREM E. — *Any compact orientable* 3*-manifold which has a* $K(\pi, 1)$*-factor in its prime decomposition cannot carry a metric of positive scalar curvature. Furthermore, any metric with scalar curvature* $\geqq 0$ *on such a manifold must be flat. (And the manifold must be, in particular, prime.)*

This result is almost sharp since any 3-manifold of the form

$$X = (S^3/\Gamma_1) \# \ldots \# (S^3/\Gamma_m) \# (S^1 \times S^2) \# \ldots \# (S^1 \times S^2)$$

does carry positive scalar curvature (see [GL_1]). The question of the existence of positive scalar curvature metrics on homotopy 3-spheres remains open.

For non-compact 3-manifolds we recover the following result of Schoen and Yau [SY_3]. A surface Σ embedded in a manifold X is called *incompressible* if Σ is compact with $\chi(\Sigma) \leqq 0$ and if the homomorphism $\pi_1 \Sigma \to \pi_1 X$ is injective.

THEOREM E′. — *Any 3-manifold which admits an incompressible surface cannot carry a complete metric of positive scalar curvature.*

Let X be a 3-manifold. An embedded circle $\gamma \subset X$ is said to be *small* if it has infinite order in $H_1 X$ and if the normal circle to γ has infinite order in $H_1(X - \gamma)$. The circles $S^1 \times \{x\}$ are small in $S^1 \times \mathbf{R}^2$ but not in $S^1 \times S^2$. If X is any $K(\pi, 1)$ 3-manifold, then the representative of a non-zero element in $\pi_1(X)$ is small in an appropriate covering manifold. Note that small circles remain small after taking connected sums of 3-manifolds.

THEOREM E″. — *An open 3-manifold which admits a small circle cannot carry a complete metric of uniformly positive scalar curvature.*

There are some similar results for 4-manifolds. We say that a compact incompressible surface Σ embedded in a 4-manifold X is *small* if $|\pi_1 X / \pi_1 \Sigma| = \infty$ and if the normal circle about Σ is of infinite order in $H_1(\widetilde{X} - \Sigma)$, where $\widetilde{X}$ is the covering of X with $\pi_1 \widetilde{X} \cong \pi_1 \Sigma$.

THEOREM F. — *A compact spin 4-manifold which admits a small incompressible surface cannot carry a metric of positive scalar curvature.*

Note that if X carries a small incompressible surface, so does $X \# Y$ for any 4-manifold Y. Theorem F holds also for non-compact 4-manifolds, if we weaken the conclusion to the non-existence of complete metrics with $\kappa \geqq \kappa_0 > 0$ and $\text{Ricci} \geqq -c^2$.

Chapter 10 of the paper represents a by-product of our thinking about 3-manifolds. *This section does not at all involve the Dirac operator and can be read independently from the rest of the paper.* These results concern 3-manifolds which are not necessarily complete. The methods involve only stable minimal surfaces.

From a philosophical point of view, our main breakthrough here was the discovery that the strength of the stability inequality is best used by choosing deformations supported on non-simply-connected domains.

Let Σ be a surface with a riemannian metric. Given a compact subset $\Omega \subset \Sigma$ and a number $\rho > 0$, set

$$\Omega(\rho) = \{x \in \Sigma : \text{dist}_\Sigma(x, \Omega) \leqq \rho\}.$$

THEOREM G_1. — *Let Σ be a compact stable minimal surface in a 3-manifold with scalar curvature $\geqq \kappa_0 > 0$. Let $\Omega \subset \Sigma$ be a compact connected domain, and let $\rho > 0$ be chosen so that:*

(1) $\qquad \Omega(\rho) \cap \partial\Sigma = \varnothing,$

(2) $\qquad \operatorname{Image}[H_1(\Omega) \to H_1(\Omega(\rho))] \neq 0.$

Then

$$\boxed{\rho < \frac{\pi}{\sqrt{\kappa_0}}.}$$

The hypotheses here can be relaxed. (See § 10.) This result has the following major consequence.

THEOREM G_2. — *Let X be a compact 3-manifold with a possibly empty boundary, and suppose X is equipped with a metric of scalar curvature $\geqq 1$. Then any closed curve $\gamma \subset X$ such that*

(1) $\qquad [\gamma] = 0 \quad \text{in } H_1(X, \partial X)$

and (2) $\qquad \operatorname{dist}(\gamma, \partial X) > 2\pi,$

must already bound in its 2π-neighborhood $U_{2\pi}(\gamma) \equiv \{x \in X : \operatorname{dist}(x, \gamma) \leqq 2\pi\}$. *That is,* $[\gamma] = 0$ *in* $H_1(U_{2\pi}(\gamma))$.

COROLLARY G_3. — *Let X be an oriented 3-manifold which is diffeomorphic to the interior of a compact 3-manifold $\overline{X}$ with $H_1(\partial\overline{X}) \neq 0$. Then X carries no complete metric with uniformly positive scalar curvature.*

An example, of course, is the manifold $X = S^1 \times \mathbf{R}^2$. Thus we retrieve some of the results in the E-series above.

COROLLARY G_4. — *Let X be a compact riemannian 3-manifold with boundary, and consider $\gamma \subset \partial X$ such that $[\gamma] \neq 0$ in $H_1(X)$. If $\kappa \geqq 1$, then any curve γ' in X which is homologous to γ must satisfy*

$$\operatorname{dist}(\gamma, \partial X) \leqq 2\pi.$$

COROLLARY G_5. — *Suppose $S^1 \times \mathbf{R}^2$ is given a complete riemannian metric. Fix a circle $\gamma \subset S^1 \times \mathbf{R}^2$ generating H_1 and for each $R > 0$, set*

$$\kappa(R) = \inf\{\kappa(x) : \operatorname{dist}(x, \gamma) \leqq R\}.$$

Then $\qquad \kappa(R) \leqq \dfrac{4\pi^2}{R^2}.$

A similar result holds for other non-simply-connected 3-manifolds.

There is also a basic result for 3-manifolds with $\inf_x \kappa(x) > -\infty$.

THEOREM G_6. — *Let* X *be a complete* 3*-manifold with*

$$\kappa \geq -1.$$

Let $\Sigma \subset X$ *be a complete (connected) stable minimal surface. Fix* $x_0 \in \Sigma$ *and let* $B(R) = \{x \in \Sigma : \mathrm{dist}_\Sigma(x, x_0) \leq R\}$. *For each* $R > 0$, *let*

$$\alpha(R) = \mathrm{rank}\{\mathrm{image}[H_1(B(R)) \to H_1(\Sigma)]\} - 1.$$

Then
$$\mathrm{Area}(B(R)) \geq \frac{4\pi}{1 + (\pi/R)^2}\,\alpha(R).$$

In particular, if $\chi(\Sigma) \leq 0$, *then*

$$\mathrm{Area}(\Sigma) \geq 4\pi\,|\chi(\Sigma)|.$$

Thus, if $\chi(\Sigma) = -\infty$, *then* $\mathrm{Area}(\Sigma) = \infty$.

The internal estimates above for 3-manifolds can also be made in higher dimensions. Using a symmetrization process inspired by the last result in [FCS], we prove the following result in § 12.

THEOREM H. — *Let* X *be a compact riemannian n-manifold,* $n \leq 7$, *with scalar curvature* ≥ 1. *(Here* X *may have boundary and need not be spin.) Then there exist no* ε_n*-contracting maps* $X \to S^n$ *which are constant on* ∂X *and of non-zero degree, where*

$$\varepsilon_n = \sqrt{\frac{n}{n-1}} \cdot \frac{1}{2^n \pi}.$$

Our results on the non-existence of complete metrics with positive scalar curvature have several interesting applications to the theory of minimal varieties. These are explored in § 11. Some of the principal results are the following. Recall that a minimal submanifold is *stable* if the second derivative of area with respect to compactly supported variations is always ≥ 0.

THEOREM I_1. — *A complete stable hypersurface in a manifold of non-negative scalar curvature cannot be* Λ^2*-enlargeable, unless it is totally geodesic.*

This holds in particular for stable hypersurfaces in Euclidean space. It shows that no such hypersurface can be of the form $X_0 \times \mathbf{R}$ where X_0 is enlargeable (a torus, say).

The results give estimates for " Bernstein " questions.

THEOREM I_2. — *Let* $F : \mathbf{R}^n \to \mathbf{R}$ *be a global solution to the minimal surface equation* $(n \geq 8)$, *and let* A *be the second fundamental form of the graph of* F *in* $\mathbf{R}^{n+1}$. *Set*

$$a(R) = \inf_{\|x\| \leq R} \|A\|^2.$$

Then there is a constant c_n depending only on dimension so that

$$a(\mathrm{R}) < \frac{c_n}{\mathrm{R}^2}.$$

Our results also give restrictions on the topological type of stable minimal cones. Such cones form the " tangent spaces " to minimizing varieties in a riemannian manifold. The following observation was first made by Rick Schoen [S].

THEOREM I_3. — *Let* $\mathrm{M} \subset \mathrm{S}^n$ *be a compact (minimal) hypersurface such that the cone*

$$\mathrm{C}(\mathrm{M}) \equiv \{tx \in \mathbf{R}^{n+1} : x \in \mathrm{M} \quad \text{and} \quad t \geqq 0\}$$

is a stable-variety in $\mathbf{R}^{n+1}$. *(That is, the second variation of area is* > 0 *on compact subdomains of* $\mathrm{C}(\mathrm{M}) - \{0\}$.*) Then* M *carries a metric of positive scalar curvature. In particular,* M *cannot be enlargeable.*

This gives the first known restrictions on the topology of stable cones in dimensions $\geqq 8$. For example, note that there are infinitely many isotopy classes of embeddings of T^{n-1} into S^n for any $n > 2$. However, none of these tori can appear as the link of a stable cone in $\mathbf{R}^{n+1}$.

There are also applications to the " placement " question for (not necessarily stable) minimal hypersurfaces.

THEOREM I_4. — *Let* M *be a compact, minimal hypersurface in a complete manifold* X *of positive scalar curvature. Suppose* X *is* 2*-sided (*i.e. *the normal line bundle is trivial), and let* $\hat{\mathrm{X}}$ *be the manifold with boundary formed by " separating "* X *along* M. *Then the double of each component of* $\hat{\mathrm{X}}$ *carries a complete metric of positive scalar curvature.*

This has powerful implications for the possible placement of M in X. For example, we have the following generalization of results in $[\mathrm{L}_2]$.

COROLLARY I_5. — *Let* $\Sigma \subset \mathrm{S}^3$ *be a compact minimal surface for some metric on* S^3 *of positive scalar curvature. Then* Σ *is isotopic to a standard embedding (as the boundary of a " pretzel ").*

As an example, consider $\mathrm{T}^2 \subset \mathrm{S}^3$ as the boundary of a tubular neighborhood of a knot. One component of $(\widehat{\mathrm{S}^3 - \mathrm{T}^2})$ is $\mathrm{S}^1 \times \mathrm{D}^2$, whose double $\mathrm{S}^1 \times \mathrm{S}^2$ carries $\kappa > 0$. The double of the other component, however, cannot carry $\kappa > 0$ by results mentioned above (see S. de C. Almeida, Thesis, Stony Brook, 1982).

Corollary I_5 also appears in a series of beautiful results recently proved by Meeks, Simon and Yau [MSY]. The implications in higher dimensions are currently being examined by Sebastião de Carneiro Almeida.

Theorem I_4 of course also applies to the complete case, and should have some interest in general relativity. For example, it seems to rule out certain topologically complicated horizons ("worm holes").

Using minimal boundaries as barriers to solve the Plateau problem gives the following result.

THEOREM I_6. — *Let X be a compact riemannian n-manifold* ($n \leqq 7$) *with* $\kappa > 0$, *and with boundary* ∂X *of mean curvature* $\geqq 0$ *(with respect to the interior). Suppose* ∂X *is an enlargeable* $K(\pi, 1)$*-manifold, or more generally, suppose* ∂X *admits a map of non-zero degree onto such a manifold. Then*

$$\pi_1(\partial X) \to \pi_1(X)$$

is not injective.

As an example, consider the interesting component of the complement of a "knotted" $T^{n-1} \subset S^n$.

If $\dim X = 3$ and if ∂X is connected and of genus > 0, then the hypotheses of I_6 are automatically satisfied and we retrieve the results above.

The conclusion of Theorem I_6 can be strengthened somewhat. Suppose K is the $K(\pi, 1)$-manifold onto which ∂X maps with non-zero degree. Then the map $\pi_1(\partial X) \to \pi_1(K)$ must factor through $\pi_1(X)$.

It is a possibility that no compact $K(\pi, 1)$-manifold, and in fact, no manifold which represents a non-trivial homology class in such a space, can carry positive scalar curvature. In § 13 we give a proof of this fact for $K(\pi, 1)$-spaces of non-positive sectional curvature.

THEOREM J. — *Let* K *be a compact manifold which admits a metric with sectional curvature* $\leqq 0$. *Then any compact n-dimensional spin manifold* X *which represents a non-zero class in* $H_n(K; \mathbf{Q})$ *cannot carry a metric of positive scalar curvature.*

By "representing" a class $\alpha \in H_n(K; \mathbf{Q})$ we mean that there exists a continuous map $f: X \to K$ such that $f_*[X] = \alpha$.

This theorem appears to be difficult to prove without Dirac operator methods, even for $n \leqq 7$.

The case of general $K(\pi, 1)$ manifolds is completely open.

It should be noted that in general it is not the homology, but more precisely, the *spin bordism* of a $K(\pi, 1)$ that is relevant to this question.

Note that any compact spin manifold X of dimension n canonically determines a spin bordism class

$$[X] \in \Omega_n^{\text{spin}}(K(\pi, 1))$$

where $\pi = \pi_1 X$. (Take the map $X \to K(\pi, 1)$ which is an isomorphism on π_1.) It can now be shown that the question of the existence of a metric of positive scalar curvature on X is entirely determined by this bordism class.

There is a transformation of generalized homology theories

$$\Omega_*^{\mathrm{spin}} \xrightarrow{\hat{\mathscr{A}}} \mathrm{KO}_{-*}$$

which on the coefficients is just the $\hat{\mathscr{A}}$-homomorphism referred to in the preface. It seems reasonable to conjecture that for spin manifolds these $\hat{\mathscr{A}}$-classes in $\mathrm{KO}_*(\mathrm{K}(\pi, 1))$ constitute a complete set of obstructions to the existence of positive scalar curvature metrics. This is nearly proved in the simply-connected case (cf. [GL_2]), and one of the themes of this paper is a marshalling of evidence for the conjecture in general.

Many of the results concerning the non-existence of positive scalar curvature metrics can be collected in the following general framework. Let K be a compact oriented $\mathrm{K}(\pi, 1)$-manifold. Then given a compact spin manifold X and a smooth map $f: \mathrm{X} \to \mathrm{K}$, we can consider the "fibre" $\mathrm{X}_0 \equiv f^{-1}$ (a regular value). *The spin cobordism class of* X_0 *depends only on the spin bordism class* $[\mathrm{X}] \in \Omega^{\mathrm{spin}}(\mathrm{K})$. Hence, $\hat{\mathscr{A}}(\mathrm{X}_0) \in \mathrm{KO}_*(\mathrm{pt.})$ is a well defined invariant of the class [X] which we denote by $\hat{a}([\mathrm{X}])$. In succinct form, the results say that for "geometric" $\mathrm{K}(\pi, 1)$-manifolds K, *any spin manifold representing a class* $x \in \Omega^{\mathrm{spin}}(\mathrm{K})$, *such that* $\hat{a}(x) \neq 0$, *cannot carry positive scalar curvature.*

This "$\hat{\mathscr{A}}$-degree" class $\hat{a}$ is something like the cap product of the $\hat{\mathscr{A}}$-class with the "fundamental class" of K.

1. GENERALIZED DIRAC OPERATORS ON A COMPLETE MANIFOLD

In this section we shall recall some basic properties of an important class of first-order elliptic operators on a complete riemannian manifold X. Let Cl(X) denote the *Clifford bundle* of X. This is the bundle over X whose fibre at a point $x \in X$ is the Clifford algebra $Cl(T_xX)$ of the tangent space at x. (See [GL_1], [H] or [LM], for example.) There is a canonical embedding $T(X) \subset Cl(X)$. Furthermore, the riemannian metric and connection extend to Cl(X) with the properties that: covariant differentiation ∇ preserves the metric, and

$$(1.1) \qquad \nabla(\varphi.\psi) = (\nabla\varphi).\psi + \varphi.(\nabla\psi)$$

for all sections $\varphi, \psi \in \Gamma(Cl(X))$.

We now suppose that $S \to X$ is a bundle of left modules over the bundle of algebras Cl(X) (*i.e.* we assume that at each point x, the fibre S_x is a module over the algebra $Cl_x(X)$, and that the multiplication maps vary smoothly with x). We assume, furthermore, that S is furnished with a metric and an orthogonal connectiont [1] ∇ such that:

(1.2) The module multiplication $e: S_x \to S_x$ by any unit vector $e \in T_xX$ is an isometry at each point x;

$$(1.3) \qquad \nabla(\varphi.\sigma) = (\nabla\varphi).\sigma + \varphi.(\nabla\sigma) \quad \text{for all } \varphi \in \Gamma(Cl(X)) \text{ and } \sigma \in \Gamma(S).$$

Under these assumptions we define a first-order operator $D : \Gamma(S) \to \Gamma(S)$, called the *(generalized) Dirac operator of* S, by setting

$$(1.4) \qquad D \equiv \sum_{k=1}^{n} e_k \cdot \nabla_{e_k}$$

where $\{e_1, \ldots, e_n\}$ denotes any orthonormal basis of the space T_xX at each point x. Since the multiplication is linear, the expression (1.4) is clearly independent of the choice of orthonormal basis. The principal symbol of D at a cotangent vector $\xi = \Sigma\xi_k e_k^* \cong \Sigma\xi_k e_k$, is

$$(1.5) \qquad \sigma_\xi(D) = \xi\cdot$$

[1] If X is complex, the metric and connection are assumed to be hermitian.

where the " $\cdot$ " denotes module multiplication. Note that

$$\sigma_\xi(D^2) = \sigma_\xi(D)^2 = -\|\xi\|^2, \tag{1.6}$$

and so $\sigma_\xi(D)$ is a linear isomorphism for all $\xi \neq 0$, *i.e.* D is elliptic.

This argument proving ellipticity demonstrates a general phenomenon, namely, that many basic properties of Dirac operators are independent of the bundle S and follow formally from the " axioms " (1.2) and (1.3). Nevertheless, there is a wealth of such operators.

EXAMPLE **1.7.** — Let $S = Cl(X)$. Then it is well known that under the canonical vector bundle isomorphism $Cl(X) \cong \Lambda^*(X)$, one has

$$D \cong d + d^*$$

where d^* denotes the formal adjoint of exterior differentiation.

EXAMPLE **1.8.** — Suppose X is a spin manifold of dimension $2n$ and let $S = \mathbf{S}$ be the complex bundle of spinors over X with its canonical riemannian connection. To be more specific, let $P_{SO}(X)$ denote the bundle of oriented orthonormal frames on X, and let $P_{Spin}(X)$ be a principal $Spin_{2n}$-bundle over X with a given $Spin_{2n}$-equivariant covering map $\xi : P_{Spin}(X) \to P_{SO}(X)$. (The map ξ is called a *spin-structure* on X.) Consider $Spin_{2n} \subset Cl(\mathbf{R}^{2n})$, and recall that $Cl(\mathbf{R}^{2n}) \otimes \mathbf{C} \cong Hom(\mathbf{C}^{2^n})$ (cf. [ABS]). Restriction gives a representation $\Delta : Spin_{2n} \to Hom(\mathbf{C}^{2^n})$, and $\mathbf{S}$ is defined to be the associated vector bundle

$$\mathbf{S} \equiv P_{Spin}(X) \times_\Delta \mathbf{C}^{2^n}.$$

Here D is the " classical " Dirac operator.

EXAMPLE **1.9.** — Suppose S_0 is any bundle of modules over $Cl(X)$ with a metric and connection ∇^{S_0} satisfying (1.2) and (1.3). Let E be any complex vector bundle over X with a hermitian connection ∇^E. Then the tensor product $S \equiv S_0 \otimes E$ is again a bundle of modules over $Cl(X)$, and the tensor product metric and connection:

$$\nabla^{S_0 \otimes E} \equiv \nabla^{S_0} \otimes 1 + 1 \otimes \nabla^E \tag{1.10}$$

again satisfy conditions (1.2) and (1.3). Combined with 1.7 and 1.8, this construction gives many non-trivial examples of generalized Dirac operators.

It is an interesting and important fact that any Dirac operator on a complete riemannian manifold is essentially self-adjoint. The next few paragraphs are devoted to proving this fact. We begin by observing that D is always formally self-adjoint.

PROPOSITION **1.11.** — *Let* $D : \Gamma(S) \to \Gamma(S)$ *be a Dirac operator defined over a riemannian manifold* X, *and consider the usual inner product*

$$(\sigma_1, \sigma_2) \equiv \int_X \langle \sigma_1, \sigma_2 \rangle \tag{1.12}$$

on the space $\Gamma(S)$ *of* C^∞ *cross-sections. Then*

$$(\mathbf{1.13}) \qquad (D\sigma_1, \sigma_2) = (\sigma_1, D\sigma_2)$$

for any pair of sections $\sigma_1, \sigma_2 \in \Gamma(S)$ *of which at least one has compact support.*

Proof. — Fix $x \in X$ and choose local pointwise orthonormal tangent vector fields $e_1, \ldots, e_n$ near x such that $(\nabla e_k)_x = 0$ for each k. Then at the point x,

$$(\mathbf{1.14}) \qquad \begin{aligned} \langle D\sigma_1, \sigma_2\rangle_x &= \Sigma\langle e_k.\nabla_{e_k}\sigma_1, \sigma_2\rangle_x = -\Sigma\langle\nabla_{e_k}\sigma_1, e_k.\sigma_2\rangle_x \\ &= -\Sigma\{\nabla_{e_k}\langle\sigma_1, e_k.\sigma_2\rangle - \langle\sigma_1, (\nabla_{e_k}e_k).\sigma_2 + e_k.\nabla_{e_k}\sigma_2\rangle\}_x \\ &= \operatorname{div}(V)_x + \langle\sigma_1, D\sigma_2\rangle_x \end{aligned}$$

where V is the tangent vector field on X defined by the condition:

$$\langle V, W\rangle = -\langle\sigma_1, W.\sigma_2\rangle$$

for any tangent vector field W. Note that

$$\operatorname{div}(V)_x = \Sigma\langle\nabla_{e_k}V, e_k\rangle_x = \Sigma\{\nabla_{e_k}\langle V, e_k\rangle\}_x = -\Sigma\{\nabla_{e_k}\langle\sigma_1, e_k.\sigma_2\rangle\}_x.$$

Since V has compact support, integration of (1.14) gives the result. ∎

Lemma **1.15**. — *Let* $D : \Gamma(S) \to \Gamma(S)$ *be a Dirac operator on a riemannian manifold* X. *Then*

$$D(f\sigma) = (\nabla f).\sigma + fD\sigma$$

for each section $\sigma \in \Gamma(S)$ *and each function* $f \in C^\infty(X)$.

Proof. — Let $e_1, \ldots, e_n$ be a local orthonormal tangent frame field. Then

$$\begin{aligned} D(f\sigma) &= \sum_k e_k.\nabla_{e_k}(f\sigma) \\ &= \sum_k e_k.\{(\nabla_{e_k}f)\,\sigma + f\nabla_{e_k}\sigma\} \\ &= \sum_k (\nabla_{e_k}f)\, e_k.\sigma + f\sum_k e_k.\nabla_{e_k}\sigma \\ &= (\operatorname{grad} f).\sigma + fD\sigma. \ \blacksquare \end{aligned}$$

We now consider extensions of D to L^2-sections. Let $\Gamma_{\text{cpt}}(S) \subset \Gamma(S)$ denote the space of C^∞ sections of S with compact support, and let $L^2(S)$ denote the Hilbert space completion of $\Gamma_{\text{cpt}}(S)$ in the norm (1.12). The operator $D : \Gamma_{\text{cpt}}(S) \to \Gamma_{\text{cpt}}(S)$ has two natural extensions as an unbounded operator on $L^2(S)$. The *minimal extension* is obtained by taking the closure of the graph of D. Thus we say σ is in the *minimal domain* of D on $L^2(S)$ if there is a sequence $\{\sigma_k\}_{k=1}^\infty \subset \Gamma_{\text{cpt}}(S)$ such that $\sigma_k \to \sigma$ and $D\sigma_k \to \tau$ (= some element) in $L^2(S)$. This limit $\tau \equiv D\sigma$ is independent of the sequence chosen. The *maximal extension* of D is obtained by taking the domain to be all $\sigma \in L^2(S)$ such that the distributional image $D\sigma$ is also in $L^2(S)$. That is, σ is in the *maximal domain* of D if the linear functional $L(\sigma') \equiv (\sigma, D\sigma')$ on $\Gamma_{\text{cpt}}(S)$ is bounded in the L^2-norm. (Here we are using Proposition 1.11, *i.e.* that D is its own formal adjoint.) The boundedness of L implies that there exists an element $D\sigma \in L^2(S)$ such that

$$(D\sigma, \sigma') = (\sigma, D\sigma') \tag{1.16}$$

for all $\sigma' \in \Gamma_{\mathrm{opt}}(S)$, and hence *for all σ' in the minimal domain of* D. The minimal domain is clearly contained in the maximal domain.

THEOREM **1.17**. — *Let* $D : \Gamma(S) \to \Gamma(S)$ *be any (generalized) Dirac operator on a complete riemannian manifold. Then the minimal and maximal extensions of* D *coincide. In particular, this is the unique closed, self-adjoint extension of* D, i.e. D *is "essentially self-adjoint."*

PROOF. — Let $f: [0, \infty) \to [0, 1]$ be a C^∞ function such that: $f \equiv 1$ on $[0, 1]$, $f \equiv 0$ on $[2, \infty]$, and $f' \approx -1$ on $[1, 2]$. Fix a point $x_0 \in X$ and let $d: X \to \mathbf{R}^+$ be a regularization of the function $\mathrm{dist}(x, x_0)$, such that $\|\nabla d\| \leqq 3/2$. Then for each positive integer m, we set

$$f_m(x) \equiv f\left(\frac{1}{m} d(x)\right). \tag{1.18}$$

We observe that

$$\|\nabla f_m\| \leqq \frac{2}{m} \tag{1.19}$$

and

$$\mathrm{supp}(f_m) \subset B_{2m} - B_m \tag{1.20}$$

where $B_\rho \equiv \{x \in X : d(x) \leqq \rho\}$ is approximately the ball of radius ρ centered at x_0. Completeness implies that B_ρ is compact for all ρ.

We want to prove that an element σ in the maximal domain of D is also in the minimal domain. This will be done by first cutting down the support of σ and then smoothing.

We begin by noting that Lemma 1.15 holds for any $\sigma \in L^2(S)$ and any $f \in C^\infty(X)$ provided that $D\sigma$ is interpreted in the distributional sense. Consequently, we can define sections

$$\sigma_m = f_m \sigma \in L^2_{\mathrm{opt}}(S)$$

with the property that

$$D\sigma_m = (\nabla f_m) \cdot \sigma + f_m D\sigma$$

for each m. In particular, σ_m is in the maximal domain of D. Furthermore, from (1.19) and (1.20) we see that $\|(\nabla f_m) \cdot \sigma\| \to 0$, and so $D\sigma_m \to D\sigma$ in $L^2(S)$.

By the above argument, it sufficies to consider the case where both σ and $D\sigma$ are in $L^2(S)$ and have compact support. We now consider a *local parametrix* for D, that is, a bounded pseudo-differential operator $Q: L^2(S) \to L^2(S)$ such that

$$DQ = 1 - \mathscr{S} \quad \text{and} \quad QD = 1 - \mathscr{S}' \tag{1.21}$$

where $\mathscr{S}$ and $\mathscr{S}'$ are infinitely smooting operators and where Q, $\mathscr{S}$ and $\mathscr{S}'$ all have Schwartz kernels supported near the diagonal. We choose a sequence $\{\psi_k\}_{k=1}^{\infty} \subset \Gamma_{\text{opt}}(S)$ with uniformly bounded supports, such that $\psi_k \to D\sigma$ in $L^2(S)$, and set

$$\sigma_k = Q\psi_k + \mathscr{S}'\sigma.$$

Since Q is pseudo-differential and since Q and $\mathscr{S}'$ have kernels supported near the diagonal, we have $\{\sigma_k\}_{k=1}^{\infty} \subset \Gamma_{\text{opt}}(S)$. Clearly, we have that $\sigma_k \to QD\sigma + \mathscr{S}'\sigma = \sigma$, and $D\sigma_k = DQ\psi_k + D\mathscr{S}'\sigma = \psi_k - \mathscr{S}\psi_k + D\mathscr{S}'\sigma \to D\sigma - \mathscr{S}D\sigma + D\mathscr{S}'\sigma = D\sigma$ in $L^2(S)$. Hence, σ lies in the minimal domain of D. ∎

REMARK 1.22. — On a complete manifold we shall always work with this unique closed, self-adjoint extension of a given Dirac operator D. This extension will also be denoted by D. Of course for any two sections $\sigma_1, \sigma_2 \in$ domain (D), we have that

$$(D\sigma_1, \sigma_2) = (\sigma_1, D\sigma_2).$$

This proves, in particular, that if $\sigma \in \text{domain}(D)$, then $D^2\sigma = 0$ if and only if $D\sigma = 0$. This result extends to any $\sigma \in L^2(S)$.

THEOREM 1.23. — *Let* $D : \Gamma(S) \to \Gamma(S)$ *be a Dirac operator on a complete riemannian manifold* X, *and let* σ *be any element of* $L^2(S)$. *Then*

$$D^2\sigma = 0 \Leftrightarrow D\sigma = 0.$$

PROOF. — The non-trivial part is of course to prove that $D^2\sigma = 0 \Rightarrow D\sigma = 0$. Since D^2 is elliptic, the equation $D^2\sigma = 0$ implies that σ is of class C^∞. Choosing f_m as above, we have that $0 = (D^2\sigma, f_m^2\sigma) = (D\sigma, D(f_m^2\sigma)) = (f_m D\sigma, 2(\nabla f_m)\cdot\sigma + f_m D\sigma)$. Therefore, using (1.19) we have

$$\|f_m D\sigma\|^2 = -2(f_m D\sigma, (\nabla f_m)\cdot\sigma) \leqq \frac{2}{m}(\|f_m D\sigma\|^2 + \|\sigma\|^2),$$

from which follows that $\|D\sigma\|^2 = \lim_m \|f_m D\sigma\|^2 = 0$. ∎

This is a reasonable place to insert a note concerning parametrices. We recall that a *parametrix* for a generalized Dirac operator $D : \Gamma(S) \to \Gamma(S)$ is a bounded operator $Q : L^2(S) \to L^2(S)$ such that

$$DQ = 1 - \mathscr{S} \quad \text{and} \quad QD = 1 - \mathscr{S}'$$

where $\mathscr{S}$ and $\mathscr{S}'$ are infinitely smoothing operators, *i.e.* $\mathscr{S}$ and $\mathscr{S}'$ have C^∞ Schwartzian kernels. The parametrix is said to be *semi-local* if Q, $\mathscr{S}$ and $\mathscr{S}'$ have Schwartzian kernels supported in a small neighborhood of the diagonal. The Green's operator, when it exists, provides a parametrix for D. Here $\mathscr{S}$ and $\mathscr{S}'$ are harmonic projection operators. However, if X is not compact, the Green's operator is not semi-local.

Our main observation here is that parametrices for a Dirac operator are easily spliced together.

PROPOSITION 1.24. — *Let Q_0 and Q_1 be parametrices for a generalized Dirac operator* D *on a riemannian manifold* X, *and let f be any compactly supported* C^∞ *function on* X. *Then* $Q \equiv fQ_1 + (1-f)\,Q_0$ *is also a parametrix for* D. *In fact, if* $DQ_k = 1 - \mathscr{S}_k$ *and* $Q_k D = 1 - \mathscr{S}'_k$ *for* $k = 0, 1$, *then*

$$DQ = 1 - \mathscr{S} \quad \textit{and} \quad QD = 1 - \mathscr{S}'$$

where

$$\begin{cases} \mathscr{S} = f\mathscr{S}_1 + (1-f)\,\mathscr{S}_0 + (\nabla f)\cdot(Q_1 - Q_0) \\ \mathscr{S}' = f\mathscr{S}'_1 + (1-f)\,\mathscr{S}'_0. \end{cases} \tag{1.25}$$

If Q_0 and Q_1 are semi-local, so is Q.

A similar result holds for $Q' \equiv Q_1 f + Q_0(1-f)$.

PROOF. — This result follows directly from Lemma 1.15.

Since the Dirac operators are self-adjoint, their index by any definition must be zero. However, if we assume that the manifold X is even-dimensional and orientable, then each Dirac operator $D: \Gamma(S) \to \Gamma(S)$ canonically gives rise to a restricted operator D^+ whose index in the compact case is not zero. The construction is as follows. Suppose $\dim(X) = 2m$ and consider the "volume form" $\omega \in \Gamma(Cl(X))$ defined by

$$\omega = i^m e_1 \ldots e_{2m} \tag{1.26}$$

where $e_1, \ldots, e_{2m}$ is a local tangent frame field. One can easily see that ω has the following properties (cf. [LM]):

$$\omega^2 = 1 \tag{1.27}$$

$$\nabla\omega = 0 \tag{1.28}$$

$$\omega e = -e\omega \quad \text{for any } e \in T^*X. \tag{1.29}$$

It follows that there is a parallel orthogonal splitting

$$S = S^+ \oplus S^- \tag{1.30}$$

into the $+1$ and -1 eigenbundles for left multiplication by ω. (If m is odd, we assume the bundle S is complex.) In fact, we can write

$$S^\pm = \frac{1}{2}(1 \pm \omega).S. \tag{1.31}$$

It follows from (1.28) and (1.29) that $D\Gamma(S^\pm) \subseteq \Gamma(S^\mp)$, and that for each non-zero cotangent vector $\xi \in T^*_x X$, the symbol $\sigma_\xi(D) = \xi : S^\pm_x \to S^\mp_x$ is an isomorphism. Hence, the restriction of D gives a pair of elliptic operators

$$D^+ : \Gamma(S^+) \to \Gamma(S^-) \quad \text{and} \quad D^- : \Gamma(S^-) \to \Gamma(S^+) \tag{1.32}$$

which are formal adjoints of one-another. The splitting (1.30) gives rise to an orthogonal splitting $\Gamma(S) = \Gamma(S^+) \oplus \Gamma(S^-)$ and thereby to an orthogonal decomposition

$$L^2(S) = L^2(S^+) \oplus L^2(S^-). \tag{1.33}$$

Since this splitting respects $\Gamma_{\text{opt}}(S)$ we see easily that it also respects the maximal domain of D. That is, the $+$ and $-$ components of an element in the domain of D are also in the domain of D. Hence, *we may split the extended operator* D on $L^2(S)$ *into a direct sum*

$$D = D^+ \oplus D^- \tag{1.34}$$

where D^- *is the adjoint of* D^+. *Furthermore, if* X *is complete, then the maximal and minimal domains of* D^+ *(and of* D^-*) coincide.*

In a similar way any parametrix Q for D with associated smoothing operators $\mathcal{S}$ and $\mathcal{S}'$ (cf. 1.21) can be decomposed as: $Q = Q^+ \oplus Q^-$, $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ and $\mathcal{S}' = (\mathcal{S}')^+ \oplus (\mathcal{S}')^-$, where

$$D^{\mp}Q^{\pm} = 1 - \mathcal{S}^{\pm} \quad \text{and} \quad Q^{\mp}D^{\pm} = 1 - (\mathcal{S}')^{\pm}. \tag{1.35}$$

When X is compact, the operator D^+ is Fredholm and has an index which is computable from the Atiyah-Singer Formula [AS III].

Example 1.36. — Let $S = Cl(X) \otimes E$ as in Example 1.7 and 1.9 (with X compact). Then

$$\operatorname{Index}(D^+) = \{\operatorname{ch} E \cdot \mathbf{L}(X)\}[X]$$

where $\mathbf{L}$ is the total L-class of Hirzebruch (cf. [H]). In particular, if E is the trivial line bundle, we have $\operatorname{Index}(D^+) = \operatorname{signature}(X)$.

Example (1.37). — Suppose X is a compact spin-manifold, and let $S = \mathbf{S} \otimes E$ be a twisted bundle of spinors as in examples 1.8 and 1.9. Then

$$\operatorname{Index}(D^+) = \{\operatorname{ch} E \cdot \hat{\mathbf{A}}(X)\}[X]$$

where $\hat{\mathbf{A}}$ is the total $\hat{A}$-class. (See again [H].) In particular, when E is the trivial line bundle, $\operatorname{Index}(D^+) = \hat{A}(X)$, the " $\hat{A}$-genus " of X.

2. SOME VANISHING THEOREMS

In this section we prove that, under appropriate curvature assumptions, the kernel (and cokernel) of a Dirac operator with various domains on a complete manifold must vanish. The key to all arguments presented here is the general Bochner-Weitzenböck formula for a Dirac operator $D: \Gamma(S) \to \Gamma(S)$ of the type discussed in § 1.

We recall that the connection ∇ on the bundle S gives rise to an elliptic second-order operator

$$\nabla^*\nabla : \Gamma(S) \to \Gamma(S)$$

defined by

$$(2.1) \qquad \nabla^*\nabla = -\sum_k \nabla_{e_k, e_k}$$

where $e_1, \ldots, e_n$ denotes a local basis of pointwise orthonormal tangent vector fields, and where $\nabla_{e_i, e_j} \equiv \nabla_{e_i}\nabla_{e_j} - \nabla_{\nabla_{e_i} e_j}$ denotes the invariant second covariant derivative. The following is well known (cf. [Si]).

Proposition 2.2. — *For any two sections $\sigma_1, \sigma_2 \in \Gamma(S)$, at least one of which has compact support, the following holds*:

$$\int_X \langle \nabla^*\nabla\sigma_1, \sigma_2 \rangle = \int_X \langle \nabla\sigma_1, \nabla\sigma_2 \rangle$$

where, by definition, $\langle \nabla\sigma_1, \nabla\sigma_2 \rangle \equiv \sum_k \langle \nabla_{e_k}\sigma_1, \nabla_{e_k}\sigma_2 \rangle$. Hence, $\nabla^*\nabla$ is a formally self-adjoint, non-negative operator.

This fact motivates the following choice of a Sobolev 1-norm on S. For $\sigma \in \Gamma_{\text{cpt}}(S)$ we set

$$(2.3) \qquad \|\sigma\|_1^2 \equiv \int_X (\langle \sigma, \sigma \rangle + \langle \nabla\sigma, \nabla\sigma \rangle)$$

and denote by $L^{1,2}(S)$ the completion of $\Gamma_{\text{cpt}}(S)$ in this norm. Using Proposition 2.2 and arguing as in § 1 proves the following.

Proposition 2.4. — *If X is complete, the operator $\nabla^*\nabla$ is essentially self-adjoint. The domain of its closed self-adjoint extension is exactly the subspace $L^{1,2}(S)$. Furthermore, for any $\sigma \in L^2(S)$, $\nabla^*\nabla\sigma = 0$ if and only if $\nabla\sigma = 0$,* i.e. *if and only if σ is parallel.*

We now recall the fundamental Bochner-Weitzenböck formula.

PROPOSITION 2.5. — *Let* $D : \Gamma(S) \to \Gamma(S)$ *be any generalized Dirac operator. Then*

$$D^2 = \nabla^*\nabla + \mathscr{R} \tag{2.6}$$

where

$$\mathscr{R} \equiv \frac{1}{2}\sum_{i,j=1}^{n} e_i . e_j . R_{e_i, e_j} \tag{2.7}$$

and where $R_{v,w} \equiv [\nabla_v, \nabla_w] - \nabla_{[v,w]}$ *denotes the curvature tensor of* S.

PROOF. — Fix $x \in X$ and choose a local orthonormal frame field $e_1, \ldots, e_n$ near x such that $(\nabla e_k)_x = 0$ for all k. Then at x,

$$\begin{aligned} D^2\sigma &= \sum_{j,k} e_j . e_k . \nabla_{e_j}\nabla_{e_k} = \sum_{j,k} e_j . e_k . \nabla_{e_j, e_k}\sigma \\ &= -\sum_j \nabla_{e_j, e_j}\sigma + \sum_{j<k} e_j . e_k . (\nabla_{e_j, e_k} - \nabla_{e_k, e_j})\,\sigma \\ &= \nabla^*\nabla\sigma + \mathscr{R}\sigma \quad \blacksquare \end{aligned}$$

The first important consequence of this formula is the following.

THEOREM 2.8. — *Suppose that* $D : \Gamma(S) \to \Gamma(S)$ *is a generalized Dirac operator on a complete manifold* X, *and suppose further that the operator* $\mathscr{R}$ *in formula* (2.6) *is uniformly bounded on* X. *Then the (maximal) domain of* D *on* $L^2(S)$ *is exactly* $L^{1,2}(S)$, *and for any* σ *in this space, we have that*

$$\|D\sigma\|^2 = \|\nabla\sigma\|^2 + (\mathscr{R}\sigma, \sigma). \tag{2.9}$$

PROOF. — It follows immediately from (2.2) and (2.6) that formula (2.9) holds for any $\sigma \in \Gamma_{cpt}(S)$. However, by Theorem 1.17, for any element σ in the domain of D, there is a sequence $\{\sigma_k\}_{k=1}^{\infty} \subset \Gamma_{cpt}(S)$ such that $\sigma_k \to \sigma$ and $D\sigma_k \to D\sigma$ in $L^2(S)$. Since $\mathscr{R}$ is a bounded operator on $L^2(S)$, it is clear that formula (2.9) holds for σ, and in particular that $\sigma \in L^{1,2}(S)$. $\blacksquare$

COROLLARY 2.10. — *Let* D, $\mathscr{R}$ *and* X *be as in Theorem* 2.8, *and suppose that* $\mathscr{R} > 0$ *pointwise on* X. *Then* $\ker(D)$ $(= \operatorname{coker}(D)) = \{0\}$. *If, furthermore,* $\mathscr{R} \geq c\,\mathrm{Id}.$ *for some constant* $c > 0$, *then* $D : L^{1,2}(S) \xrightarrow{\approx} L^2(S)$ *is an isomorphism of Hilbert spaces.*

When $\mathscr{R} > 0$, it is easy to see that these results hold without requiring $\mathscr{R}$ to be bounded above. In this case the domain of D can be a proper subspace of $L^{1,2}(S)$.

THEOREM 2.11. — (Vanishing Theorem I) *Let* $D : \Gamma(S) \to \Gamma(S)$ *be a generalized Dirac operator on a complete riemannian manifold* X, *and suppose* $\mathscr{R} > 0$. *Then formula* (2.9) *holds for all* σ *in the domain of* D. *In particular* $\ker(D) = \operatorname{coker}(D) = 0$. *If, furthermore,* $\mathscr{R} \geq c\,\mathrm{Id}.$ *for some constant* $c > 0$, *then range*$(D) = L^2(S)$ *and* $D^{-1} : L^2(S) \to L^{1,2}(S)$ *is a bounded operator.*

COROLLARY 2.12. — *Let* D *and* X *be as in Theorem* 2.11, *and suppose* X *is even-dimensional and oriented. Let* $D = D^+ \oplus D^-$ *be the splitting of* D *on* $L^2(S)$ *given in* § 1. *Then if* $\mathscr{R} > 0$, *we have that* $\ker(D) = \operatorname{coker}(D) = \{0\}$. *If, furthermore* $\mathscr{R} \geqq c\,\mathrm{Id}$. *for some* $c > 0$, *then both* D^+ *and* D^- *are surjective and have bounded inverses on* L^2.

We now turn our attention to a completely different domain for D, namely, the space $\Gamma_{\mathrm{bdd}}(S)$ of uniformly bounded, smooth sections of S.

THEOREM 2.13 (Vanishing Theorem II). — *Let* X *be a complete riemannian manifold whose Ricci curvature tensor is uniformly bounded from below. Suppose* $D : \Gamma(S) \to \Gamma(S)$ *is a generalized Dirac operator over* X *such that* $\mathscr{R} > 0$. *Then*

$$\ker(D) \cap \Gamma_{\mathrm{bdd}}(S) = \{0\}.$$

PROOF. — Suppose $\sigma \in \Gamma_{\mathrm{bdd}}(S)$ and $D\sigma = 0$. Then from (2.6) we see that (pointwise on X),

$$\begin{aligned}\frac{1}{2}\Delta\,\|\sigma\|^2 &= -\langle \nabla^*\nabla\sigma, \sigma\rangle + \|\nabla\sigma\|^2 \\ &= \langle \mathscr{R}\sigma, \sigma\rangle + \|\nabla\sigma\|^2 \\ &> \|\nabla\sigma\|^2\end{aligned}$$

where $\Delta = \Sigma(e_i e_i - \nabla_{e_i} e_i) = -\delta d$ is the negative Laplace operator on X. The result now follows from the maximum principle of Cheng and Yau [CY]. ∎

We now specialize our bundles and make a more detailed calculation of the term $\mathscr{R}$ appearing in the Bochner-Weitzenböck formula. Suppose X is an even-dimensional spin manifold, and let **S** be the complex bundle of spinors over X. (See 1.8.) Then a fundamental result of A. Lichnerowicz [Li] states that

$$D^2 = \nabla^*\nabla + \frac{1}{4}\kappa \tag{2.14}$$

where $\kappa = \sum_{i,j} R_{ijji}$ is the unnormalized *scalar curvature function* on X. That is, for the classical Dirac operator on spinors **S**, the term appearing in formula (2.6) is

$$\mathscr{R}^{\mathbf{S}} = \frac{1}{4}\kappa\,\mathrm{Id}.$$

Suppose we now consider a bundle $S = \mathbf{S} \otimes E$ with the tensor product connection, as in Example 1.9. Then the curvature tensor of $\mathbf{S} \otimes E$ is simply a derivation, *i.e.* $R^{\mathbf{S}\otimes E} = R^{\mathbf{S}} \otimes 1 + 1 \otimes R^E$. It follows that the term $\mathscr{R}^{\mathbf{S}\otimes E}$ appearing in formula (2.6) for the Dirac operator on $\mathbf{S} \otimes E$ is (cf. [GL₁], [LM])

$$\mathscr{R}^{\mathbf{S}\otimes E} = \frac{1}{4}\kappa + \mathscr{R}^E \tag{2.15}$$

where

(2.16) $$\mathscr{R}^{\mathrm{E}}(\sigma \otimes \varphi) \equiv \frac{1}{2} \sum_{i,j=1}^{n} (e_i e_j \sigma) \otimes (\mathrm{R}^{\mathrm{E}}_{e_i, e_j} \varphi)$$

on simple elements $\sigma \otimes \varphi \in \mathbf{S} \otimes \mathrm{E}$. Note that $\mathscr{R}^{\mathrm{E}}$ depends linearly on the curvature tensor R^{E} of E, and that there exists a constant c_n depending only on the dimension n of X, such that

(2.17) $$\|\mathscr{R}^{\mathrm{E}}\| \leqq c_n \|\mathrm{R}^{\mathrm{E}}\|.$$

Consequently, if the scalar curvature κ of X satisfies: $\kappa > 4\mathscr{R}^{\mathrm{E}}$, and in particular, if

(2.18) $$\kappa > 4c_n \|\mathrm{R}^{\mathrm{E}}\|,$$

then $\mathscr{R}^{\mathbf{S}\otimes\mathrm{E}} > 0$ and the Vanishing Theorems apply.

3. ESTIMATES FOR THE DIMENSION OF THE KERNEL AND COKERNEL

Let $D : \Gamma(S) \to \Gamma(S)$ be a generalized Dirac operator on a complete riemannian manifold X, and let $\mathscr{R}$ be the symmetric zero-order term appearing in formula (2.6). Throughout this section we shall make the following hypothesis.

ASSUMPTION 3.1. — There exists a compact subset $K \subset X$ and a constant $\kappa_0 > 0$ such that

$$\mathscr{R} \geqq \kappa_0 \,\mathrm{Id}.$$

in $X - K$.

The first main result of this section is the following.

THEOREM 3.2. — *There exists an integer d, depending only on* D *in a neighborhood of* K *and on κ_0, such that*

$$\dim(\ker D) \leqq d.$$

In particular, if $D = D^+ \oplus D^-$ *is the splitting* (1.34), *obtained when* $\dim(X)$ *is even, then*

$$\dim(\ker D^+) + \dim(\operatorname{coker} D^+) \leqq d.$$

Consequently, the operator $\widetilde{D}^+ \equiv (1 + D^+D^-)^{-1/2}D^+ = D^+(1 + D^-D^+)^{-1/2}$ *is Fredholm with index* $\leqq d$.

PROOF. — Since K is compact, there exists $\kappa_1 > 0$ such that $\kappa_1 \,\mathrm{Id} \geqq -\mathscr{R}$ on K. Suppose now that $\sigma \in L^2(S)$ satisfies $D\sigma = 0$. Then from the basic formula (2.9) we know that

$$||\nabla\sigma||^2 + (\mathscr{R}\sigma, \sigma) = 0.$$

Since $\mathscr{R} \geqq \kappa_0$ on $X - K$, this implies that

$$||\nabla\sigma||^2 + \int_K \langle \mathscr{R}\sigma, \sigma\rangle + \kappa_0 \int_{X-K} ||\sigma||^2 \leqq 0$$

and consequently that

$$||\nabla\sigma||^2 + \kappa_0 ||\sigma||^2_{X-K} \leqq \kappa_1 ||\sigma||^2_K \tag{3.3}$$

where for $A \subseteqq X$, the symbol $||\sigma||^2_A$ denotes $\int_A ||\sigma||^2$.

We now make the assumption that $||\sigma||^2 = ||\sigma||^2_X = ||\sigma||^2_K + ||\sigma||^2_{X-K} = 1$. Then equation (3.3) becomes

$$\frac{1}{\kappa_0+\kappa_1}||\nabla\sigma||^2 + \frac{\kappa_0}{\kappa_0+\kappa_1} \leqq ||\sigma||^2_K. \tag{3.4}$$

We shall now appeal to the following basic fact concerning elliptic operators. (See [Ag], for example.)

THEOREM (3.5) (The Friedrichs Lemma). — *Let Ω be any open subset of* X *with* $K \subset \Omega$, *and let* $||\,.\,||_{C^k,K}$ *denote the uniform* C^k*-norm for* C^k *sections of* S *over the compact set* K. *Then there exists a constant c, depending only on* K, Ω *and k, such that for any* $\sigma \in \Gamma\left(S\big|_{\Omega}\right)$ *with* $D\sigma = 0$,

$$||\sigma||_{C^k,K} \leqq c||\sigma||_{\Omega}. \tag{3.6}$$

We now fix a neighborhood Ω of K and let c be the constant appearing in Theorem 3.5 for $k = 1$. We also fix an $\varepsilon > 0$ and choose an ε-dense subset $\{x_m\}_{m=1}^d$ of K. (That is, every point of K is within distance ε of some x_m.) Let H denote the kernel of D on $L^2(S)$ (over all of X), and suppose $\dim(H) > d$. Then there exists an element $\sigma \in H$ such that $||\sigma|| = 1$ and $\sigma(x_m) = 0$ for $m = 1, \dots, d$. It follows from (3.6) and ε-density, that $||\sigma(x)|| \leqq c\varepsilon$ for all $x \in K$. For ε sufficiently small, this is a direct violation of (3.4) and we have established the bound on the dimension H.

The remainder of the proof of Theorem 3.2 follows immediately.

Similar arguments also prove the following.

THEOREM (3.7). — *Let* D *be a generalized Dirac operator on a complete riemannian manifold* X, *which satisfies Assumption* 3.1. *Let* H *be the (finite dimensional) kernel of* D *on* $L^2(S)$, *and let* $H^\perp$ *denote its orthogonal complement. Then there is a* $c > 0$ *so that*

$$||D\sigma||^2 \geqq c^2||\sigma||^2$$

for $\sigma \in H^\perp$. *Thus, the operator* D, *and also, when* $\dim(X)$ *is even, the operators* $D^\pm$ *admit bounded Green's operators.*

PROOF. — Let E_λ denote the eigenspace of D on $L^2(S)$ with eigenvalue λ. It will suffice to prove that there exists $c > 0$ such that the space

$$H_c \equiv \bigoplus_{|\lambda| \leqq c} E_\lambda$$

is finite dimensional.

Let κ_1 be as above, and observe that if $||D\sigma||^2 \leqq c^2||\sigma||^2$, then formula (2.9) and the above arguments show that, for $\sigma \in H_c$,

$$||\nabla\sigma||^2 + (\kappa_0 - c^2)\,||\sigma||^2 \leqq (\kappa_0+\kappa_1)\,||\sigma||^2_K. \tag{3.8}$$

We choose $c > 0$ with $c^2 \ll \kappa_0$, and observe that if $\sigma \in H_c$ with $\|\sigma\|^2 \neq 0$, then

$$c' \equiv \frac{\kappa_0 - c^2}{\kappa_0 + \kappa_1} \leq \frac{\|\sigma\|_K^2}{\|\sigma\|^2}. \tag{3.9}$$

Consider now a parametrix Q for D so that $QD = 1 - \mathscr{S}$ were $\mathscr{S}$ is smoothing. Let ρ denote restriction to K, and consider the compact operator $\widetilde{\mathscr{S}} \equiv \rho \circ \mathscr{S}$. Then for $\sigma \in H_c$ we have

$$\widetilde{\mathscr{S}}\sigma = \rho\sigma - \rho QD\sigma.$$

For $\sigma \in H_c$ we have $\|D\sigma\| \leq c\|\sigma\|$. Hence, setting $q = \|Q\|$ and using (3.9), we have that

$$\|\widetilde{\mathscr{S}}\sigma\| \geq \|\rho\sigma\| - cq\|\sigma\| = \|\sigma\|_K - cq\|\sigma\| \geq (c' - cq)\,\|\sigma\|. \tag{3.10}$$

Choosing c sufficiently small, we see that $\|\widetilde{\mathscr{S}}\sigma\| \geq c''\|\sigma\|$ for all $\sigma \in H_c$, where c'' is some positive constant. Since $\widetilde{\mathscr{S}}$ is a compact operator, we conclude that $\widetilde{\mathscr{S}}H_c \cong H_c$ is finite dimensional. ∎

REMARK (3.11). — Of course the index of the Fredholm operator $\widetilde{D}^+$, presented in Theorem 3.2, is just $\ker(D^+) - \ker(D^-)$. This index is invariant under continuous deformations of D^+ on compact subsets of X.

Using this remark, we are able to define a simple and useful invariant for studying spaces of positive scalar curvature metrics. Suppose X is a compact odd-dimensional spin manifold, and let g_0, g_1 be two riemannian metrics of positive scalar curvature on X. We than construct a complete metric g on $X \times \mathbf{R}$ by setting

$$g = \begin{cases} g_0 + dt^2 & \text{for } t \leqq 0 \\ g_1 + dt^2 & \text{for } t \geqq 1 \\ \text{anything} & \text{for } 0 \leqq t \leqq 1. \end{cases} \tag{3.12}$$

Let D^+ denote the canonical Dirac operator on spinors associated to this metric. Then, by (2.14) and Theorem 3.2 the index of D^+ is well defined. Moreover, by Remark 3.11 this index depends only on g_0 and g_1 since any two metrics of type 3.12 can be joined by a homotopy which is supported on the compact subset $X \times [0, 1]$. (In fact, the linear homotopy has this property.) Therefore, we can write

$$i(g_0, g_1) \equiv \operatorname{index}(D^+). \tag{3.13}$$

We shall see in the next section that $i(g_0, g_1)$ depends only on the connectedness components of g_0 and g_1 in the space $\mathscr{P}(X)$ of positive scalar curvature metrics on X. In particular, if g_0 is homotopic to g_1 in $\mathscr{P}(X)$, then $i(g_0, g_1) = 0$.

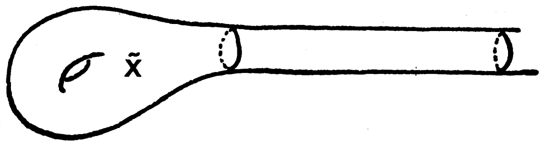
Clearly, $i(g_0, g_1) = -i(g_1, g_0)$. Furthermore, we shall show that

$$i(g_0, g_1) + i(g_1, g_2) = i(g_0, g_2)$$

for any triple of such metrics on X.

This index is certainly non-trivial. Even when $X = S^7$ and $g_0 =$ the canonical metric, this index can assume all integer values. (See § 4.)

There are many similar constructions that can be made. For example, suppose X is the boundary of a compact spin manifold $\tilde{X}$, and suppose that g is a metric of positive scalar curvature on X. Then we can construct a complete metric $\tilde{g}$ on interior($\tilde{X}$) by letting $\tilde{g}$ be the half-infinite cylinder $g + dt^2$ on $\partial\tilde{X} \times [0, \infty)$ at the boundary and extending in any manner to the compact piece.



As above the canonical Dirac operator D^+ has an index which is independent of this extension of the metric over the compact piece, and we set

$$i(g, \tilde{X}) \equiv \operatorname{index}(D^+). \tag{3.14}$$

Note that *if g can be extended to a metric of positive scalar curvature on $\tilde{X}$, then $i(g, \tilde{X}) = 0$.* This follows from Theorem 2.11.

There are many further constructions of this kind. They are particularly useful when put into families. We shall take up this subject again in § 4, and in a forthcoming paper of the authors. However, for calculations with these objects we shall need a certain non-compact index theorem. This is the subject of the next section.

4. THE INDEX THEOREM

In this section we state and prove the central analytic result of the paper: a relative index theorem for pairs of elliptic operators which agree outside a compact set.

Suppose that $D_k : \Gamma(S_k) \to \Gamma(S_k)$ is a generalized Dirac operator on a complete riemannian manifold X_k for $k = 0, 1$. We make the following hypotheses:

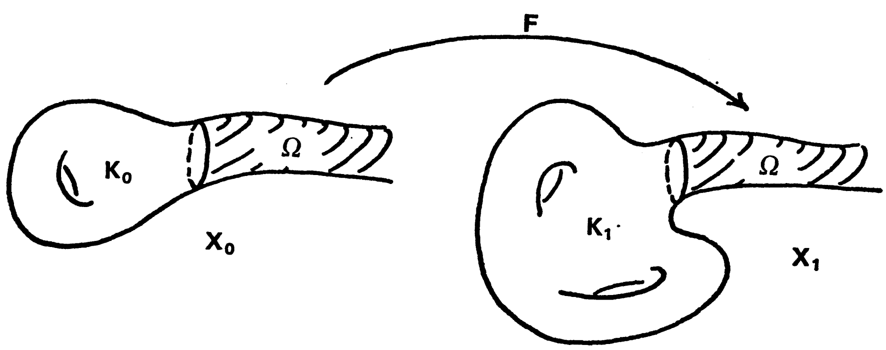
ASSUMPTION 4.1. — *The operators coincide outside a compact set.* That is, there exist compact subsets $K_0 \subset X_0$ and $K_1 \subset X_1$ and an isometry $F : (X_0 - K_0) \xrightarrow{\approx} (X_1 - K_1)$ which is covered by a bundle isometry

$$\widetilde{F} : S_0\Big|_{(X_0 - K_0)} \xrightarrow{\approx} S_1\Big|_{(X_1 - K_1)} \qquad \text{so that} \qquad D_1 = \widetilde{F} \circ D_0 \circ \widetilde{F}^{-1}$$

in $X_1 - K_1$.

It will be easier to drop the cumbersome notation of equivalence and actually identify the manifolds and the operators on the set

$$\Omega \equiv X_0 - K_0 \cong X_1 - K_1. \tag{4.2}$$



EXAMPLE 4.3. — A simple and important case of such a pair of operators occurs when D_0 and D_1 are defined on the *same* manifold X. For example, suppose X is spin and let D be the canonical Dirac operator on spinors $\mathbf{S}$ over X. Suppose then that E is a complex vector bundle which is *trivialized* outside a compact subset of X. Let $\mathbf{C}^k$ denote the trivial complex k-plane bundle, and as in Example 1.9 construct operators

$$D_0 : \Gamma(\mathbf{S} \otimes \mathbf{C}^k) \to \Gamma(\mathbf{S} \otimes \mathbf{C}^k),$$

$$D_1 : \Gamma(\mathbf{S} \otimes E) \to \Gamma(\mathbf{S} \otimes E).$$

(We assume that E and $\mathbf{C}^k$ carry connections compatible with the trivializations.) Clearly

$$D_0 \cong D_1 \cong \underbrace{D \oplus \ldots \oplus D}_{k\text{-times}}$$

outside a compact subset.

EXAMPLE 4.4. — Another basic example of such a pair of operators occurs when X_0 and X_1 are spin manifolds, and when D_0 and D_1 are the associated canonical Dirac operators on spinors. Here, of course, we must assume that the isometry of X_0 with X_1 at infinity preserves the spin structure.

Returning to the general case, we now suppose that the manifolds X_0 and X_1 are even-dimensional and (compatibly) oriented so that we can consider the operators D_0^+ and D_1^+. These operators also agree at infinity, and their difference has a well-defined topological index which we shall now discuss.

There are several possible equivalent definitions, and we begin with the most direct and conceptual one. If X_0 and X_1 are compact, then the relative topological index is simply defined to be the difference, $\operatorname{index}(D_1^+) - \operatorname{index}(D_0^+)$. If they are not compact we proceed as follows. Chop off the manifolds X_0 and X_1 along a compact hypersurface H contained in the set Ω where the manifolds agree. Compactify X_0 and X_1 by sewing on another compact manifold with boundary H. Extend the operators D_0^+ and D_1^+ by an elliptic pseudo-differential operator defined on the new piece. (This compactification is always possible.) Let $\widetilde{D}_0^+$ and $\widetilde{D}_1^+$ denote the elliptic operators so obtained. Then we set

$$\textbf{(4.5)} \qquad \operatorname{ind}_t(D_1^+, D_0^+) \equiv \operatorname{index}(\widetilde{D}_1^+) - \operatorname{index}(\widetilde{D}_0^+).$$

(The expression on the right in (4.5) is independent of the choice of extensions $\widetilde{D}_0^+$ and $\widetilde{D}_1^+$, as we shall soon prove.)

This index can be reexpressed in terms of local formulas. In fact we have the following. Given any smoothing operator $\mathscr{S}$ on a manifold X, the *local trace of* $\mathscr{S}$ is the C^∞ function

$$\mathfrak{k}^{\mathscr{S}}(x) = \operatorname{trace} K^{\mathscr{S}}(x, x)$$

where $K^{\mathscr{S}}(x,y)$ is the Schwartizian kernel of $\mathscr{S}$ on $X \times X$. The topological index can be computed in terms of certain local trace functions.

PROPOSITION 4.6. — *Suppose that* Q_0 *and* Q_1 *are semi-local parametrices for the operators* D_0^+ *and* D_1^+, *and suppose that* Q_0 *and* Q_1 *agree in a neighborhood of infinity. Write*

$$D^+ Q_j = 1 - \mathscr{S}_j \quad \textit{and} \quad Q_j D_j^+ = 1 - \mathscr{S}_j'$$

where $\mathscr{S}_j$, $\mathscr{S}_j'$ *are the associated semi-local smoothing operators for* $j = 0, 1$. *Then the local trace functions satisfy*

$$\textbf{(4.7)} \qquad \mathfrak{k}^{\mathscr{S}_0} = \mathfrak{k}^{\mathscr{S}_1} \quad \textit{and} \quad \mathfrak{k}^{\mathscr{S}_0'} = \mathfrak{k}^{\mathscr{S}_1'}$$

near infinity, and

$$\text{(4.8)} \qquad \operatorname{ind}_t(D_1^+, D_0^+) = \int_{X_1} (\mathfrak{k}^{\mathscr{S}'_1} - \mathfrak{k}^{\mathscr{S}_1}) - \int_{X_0} (\mathfrak{k}^{\mathscr{S}'_0} - \mathfrak{k}^{\mathscr{S}_0}).$$

(Note that this integral is well defined since the function $\mathfrak{k}^{\mathscr{S}'_1} - \mathfrak{k}^{\mathscr{S}_1} - \mathfrak{k}^{\mathscr{S}'_0} + \mathfrak{k}^{\mathscr{S}_0}$ *vanishes outside a compact subset of the domain* Ω *where the manifolds " coincide ".)*

PROOF. — Let $\Omega_0 \subset \Omega$ be an open neighborhood of infinity where the local trace functions coincide, *i.e.* where (4.7) holds. We chop off the manifolds along a compact hypersurface $H \subset \Omega_0$ and compactify as above. Let $\tilde{Q}$ be a local parametrix for the new operator $\tilde{D}_0^+ = \tilde{D}_1^+$ on $\tilde{\Omega}_0$, with Schwartzian kernel supported in a small neighborhood of the diagonal in $\tilde{\Omega}_0 \times \tilde{\Omega}_0$. Patch $\tilde{Q}$ onto Q_0 and onto Q_1 as in 1.24 using a function f such that: $f \equiv 0$ outside Ω_0, $f \equiv 1$ near infinity, and ∇f is supported in a small neighborhood of H. This produces local parametrices $\tilde{Q}_0$ and $\tilde{Q}_1$ for the new operators $\tilde{D}_0^+$ and $\tilde{D}_1^+$, and writing

$$\tilde{D}_j^+ \tilde{Q}_j = 1 - \tilde{\mathscr{S}}_j \quad \text{and} \quad \tilde{Q}_j \tilde{D}_j^+ = 1 - \tilde{\mathscr{S}}'_j,$$

we know that (cf. [A])

$$\text{(4.9)} \qquad \operatorname{index}(\tilde{D}_j^+) = \int_{\tilde{X}_j} (\mathfrak{k}^{\tilde{\mathscr{S}}'_j} - \mathfrak{k}^{\tilde{\mathscr{S}}_j})$$

for $j = 0, 1$. However, by construction we know that

$$\text{(4.10)} \qquad \mathscr{S} = \tilde{\mathscr{S}}_j \quad \text{and} \quad \mathscr{S}'_j = \tilde{\mathscr{S}}'_j \quad \text{in } X_j - \Omega_0 = \tilde{X}_j - \tilde{\Omega}_0$$

and that

$$\text{(4.11)} \qquad \mathfrak{k}^{\tilde{\mathscr{S}}'_1} - \mathfrak{k}^{\tilde{\mathscr{S}}_1} - \mathfrak{k}^{\tilde{\mathscr{S}}'_0} + \mathfrak{k}^{\tilde{\mathscr{S}}_0} \equiv 0 \quad \text{in } \tilde{\Omega}_0$$

(since we modified the two parametrices Q_0 and Q_1 in exactly the same way in $\tilde{\Omega}_0$). Consequently, from (4.5), (4.9), (4.10) and (4.11) we see that

$$\operatorname{ind}_t(\tilde{D}_1^+, \tilde{D}_0^+) = \int_{X_1 - \Omega_0} (\mathfrak{k}^{\mathscr{S}'_1} - \mathfrak{k}^{\mathscr{S}_1}) - \int_{X_0 - \Omega_0} (\mathfrak{k}^{\mathscr{S}'_0} - \mathfrak{k}^{\mathscr{S}_0})$$

which, by (4.7), gives the result.

Proposition 4.6 shows that the index defined in (4.5) is independent of the choice of the extension.

It is useful to examine the index associated to the examples discussed above. Let D_0 and D_1 be the two operators from Example 4.3 defined on the same manifold X. Using the equivalence of the operators outside a compact set, we see that the difference of the symbol mappings gives an element in the K-Theory of TX with compact supports. (See [ABS].) The index $\operatorname{ind}_t(D_1^+, D_0^+)$ is equal to the image of this element in $K_*(\text{pt.})$ under the map $f_!$, where $f: TX \to \text{pt}$.

Alternatively, we could do the following. Using Chern-Weil's Theory (cf. [KN]) we can express the total $\hat{A}$-class of X (formally) as a sum of differential forms $\hat{\mathbf{A}}(X)$

constructed canonically out of the curvature tensor of X. Similarly, we can express the Chern character of E

$$\mathrm{ch}(\mathrm{E}) = k + \mathrm{ch}_1(\mathrm{E}) + \mathrm{ch}_2(\mathrm{E}) + \dots$$

canonically in terms of the curvature tensor of E. (Recall that $k = \dim \mathrm{E}$.) Since E is flat outside a compact subset, we see that the *reduced Chern character*

$$(4.12) \qquad \widehat{\mathrm{ch}}(\mathrm{E}) \equiv \mathrm{ch}(\mathrm{E}) - k = \mathrm{ch}(\mathrm{E} - \mathbf{C}^k)$$

has compact support in X.

PROPOSITION (4.13). — *For the pair of operators* D_0 *and* D_1 *given in Example* 4.3,

$$(4.14) \qquad \mathrm{ind}_t(\mathrm{D}_1^+, \mathrm{D}_0^+) = \int_{\mathrm{X}} \widehat{\mathrm{ch}}(\mathrm{E}) \cdot \widehat{\mathbf{A}}(\mathrm{X}).$$

PROOF. — Let $\widetilde{\mathrm{X}}$ be a compactification of X obtained by modification outside the support of E. Let D and $\widetilde{\mathrm{D}}$ be the canonical Dirac operators of X and $\widetilde{\mathrm{X}}$ respectively. Then

$$\begin{aligned}\mathrm{ind}_t(\mathrm{D}_1^+, \mathrm{D}_0^+) &= \mathrm{ind}(\widetilde{\mathrm{D}}^+ \otimes \mathrm{E}) - \mathrm{ind}(\widetilde{\mathrm{D}}^+ \otimes \mathbf{C}^k)\\ &= \{\mathrm{ch}\,\mathrm{E} \cdot \widehat{\mathbf{A}}(\widetilde{\mathrm{X}})\}[\widetilde{\mathrm{X}}] - \{\mathrm{ch}\,\mathbf{C}^k \cdot \widehat{\mathbf{A}}(\widetilde{\mathrm{X}})\}[\widetilde{\mathrm{X}}]\\ &= \{\widehat{\mathrm{ch}}\,\mathrm{E} \cdot \widehat{\mathbf{A}}(\widetilde{\mathrm{X}})\}[\widetilde{\mathrm{X}}] = \{\widehat{\mathrm{ch}}\,\mathrm{E} \cdot \widehat{\mathbf{A}}(\mathrm{X})\}[\mathrm{X}].\end{aligned}$$

The last equality holds since the support of $\widehat{\mathrm{ch}}$ E is contained in the support of E where X and $\widetilde{\mathrm{X}}$ coincide. ■

We now focus attention on Example 4.4 Let X_0 and X_1 be two spin manifolds with a spin-structure-preserving isometry $(\mathrm{X}_0 - \mathrm{K}_0) \overset{\approx}{\to} (\mathrm{X}_1 - \mathrm{K}_1)$. Let D_0 and D_1 be the canonical Dirac operator on spinors on X_0 and X_1 respectively. We now chop off X_0 and X_1 along a compact hypersurface $\mathrm{H} \subset \mathrm{X}_0 - \mathrm{K}_0 \cong \mathrm{X}_1 - \mathrm{K}_1$, producing compact manifolds $\hat{\mathrm{X}}_0$ and $\hat{\mathrm{X}}_1$ with boundary H. We then modify the (common) metric near the boundary so that it becomes a product in the collar $\mathrm{H} \times (0, \varepsilon)$. The manifold $\hat{\mathrm{X}}_0$ can now be joined metrically to $\hat{\mathrm{X}}_1$ and $\hat{\mathrm{X}}_0$ (after a change of orientation). The extended operators $\widetilde{\mathrm{D}}_0$ and $\widetilde{\mathrm{D}}_1$ are just the canonical Dirac operators of $\hat{\mathrm{X}}_0 \cup (-\hat{\mathrm{X}}_0)$ ($=$ the double of $\hat{\mathrm{X}}_0$) and $\hat{\mathrm{X}}_1 \cup (-\hat{\mathrm{X}}_0)$. Since the $\hat{\mathrm{A}}$-genus of the double of a manifold is zero, we have that

$$(4.15) \qquad \begin{aligned}\mathrm{ind}_t(\mathrm{D}_1^+, \mathrm{D}_0^+) &= \hat{\mathrm{A}}[\hat{\mathrm{X}}_1 \cup (-\hat{\mathrm{X}}_0)]\\ &= \int_{\mathrm{X}_1} \widehat{\mathbf{A}}(\mathrm{X}_1) - \int_{\mathrm{X}_0} \widehat{\mathbf{A}}(\mathrm{X}_0)\end{aligned}$$

where again $\widehat{\mathbf{A}}(\mathrm{X}_j)$ denotes the canonical $\hat{\mathrm{A}}$-form associated to the riemannian metric on X_j. (The difference of integrals here is defined in the obvious manner, using the identification outside a compact set.)

From the two examples above it should be clear how to compute the index in many more complicated situations.

We now come to the fundamental result of this section. The theorem will be proved under the following hypothesis on D_0 and D_1.

ASSUMPTION **4.16**. — *The operators are strictly positive at infinity.* That is, if $\mathscr{R}$ denotes the symmetric, zero-order term appearing in formula (2.6), for the operator $D_0 \cong D_1$ in Ω, then there exists a constant $\kappa_0 > 0$, such that

$$\mathscr{R} \geqq \kappa_0 \, \mathrm{Id}.$$

in Ω.

We know from Theorem 3.2 that under this hypothesis, each of the operators D_0^+ and D_1^+ has a well defined analytic index, $\mathrm{index}(D_j^+) \equiv \dim(\ker D^+) - \dim(\mathrm{coker}\, D_j^+)$. This enables us to define a *relative analytic index* for the pair

$$\mathrm{ind}_a(D_1^+, D_0^+) \equiv \mathrm{index}(D_1^+) - \mathrm{index}(D_0^+). \tag{4.17}$$

The main result is the following:

THEOREM **4.18** (The relative Index Theorem). — *Let* D_0 *and* D_1 *be generalized Dirac operators on complete riemannian manifolds* X_0 *and* X_1, *and suppose these operators are strictly positive at infinity and coincide outside a compact set* (*cf.* 4.1 *and* 4.16). *Then*

$$\mathrm{ind}_a(D_1^+, D_0^+) = \mathrm{ind}_t(D_1^+, D_0^+).$$

REMARK. — A result of this kind holds in greater generality, but the above statement will suffice for the purposes of this paper.

PROOF. — We begin with a discussion of harmonic sections. Suppose $D : \Gamma(S) \to \Gamma(S)$ is a generalized Dirac operator on a (not necessarily compact) riemannian manifold X, and let $H \equiv \{\sigma \in L^2(S) : D\sigma = 0\}$. Let $(\sigma_m)_m$ denote an orthonormal basis for H. Then it is a general fact (cf. [A]) that the *Bergman kernel*

$$K^H(x,y) \equiv \sum_m \sigma_m(x) \otimes \bar{\sigma}_m(y)$$

converges uniformly in the C^k-norm on compact subsets of $X \times X$ for any $k \geqq 0$. This is the Schwartzian kernel of the orthogonal projection operator $\mathscr{H} : L^2(S) \to H$. The associated local trace function

$$\mathfrak{k}^{\mathscr{H}}(x) \equiv \sum_m \|\sigma_m(x)\|^2$$

is C^∞ on X, and clearly we have that

$$\dim H = \int_X \mathfrak{k}^{\mathscr{H}}. \tag{4.19}$$

We suppose now that X is complete and that D is strictly positive outside a compact subset $K \subset X$ (*i.e.* 3.1 is satisfied). Let $F : X \to \mathbf{R}^+$ be a smooth exhaustion function and set $X(t) \equiv \{x \in X : F(x) > t\}$ for each $t \in \mathbf{R}^+$. Fix t_0 so that $F\big|_K < t_0$, *i.e.* so that D is strictly positive on $X(t_0)$.

Consider now the operator D restricted to $X(t_0)$, and set

$$H(t_0) \equiv \left\{\sigma \in L^2\left(S\big|_{X(t_0)}\right) : D\sigma = 0\right\}.$$

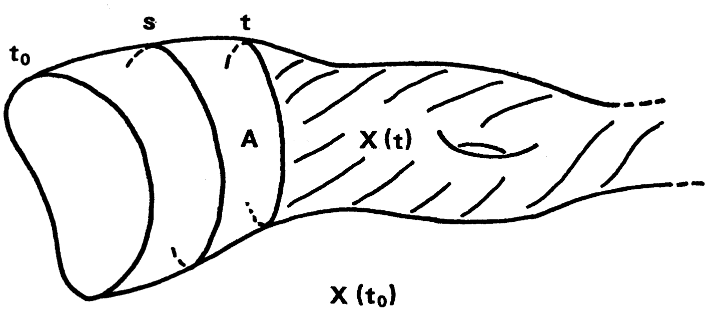
Let $\mathfrak{k}^{\mathscr{H}(t_0)}$ denote the local trace function on $X(t_0)$ associated to the orthogonal projection $\mathscr{H}(t_0) : L^2\left(S\big|_{X(t_0)}\right) \to H(t_0)$.

Then we have the following

LEMMA 4.20. — *For any* $t > t_0$

$$\int_{X(t)} \mathfrak{k}^{\mathscr{H}(t_0)} < \infty.$$

PROOF. — Fix $t > t_0$. Choose s so that $t_0 < s < t$, and consider the compact "annulus" $A \equiv \text{Closure}(X(s) - X(t))$. Recall



from above that the local trace function $\mathfrak{k}^{\mathscr{H}(t_0)} = \sum_m ||\sigma_m||^2$, where (σ_m) is an orthonormal basis of $H(t_0)$, converges uniformly in the C^1 norm on A. In particular, the sum of the Sobolev 1-norms on A is finite, *i.e.*,

$$\sum_m ||\sigma_m||^2_{1,A} < \infty. \tag{4.21}$$

We now claim that there exists a constant c so that for each $\sigma \in H(t_0)$,

$$||\sigma||^2_{1,X(t)} \leqq c\, ||\sigma||^2_{1,A}. \tag{4.22}$$

To see this we choose a "cut-off" function $f \in C^\infty(X(t_0))$ so that: $0 \leqq f \leqq 1$, $f \equiv 1$ on $X(t)$, and $f \equiv 0$ on $X(t_0) - X(s)$. Clearly there exists a $c_0 > 0$, such that $||\nabla f|| < c_0$. Applying 1.15 and 2.5, and integrating by parts (cf. Theorem 2.8), we have that

$$\begin{aligned} 0 \equiv (D^2\sigma, f^2\sigma) &= ((\nabla^*\nabla + \mathscr{R})\,\sigma, f^2\sigma) \\ &= (f\nabla\sigma, f\nabla\sigma) + 2(f\nabla\sigma, \nabla f \otimes \sigma) + (\mathscr{R}f\sigma, f\sigma). \end{aligned}$$

From the positivity of $\mathscr{R}$ (Assumption 3.1) and the properties of f, we conclude that

$$\begin{aligned}||\nabla\sigma||^2_{X(t)} + \kappa_0||\sigma||^2_{X(t)} \leqq 2|(f\nabla\sigma, \nabla f\otimes\sigma)| &\leqq 2c_0||\nabla\sigma||_A\,||\sigma||_A \\ &\leqq c_0(||\nabla\sigma||^2_A + ||\sigma||^2_A).\end{aligned}$$

Thus
$$||\sigma||^2_{1,X(t)} \leqq c_0\left(1+\frac{1}{\kappa_0}\right)||\sigma||^2_{1,A}$$

as claimed.

Combining (4.21) and (4.22) proves the Lemma.

Consider now the operators D_0 and D_1 which agree in $\Omega = X_0 - K_0 \cong X_1 - K_1$ and are strictly positive there (*i.e.* which satisfy Assumptions 4.1 and 3.1). Let G_0 and G_1 be the Green's operators for D_0^+ and D_1^+ respectively. Recall from Theorem 3.7 that for each j, G_j is a bounded operator. It satisfies the relations

$$\textbf{(4.23)}\qquad D_j^+G_j = 1 - \mathscr{H}_j^+ \quad\text{and}\quad G_jD_j^+ = 1 - \mathscr{H}_j^-$$

where $\mathscr{H}_j^\pm : L^2(S^\pm) \to L^2(S^\pm)$ denote orthogonal projections onto the finite dimensional subspaces

$$\textbf{(4.24)}\qquad \begin{cases} H_j^+ = \ker(D_j^+) \\ H_j^- = \ker(D_j^-) \cong \operatorname{coker}(D_j^+). \end{cases}$$

Each of these Green's operators has a locally-L^1 Schwartzian kernel $K^{G_j}(x,y)$ which is smooth off the diagonal. These operators can be easily restricted to the open set Ω where

$$S_0\Big|_\Omega \cong S_1\Big|_\Omega \overset{\text{def}}{=} S$$

and
$$D_0^+\Big|_\Omega \cong D_1^+\Big|_\Omega \overset{\text{def}}{=} D^+.$$

This restriction gives bounded operators

$$\hat{G}_j : L^2(S^-) \to L^2(S^+)$$

defined by setting $\hat{G}_j = \chi G_j \chi$ where χ is the characteristic function of Ω. The difference of these operators satisfies the equation

$$\textbf{(4.25)}\qquad D^+(\hat{G}_1 - \hat{G}_0) = \hat{\mathscr{H}}_1^+ - \hat{\mathscr{H}}_0^+ \quad \text{in } \Omega$$

where $\hat{\mathscr{H}}_j^+ = \chi\mathscr{H}_j^+\chi$ is a bounded operator with finite dimensional range. This implies that the range of $\hat{G}_1 - \hat{G}_0$ is nearly contained in the kernel $H(\Omega)$ of D^+ on Ω. To be more specific, let $V = \ker(\hat{\mathscr{H}}_1^+ - \hat{\mathscr{H}}_0^+)$. Then V is a closed subspace of *finite codimension* in $L^2(S^+)$, and

$$\textbf{(4.26)}\qquad (\hat{G}_1 - \hat{G}_0)(V) \subseteq \ker(D^+).$$

We now claim that the local trace function of $\hat{G}_1 - \hat{G}_0$ is integrable at infinity. Recall that $\Omega = X_0 - K_0$. Choose $\Omega' = X_0 - K_0'$ where K_0' is compact and

$K_0 \subset \text{interior}(K_0')$. Then $\Omega' \subset \Omega$ is a "strictly smaller" neighborhood of infinity, and by Lemma 4.20 we have that if

$$\mathscr{H}(\Omega) : L^2(S^+) \to H(\Omega)$$

denotes orthogonal projection onto the kernel of D^+ on Ω, then

$$\int_{\Omega'} \mathfrak{t}^{\mathscr{H}(\Omega)} < \infty. \tag{4.27}$$

LEMMA **4.28**. — *Let* $\mathscr{E} = \hat{G}_1 - \hat{G}_0$. *Then*

$$\int_{\Omega'} |\mathfrak{t}^{\mathscr{E}}| < \infty.$$

PROOF. — From the discussion above (cf. (4.26)) we know that $\text{range}(\mathscr{E}) \subseteq H(\Omega) + F$ where F is a finite dimensional subspace of $L^2(S^-)$. Let $(\sigma_m)_{m=1}^{\infty}$ be an orthonormal basis of $H(\Omega) + F$ such that $(\sigma_m)_{m=M}^{\infty}$ is an orthonormal basis of $H(\Omega)$. Then the Schwartzian kernel of $\mathscr{E}$ can be written as

$$K^{\mathscr{E}}(x,y) = \sum_m \sigma_m(x) \otimes (\mathscr{E}^* \sigma_m)(y)$$

where $\mathscr{E}^*$ denotes the adjoint of $\mathscr{E}$. The local trace function satisfies

$$|\mathfrak{t}^{\mathscr{E}}(x)| \leqq \sum_m |\langle \sigma_m(x), \mathscr{E}^* \sigma_m(x) \rangle|.$$

Let $\mathscr{E}' = \chi_{\Omega'} \mathscr{E} \chi_{\Omega'}$ denote the restriction of $\mathscr{E}$ to Ω', and note that $||\mathscr{E}'|| \leqq ||\mathscr{E}||$. Then

$$\begin{aligned}
\int_{\Omega'} |\mathfrak{t}^{\mathscr{E}}| &\leqq \sum_{m=0}^{\infty} \int_{\Omega'} |\langle \mathscr{E} \chi_{\Omega'} \sigma_m, \sigma_m \rangle| \\
&\leqq \sum_{m=0}^{\infty} ||\mathscr{E}' \sigma_m||_{\Omega'} \, ||\sigma_m||_{\Omega'} \\
&\leqq ||\mathscr{E}|| \sum_{m=0}^{\infty} ||\sigma_m||^2_{\Omega'} \\
&\leqq ||\mathscr{E}|| \Big(\sum_{m=0}^{M} ||\sigma_m||^2_{\Omega'} + \int_{\Omega'} \mathfrak{t}^{\mathscr{H}(\Omega)} \Big) \\
&< \infty
\end{aligned}$$

by (4.27). This completes the proof of the Lemma.

We now construct local parametrices Q_j for D_j^+ by cutting off the Green operators in a small neighborhood of the diagonal. That is, we choose $\psi_j \in C^\infty(X_j \times X_j)$ with support in a small neighborhood of the diagonal so that $0 \leqq \psi_j \leqq 1$ and so that $\psi_j \equiv 1$ near the diagonal. Let Q_j be the operator whose Schwartzian kernel is

$$K^{Q_j}(x,y) = \psi_j(x,y) \, K^{G_j}(x,y)$$

on $X_j \times X_j$ (for $j = 0, 1$). The operator Q_j is a semi-local parametrix for D_j^+ with

$$D_j^+ Q_j = 1 - \mathscr{S}_j^- \quad \text{and} \quad Q_j D_j^+ = 1 - \mathscr{S}_j^+$$

where the semi-local smoothing operators satisfy

$$(4.29) \qquad \mathfrak{i}^{\mathscr{S}_j^+} \equiv \mathfrak{i}^{\mathscr{H}_j^+} \quad \text{and} \quad \mathfrak{i}^{\mathscr{S}_j^-} \equiv \mathfrak{i}^{\mathscr{H}_j^-}$$

since Q_j and H_j agree near the diagonal.

Recall from Proposition 4.6 that to compute the topological index we need to use semi-local parametrices which agree in some neighborhood of ∞. We do this by splicing Q_0 onto Q_1 in Ω as follows.

Let $f_m : X_1 \to [0, 1]$ be a sequence of functions defined as in (1.18), and suppose always that if, m is sufficiently large, $f_m \equiv 1$ on K_1. Define a sequence of semi-local parametrices $Q_{1,m}$ for D_1^+ by setting

$$Q_{1,m} = f_m Q_1 + (1 - f_m)\, Q_0.$$

From Proposition 1.24 we know that the associated smoothing operators $\mathscr{S}_{1,m}^-$ and $\mathscr{S}_{1,m}^+$, where $D_1^+ Q_{1,m} = 1 - \mathscr{S}_{1,m}^-$ and $Q_{m,1} D_1^+ = 1 - \mathscr{S}_{1,m}^+$ are given by

$$(4.30) \qquad \begin{cases} \mathscr{S}_{1,m}^- = f_m \mathscr{S}_1^- + (1 - f_m)\, \mathscr{S}_0^- + (\nabla f_m)(Q_1 - Q_0) \\ \mathscr{S}_{1,m}^+ = f_m \mathscr{S}_1^+ + (1 - f_m)\, \mathscr{S}_0^+. \end{cases}$$

We can now apply (4.6), (4.29) and (4.30) to conclude that

$$(4.31) \qquad \begin{aligned} \operatorname{ind}_t(D_1^+, D_0^+) &= \int_{X_1} (\mathfrak{i}^{\mathscr{S}_{1,m}^+} - \mathfrak{i}^{\mathscr{S}_{1,m}^-}) - \int_{X_0} (\mathfrak{i}^{\mathscr{S}_0^+} - \mathfrak{i}^{\mathscr{S}_0^-}) \\ &= \int_{X_1} f_m(\mathfrak{i}^{\mathscr{S}_1^+} - \mathfrak{i}^{\mathscr{S}_1^-}) - \int_{X_0} f_m(\mathfrak{i}^{\mathscr{S}_0^+} - \mathfrak{i}^{\mathscr{S}_0^-}) - \int_\Omega \mathfrak{i}^{\nabla f_m \cdot (Q_1 - Q_0)} \\ &= \int_{X_1} f_m(\mathfrak{i}^{\mathscr{H}_1^+} - \mathfrak{i}^{\mathscr{H}_1^-}) - \int_{X_0} f_m(\mathfrak{i}^{\mathscr{H}_0^+} - \mathfrak{i}^{\mathscr{H}_0^-}) - \int_\Omega \mathfrak{i}^{\nabla f_m \cdot (Q_1 - Q_0)}. \end{aligned}$$

(Here we consider f_m to be also defined on X_0 by letting $f_m \equiv 1$ on K_0.) Now the Schwartz kernel for the operator $\mathscr{E}_m \equiv \nabla f_m \cdot (Q_1 - Q_0)$ is

$$K^{\mathscr{E}_m}(x, y) = (\nabla f_m)_x \cdot K^{Q_1 - Q_0}(x, y).$$

Since $Q_j = G_j = G_j'$ near the diagonal we see that

$$|\mathfrak{i}^{\mathscr{E}_m}(x)| \leq \|\nabla f_m(x)\|\, |\mathfrak{i}^{\mathscr{E}}(x)|$$

where as above $\mathscr{E} = \hat{G}_1 - \hat{G}_0$. We now choose $\Omega' \subset \Omega$ as above and observe that for all m sufficiently large (cf. (1.19) and (1.20)), we have

$$\operatorname{supp}(\nabla f_m) \subset \Omega' \quad \text{and} \quad \|\nabla f_m\| \leq \frac{2}{m}.$$

It follows that

$$\int_\Omega |\mathfrak{i}^{\nabla f_m \cdot (Q_1 - Q_0)}| \leq \frac{2}{m} \int_{\Omega'} |\mathfrak{i}^{\mathscr{E}}| \xrightarrow[m \to \infty]{} 0$$

by Lemma 4.28. Consequently, from (4.31) and (4.19) we see that when $m \to \infty$, we have

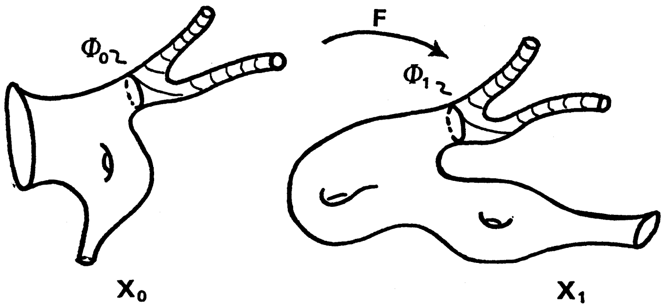
$$\begin{aligned} \operatorname{ind}_t(D_1^+, D_0^+) &= \int_{X_1} (\mathfrak{i}^{\mathscr{H}_1^+} - \mathfrak{i}^{\mathscr{H}_1^-}) - \int_{X_0} (\mathfrak{i}^{\mathscr{H}_0^+} - \mathfrak{i}^{\mathscr{H}_0^-}) \\ &= \operatorname{index}(D_1^+) - \operatorname{index}(D_0^+). \end{aligned}$$

This completes the proof of the index theorem. ■

The argument presented above has a number of useful generalizations. One such generalization allows us to compare operators which agree only on *some* of the ends of the manifolds where they are defined.

Suppose that D_0 and D_1 are generalized Dirac operators on complete riemannian manifolds X_0 and X_1, and suppose that both operators are strictly positive at ∞, *i.e.* both satisfy Assumption 3.1. Let $K_j \subset X_j$ be a compact subset, and suppose that $\Phi_j \subseteq X_j \backslash K_j$ is a union of connected components of $X_j - K_j$ for $j = 0, 1$. If there exists an isometry $F : \Phi_0 \to \Phi_1$ which is covered by a bundle isometry $\widetilde{F} : S_0\big|_{\Phi_0} \to S_1\big|_{\Phi_1}$ such that $D_1 = \widetilde{F} \circ D_0 \circ \widetilde{F}^{-1}$, we say that *the operators* D_0 *and* D_1 *agree on the family of ends*

$$\Phi \equiv \Phi_0 \cong \Phi_1$$



(Of course if $\Phi_j = X_j - K_j$ for $j = 0, 1$, then the operators agree at infinity (cf. 4.1).)

We can define a relative index in this situation by proceeding very much as we did above. We chop off X_0 and X_1 along a compact hypersurface $H \subset \Phi$ which separates off the infinite part of Φ. (To be specific, there should be a constant c so that every point of $\Phi - H$ is a distance $\leq c$ from K_0 or cannot be connected to K_0 by a path in $X_0 - H$.) We now deform the metric and the operator in a small neighborhood of H in Φ so that the metric becomes a product metric on $H \times (-\varepsilon, \varepsilon)$ and so that the operator $D_0 \cong D_1$ is "constant" along the lines $\{x\} \times (-\varepsilon, \varepsilon)$. (This is not hard to do.) The isometry $(x, t) \to (x, -t)$ is then covered by an operator equivalence.

Let X_0' and X_1' denote the chopped-off manifolds, and attach X_0' to X_1' along H to give a complete riemannian manifold

$$X \equiv X_1' \cup_H X_0'.$$

From the above, the operators D_0 and D_1 naturally join to give a generalized Dirac operator D on X which is strictly positive at infinity. We now assume that X_0 and X_1 are even-dimensional and oriented, and that the isometry $F : \Phi_0 \to \Phi_1$ is orientation-preserving. The manifold X is given the orientation which agrees with that of X_1 (and disagrees with that of X_0). We then define the Φ-*relative index of* D_0^+ *and* D_1^+ to be

$$\text{ind}(D_1^+, D_0^+ \mid \Phi) \equiv \text{index}(D^+). \tag{4.32}$$

Since the index of D^+ is invariant under deformations of D^+ on compact subsets, the above definition is independent of the details of our construction near H. Furthermore, we have the following.

PROPOSITION **4.33**. — *The* Φ*-relative index of* D_0 *and* D_1 *is independent of the choice of the hypersurface* H.

PROOF. — Let Φ' denote the bounded component of $\Omega - H$ and note that we have the double $\mathbf{D}(\Phi') \equiv \Phi' \cup (-\Phi')$ contained in X. The "flip" isometry σ of $\mathbf{D}(\Phi')$ is covered by a bundle isometry that carries D^+ to D^- = the adjoint of D^+. Thus we can construct a local parametrix $\mathcal{Q}$ for D^+ with smoothing operators $\mathcal{S}^+$ and $\mathcal{S}^-$ so that for any given σ-invariant compact domain $\mathcal{D} \subset \text{interior}(\mathbf{D}(\Phi'))$, we have

$$\int_{\mathcal{D}} (\mathfrak{k}^{\mathcal{S}^+} - \mathfrak{k}^{\mathcal{S}^-}) = 0.$$

Suppose we had chosen two different hypersurfaces H and $\widetilde{H}$ in Φ with resulting manifolds X and $\widetilde{X}$. We can choose $\mathcal{D} \subset X$ and $\widetilde{\mathcal{D}} \subset \widetilde{X}$ as above so that $X - \mathcal{D} \cong \widetilde{X} - \widetilde{\mathcal{D}}$. Since the parametrices $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ can be constructed locally out of the operators, we can assume that $\mathfrak{k}^{\mathcal{S}^+} = \mathfrak{k}^{\widetilde{\mathcal{S}}^+}$ and $\mathfrak{k}^{\mathcal{S}^-} = \mathfrak{k}^{\widetilde{\mathcal{S}}^-}$ in $X - \mathcal{D} \cong \widetilde{X} - \widetilde{\mathcal{D}}$. It is now clear that the topological index $\text{ind}_t(\widetilde{D}^+, D^+) = 0$. (See Proposition 4.6.) Hence, by the Relative Index Theorem, we have $\text{ind}(\widetilde{D}^+) = \text{ind}(D^+)$ as desired. ∎

A similar application of the Relative Index Theorem proves the following.

PROPOSITION **4.34**. — *Let* $H \subset \Phi$ *be a compact hypersurface as above, and assume that* $H = \partial Y$ *where* Y *is a compact oriented manifold. Let* $\hat{X}_j = X'_j \cup_H Y$ *be the manifold obtained by attaching* Y *to the chopped-off manifold* X'_j *along* H. *Let* $\hat{D}_0$ *and* $\hat{D}_1$ *represent simultaneous extensions of* D_0 *and* D_1 *over the compact piece* Y. *Then*

$$\text{ind}(D_1^+, D_0^+ \mid \Phi) = \text{index}(\hat{D}_1^+) - \text{index}(\hat{D}_0^+).$$

We now come to our next major result. It is, of course, a strict generalization of the Relative Index Theorem.

THEOREM **4.35** (The Φ-relative index theorem).

$$\text{ind}(D_1^+, D_0^+ \mid \Phi) = \text{index}(D_1^+) - \text{index}(D_0^+).$$

PROOF. — Let $X = X_1 \cup_H X_0$ be a manifold used to define the Φ-relative index. Let Q_0, Q_1 and Q denote semi-local parametrices for D_0^+, D_1^+ and D^+, obtained by cutting off the Schwartzian kernels of the respective Green operators in a small neighborhood of the diagonal. Let $\mathcal{S}_0^\pm$, $\mathcal{S}_1^\pm$ and $\mathcal{S}^\pm$ be the associated smoothing operators (as, for example, in 4.6). Then the Φ-relative index of D_0^+ and D_1^+ is given by

$$\text{index}(D^+) = \int_X (\mathfrak{k}^{\mathcal{S}^+} - \mathfrak{k}^{\mathcal{S}^-}). \tag{4.36}$$

Furthermore, standard arguments show that this formula continues to hold if we modify Q on a compact subset (*i.e.* if we replace Q by another semi-local parametrix for D^+ which agrees with Q outside a compact subset).

The arguments used to prove the Relative Index Theorem apply directly to prove the following:

$$(4.37) \qquad \mathrm{index}(D_1^+) - \mathrm{index}(D_0^+) = \int_{X_1} (\mathfrak{k}^{\tilde{\mathscr{S}}_1^+} - \mathfrak{k}^{\tilde{\mathscr{S}}_1^-}) - \int_{X_0} (\mathfrak{k}^{\tilde{\mathscr{S}}_0^+} - \mathfrak{k}^{\tilde{\mathscr{S}}_0^-})$$

where $\tilde{\mathscr{S}}_j^{\pm}$ are the smoothing operators associated to semi-local parametrices $\tilde{Q}_j$ obtained from Q_j by splicing on the same semi-local parametrix in a neighborhood of ∞ in Φ. (Consequently $\mathfrak{k}^{\tilde{\mathscr{S}}_0^{\pm}} = \mathfrak{k}^{\tilde{\mathscr{S}}_1^{\pm}}$ near infinity in Φ and the integral is well defined.) We may assume that $\tilde{Q}_0$ and $\tilde{Q}_1$ agree in a neighborhood containing H and all points beyond (*i.e.* in all unbounded components of $\Phi - H$). Arguments similar to those presented in 4.6 now show that the integrals in (4.37) can be replaced by

$$(4.38) \qquad \mathrm{index}(D_1^+) - \mathrm{index}(D_0^+) = \int_X (\mathfrak{k}^{\tilde{\mathscr{S}}^+} - \mathfrak{k}^{\tilde{\mathscr{S}}^-})$$

where $\tilde{\mathscr{S}}^{\pm}$ are the smoothing operators associated to a semi-local parametrix $\tilde{Q}$ which agrees with $\tilde{Q}_j (= Q_j)$ near infinity in the X_j' piece of X for $j = 0$ and 1.

It remains to show that the integrals in (4.36) and (4.38) coincide. To see this, choose a family of functions $(f_m)_{m=1}^{\infty}$ as in (1.18) and set

$$Q_m = f_m \tilde{Q} + (1 - f_m) Q.$$

Let

$$\begin{cases} \mathscr{S}_m^- = f_m \tilde{\mathscr{S}}^- + (1 - f_m)\mathscr{S}^- + (\nabla f_m)(\tilde{Q} - Q) \\ \mathscr{S}_m^+ = f_m \tilde{\mathscr{S}}^+ + (1 - f_m)\mathscr{S}^+ \end{cases}$$

denote the associated smoothing operators. Since Q_m differs from Q only on a compact subset, we have that

$$(4.39) \qquad \begin{aligned} \mathrm{index}(D^+) &= \int_X (\mathfrak{k}^{\mathscr{S}_m^+} - \mathfrak{k}^{\mathscr{S}_m^-}) \\ &= \int_X f_m(\mathfrak{k}^{\tilde{\mathscr{S}}^+} - \mathfrak{k}^{\tilde{\mathscr{S}}^-}) + \int_X (1 - f_m)(\mathfrak{k}^{\tilde{\mathscr{S}}^+} - \mathfrak{k}^{\tilde{\mathscr{S}}^-}) - \int_X \mathfrak{k}^{\nabla f_m \cdot (Q_1 - Q_0)} \end{aligned}$$

For exactly the same reasons as above (cf. 4.30 forward), the third integral in (4.39) goes to zero as $m \to \infty$. Consequently, in the limit we have

$$\mathrm{index}(D^+) = \int_X (\mathfrak{k}^{\tilde{\mathscr{S}}^+} - \mathfrak{k}^{\tilde{\mathscr{S}}^-})$$

which, when combined with (4.38), completes the proof. ∎

We now make some immediate applications of Theorem 4.35 to the invariants defined at the end of § 3. Let g_0, g_1 and g_2 be metrics of positive scalar curvature on a compact, odd-dimensional spin manifold X, and let $i(g_i, g_j)$ be the index defined in (3.13). It is obvious from the definition that

$$(4.40) \qquad i(g_i, g_j) = -\, i(g_j, g_i).$$

Furthermore, we have the following.

THEOREM 4.41.

$$i(g_0, g_1) + i(g_1, g_2) + i(g_2, g_0) = 0.$$

PROOF. — Let $X_0 = X \times \mathbf{R}$ have metric $g_0 + dt^2$ for $t \leqq 0$ and $g_1 + dt^2$ for $t \geqq 1$. Let $X_1 = X \times \mathbf{R}$ have metric $g_2 + dt^2$ for $t \leqq 0$ and $g_1 + dt^2$ for $t \geqq 1$. Apply the Φ-relative Index Theorem to the end $\Phi \equiv X \times (1, \infty)$.

COROLLARY 4.42. — *The index $i(g_0, g_1)$ depends only on the homotopy classes of the metrics g_0 and g_1 in the space $\mathscr{P}(X)$ of positive scalar curvature metrics on X.*

PROOF. — This corollary is a direct consequence of Theorems 4.41 and 2.11 and the following proposition proved in [GL_2, Lemma 3].

PROPOSITION 4.43. — *Suppose g_0 and g_1 can be joined by a family g_t, $0 \leq t \leq 1$, of metrics of positive scalar curvature on X. Then there exists a metric g on $X \times \mathbf{R}$ of positive scalar curvature which agrees with $g_0 + dt^2$ for $t \ll 0$ and with $g_1 + dt^2$ for $t \gg 0$. In particular, $i(g_0, g_1) = 0$.*

Note that $i(g_0, g_1) = 0$ not only if g_0 and g_1 are homotopic in $\mathscr{P}(X)$, but even if they are " concordant " by a metric of positive scalar curvature.

Suppose now that X bounds a compact spin manifold $\widetilde{X}$, and let $i(g, \widetilde{X})$ be the index defined in (3.14). Then arguing as above proves the following.

THEOREM 4.44.

$$i(g_0, \widetilde{X}) + i(g_0, g_1) = i(g_1, \widetilde{X}).$$

COROLLARY 4.45. — *The index $i(g, \widetilde{X})$ depends only on the homotopy class of g in $\mathscr{P}(X)$.*

A thorough study of these related invariants will be made in another paper. However, before leaving the subject here we should at least demonstrate that these indices are non-trivial. To do this we consider the 4-dimensional real vector bundles over S^4. Each such bundle E will be given a riemannian inner product $\langle \cdot, \cdot \rangle$, and we will denote by X_E and $\Sigma_E = \partial X_E$ the unit disk bundle and the unit sphere bundles of E respectively.

Using the standard metric on the base, an orthogonal connection in E, and a rotationally invariant metric g_0 in the fibres, we construct a riemannian metric on the total space of E in the standard way (cf. [LY] or [GL_2]). The fibres will be totally geodesic. The metric g_0 we choose on them is the " torpedo " metric: a smoothing of the S^4 hemispherical metric attached along the equator to the cylindrical metric on $S^3 \times \mathbf{R}$. We can multiply g_0 by a small constant $\varepsilon > 0$ to guarantee that the scalar curvature (on the total space of E) is everywhere positive.

Observe that outside the tubular neighborhood X_E of the zero section, this metric is a riemannian product $g_E + dt^2$ on $E - X_E \cong \partial X_E \times \mathbf{R} = \Sigma_E \times \mathbf{R}$. The metric g_E

on $\Sigma_E = \partial X_E$ has positive scalar curvature, and since it extends over X_E with positive scalar curvature (so that it is a product metric near the boundary), we have that

$$(4.46) \qquad i(g_E, X_E) = 0.$$

Recall now that oriented 4-plane bundles over S^4 are classified by two integer invariants: the Euler number χ and the Pontrjagin number p_1. It is an elementary calculation to show that Σ_E is a homotopy sphere if and only if $\chi_E = \pm 1$. It is a classical result of Milnor [M_1] that Σ_E is diffeomorphic to the standard S^7 if $p_1^2 \equiv 4 \pmod{896}$. Thus we have constructed above a sequence $\{g_k\}_{k=0}^{\infty}$ of metrics of positive scalar curvature on S^7. The metric g_k was constructed using E_k with $\chi(E_k) = 1$ and $p_1(E_k)^2 = 4 + 896k$.

Let $g_{can}(\sim g_0)$ denote the canonical constant curvature metric on S^7, and set $i_k = i(g_k, g_{can})$. Then by Theorem 4.44 and 4.46 above we see that

$$i(X_k, g_{can}) = i(X_k, g_k) + i_k = i_k$$

where $X_k \equiv X_{E_k}$. On the other hand, by the Relative Index Theorem, we see that $i(X_k, g_{can}) = \hat{A}(\hat{X}_k)$ where $\hat{X}_k = X_k \cup_{S^7} D^8$ is the compact 8-manifold obtained by attaching and 8-disk along the boundary S^7. Thus, for all k we have that

$$i_k = \hat{A}(\hat{X}_k).$$

However, following Milnor [M_2, page 57] we compute that $\hat{A}(\hat{X}_k) = (p_1^2 - 4)/896 = k$. This gives the following result.

THEOREM 4.47. — *For metrics g of positive scalar curvature on* S^7, *the invariant* $i(g, g_{can})$ *takes on infinitely many distinct integer values. In particular, the space of positive scalar curvature metrics on* S^7 *has infinitely many connected components.*

We should note the second part of Theorem 4.47 can be proved using only the elementary constructions above, that is, without using the Relative Index Theorem.

We now observe that the index $i(g, g')$ behaves nicely with respect to the group of diffeomorphisms. Fix any metric g on the compact spin manifold X, and for each element $F \in \mathrm{Diff}^\infty(X)$, set

$$i_g(F) \equiv i(g, F^* g).$$

THEOREM 4.48. — *For each positive-scalar-curvature metric g on* X, *the map*

$$i_g : \mathrm{Diff}^\infty(X) \to \mathbf{Z}$$

is a group homomorphism.

PROOF. — Choose $F_1, F_2 \in \mathrm{Diff}^\infty(X)$. Then by Theorem 4.41 we have that

$$\begin{aligned} i_g(F_2 \circ F_1) &= i(g, F_1^* F_2^* g) \\ &= i(g, F_1^* g) + i(F_1^* g, F_1^* F_2^* g) \\ &= i_g(F_1) + i_g(F_2) \end{aligned}$$

where we have used the obvious fact that for any diffeomorphism F,

$$i(F^* g_0, F^* g_1) = i(g_0, g_1). \blacksquare$$

Since i_g is constant on connected components of $\mathrm{Diff}^\infty(X)$, we conclude the following.

COROLLARY 4.49. — *Each g of positive scalar curvature gives a homomorphism*

$$i_g : \Gamma(X) \to \mathbf{Z}$$

where $\Gamma(X) = \mathrm{Diff}^\infty(X)/\mathrm{Diff}_0^\infty(X)$ *is the component group of the group of diffeomorphisms of* X.

If $\Gamma(X)$ is finite, then i_g must be zero for all g. Consequently the invariant $i(g, g_{can})$ on S^{4k+3} is always a function on the space of diffeomorphism-equivalent metrics.

Note that Theorem 4.47 applies equally well to exotic Milnor 7-spheres. (Let g_{can} be any metric with $\kappa > 0$.) The construction above can be greatly generalized using Browder-Novikov Theory. Similar construction detecting higher homotopy groups of the space of positive scalar curvature metrics can also be made. These results are related to work of N. Hitchin [H].

5. HYPERSPHERICAL AND ENLARGEABLE MANIFOLDS

The results of the previous section were specifically developed to study certain large and interesting classes of manifolds. For the remainder of this paper we will be concerned with defining these classes, examining their properties, and applying the index theorem to establish the non-existence of complete, positive scalar curvature metrics.

One of our fundamental concepts is the following. Let S^n denote the euclidean n-sphere of curvature 1, and assume from this point on that all manifolds are connected.

Definition 5.1. — A complete oriented (connected) riemannian n-manifold X is said to be *ε-hyperspherical* if there exists an ε-contracting map $f: X \to S^n$ which is constant outside a compact set and is of non-zero degree. The manifold is simply called *hyperspherical* if it is ε-hyperspherical for all $\varepsilon > 0$.

Note. — A map f between riemannian manifolds is said to be *ε-contracting* if $||f_* V|| \leqq \varepsilon ||V||$ for all tangent vectors V. If f is constant at ∞, the notion of " degree " is the usual one, e.g.

$$\deg(f) = \sum_{p \in f^{-1}(q)} \operatorname{sign}(\det f_*)_p \tag{5.2}$$

where q is a regular value of f.

Note that the euclidean space is hyperspherical. To see this, choose a map

$$F : \mathbf{R}^n \to S^n \tag{5.3}$$

of degree 1 which sends everything outside the unit ball to the " south pole " of S^n. This map will have bounded dilation, that is, it will satisfy $||F_* V|| \leqq c\, ||V||$ for some $c > 0$. Then for each positive ε, the map $F_\varepsilon(x) \equiv F((\varepsilon/c)\, x)$ will be ε-contracting.

Similarly, *any complete, simply-connected manifold* X *with sectional curvatures* $\leqq 0$ is *hyperspherical*. To see this recall the standard fact that at any point $p \in X$, the map $\exp_p^{-1} : X \to T_p X$ is 1-contracting. Hence, $F_\varepsilon \circ \exp_p^{-1}$ is ε-contracting for all $\varepsilon > 0$.

Note that *products of hyperspherical manifolds are hyperspherical.* To see this, fix a degree 1 map $S^n \times S^m \to S^{n+m}$.

The property of being hyperspherical depends only on the quasi-isometry class of X. That is, if we change the metric by a uniformly bounded amount, the manifold remains hyperspherical. More generally, we have the following. We say that a C^1 map between riemannian manifolds is *bounded* if it is c-contracting for some (possibly large) $c > 0$. A proper map between oriented n-manifolds has a well-defined degree given by formula (5.2) for any regular value q. (That this definition is independent of the choice of q follows as in [M_3].)

PROPOSITION 5.4. — *Let* X *and* Y *be complete oriented riemannian n-manifolds, and suppose there exists a bounded proper map of non-zero degree from* X *to* Y. *Then if* Y *is hyperspherical,* X *is also hyperspherical.*

PROOF. — Suppose $g: X \to Y$ is c-contracting and proper, and that $f: Y \to S^n$ is ε-contracting and constant at ∞. Then $f \circ g: X \to S^n$ is εc-contracting, constant at ∞, and of degree $= \deg(f).\deg(g)$. ■

Thus, given a hyperspherical manifold X, we have tremendous freedom to " enlarge " the manifold while preserving this property. For example let $\{D^n_\alpha\}_{\alpha \in A}$ be a family of disjointly embedded disks in X with the property that there is a bounded proper map $f: X \to X$ of degree 1 such that $f(D^n_\alpha) = \{x_\alpha\}$ for each α. (That is, each disk is shrunk to a point.) Then we may alter the manifold X any way we please on the union: $\bigcup_\alpha D^n_\alpha$. In particular, we can take connected sum in each D^n_α with a compact connected manifold M^n_α, and extend the metric arbitrarily. The resulting manifold X is again hyperspherical since f determines a bounded proper map $\hat{f}: \hat{X} \to X$, of degree 1.

Suppose now that X is a compact manifold and $\widetilde{X} \to X$ is some covering manifold of $\widetilde{X}$. Then each riemannian metric on X can be lifted to $\widetilde{X}$, and the quasi-isometry class of this lifted metric is clearly independent of the choice of metric on X.

We now introduce a notion which strictly generalizes the one given in [GL_1].

DEFINITION 5.5. — A compact riemannian manifold X is said to be *enlargeable* if for each $\varepsilon > 0$, there exists an oriented covering manifold $\widetilde{X} \to X$ which is spin and ε-hyperspherical.

Clearly if some covering space of X is spin and hyperspherical, then X is enlargeable. Thus by considering the universal covering, we see that any compact manifold of non-positive sectional curvature is enlargeable.

PROPOSITION 5.6. — *Enlargeability is an invariant of the homotopy type of a manifold. Products of enlargeable manifolds are enlargeable. The connected sum of any spin manifold with an enlargeable manifold is again enlargeable.*

PROOF. — We begin with the following. Recall that a map $F: X \to Y$ between manifolds is said to be *spin* if either X is spin or $F^*w_2(Y) = w_2(X)$, where w_2 denotes the second Stiefel-Whitney class.

PROPOSITION 5.7. — *Let* X *and* Y *be compact oriented riemannian n-manifolds, and suppose there exists a spin map* $F: X \to Y$ *of non-zero degree. Then if* Y *is enlargeable,* X *is also enlargeable.*

PROOF. — The map F is bounded, *i.e.* c-contracting for some c. Furthermore, suppose $\widetilde{Y} \to Y$ is a covering space, and let $\widetilde{X} \to X$ be the covering corresponding to the subgroup $F_*^{-1}(\pi_1 \widetilde{Y})$. Then F lifts to a *proper* map $\widetilde{F}: \widetilde{X} \to \widetilde{Y}$ which is of course, still c-contracting. Thus if $\widetilde{Y}$ admits an ε-contracting map $f: \widetilde{Y} \to S^n$ which is constant at ∞ and of non-zero degree, then $f \circ \widetilde{F}$ is εc-contracting, constant at ∞ and also of non-zero degree. ∎

Suppose now that $F: X \to X'$ is a homotopy equivalence of compact n-manifolds. Then (after possibly passing to the orientable 2-fold covering manifolds), F is of degree 1. This proves the first statement of 5.6. The remainder of the Proposition is straightforward. ∎

We recall that it was proved in [GL_1] that compact solvmanifolds, and sufficiently large compact 3-manifolds with residually finite fundamental group, are enlargeable.

We now come to the principal result of this section.

THEOREM 5.8. — *A (compact) enlargeable manifold cannot carry a riemannian metric of positive scalar curvature. In fact, any metric with non-negative scalar curvature on an enlargeable manifold must be flat.*

This has the following immediate consequence (announced in [GL_1]). Recall from above that any compact manifold of non-positive curvature is enlargeable.

COROLLARY 5.9. — *Any compact manifold* X *which carries a metric with sectional curvatures* $\leqq 0$, *cannot carry a metric with scalar curvature* $\kappa > 0$. *Moreover, any metric with* $\kappa \geqq 0$ *on* X *is flat.*

Recall from Proposition 5.6 that if X is enlargeable and Y is spin, then the connected sum $X \# Y$ is again enlargeable. Thus, it follows also that no compact manifold of the form $X \# Y$, where Y is spin and X admits non-positive sectional curvature, can carry a metric with positive scalar curvature.

PROOF OF THEOREM 5.8. — Suppose X is enlargeable and carries a metric with scalar curvature $\kappa > 0$. As shown in [GL_1], if X is not flat then by results of J. P. Bourguignon [KW] and Cheeger-Gromoll [CG] we can modify the metric so that

$$\kappa \geqq \kappa_0 > 0 \tag{5.10}$$

where κ_0 is a constant. We may also assume that the dimension of X is even, since if it is not, we can replace X by $X \times S^1$ (with the product metric). Note that S^1 is enlargeable and therefore by 5.6, so is $X \times S^1$.

Let $2m = \dim(X)$, and choose a complex vector bundle E_0 over S^{2m} such that $c_m(E_0) \neq 0$. This is always possible since, under the Chern character, the groups $\widetilde{K}(S^{2m})$ and $\widetilde{H}(S^{2m}; \mathbf{Z})$ are isomorphic (cf. [AH]). Fix a unitary connection ∇^0 in E_0 and denote the corresponding curvature tensor by R^0.

The proof now proceeds very much like the arguments given in [GL$_1$, § 3]. Roughly, the idea is as follows. Choose a covering space $\widetilde{X}$ with an ε-contracting map $f: \widetilde{X} \to S^{2m}$ and set $E = f^*E_0$. If ε is sufficiently small, the curvature of E becomes uniformly small, and the Dirac operator on spinors with coefficients in E becomes strictly positive by (5.10). Comparing this operator with the untwisted Dirac operator and applying the Relative Index Theorem leads to a contradiction. This is because the relative topological index of these operators is, by construction, not zero, whereas the relative analytic index, by vanishing theorems, must be zero. The details are as follows.

Let $\varepsilon > 0$ be given. Since X is enlargeable, there exists a covering space $\widetilde{X} \to X$ and (with the metric lifted from X) an ε-contracting map $f: \widetilde{X} \to S^{2m}$ which is constant outside a compact subset $K \subset \widetilde{X}$ and is of non-zero degree. Let $E = f^*E_0$ be the pull-back of E_0 by f and let $\nabla = f^*\nabla^0$ be the induced connection. For tangent vectors V, $W \in T_x\widetilde{X}$, the curvature of ∇ is given by

$$(5.11) \qquad R_{V,W} = R^0_{f_*V, f_*W}$$

where $R^0_{\cdot,\cdot}$ is considered as an endomorphism of $E_x \equiv (E_0)_{f(x)}$. It follows immediately that

$$(5.12) \qquad R \equiv 0 \quad \text{in } \widetilde{X} - K.$$

Furthermore, if $e_1, \ldots, e_{2m}$ is any orthonormal basis of $T_x\widetilde{X}$, then

$$(5.13) \qquad \begin{aligned} \|R\|_x^2 = \sum_{i<j} \|R_{e_i, e_j}\|^2 &= \sum_{i<j} \|R^0_{f_*e_i, f_*e_j}\|^2 \\ &\leqq \|f_*(e_i \wedge e_j)\|^2 \, \|R^0\|^2 \end{aligned}$$

where $\|R^0\|^2 = \sup\{\|R^0\|_p^2 : p \in S^{2m}\}$. In particular, since f is ε-contracting, we have

$$(5.14) \qquad \|R\|_x \leqq \varepsilon^2 \|R^0\|$$

for all $x \in \widetilde{X}$.

Let $\mathbf{S}$ be the canonical complex spin bundle over $\widetilde{X}$ and let

$$D_1 : \Gamma(\mathbf{S} \otimes E) \to \Gamma(\mathbf{S} \otimes E)$$

be the generalized Dirac operator constructed as in 1.8 and 1.9. Let $\mathscr{R}^{\mathbf{S} \otimes E}$ be the term appearing in the generalized Bochner-Lichnerowicz-Weitzenböck formula 2.6 for D_1. Then, as noted in (2.15)-(2.17), we can write

$$(5.15) \qquad \mathscr{R}^{\mathbf{S} \otimes E} = \frac{1}{4}\kappa + \mathscr{R}^E$$

where $\mathscr{R}^E$ satisfies the pointwise inequality

$$\|\mathscr{R}^E\|_x \leqq k_{2m} \|R\|_x$$

for a constant k_{2m} depending only on m. It therefore follows directly from (5.12) and (5.14) that

$$\text{(5.16)} \qquad \begin{cases} \mathscr{R}^{\mathrm{E}} \equiv 0 & \text{in } \widetilde{\mathrm{X}} - \mathrm{K} \\ \|\mathscr{R}^{\mathrm{E}}\| \leqq \varepsilon^2 k_{2m} \|\mathrm{R}^0\| & \text{in } \mathrm{K}. \end{cases}$$

Suppose now that ε^2 is chosen smaller than $\kappa_0/k_{2m}\|\mathrm{R}^0\|$. Then from (5.16), (5.15) and from our assumption (5.10) on the scalar curvature, we see that $\mathscr{R}^{\mathbf{S}\otimes\mathrm{E}} > 0$ in $\widetilde{\mathrm{X}}$ and, in fact, $\mathscr{R}^{\mathbf{S}\otimes\mathrm{E}} \geqq \kappa_0/4$ in $\widetilde{\mathrm{X}} - \mathrm{K}$. It follows immediately from the vanishing theorem 2.11, that the operator $\mathrm{D}_1^+ : \Gamma(\mathbf{S}^+ \otimes \mathrm{E}) \to \Gamma(\mathbf{S}^- \otimes \mathrm{E})$ obtained by restriction of D_1 has

$$\text{(5.17)} \qquad \operatorname{ind}(\mathrm{D}_1^+) = 0.$$

Recall now that the map $f : \widetilde{\mathrm{X}} \to \mathrm{S}^{2m}$ is constant outside the compact set K. Choosing a basis for $\mathrm{T}_q\mathrm{S}^{2m}$, where $q = f(\widetilde{\mathrm{X}} - \mathrm{K})$, gives a trivialization

$$\mathrm{E}\Big|_{\widetilde{\mathrm{X}}-\mathrm{K}} \cong \mathbf{C}^k$$

which is, of course, compatible with the flat connection induced on E in $\widetilde{\mathrm{X}} - \mathrm{K}$. Hence, if we let $\mathrm{D}_0 : \Gamma(\mathbf{S} \otimes \mathbf{C}^k) \to \Gamma(\mathbf{S} \otimes \mathbf{C}^k)$ denote k-copies of the canonical Dirac operator, then we have that $\mathrm{D}_0^+ \cong \mathrm{D}_1^+$ in $\widetilde{\mathrm{X}} - \mathrm{K}$. This is exactly the case of Example 4.3.

From the classical Lichnerowicz formula (2.14) we have that $\mathscr{R}^{\mathbf{S}\otimes\mathbf{C}^k} \geqq \kappa_0/4$ on $\widetilde{\mathrm{X}}$, and so

$$\operatorname{ind}(\mathrm{D}_0^+) = 0.$$

Combined with (5.17) and the Relative Index Theorem 4.18, this implies that

$$\text{(5.18)} \qquad \operatorname{ind}_t(\mathrm{D}_1^+, \mathrm{D}_0^+) = \operatorname{ind}(\mathrm{D}_1^+) - \operatorname{ind}(\mathrm{D}_0^+) = 0.$$

It was shown in 4.14 that this topological index can be computed in terms of the reduced Chern character:

$$\begin{aligned} \widehat{\mathrm{ch}}\, \mathrm{E} &= \mathrm{ch}^1\mathrm{E} + \dots + \mathrm{ch}^m\mathrm{E} \\ &= f^*(\mathrm{ch}^1\mathrm{E}_0 + \dots + \mathrm{ch}^m\mathrm{E}_0) \\ &= \frac{1}{(m-1)!} f^* c_m(\mathrm{E}_0), \end{aligned}$$

where $c_{2m}(\mathrm{E}_0)$ is considered as a differential $2m$-form on S^{2m}. Thus, our conclusion (5.18) together with Proposition 4.14 imply that

$$\begin{aligned} 0 &= \int_{\widetilde{\mathrm{X}}} \widehat{\mathrm{ch}}\, \mathrm{E}.\hat{\mathbf{A}}(\widetilde{\mathrm{X}}) \\ &= \frac{1}{(m-1)!} \int_{\widetilde{\mathrm{X}}} f^* c_m(\mathrm{E}_0) \\ &= \frac{1}{(m-1)!} \deg(f) \int_{\mathrm{S}^{2m}} c_m(\mathrm{E}_0). \end{aligned}$$

This contradicts the assumptions that $c_m(E_0) \neq 0$ and $\deg(f) \neq 0$, and completes the proof. ∎

Note that if X is enlargeable and Y is spin with $\hat{A}(Y) \neq 0$, then neither X nor Y can carry a metric of positive scalar curvature, but it is still unclear whether the product $X \times Y$ can carry such a metric. The results above can be suitably generalized to cover this case. The generalization follows essentially from observing that the arguments already given actually prove much more than has thus far been claimed. We need only modify our definitions.

DEFINITION **5.19**. — A complete oriented connected riemannian manifold X is said to be *ε-hyperspherical in dimension n* (where $0 \leq n \leq \dim X$) if there exists an ε-contracting map $f: X \to S^n$ which is constant outside a compact set and is of non-zero $\hat{A}$-degree.

The *$\hat{A}$-degree* of f is defined, as in [GL_1], to be $\hat{A}(f^{-1}(q))$ where q is any regular value of f. Note that the oriented cobordism class of $f^{-1}(q)$ is independent of the choice of regular value q (cf. [M_3]).

A manifold which is ε-hyperspherical in dimension n for all $\varepsilon > 0$, is simply called *hyperspherical in dimension n.*

The property of being hyperspherical depends only on the quasi-isometry class of the metric and is induced by any bounded proper map of non-zero $\hat{A}$-degree (cf. Proposition 5.4).

In analogy with earlier discussion we make the following.

DEFINITION **5.20**. — A compact riemannian manifold is said to be *enlargeable in dimension n* if for each $\varepsilon > 0$, there exists an oriented covering manifold $\widetilde{X} \to X$ which is spin and ε-hyperspherical in dimension n.

This property is an invariant of the homotopy type of the manifold, and it is stable under products. That is, *if* X *is enlargeable in dimension n and* Y *is enlargeable in dimension m, then* $X \times Y$ *is enlargeable in dimension* $n + m$. A compact spin manifold Y with $\hat{A}(Y) \neq 0$ is enlargeable in dimension 0. Thus $Y \times X$, where X is an enlargeable n-manifold, is enlargeable in dimension n.

The main result is the following.

THEOREM **5.21**. — *A compact manifold* X *which is enlargeable in some dimension* $n \geqq 0$, *cannot carry a metric of positive scalar curvature. In fact, any metric with non-negative scalar curvature on* X *must be flat.*

COROLLARY **5.22**. — *Let* X *be a compact manifold which admits a spin map of non-zero* $\hat{A}$-*degree onto a manifold* X_0 *of non-negative sectional curvature. Then* X *carries no metric of positive scalar curvature.*

In particular, the product $X_0 \times Y$, where Y is any compact spin manifold with $\hat{A}(Y) \neq 0$, carries no metric of positive scalar curvature.

Of course, Corollary 5.22 remains true if X_0 is any enlargeable manifold, for example a solvmanifold, a sufficiently large 3-manifold, or any product of such manifolds.

Proof of Theorem 5.21. — We proceed exactly as in the proof Theorem 5.8. We assume $\kappa \geqq \kappa_0 > 0$ and that $n = 2m$ is even. For given $\varepsilon > 0$, we pass to the covering space $\widetilde{X}$ where there exists an ε-contracting map $f : X \to S^n$ of non-zero $\hat{A}$-degree which is constant outside a compact set $K \subset \widetilde{X}$. We fix a bundle E_0 with connection over S^n and consider the induced bundle $E = f^* E_0$ which is trivialized in $\widetilde{X} - K$. This gives us two generalized Dirac operators

$$D_0^+ : \Gamma(\mathbf{S}^+ \otimes \mathbf{C}^k) \to \Gamma(\mathbf{S}^- \otimes \mathbf{C}^k)$$

$$D_1^+ : \Gamma(\mathbf{S}^+ \otimes E) \to \Gamma(\mathbf{S}^- \otimes E)$$

which are positive at infinity and agree outside the compact set K. For ε sufficiently small, the index of both operators is zero. Hence, by Relative Index Theorem and Proposition 4.4 we have the following. Write $c_m(E_0) = c.\omega$ where $c \neq 0$ and ω denotes the normalized volume form of S^{2m}. Set $4k = \dim(X) - 2m$. Then

$$0 = \int_{\widetilde{X}} \hat{\mathrm{ch}}\, E . \hat{\mathbf{A}}(\widetilde{X}) = \int_{\widetilde{X}} \hat{\mathrm{ch}}(f^* E_0) . \hat{\mathbf{A}}(\widetilde{X})$$

$$= \int_{\widetilde{X}} f^* \left(\frac{1}{(m-1)!} c_m E_0 \right) . \hat{A}_k(\widetilde{X}) = \frac{c}{(m-1)!} \int_{\widetilde{X}} f^*(\omega) . \hat{A}_k(\widetilde{X})$$

$$= \frac{c}{(m-1)!} \int_{S^{2m}} \hat{A}(f^{-1}(p))\, \omega_p = \frac{c}{(m-1)} (\hat{A}\text{-deg})(f)$$

in contradiction to the fact that f had non-zero $\hat{A}$-degree. ■

6. MANIFOLDS WHICH ADMIT NO COMPLETE METRICS OF POSITIVE SCALAR CURVATURE

In this section we refine somewhat the arguments presented in the preceding section. Our main observation is that all arguments of § 5 remain valid if the notion of being ε-contracting on tangent vectors is replaced with the strictly weaker hypothesis of being ε-contracting on 2-forms. Together with a new construction, this will allow us to establish a large class of non-compact manifolds which admit no complete metrics with positive scalar curvature. In fact any manifold of the form $X \times \mathbf{R}$, where X is enlargeable, has this property. So does the manifold $T^n - \{\text{pt.}\}$.

It should perhaps be emphasized that here the scalar curvature is not assumed to be bounded away from zero. These manifolds carry no complete metrics with $\kappa > 0$ even if $\kappa \to 0$ rapidly at infinity. In fact, by recent results of J. Kazdan [K], any complete metric with $\kappa \geqq 0$ which is not Ricci flat, can be deformed to a complete metric with $\kappa > 0$. Consequently, by applying results of Cheeger-Gromoll, we can assert that for certain classes of non-compact manifolds, there exist no complete metrics with $\kappa \geqq 0$, and, in somewhat larger classes, any complete metric with $\kappa \geqq 0$ is flat.

We begin with the following basic notion. Recall that the *comass norm* of a 2-form φ on a riemannian manifold is given by setting $\|\varphi\| = \sup |\varphi(e_1 \wedge e_2)|$, where e_1 and e_2 range over all unit tangent vectors. This norm is, of course, uniformly equivalent to the standard pointwise norm on 2-forms.

Definition **6.1**. — A C^1-map $f: X \to Y$ between riemannian manifolds is said to be *ε-contracting on 2-forms*, or simply (ε, Λ^2)*-contracting*, if $\|f^*\varphi\| \leqq \varepsilon \|\varphi\|$ for all 2-forms φ on Y. This just means that for all $x \in X$, the map $f^*: \Lambda^2 T^*_{f(x)} Y \to \Lambda^2 T^*_x X$ is ε-contracting in the comass norm.

Note that an ε- (or " (ε, Λ^1)- ") contracting map is always (ε, Λ^2)-contracting (for $\varepsilon \leqq 1$). In fact, if $f: X \to Y$ is ε-contracting and if φ is any 2-form on Y, then

$$\textbf{(6.2)} \qquad |(f^*\varphi)(e_1 \wedge e_2)| = |\varphi(f_* e_1 \wedge f_* e_2| \leqq \|\varphi\| \, \|f_* e_1\| \, \|f_* e_2\| \leqq \varepsilon \|\varphi\|$$

and so f is ε^2-contracting on 2-forms. This suggests a useful class of mappings which are (ε, Λ^2)-contracting, namely those which are bounded and, at each point, are ε-contracting on some tangent hyperplane. More specifically, we have the following.

Proposition **6.3**. — *Let $f: X \to Y$ be a c-contracting map between riemannian manifolds, and suppose that at each point $x \in X$, all but possibly one of the eigenvalues of $(f_*^t f_*)_x : T_x X \to T_x X$ are smaller than ε^2. Then f is εc contracting on 2-forms.*

Proof. — Fix $x \in X$ and let $H \subset T_x X$ be a hyperplane with the property that $\|f_x e\| \leqq \varepsilon \|e\|$ for $e \in H$. Given a unit simple 2-vector $e_1 \wedge e_2$, we may assume by a change of basis that $e_1 \in H$. The proposition now follows directly as in (6.2). ■

Our first observation is that the arguments of § 5 only require maps to be ε-contracting on 2-forms (cf. (5.13)). Consequently, we are led to modify Definitions 5.1 and 5.19 by replacing the term " ε-contracting " with the term " (ε, Λ^2)-contracting. " We shall designate the resulting notions in the obvious way by the terms: *(ε, Λ^2)-hyperspherical (in dimension n) and Λ^2-hyperspherical (in dimension n).*

All the results of the previous section remain true under these weaker hypotheses.

We shall now substantially expand the class of manifolds we are considering, by introducing the following concept.

Definition 6.4. — A connected (not necessarily compact) manifold X is said to be *Λ^2-enlargeable* if given any riemannian metric on X and any $\varepsilon > 0$, there exists a covering manifold $\widetilde{X} \to X$ which is spin and (ε, Λ^2)-hyperspherical in the lifted metric. The concept of *Λ^2-enlargeability in dimension n* is defined analogously.

Clearly any enlargeable manifold is Λ^2-enlargeable. Furthermore, the product of an enlargeable manifold with a Λ^2-enlargeable manifold is Λ^2-enlargeable. In fact, if X is enlargeable in dimension n and if Y is enlargeable in dimension m, then $X \times Y$ is Λ^2-enlargeable in dimension $n + m$. Thus, for example, if X is Λ^2-enlargeable in dimension n and if Y is a compact spin manifold with $\hat{A}(Y) \neq 0$, then $X \times Y$ is also Λ^2-enlargeable in dimension n.

The property of being Λ^2-enlargeable depends on the " proper homotopy type " of the manifold. In fact, we have the following.

Proposition 6.5. — *Let X and Y be connected oriented manifolds, and suppose there exists a proper spin mapping $f: X \to Y$ of non-zero $\hat{A}$-degree. Then if Y is Λ^2-enlargeable in dimension n, so is X.*

Proof. — Suppose we are given a riemannian metric g on X. We choose a metric h on Y so that the mapping $f: X \to Y$ is bounded (say 1-contracting). The argument now proceeds as in the proof of Proposition 5.7. ■

As a particular consequence of this, suppose $\dim(X) = \dim(Y)$ and that $f: X \to Y$ is a proper map of non-zero degree. Then if Y is Λ^2-enlargeable, X is also Λ^2-enlargeable. It follows that the connected sum of any compact spin manifold with a Λ^2-enlargeable manifold is again Λ^2-enlargeable.

The next result is an easy and illuminating consequence of the definitions.

Proposition 6.6. — *Let X be a connected spin manifold. If X contains an open, Λ^2-enlargeable submanifold $U \subset X$ such that the homomorphism $\pi_1 U \to \pi_1 X$ is injective, then X is Λ^2-enlargeable.*

PROOF. — Let X be given any riemannian metric, and fix an $\varepsilon > 0$. Since U is Λ^2-enlargeable, there exists a covering $\widetilde{U}$ of U which admits an (ε, Λ^2)-contracting map $f: \widetilde{U} \to S^n$ which is constant outside a compact subset and of non-zero degree. (Here $\widetilde{U}$ carries the induced metric.) Since $\pi_1 U \to \pi_1 X$ is injective, the covering $\widetilde{U} \to U$ is the restriction of a covering $\widetilde{X} \to X$. Since the map f is constant outside a compact subset of $\widetilde{U}$ it extends trivially (as a constant) to a map $f: \widetilde{X} \to S^n$. This extended map is obviously also (ε, Λ^2)-contracting and of non-zero degree. ∎

It seems an appropriate time to demonstrate that there exist many interesting non-compact manifolds which are Λ^2-enlargeable.

PROPOSITION **6.7**. — *Let* X_0 *be any (compact) enlargeable manifold. Then the product* $X = X_0 \times \mathbf{R}$ *is* Λ^2*-enlargeable.*

PROOF. — Fix a metric on X and choose $\varepsilon > 0$. Fix a "degree-one" mapping $g: \mathbf{R} \to S^1$ such that $g(\mathbf{R} - (-1, 1)) = \{\text{point}\}$, and consider the product mapping

$$X_0 \times \mathbf{R} \xrightarrow{\text{Id.} \times g} X_0 \times S^1.$$

If we introduce on $X_0 \times S^1$ a riemannian product metric $ds_0^2 + d\theta^2$, then the map $\text{Id.} \times g$ will be bounded, *i.e.* c-contracting for some $c > 0$. Since X_0 is enlargeable, there exists a spin covering manifold $\widetilde{X}_0 \to X_0$ with a map $f: \widetilde{X}_0 \to S^n$, which is ε-contracting with respect to the lifted metric $d\widetilde{s}_0$, and which is also constant outside a compact set and of non-zero degree. By Proposition 6.3, the composition

$$\widetilde{X}_0 \times S^1 \xrightarrow{f \times \text{Id}} S^n \times S^1 \longrightarrow S^n \wedge S^1 \cong S^{n+1}$$

is $(c'\varepsilon, \Lambda^2)$-contracting for some c' independent of ε. It follows easily that the composition

$$\widetilde{X}_0 \times \mathbf{R} \xrightarrow{f \times g} S^n \times S^1 \longrightarrow S^{n+1}$$

is $(cc'\varepsilon, \Lambda^2)$-contracting with respect to the metric lifted from $X_0 \times \mathbf{R}$. This composition map is constant outside a compact set and is of non-zero degree. Hence, $X_0 \times \mathbf{R}$ is Λ^2-enlargeable. ∎

COROLLARY **6.8**. — *Suppose* X *is a connected manifold. If there exists a compact enlargeable hypersurface* $X_0 \subset X$ *such that the induced map* $\pi_1(X_0) \to \pi_1(X)$ *is injective, then* X *is* Λ^2*-enlargeable.*

PROOF. — Take a tubular neighborhood of X_0 in X and apply Propositions 6.6 and 6.7. (If the normal line bundle to X_0 in X is not trivial, first pass to the appropriate 2-sheeted covering of the pair.) ∎

This corollary gives us the following.

EXAMPLE **6.9**. — Let $T^k \subset T^n$ denote a linear subtorus of the n-torus for $0 \leqq k < n$ and $n \geqq 2$. Then *the manifold* $X = T^n - T^k$ *is* Λ^2*-enlargeable.* To see this observe that there exists an $(n-1)$-torus $T^{n-1} \subset X$ such that $\pi_1 T^{n-1} \to \pi_1 X$ is injective. (Choose a linear T_0^{n-1} with $T^k \subset T_0^{n-1} \subset T^n$, and translate T_0^{n-1} slightly in the normal direction.)

In fact, it is clear that any manifold of the form $X = T^n - K$, where K is any compact subset of $T^n - T^{n-1}$, is Λ^2-enlargeable. These examples generalize.

EXAMPLE **6.10**. — Let X be a compact manifold of non-positive sectional curvature which carries a connected totally geodesic hypersurface $X_0 \subset X$. Let K be an arbitrary compact subset of $X - X_0$. Then *the manifold* $X - K$ *is Λ^2-enlargeable.* This follows immediately from Corollary 6.7, since X_0 is enlargeable and $\pi_1(X_0) \to \pi_1(X)$ is injective.

EXAMPLE **6.11**. — *Any hyperbolic manifold* X *of finite volume is Λ^2-enlargeable.* (*Proof*: If X is not compact, it has an end of the form $X_0 \times \mathbf{R}^+$ where X_0 is a compact infranil-manifold, and $\pi_1(X_0) \to \pi_1(X)$ is injective. But X_0 is enlargeable by [GL_1, Thm. 4.5].) Here " hyperbolic " means complete with curvature bounded between two negative constants.

By now the reader can easily see that Corollary 6.7 provides an enormous class of non-compact manifolds which are Λ^2-enlargeable. This makes the following result quite interesting.

THEOREM **6.12**. — *A manifold* X *which is Λ^2-enlargeable in any dimension, cannot carry a complete metric of positive scalar curvature. In fact, any metric of non-negative scalar curvature on* X *must be Ricci flat.*

COROLLARY **6.13**. — *There exists no complete metric of positive scalar curvature on manifolds of the form:*

(1) $X_0 \times \mathbf{R}$ *where* X_0 *is enlargeable (e.g., where* X_0 *carries sectional curvature* $\leqq 0$ *or where* X_0 *is a compact solvmanifold);*
(2) *described in Corollary* 6.8; *(See the examples above.)*
(3) X *where* X *carries a hyperbolic metric of finite volume;*
(4) $X \times Y$ *where* X *is as in* (1) (2) *or* (3), *and where* Y *is a compact spin manifold with* $\hat{A}(Y) \neq 0$. *(This includes the manifold* $\mathbf{R} \times Y$.*)*

PROOF OF THEOREM **6.12**. — We first observe that the proof of Theorem 5.21 given in the previous section carries over immediately to this case *provided* we assume that the metric in question has scalar curvature $\kappa \geqq \kappa_0$ for some constant $\kappa_0 > 0$.

To handle the more general case, we must multiply our manifold by a large euclidean sphere $S^2(R)$. This guarantees that $\kappa \geqq \kappa_0$. However, the radius R must be chosen large enough so that the vanishing arguments still work.

For clarity of exposition we shall first treat the basic case where $X = X_0 \times \mathbf{R}$ for some (compact) enlargeable manifold X_0. Suppose X_0 carries a complete metric with scalar curvature $\kappa > 0$, and note that $\kappa \geqq \kappa_0$, for some $\kappa_0 > 0$, on the compact subset

$X_0 \times [-1, 1]$. We may assume, by multiplying with a circle if necessary, that the dimension of X is even, say, $2m$.

We showed in the proof of Proposition 6.7 that for any $\varepsilon > 0$, there is a covering space $\widetilde{X}_0 \to X_0$ and an $(\varepsilon, \Lambda^2($-contracting map $f : \widetilde{X}_0 \times \mathbf{R} \to S^{2m}$ which is constant outside the subset $\widetilde{X}_0 \times (-1, 1)$. We now consider the composition

$$\widetilde{X} \times S^2(R) \xrightarrow{f \times \left(\frac{1}{R}\right)} S^{2m} \times S^2 \longrightarrow S^{2m+2}$$

which we denote by F. The second map in this composition is given by a fixed "smashing" $S^{2m} \times S^2 \to S^{2m} \wedge S^2 \cong S^{2m+2}$. From the definition of the smash product we see that the map F is constant outside the subset $\widetilde{X}_0 \times (-1, 1) \times S^2(R)$. On this subset the scalar curvature of $\widetilde{X} \times S^2(R)$ is $\kappa + (1/R^2) > \kappa \geqq \kappa_0$. Of course, the scalar curvature of $\widetilde{X} \times S^2(R)$ is everywhere $\geqq (1/R^2)$.

Note that if R is chosen sufficiently large (of the order of $\varepsilon^{-1/2}$), then F will be (ε, Λ^2)-contracting.

We choose a complex vector bundle E_0 over S^{2m+2} with non-trivial Chern character and we equip E_0 with a fixed unitary connection ∇^0. We have shown that for every $\varepsilon > 0$ we can find a covering $\widetilde{X} \to X$ and an (ε, Λ^2)-contracting map $F : \widetilde{X} \times S^2(R) \to S^{2m+2}$ for appropriate R, which is constant outside a compact subset and has non-zero degree. The manifolds $\widetilde{X} \times S^2(R)$ are complete and have strictly positive scalar curvature. Hence, the Relative Index Theorem applies, and considering the Dirac operators $D : \Gamma(\mathbf{S}^+) \to \Gamma(\mathbf{S}^-)$ and $D_E : \Gamma(\mathbf{S}^+ \otimes E) \to \Gamma(\mathbf{S}^- \otimes E)$, where $E = F^* E_0$, leads to a contradiction exactly as in the proof of Theorem 5.8.

We consider now the case where X is any Λ^2-enlargeable $2m$-manifold. We assume that X is provided with a complete metric ds^2 with strictly positive scalar curvature function κ. We equip X with a second (not necessary complete) metric

$$ds_0^2 = \kappa ds^2.$$

Now for any $\varepsilon > 0$, there exists by definition a spin covering $\widetilde{X} \to X$ and a mapping $f : \widetilde{X} \to S^{2m}$ which is constant outside a compact subset and of non-zero degree, and which is (ε, Λ^2)-contracting *for the metric* ds_0^2.

Consider, as above, the composition F of the maps

$$\widetilde{X} \times S^2(R) \xrightarrow{f \times \left(\frac{1}{R}\right)} S^{2m} \times S^2 \xrightarrow{\wedge} S^{2m+2},$$

and fix a vector bundle E_0 with connection over S^{2m+2}. Set $E = F^* E_0$ and consider the Dirac operator D_E for the twisted spin bundle $\mathbf{S} \otimes E$. (The metric used here on the $\widetilde{X}$-factor is, of course, the lift of ds^2.) The zero-order term in the Bochner-Lichnerowicz-Weitzenböck formula is of the form

$$\frac{1}{4} \hat{\kappa} + \mathscr{R}^E$$

where $\hat{\kappa} = \kappa + (1/R^2)$ and where $\|\mathscr{R}^E\| \leqq k_{2m} \|R^E\|$. The mapping f is supported in a compact subset K of $\widetilde{X}$ and is *pointwise* $(\varepsilon\kappa, \Lambda^2)$-*contracting* with respect to the metric ds^2. Set $c = \sup \|f_*\|$. Now the mapping F is constant outside the set $K \times S^2(R)$ and is pointwise $(\max(c/R, \varepsilon\kappa), \Lambda^2)$-contracting (with perhaps a small uniform adjustment for the fixed smashing map $S^{2m} \times S^2 \to S^{2m+2}$). If ε is chosen smaller than k_{2m}, and if cR^{-1} is chosen smaller than $\min\{\varepsilon\kappa(x) : x \in K\}$, then the zero-order term in the Bochner-Lichnerowicz-Weitzenböck formula will be everywhere positive on $\widetilde{X} \times S^2(R)$. The argument now proceeds as before.

The general case of manifolds which are Λ^2-enlargeable in some dimension n follows trivially by replacing the word " degree " with " $\hat{A}$-degree " in the arguments above, and by using in the end the relative index calculation given in the proof of Theorem 5.21. ■

7. MANIFOLDS WHICH ADMIT NO COMPLETE METRIC OF UNIFORMLY POSITIVE SCALAR CURVATURE (INTRODUCTION OF SOME FUNDAMENTAL TECHNIQUES)

In the previous section we proved that if X_0 is enlargeable, then the manifold $X_0 \times \mathbf{R}$ carries no complete metric with scalar curvature $\kappa > 0$. This statement is false in general for the manifold $X_0 \times \mathbf{R}^2$. (Consider, for example, the riemannian product of a flat torus and a paraboloid of revolution.) Nevertheless, we shall prove here that essentially the manifold $X_0 \times \mathbf{R}^2$ cannot carry a complete metric with *uniformly* positive scalar curvature, *i.e.* with $\kappa \geqq \kappa_0$ for some positive constant κ_0.

The results of this section will apply to manifolds considerably more general than $X_0 \times \mathbf{R}^2$ (X_0 enlargeable). There are particularly interesting applications to 3-manifolds. It will be shown in § 8 that *no compact 3-manifold which contains a* $K(\pi, 1)$*-factor in its prime decomposition can carry a metric of positive scalar curvature.* (In fact any $\kappa \geqq 0$ metric is flat.) There are, in fact, quite strong results for non-compact 3-manifolds.

We begin our discussion by refining somewhat the ideas of the previous sections. It will be useful here to narrow the notion of enlargeability from the manifold to a class of metrics on a manifold.

Definition 7.1. — A riemannian metric on a connected manifold X is said to be Λ^2*-enlargeable* if given any $\varepsilon > 0$, there exists a covering manifold $\widetilde{X} \to X$ which is spin and (ε, Λ^2)-hyperspherical in the lifted metric. (The notion of being Λ^2*-enlargeable in dimension* n is defined similarly.)

A Λ^2-enlargeable manifold is one for which every riemannian metric is Λ^2-enlargeable.

The property of being Λ^2-enlargeable (in any dimension) depends only on the quasi-isometry class of the metric. In fact, the arguments of Proposition 5.7 prove the following.

Proposition 7.2. — *Let* X *and* Y *be connected riemannian manifolds and suppose there exists a bounded proper spin map* $f: X \to Y$ *of non-zero degree. Then if the metric on* Y *is* Λ^2*-enlargeable, so is the metric on* X.

Note that a compact manifold which admits a Λ^2-enlargeable metric is Λ^2-enlargeable (in any metric).

The following result is an immediate consequence of the methods of section 5.

THEOREM 7.3. — *No complete riemannian metric which is Λ^2-enlargeable can have uniformly positive scalar curvature.*

It is the key observation of this section that if X_0 is enlargeable, then most complete metrics on $X_0 \times \mathbf{R}^2$ are Λ^2-enlargeable. (Not every complete metric is Λ^2-enlargeable, however. A riemannian product metric where $\operatorname{area}(\mathbf{R}^2) < \infty$, does not have this property.) The basic constructions have important generalizations. To state these we focus on the following concept.

DEFINITION 7.4. — Let X be a connected oriented manifold and $X_0 \subset X$ a compact oriented submanifold of codimension 2. A *transversal* to X_0 is a properly embedded, oriented surface $\Sigma \subset X$ which meets X_0 transversely and with non-zero intersection number. (Thus, for any compact domain with boundary $\Omega \subset X$ such that $X_0 \subset \operatorname{int}(\Omega)$, the surface $\Sigma \cap \Omega$ pairs non-trivially with X_0 under Lefschetz duality: $H_2(\Omega, \partial\Omega) \cong H^{n-2}(\Omega)$.)

The following theorem constitutes the main result of this section. There are three distinct statements in the theorem, and each statement is proved by a different technique. These methods of proof have a fairly broad applicability to the study of positive scalar curvature, and we would like to emphasize the methods as much as the particular results established here.

THEOREM 7.5. — *Let* X *be a connected n-dimensional manifold, and suppose there exists a (compact) enlargeable $(n-2)$-manifold X_0 embedded in* X *with trivial normal bundle. Furthermore, suppose that the inclusion*

$$X_0 \times S^1 \hookrightarrow X - X_0 \tag{7.6}$$

obtained by taking the boundary of a small tubular neighborhood of X_0, is a homotopy equivalence. Then a complete riemannian metric on X *having any one of the following properties:*

(A) *there is an inverse homotopy equivalence $H : X - X_0 \to X_0 \times S^1$ which is bounded outside a compact subset of* X,

(B) *there are no transversals to X_0 of finite area,*

(C) *the Ricci curvature is uniformly bounded from below,*

cannot have uniformly positive scalar curvature.

REMARK 7.7. — The mildest of these properties is (B) which is close to being both necessary and sufficient for our method of proof to work. The failure of this condition is equivalent to the existence of complete proper minimal surfaces of finite area (transversal to X_0). When $\dim(X) = 3$, these surfaces are regular and stable (of positive second variation), and we find a relationship between our approach and that of Schoen-Yau [$SY_{1,2}$].

Note that properties (A) and (B) always hold when the geometry of X is bounded.

PROOF OF THEOREM **7.6.** — We suppose X carries a complete riemannian metric with scalar curvature $\kappa \geqq \kappa_0$ where κ_0 is a positive constant. We shall show that none of the properties (A), (B) or (C) can hold.

Let $\delta : X \to \mathbf{R}^+$ be a smooth approximation to the distance function $d(x) \equiv \operatorname{dist}(x, X_0)$. We can assume that

$$(7.8) \qquad \|\nabla\delta\| \leqq 2$$

and, furthermore, that $d = \delta$ near X_0, so that for all $a_0 > 0$ sufficiently small, the manifold

$$(7.9) \qquad X_{a_0} \equiv \delta^{-1}(a_0) \cong X_0 \times S^1$$

has (by hypothesis) the property that the inclusion $X_{a_0} \subset X - X_0$ is a homotopy equivalence.

We fix now an inverse homotopy equivalence

$$(7.10) \qquad H : X - X_0 \to X_0 \times S^1$$

and make the following observation. (We may assume without loss of generality that X_0 is oriented.)

LEMMA **7.11.** — *Let $b > a > 0$ be regular values of δ, and consider the compact manifold*

$$X_{[a,b]} \equiv \delta^{-1}([a, b])$$

with boundary $\partial X_{[a,b]} = X_b - X_a$. Then the mapping

$$(7.12) \qquad X_{[a,b]} \xrightarrow{H\times\delta} X_0 \times S^1 \times [a, b]$$

is a degree-1 map of manifolds with boundary.

PROOF. — Choose $a_0 > 0$ sufficiently small in order that (7.9) hold, and let $t > 0$ be any regular value of δ. Then the manifold $X_t = \delta^{-1}(t)$ is homologous to X_a in S. Consequently, the map $H : X_t \to X_0 \times S^1$ (given by restriction) has the same degree as the map $H : X_a \to X_0 \times S^1$. This latter map is just H composed with the inclusion $X_a \subset X - X_0$, and so by hypothesis it is homotopic to the identity. Hence, the map $H : X_t \to X_0 \times S^1$ has degree 1 for any regular value t of δ.

Suppose now that $(x, t) \in (X_0 \times S^1) \times (a, b)$ is a regular value of the map $H \times \delta$. By Sard's Theorem we may choose (x, t) so that t is also a regular value of the map δ. The degree of $H \times \delta$ is equal to the number of points in the inverse image of (x, t) counted with a sign depending as usual on whether $H \times \delta$ is locally orientation-preserving or reversing. Now the point x is a regular value of the map $H : X_t \to X_0 \times S^1$. Furthermore, at each point of the inverse image of x, the map $H\big|_{X_t}$ is orientation preserving if and only if the map $H \times \delta$ is. (To see this choose an oriented basis $e_1, \ldots, e_n$ of T_xX with $e_1, \ldots, e_{n-1}$ tangent to X_t and with $\delta_* e_n = \partial/\partial t$.) It follows that this weighted sum must equal the degree of the map $H : X_t \to X_0 \times S^1$, that is, it must be 1. ∎

Case (A). Suppose now that the metric has property (A), that is, we assume that the map H can be chosen to have bounded Jacobian near infinity in X (*i.e.* in the set where $\delta \geqq 1$). We shall show that the metric is then Λ^2-enlargeable, in contradiction to Theorem 7.3.

We want to show that for a given $\varepsilon > 0$, there exists a covering $\widetilde{X} \to X$ and an (ε, Λ^2)-contracting map $\widetilde{X} \to S^n$ which is constant outside a compact set and of non-zero degree.

Note that since X is complete and not compact, the function $\delta : X \to \mathbf{R}^+$ is onto. Furthermore, by (7.8) we see that δ is 2-contracting. Thus, if we fix regular values a and b with $b > a + (2/\varepsilon)$, the composition

$$X_{[a,b]} \xrightarrow{\delta} [a, b] \xrightarrow{\lambda} [0, 1], \tag{7.13}$$

where $\lambda(t) = (t-a)/(b-a)$, is onto and ε-contracting.

We now fix a metric on X_0 and observe that by compactness the mapping $H : X_{[a,b]} \to X_0 \times S^1$ onto the riemannian product is bounded, *i.e.* c-contracting for some $c \geqq 1$. In fact, by hypothesis (A) this constant c is *independent of the choice of b.* Since X_0 is enlargeable, there exists a covering space $\pi_0 : \widetilde{X}_0 \to X_0$ and an (ε/c)-contracting map $f : \widetilde{X}_0 \to S^{n+2}$ which is constant outside a compact subset and of non-zero degree.

The inclusion $X_0 \subset X$ is easily seen to be a homotopy equivalence, and therefore there exists a unique (connected) covering $\pi : \widetilde{X} \to X$ extending the covering π_0. Furthermore, there is a commutative diagram

$$\begin{tikzcd} \widetilde{X} - \widetilde{X}_0 \arrow[r, "\widetilde{H}"] \arrow[d, "\pi"'] & \widetilde{X}_0 \times S^1 \arrow[d, "\pi_0 \times \mathrm{Id}"] \\ X - X_0 \arrow[r, "H"] & X_0 \times S^1 \end{tikzcd}$$

where the lifting $\widetilde{H}$ of H is again a homotopy equivalence. We now restrict attention to the compact manifold $X_{[a,b]} \subset X - X_0$ and the corresponding covering space $\widetilde{X}_{[a,b]} = \pi^{-1}(X_{[a,b]}) = (\delta \circ \pi)^{-1}([a, b])$. This gives a commutative diagram:

$$\begin{tikzcd} \widetilde{X}_{[a,b]} \arrow[r, "\widetilde{H}"] \arrow[d] & \widetilde{X}_0 \times S^1 \arrow[d, "\pi_0 \times \mathrm{Id}"] \\ X_{[a,b]} \arrow[r, "H"] & X_0 \times S^1 \end{tikzcd}$$

Since H is proper on $X_{[a,b]}$, the lifting $\widetilde{H}$ is proper on (the fibre product) $\widetilde{X}_{[a,b]}$. Of course, the map $\widetilde{H}$ continues to be c-contracting.

We now consider the composition

$$(7.14) \qquad \widetilde{X}_{[a,b]} \xrightarrow{\widetilde{H} \times \delta} \widetilde{X}_0 \times S^1 \times [a, b] \xrightarrow{f \times \mathrm{Id} \times \lambda} S^{n-2} \times S^1 \times [0, 1]$$

which we denote by $\hat{f}$. This map is ε-contracting onto the $S^{n-2} \times [0, 1]$-factor and c-contracting onto the S^1-factor. In particular (since $c \geq 1$), the map $\hat{f}$ is $(c\varepsilon, \Lambda^2)$-contracting.

We now choose a degree-1 mapping

$$(7.15) \qquad S^{n-2} \times S^1 \times [0, 1] \xrightarrow{\sigma} S^{n-2} \wedge S^1 \wedge S^1 \cong S^n$$

which is constant near the boundary. This map remains unchanged throughout the proof. It has given dilation, *i.e.* it is γ-bounded for some fixed constant $\gamma > 0$, which for simplicity we set equal 1.

Consider the composition $\sigma \circ \hat{f} : \widetilde{X}_{[a,b]} \to S^n$. Since this map is constant near the boundary, it extends trivially to a map $F : \widetilde{X} \to S^n$. Recall that the map f_0 was constant outside a compact set and of non-zero degree, and that the map $\widetilde{H} \times \delta$ was proper. It follows easily that F is constant outside a compact subset and is of non-zero degree. Furthermore, F is $(c\varepsilon, \Lambda^2)$-contracting. Since c is independent of ε, we have proved that the given metric is Λ^2-enlargeable, as claimed. This completes the proof for case (A).

Case (B) [TECHNIQUE: " Push-down and extend "]. Observe that in the general case the argument above breaks down because the bound c for $H|_{X_{[a,b]}}$ depends on the choice of a and b which depend in turn on ε. However, the above argument does prove the following. Suppose $\varepsilon > 0$ is given and fix regular values a and $b = a + 1$ of δ. Then there exists a covering space $\widetilde{X} \to X$ which restricts to a covering $\widetilde{X}_{[a,a+1]} \to X_{[a,a+1]}$ and for which there exists a mapping

$$(7.16) \qquad \widetilde{X}_{[a,a+1]} \xrightarrow{\alpha \times \beta \times \delta} S^{n-2} \times S^1 \times [0, 1]$$

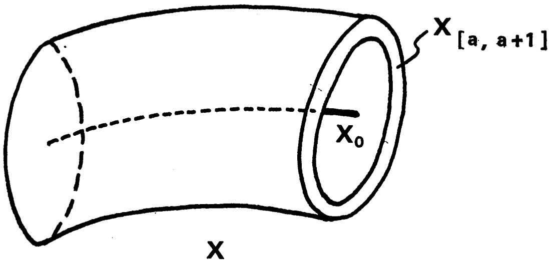
with the following properties:

(i) $\alpha \times \beta$ is constant outside a compact subset and $\alpha \times \beta \times \delta$ maps boundary to boundary;

(ii) $\alpha \times \beta \times \delta$ has non-zero degree;

(iii) α is ε-contracting, δ is 1-contracting, and β is c-contracting where the constant c depends only on a.

Observe that if we could pass to coverings of $\widetilde{X}_{[a,a+1]}$ in the " S^1-direction ", we could assume that the above map was also ε-contracting on the S^1-factor. The composition with σ would then be ε-contracting on 2-forms and we would be in business. The difficulty is that these coverings do not extend to (regular) coverings of X.



Nevertheless, suppose we do pass to a finite covering over the S^1-factor; that is, suppose we consider the N-fold cyclic covering of $\tilde{X}_{[a,a+1]}$ induced as the fibre product:

$$\begin{tikzcd} \tilde{\tilde{X}}_{[a,a+1]} \arrow[r, "\alpha \times \tilde{\beta} \times \gamma"] \arrow[d] & S^n \times \tilde{S}^1 \times [0, 1] \arrow[d, "\mathrm{Id} \times z^N \times \mathrm{Id}"] \\ \tilde{X}_{[a,a+1]} \arrow[r, "\alpha \times \beta \times \gamma"] & S^n \times S^1 \times [0, 1]. \end{tikzcd} \tag{7.17}$$

Then for N sufficiently large we can proceed as before to construct an (ε, Λ^2)-contracting map

$$\tilde{F} = \sigma \circ \left(\alpha \times \frac{\tilde{\beta}}{N} \times \delta\right) : \tilde{\tilde{X}}_{[a,a+1]} \to S^n$$

which is constant at infinity and at the boundary. Via this map we pull back a fixed bundle E_0 with connection from S^n. This induced bundle E has curvature uniformly less than $\gamma'\varepsilon$ (for some fixed γ') and is *canonically trivialized* outside a compact subset $\tilde{K} \subset \text{interior}(\tilde{\tilde{X}}_{[a,a+1]})$. That is, in the complement $\tilde{U} = \tilde{\tilde{X}}_{[a,a+1]} - \tilde{K}$ there is a connection-preserving equivalence with the (flat) product

$$E\Big|_{\tilde{U}} \cong \tilde{U} \times \mathbf{C}^k. \tag{7.18}$$

(This is because, over the region $\tilde{U}$, the bundle E is induced by the constant map.)

We now push the bundle E forward to $\tilde{X}_{[a,a+1]}$. That is, we consider the bundle $\pi_! E$ (with connection) whose fibre at a point x is

$$(\pi_! E)_x \equiv \bigoplus_{\tilde{x} \in \pi^{-1}(x)} E_{\tilde{x}}.$$

Now from (7.18) we see that this bundle is *almost* trivialized at infinity and along the boundary of $\tilde{X}_{[a,a+1]}$. Consider $K = \pi(\tilde{K}) \subset \text{interior}(\tilde{X}_{[a,a+1]})$, and set $U = \tilde{X}_{[a,a+1]} - K$. Then over U, the bundle $\pi_! E$ can be written in the form

$$\pi_! E\Big|_U \cong U \times \underbrace{\mathbf{C}^k \oplus \ldots \oplus \mathbf{C}^k}_{N} \tag{7.19}$$

where N is the degree of the covering π, and where the ordering of the factors is defined up to a cyclic permutation. The push-forward connection is flat and compatible with this local trivialization. However, the connection has *holonomy* $\cong \mathbf{Z}_N$ generated by a cyclic permutation of the factors. This holonomy is carried by cycles which map onto the "S^1-factor", *i.e.* on which β_* is not zero. (See (7.16) and (7.17).)

We now rewrite the locally flat splitting (7.19) as a direct sum of flat (holonomy-invariant) line bundles. We first rewrite $\mathbf{C}^k \oplus \ldots \oplus \mathbf{C}^k$ (N-times) as $\mathbf{C}^N \oplus \ldots \oplus \mathbf{C}^N$ (k-times), by collecting together all of the j^{th} components for $j = 1, \ldots, k$. The holonomy on each $\mathbf{C}^N$ is then generated by a cyclic permutation

$$g(z_1, \ldots, z_N) = (z_N, z_1, \ldots, z_{N-1})$$

of the coordinates. The eigenvalues of this transformation are just the distinct N^{th} roots of unity. Consequently, the bundle $\pi_1 E\big|_U$ can be written as a direct sum of flat complex line bundles

$$\textbf{(7.20)} \qquad \pi_1 E\big|_U = \ell_1 \oplus \ldots \oplus \ell_{Nk}$$

where the holonomy group of each ℓ_j is generated by scalar multiplication by $\omega_j = e^{2\pi i m_j/N}$ for some integer m_j, $0 \leqq m_j < N$. In fact, each ℓ_j can be considered to be induced by $\beta : U \to S^1$ from the flat line bundle $\lambda_j = S^1 \times_{\mathbf{Z}_N} \mathbf{C}$ whose holonomy is generated by multiplication by ω_j.

The idea now is to extend the bundle $\pi_1 E$ to all of $\widetilde{X} \supset \widetilde{X}_{[a, a+1]}$ *while keeping the curvature uniformly small* ($\leqq \varepsilon$). Note that there is no topological obstruction to extending $\pi_1 E$ since each of the line bundles ℓ_j over U is topologically trivial. Furthermore, the bundle $\pi_1 E$ extends naturally to the "outside piece" $\widetilde{X}_{[a+1, \infty)}$ of $\widetilde{X}$ since the covering (7.7) extend to this domain. In this outside piece the bundle has the same properties that it has in U, *i.e.* it will be flat with non-trivial holonomy.

The difficulty is to extend $\pi_1 E$ over the "inside piece" $\widetilde{X}_{[0, a]}$ of $\widetilde{X}$ where the covering (7.17) cannot be continued. The cycles carrying the holonomy of the flat connection are homologous to zero in this region. The idea is to extend the connection while keeping the curvature small. We do this via the Hahn-Banach Theorem.

For simplicity we shall focus on just one of the line bundles in (7.20). Suppose ℓ is such a line bundle with holonomy generated by multiplication by $\omega = e^{2\pi i m/N} (m < N)$. We can assume that ℓ is defined in the closed regular neighborhood $\widetilde{C} = \widetilde{X}_{[a-1, a]} (a \gg 1)$ of the boundary $\partial \widetilde{X}_{[0, a]}$. In fact as stated above, we can assume that ℓ is the pull-back $\beta^* \lambda$ of the flat bundle λ over S^1 with holonomy generated by a $2\pi m/N$-rotation. Moreover, examination of the definitions shows that β factors through the covering projection $\widetilde{X}_{[0, a]} \to X_{[0, a]}$. So, for convenience, we can work downstairs on our original manifold. Here we have the bundle, which we continue to denote by $\ell = \beta^* \lambda$, defined on the compact neighborhood $X_{[a-1,a]}$ of the boundary. With this representation we can explicitly construct a connection 1-form φ in ℓ by setting

$$\varphi = \beta^*\left(\frac{m}{N}\,d\theta\right). \tag{7.21}$$

Of course, $d\varphi = 0$ on the closed set C. We want to extend φ to a smooth 1-form on the entire manifold $X_{[0,a]}$ so that

$$\|d\varphi\| \leqq \varepsilon \quad \text{pointwise.} \tag{7.22}$$

(Note that φ can be considered the connection 1-form of a topologically trivial extension of the line bundle ℓ, and as such $d\varphi$ is its curvature 2-form.)

There is an obstruction to finding such an extension; for if there did exist an extension of φ satisfying (7.22), then for any compact surface $\Sigma \subset X_{[0,a]}$ with $\partial\Sigma \subset C$, we would have by Stokes' Theorem that

$$\left|\int_{\partial\Sigma}\varphi\right| \leqq \varepsilon\,\mathrm{Area}(\Sigma).$$

In fact, for a 2-dimensional current T on $X_{[0,a]}$ such that $\operatorname{supp}(\partial T) \subset C$, we would have that

$$|\partial T(\varphi)| \leqq \varepsilon \mathbf{M}(T) \tag{7.23}$$

where $\mathbf{M}(T)$ denotes the mass of the current T. (Definitions are given below.) We shall now show that this condition (7.22) is both necessary and sufficient for finding the desired extension.

We begin by setting some notation. Let Y denote the compact manifold (with boundary) $X_{[0,a]}$, and let C denote the closed tubular neighborhood of the boundary, as above. Let $\mathscr{E}^p(Y)$ and $\mathscr{E}^p(C)$ denote the spaces of C^∞ p-forms on Y and C respectively. By this we mean the germs on Y (or C) of smooth p-forms defined in a neighborhood. More explicitly, embed Y in an open manifold $\widetilde{Y}$ and let $\{U_\alpha\}_\alpha$ be a system of open neighborhoods of Y such that $\cap\, U_\alpha = Y$. For $U_\alpha \subset U_\beta$ there is a restriction map $r_{\alpha\beta} : \mathscr{E}^p(U_\beta) \to \mathscr{E}^p(U_\alpha)$ on smooth p-forms. Give $\mathscr{E}^p(U_\alpha)$ the usual C^∞-topology. Then $\mathscr{E}^p(Y) = \varinjlim \mathscr{E}^p(U_\alpha)$ with the limit topology. The definition of $\mathscr{E}^p(C)$ is similar.

The topological dual spaces $\mathscr{E}_p(Y)$ (and $\mathscr{E}_p(C)$) of $\mathscr{E}^p(Y)$ (and $\mathscr{E}^p(C)$ respectively) are called the spaces of *p-dimensional currents with support in* Y (and C). The adjoint of exterior differentiation on forms gives a continuous *boundary map* $\partial : \mathscr{E}_p(Y) \to \mathscr{E}_{p-1}(Y)$ with $\partial^2 = 0$. The *support* of a current $T \in \mathscr{E}_p(Y)$, denoted supp(T), is the smallest closed set $K \subset Y$ such that $T(\varphi) = 0$ for any form φ which vanishes along K. We set

$$\mathscr{E}_p(Y, C) \equiv \{T \in \mathscr{E}_p(Y) : \operatorname{supp}(\partial T) \subset C\}$$

and observe that there is a commutative square of continuous linear maps

$$\begin{tikzcd} \mathscr{E}_1(C) \arrow[r, hook, "i_*"] & \mathscr{E}_1(Y) \\ \mathscr{E}_2(Y, C) \arrow[u, "\partial"] \arrow[r, hook] & \mathscr{E}_2(Y) \arrow[u, "\partial"'] \end{tikzcd} \tag{7.24}$$

where $i : C \hookrightarrow Y$ denotes the inclusion.

Given a differential p-form φ on a riemannian manifold Y, we define its *comass* to be $\|\varphi\|_\infty = \sup\{\varphi(\xi) : \xi \text{ is a unit simple } p\text{-vector on } Y\}$. (This is equivalent to the sup-norm given by any other uniformly equivalent norm in the fibres of $\Lambda^p T^* Y$.) The *mass* of a current $T \in \mathscr{E}_p(Y)$ is

$$\mathbf{M}(T) \equiv \sup\{T(\varphi) : \|\varphi\|_\infty \leqq 1\}.$$

One can easily see that: $\mathbf{M}(T_1 + T_2) \leqq \mathbf{M}(T_1) + \mathbf{M}(T_2)$; $\mathbf{M}(tT) = |t|\mathbf{M}(T)$ for $t \in \mathbf{R}$; and $\mathbf{M}([\Sigma]) = \text{Area}(\Sigma)$ when $[\Sigma]$ is the current given by integration over a compact oriented p-dimensional submanifold Σ of Y.

We can now state and prove our basic extension theorem.

PROPOSITION 7.25. — *Suppose $\varphi \in \mathscr{E}^1(C)$ is a smooth d-closed 1-form (germed) on C with the property that*

$$|(\partial T)(\varphi)| \leqq \varepsilon M(T) \tag{7.26}$$

for all $T \in \mathscr{E}_2(Y, C)$. Then there exists an extension of φ to all of Y, that is, there exists $\widetilde{\varphi} \in \mathscr{E}^1(Y)$ such that $i^* \widetilde{\varphi} = \varphi$, with the property that

$$\|d\widetilde{\varphi}\|_\infty \leqq \varepsilon. \tag{7.27}$$

PROOF. — Consider the (closed) subspaces

$$\mathscr{B} \equiv \partial \mathscr{E}_2(Y) \quad \text{and} \quad \mathscr{B}_0 \equiv \mathscr{B} \cap \mathscr{E}_1(C) = \partial \mathscr{E}_2(Y, C)$$

of the space $\mathscr{E}_1(Y)$ (cf. (7.24)). Now $\mathscr{B}_0 \subset \mathscr{B}$ and we have defined on $\mathscr{B}$ the continuous semi-norm

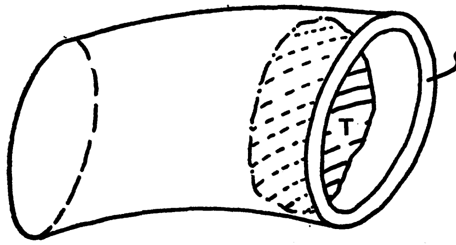
$$p(S) \equiv \inf\{\mathbf{M}(T) : T \in \mathscr{E}_2(Y) \quad \text{and} \quad \partial T = S\}. \tag{7.28}$$

Our hypothesis is that φ, considered as a linear functional on $\mathscr{B}_0$, satisfies the condition

$$\varphi(S) \leqq \varepsilon p(S) \tag{7.29}$$

for all $S \in \mathscr{B}_0$. By the Hahn-Banach Theorem (cf. [Sc], p. 49) there exists a continuous extension of φ to all of $\mathscr{B}$ so that (7.29) continues to hold. Thus, as a linear functional, we have extended $\varphi\big|_{\mathscr{B}_0}$ to all of $\mathscr{B}$. This generates an extension to the subspace $\mathscr{B} + \mathscr{E}_1(C) \subset \mathscr{E}_1(Y)$. A second application of the Hahn-Banach Theorem gives a further continuous extension to all of $\mathscr{E}_1(Y)$. Since $\mathscr{E}_1(Y)' = \mathscr{E}^1(Y)$ (*i.e.* $\mathscr{E}^1(Y)$ is reflexive) this extension is a smooth 1-form $\tilde{\varphi}$ such that $\tilde{\varphi} = \varphi$ on C. Furthermore, the fact that (7.29) continues to hold for $\tilde{\varphi}$ immediately implies that $\|d\tilde{\varphi}\|_\infty < \varepsilon$. To see this, simply choose T to be the "Dirac" current associated to a simple 2-vector ξ at some point. Then $(\partial T)(\tilde{\varphi}) \equiv T(d\tilde{\varphi}) = (d\tilde{\varphi})(\xi) \leqq \varepsilon M(T) = \varepsilon$. Since this holds for all ξ, we have $\|d\varphi\|_\infty \leqq \varepsilon$. This completes the proof of Proposition 7.25. ∎

We now return to the mainstream of our argument. We were given a closed 1-form $\varphi = \beta^*\left(\frac{m}{N}\, d\theta\right)$ on $C = X_{[a-1,a]}$, and we want to extend φ to $X_{[0,a]}$ subject to the condition (7.21) that $\|d\varphi\|_\infty \leqq \varepsilon$. Since $0 \leq m < N$, we can drop the linear factor and assume that $\varphi = \beta^*(d\theta)$. By Proposition 7.25 this will be possible if and only if $(\partial T)(\varphi) \leqq \varepsilon \mathbf{M}(T)$ for all currents T on $X_{[0,a]}$ with $\operatorname{supp}(\partial T) \subset C$. This



leads to the following "Plateau Problem" in $X_{[0,a]}$. Consider for each a, the number

$$(7.30) \qquad e(a) \equiv \inf\{\mathbf{M}(T) : T \in \mathscr{E}_2(X_{[0,a]}, X_{[a-1,a]}) \quad \text{and} \quad (\partial T)(\varphi) = 1\}.$$

As a function of a, $e(a)$ is *monotone non-decreasing*. To see this recall that $\varphi = \beta^*(d\theta)$ where β is the restriction to $X_{[a-1,a]}$ of the S^1-component $\beta : X - X_0 \to S^1$ of the homotopy equivalence H. It follows that for any $T \in \mathscr{E}_2(X, X_{[a-1,\infty)})$ we can replace T with χT where χ is the characteristic function of $X_{[0,a]}$. Then $\mathbf{M}(\chi T) \leqq \mathbf{M}(T)$ and since $\varphi = \beta^*(d\theta)$ is defined and closed in all of $X_{[a-1,\infty)}$, we have that $(\partial T)(\varphi) - \partial(\chi T)(\varphi) = ((1-\chi)\, T)(d\varphi) = 0$. It follows immediately that

$$e(a) = \inf\{\mathbf{M}(T) : T \in \mathscr{E}_2(X, X_{[a-1,\infty)}) \quad \text{and} \quad (\partial T)(\varphi) = 1\}.$$

The monotonicity of $e(a)$ is now evident.

We now claim that

$$(7.31) \qquad \lim_{a\to\infty} e(a) = \infty.$$

Suppose not, *i.e.* suppose $\lim_{a\to\infty} e(a) < \infty$. Then by general compactness theorems it is not difficult to prove that there exists a current $T \in \mathscr{E}_2(X)$ with $\mathbf{M}(T) < \infty$ such that $T(\varphi) = 1$. Approximation by rational polyhedral chains leads to a transversal surface Σ with $\text{Area}(\Sigma) < \infty$ and $\int_\Sigma \varphi > 0$.

We have assumed there are no transversals of finite mass. Hence, (7.31) must hold. It then follows that for any given $\varepsilon > 0$, there exists a corresponding $a > 0$ such that $e(a) > (1/\varepsilon)$. Therefore, given $\varepsilon > 0$, there exists an $a > 0$ such that the hypothesis (7.26) of Proposition 7.25 is satisfied for $X_{[0,a]}$.

Let us now return to the main construction. We have $\varepsilon > 0$ given, and we can suppose that $a > 0$ has been chosen as in the paragraph above. We then consider the manifold $\widetilde{X}_{[a,a+1]}$ with the bundle $\pi_! E$. This bundle has a canonical flat structure at infinity and in a neighborhood of $\partial\widetilde{X}_{[a,a+1]}$. There is a canonical flat extension of this bundle to $\widetilde{X}_{[a+1,\infty)}$. Furthermore, we have now shown that $\pi_! E$ extends over $\widetilde{X}_{[0,a]}$ with small curvature. In fact, after decomposing $\pi_! E\big|_U$ into flat line bundles (cf. (7.20)), we can extend each line bundle with a connection of curvature pointwise $\leqq \varepsilon$.

We now consider the Dirac operator on spinors $\mathbf{S} \otimes (\pi_! E)$ with coefficients in $\pi_! E$. Since the scalar curvature is $\geqq \kappa_0$, we have that for $0 < \varepsilon \ll \kappa_0$, the zero-order term $\mathscr{R}$ in the Bochner-Weitzenböck formula will be *uniformly positive* over $\widetilde{X}$. To see this easily, note that everything is operating "slotwise" with respect to the decomposition $\pi_! E \cong E \oplus E \oplus \ldots \oplus E$. That is, the formula downstairs is simply a direct sum of several copies of the formula from upstairs where we had arranged positivity to hold. This decomposition principle also applies over $\widetilde{X}_{[0,a]}$ where $\pi_! E$ decomposes into a direct sum of line bundles with connection. Consequently we conclude that the analytic index of $D^+ : \Gamma(\mathbf{S}^+ \otimes \pi_! E) \to \Gamma(\mathbf{S}^- \otimes \pi_! E)$ is zero.

We now compare this operator with the "trivial" Dirac operator, *i.e.* the Dirac operator D_0 on $\mathbf{S} \otimes F$ where F is the bundle which agrees with $\pi_! E$ over $\widetilde{X}_{[0,a]}$ and which carries the extended flat connection on $\widetilde{X}_{[a,\infty)}$. Thus $F = \pi_! E$ (and, hence $D = D_0$) outside of the compact subset K of $\widetilde{X}$. Now F is a direct sum of line bundles with connection, each of which has curvature pointwise $\leqq \varepsilon$. For the same reasons as above the operator D_0 is uniformly positive on $\widetilde{X}$ and so the analytic index of D_0^+ is zero.

The Relative Index Theorem now applies to show that the topological index $\operatorname{ind}_t(D^+, D_0^+)$ is zero. However,

$$\begin{aligned} \operatorname{ind}_t(D^+, D_0^+) &= \int_{\widetilde{X}} \widehat{\operatorname{ch}}(\pi_! E) \,.\, \hat{\mathbf{A}}(\widetilde{X}) \\ &= \int_{\widetilde{\widetilde{X}}} \widehat{\operatorname{ch}}(E) \,.\, \hat{\mathbf{A}}(\widetilde{\widetilde{X}}) \neq 0, \end{aligned}$$

as in sections 5 and 6. This contradiction completes the proof of Case (B). ■

Remark 7.32. — When X has dimension 3, and $X_0 \cong S^1$ admits transversals of finite area ($\lim_{a\to\infty} e(a) < \infty$), then the known compactness and regularity theorems of Almgren-Federer-Fleming (cf. [F]) prove that *there exists a regular properly embedded* minimal surface Σ of finite area transversal to X_0. This surface Σ can be chosen to be of *least area in its cohomology class*, in fact, we can find Σ with

$$\operatorname{Area}(\Sigma) = \lim_{a\to\infty} e(a).$$

(This is not possible in general when $\dim(X) > 3$.)

Case (C) [TECHNIQUE: " The inflating balloon "]. — The following is the simplest of the three tricks and will have interesting applications to the study of metrics with certain asymptotic behavior. For all this we will need a basic calculation of the scalar curvature of warped products.

PROPOSITION 7.33. — *Let* X *and* Y *be riemannian manifolds with metrics* ds^2_X *and* ds^2_Y *and with scalar curvature functions* κ_X *and* κ_Y *respectively. Let* $f : X \to \mathbf{R}^+$ *be a* C^2*-function, and consider on* $X \times Y$ *the " warped product " metric*

$$ds^2 = ds^2_X + f^2 \, ds^2_Y. \tag{7.34}$$

Then the scalar curvature κ *of this metric is given by the formula*

$$\boxed{\kappa = \kappa_X + \frac{1}{f^2}\kappa_Y - \frac{n(n-1)}{f^2}\|\nabla f\|^2 - \frac{2n}{f}\Delta f} \tag{7.35}$$

where $\Delta f = \Sigma \nabla_{e_i, e_i} f$ *is the (negative) Laplace-Beltrami operator on* X, *and where* $n = \dim(Y)$.

PROOF. — The proof is a straightforward exercise which we shall leave to the pleasure of the reader. ∎

This formula has the following important consequence.

THEOREM 7.36. — *Let* X *be a complete, non-compact riemannian manifold which admits a smooth exhaustion function* $F : X \to \mathbf{R}^+$ *satisfying pointwise inequalities:*

$$\|\nabla F\| \leqq C \quad \text{and} \quad \Delta F \leqq C \tag{7.37}$$

for some constant C. *Suppose further that the scalar curvature is uniformly positive outside some compact subset of* X. *Then given any* $R > 0$, *there exists a complete metric on* $X \times S^2$ *which has uniformly positive scalar curvature and which, outside a compact subset* $K \subset X$, *is the riemannian product*

$$(X - K) \times S^2(R)$$

of the given metric on $X - K$ *with the standard metric of curvature* $1/R^2$ *on* S^2.

PROOF. — Choose $r_0 > 0$ so that $\kappa \geqq \kappa_0$ outside the compact set $X(r_0) \equiv \{F \leqq r_0\}$. Choose κ_1 so that

$$\kappa_1 > \sup_{X(r_0)} |\kappa|.$$

Choose a smooth function $\varphi : \mathbf{R} \to \mathbf{R}^+$ with the following properties:

$$\begin{cases} \varphi(r) = \kappa_1^{-1/2} & \text{for } r \leqq r_0 \\ \varphi(r) = R & \text{for } r \geqq r_0 + 2R. \\ 0 \leqq \varphi' \leqq 1 \\ |\varphi''| \leqq 1 \end{cases} \tag{7.38}$$

Choose ε with $0 < \varepsilon < 1$, and consider the function

$$f_\varepsilon(x) = \varphi(\varepsilon F(x)).$$

Direct calculation using (7.37) and (7.38) shows that

$$(7.39) \qquad \|\nabla f_\varepsilon\|^2 \leqq \varepsilon^2 C^2 \quad \text{and} \quad \Delta f_\varepsilon \leqq \varepsilon^2 C^2 + \varepsilon C.$$

We now take the warped product of X with $S^2 = S^2(1)$ using this function f_ε. Since $\varepsilon < 1$, we see that $f_\varepsilon = \text{constant} = \kappa_1^{-1/2}$ on $X(r_0)$. Hence, from formula (7.35) we see that the scalar curvature κ_ε of this metric is positive on $X(r_0)$. The formula also shows that on the complement of this region,

$$(7.40) \qquad \kappa_\varepsilon = \kappa + \frac{2}{f^2}(1 - \|\nabla f\|^2 - 2f\Delta f).$$

Applying (7.39) shows that in this region

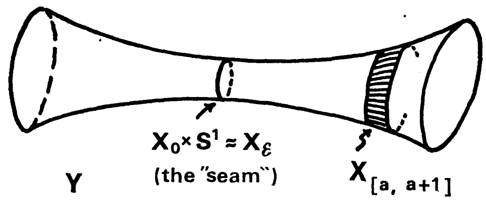
$$(7.41) \qquad \begin{aligned} \kappa_\varepsilon &\geqq \kappa_0 + \frac{2}{f^2}[1 - \varepsilon^2 C^2 - 2f(\varepsilon^2 C^2 + \varepsilon C)] \\ &\geqq \kappa_0 + \frac{2}{f^2}[1 - \varepsilon^2 C^2 - 2R\varepsilon C(1 + \varepsilon C)] \\ &\geqq \kappa_0 \end{aligned}$$

if ε is chosen sufficiently small. Thus, for ε small, this metric has uniformly positive scalar curvature and is isometric to the product $X \times S^2(R)$ at infinity. This completes the proof. ∎

Remark 7.42. — Note that we have actually proved the following. If the metric on X has $\kappa \geqq \kappa_0$ at infinity, then *the metric constructed on* $X \times S^2(R)$ *in Theorem* 7.36 *can also be made to have* $\kappa \geqq \kappa_0$ *everywhere.*

We are now in a position to prove part (C). Suppose $X \supset X_0$ is as hypothesized in 7.5, and suppose X carries a metric with $\kappa \geqq \kappa_0$ and $\mathrm{Ric} > C_0$ for constants $\kappa_0 > 0$ and C_0. Under this assumption on Ric, there exists a smooth exhaustion function $F_0 : X \to \mathbf{R}^+$ satisfying (7.37). This is obtained by taking an appropriate approximation to the distance function to a point, and then applying Chern comparison and techniques in, say, [CY].

Remove now a small tubular neighborhood of X_0 in X and double the resulting manifold along the boundary. This gives a new manifold Y which is homotopy equivalent to $X_0 \times S^1$. It is elementary to construct a smooth riemannian metric on Y which agrees with the old one outside a compact neighborhood of the "seam". It is also easy to see that there is a smooth exhaustion function $F : Y \to \mathbf{R}$ which agrees with (the double of) F_0 outside



a compact subset. This function will satisfy condition (7.37) for some constant C.

We now choose $\varepsilon > 0$ appropriately small with respect to the constant κ_0. We then fix $R > 1/\varepsilon$ and apply Theorem 7.36 (and Remark 7.42) to construct a metric on Y which has $\kappa \geqq \kappa_0$ and which is isometric to the product $Y \times S^2(R)$ at infinity, *i.e.* outside a compact subset $K \times S^2 \subset Y \times S^2$.

Now choose $a > 0$ sufficiently large in order that the subdomain $X_{[a,a+1]}$ of X defined above (cf. 7.11) be contained in $Y - K$. Since the inclusion $X_0 \times S^1 \subset Y$ is a homotopy equivalence and X_0 be enlargeable, the arguments given for Cases (A) and (B) show the following. There is a covering $\pi : \widetilde{Y} \to Y$ and an (ε, Λ^2)-contracting map $f : \widetilde{Y} \to S^n$ with compact support in $\widetilde{X}_{[a,a+1]} = \pi^{-1}(X_{[a,a+1]})$, with non-zero degree and with $\|f_*\| \leqq 1$. Lift the metric just constructed to $\widetilde{Y} \times S^2$ and consider the composition

$$(7.43) \qquad \widetilde{Y} \times S^2 \xrightarrow{f \times \mathrm{Id}} S^n \times S^2 \xrightarrow{\wedge} S^n \wedge S^2 = S^{n+2}.$$

From the property of the smash product $\wedge$, this map is constant outside

$$\mathrm{supp}(f) \times S^2 \subset\subset \widetilde{X}_{[a,a+1]} \times S^2.$$

Therefore, since the S^2 factor is contracted by $1/R < \varepsilon$ in this set, *the map is* (ε, Λ^2)-*contracting.*

We now proceed as before, via the Relative Index Theorem, to reach a contradiction. This completes the proof of Theorem 7.5. ■

We shall now make some useful observations concerning the arguments presented above. Our first remark is that the hypotheses concerning X_0 in Theorem 7.5 can be relaxed somewhat. All we really required was the existence of the enlargeable hypersurface $X_0 \times S^1 \subset X$ and a map $H : \overline{X}_+ \to X_0 \times S^1$ of a non-compact component X_+ of $X - (X_0 \times S^1)$ with the property that the composition $X_0 \times S^1 \subset X_+ \xrightarrow{H} X_0 \times S^1$ be of non-zero degree. The arguments given above actually prove the following.

THEOREM 7.44. — *Let* X *be a connected spin manifold which contains a compact hypersurface* Z *with the following properties:*

(α) $Z \approx X_0 \times S^1$ *where* X_0 *is enlargeable and the map* $\pi_1(X_0) \to \pi_1(X)$ *is injective;*

(β) *There is a non-compact component* X_+ *of* $X - Z$ *and a map* $X_+ \to Z$ *such that the composition* $Z \hookrightarrow X_+ \to Z$ *has non-zero degree.*

Then a complete metric on X *satisfying one of the conditions*:

(A) *The map* $X_+ \to Z$ *is bounded*,

(B) X_0 *has no transversal of finite area*,

cannot have uniformly positive scalar curvature.

Furthermore, suppose X *has a complete metric satisfying* (A) *or* (B). *Then any complete riemannian spin manifold* X^* *which admits a bounded proper map of non-zero* $\hat{A}$*-degree onto* X *cannot have uniformly positive scalar curvature.*

PROOF. — The first conclusion is an immediate consequence of the arguments given above. For the second conclusion, we suppose $f: X^* \to X$ is the hypothesized map. Then for any $\varepsilon > 0$, the arguments above produce a covering $\widetilde{X} \to X$ and a bundle E over $\widetilde{X}$ with $||R|| \leq \varepsilon$ and with certain topological properties (compact support, etc.). We then consider the induced covering $\widetilde{X}^* \to X^*$ with a lifting to a proper map $\widetilde{f}: \widetilde{X}^* \to \widetilde{X}$ of non-zero $\hat{A}$-degree. Using the bundle $\widetilde{f}^*E$ with induced connection as before leads to a contradiction. ∎

The method for Case (C) (the inflating balloon) has a number of interesting applications. To begin we note that the argument given above actually proves much more than claimed. To give a clear statement we introduce the following notion.

DEFINITION 7.45. — A connected manifold X is said to have a *bad end* if there exists a (compact, oriented) enlargeable hypersurface $Z \subset X$ and a non-compact component X_+ of $X - Z$ with a map $\overline{X}_+ \to Z$ whose restriction to $Z \subset \overline{X}_+$ has non-zero degree. (The set X_+ is, of course, the " bad end ".)

Note that the property of having a bad end persists after making arbitrary modifications of the manifold away from the end.

REMARK. — The map $\overline{X}_+ \to Z$ in Definition 7.45 could be replaced by a map to any enlargeable manifold.

THEOREM 7.46. — *Let* X *be a connected spin manifold with a bad end* X_+. *Then any complete metric on* X *which has Ric bounded from below on* X_+ *cannot have uniformly positive scalar curvature. In fact such a metric cannot even have uniformly positive scalar curvature at infinity on the one end* X_+ *(*i.e. *cannot have* $\kappa \geq \kappa_0 > 0$ *on* $X_+ - K$ *for* K *compact in* X*).*

Furthermore, let X^* *be a connected spin manifold which admits a proper map* $f: X^* \to X$ *of non-zero* $\hat{A}$*-degree. Then any complete metric on* X^* *with* $\mathrm{Ric} \geq -C_0^2$, *cannot have uniformly positive scalar curvature. In fact, such a metric cannot have* $\kappa \geq \kappa_0$ *at infinity in* $f^{-1}(\overline{X}_+)$.

PROOF. — The first statement follows as in the proof of 7.5 (C) by doubling the end and slowly blowing-up a balloon.

For the second statement we first deform f on a compact set so that it becomes transversal to the hypersurface Z. Then $f^{-1}(Z)$ determines an end X_+^* with a proper map $f: \overline{X}^*_+ \to \overline{X}_+$, which is a fibration at the boundary. We now double X^*_+ and X_+ to give manifolds Y^* and Y with the same metrics outside a compact set and with the "doubled" map $f: Y^* \to Y$. The proof now proceeds as in the basic case. Fix an exhaustion $F: Y^* \to \mathbf{R}$ with $\|\nabla F\|$ and $\Delta F \leqq$ constant. Given $\varepsilon > 0$, choose $R > (1/\varepsilon)$ and construct a warped product on Y^* which has $\kappa \geqq \kappa_0$ and which is a product $Y^* \times S^2(R)$ outside a compact subset $K \times S^2 \subset Y^* \times S^2$. Fix $a > 0$ so that $X_{[a,a+1]} \cap f(K) = \varnothing$. Here, $X_{[a,a+1]} \subset X \subset Y$ is constructed as above. Let c be the bound for f and pass to a covering $\widetilde{Y} \to Y$ for which there exists an $(\varepsilon/c, \Lambda^2)$-contracting map to S^n with compact support in $\widetilde{X}_{[a,a+1]}$ and non-zero degree. Let $\widetilde{Y}^* \to \widetilde{Y}^*$ be the corresponding covering with a lifting of f to a proper map $\tilde{f}: \widetilde{Y}^* \to \widetilde{Y}$. The composition

$$\widetilde{Y}^* \times S^2 \to S^n \times S^2 \to S^{n+2}$$

defined as in (7.43), is ε-contracting on 2-forms, and the proof proceeds as before. ■

The results above have the following corollary which is of particular interest in low dimensions.

THEOREM **7.47**. — *Let* X *be a compact* $K(\pi, 1)$*-manifold, and suppose* X *contains an enlargeable* $K(\pi, 1)$*-submanifold* X_0 *of codimension-2 such that the homomorphism* $\pi_1 X_0 \to \pi_1 X$ *is injective. Suppose also that the boundary* Z *of a tubular neighborhood of* X_0 *in* X *is an enlargeable manifold. Then* X *carries no metric of positive scalar curvature. In fact, any metric with* $\kappa \geqq 0$ *is flat.*

THEOREM **7.48**. — *There are no (non-flat) metrics with* $\kappa \geqq 0$ *on a compact manifold* X' *if*

(1) $X' \approx X \# Y$ *where* Y *is spin, and* X *is as in Theorem* 7.47.
(2) $X' = X \times Y$ *where* Y *is spin with* $\hat{A}(Y) \neq 0$, *and* X *is as in part* (1).

REMARK **7.49**. — The hypothesis that Z be enlargeable may always be satisfied. It is related to the following question, which appears not to be trivial. Suppose some covering of a manifold is hyperspherical, is the universal covering hyperspherical?

Of course, if $Z \approx X_0 \times S^1$, *i.e.* if the normal bundle is trivial, then Z is enlargeable. Also, whenever $\dim X_0 = 1$ or 2, the manifold Z is enlargeable. When $\dim(X_0) = 2$, this follows from the fact that the universal cover $\widetilde{X}_0$ of X_0 is hyperspherical in any lifted metric, and that the lift of Z to $\widetilde{X}_0$ is trivial, *i.e.* $\widetilde{Z} \approx \widetilde{X}_0 \times S^1$.

PROOF OF **7.47**. — Let $\widetilde{X} \to X$ be the covering space corresponding to the subgroup $\pi_1(X_0) \subset \pi_1(X)$. The embedding $X_0 \subset X$ lifts to an embedding $X_0 \subset \widetilde{X}$ which, since X and X_0 are $K(\pi, 1)$-manifolds, is a homotopy equivalence. Consider now the inclusion

$$Z \subset \widetilde{X} - X_0 \tag{7.50}$$

where Z denotes the boundary of a tubular neighborhood of X_0. It is an easy consequence of the Mayer-Vietoris sequence that the map (7.50) induces an isomorphism on homology groups.

Let $g : \widetilde{X} \to X_0$ denote the (inverse) homotopy equivalence, and consider the problem of lifting the map $g\big|_{\widetilde{X}-X_0}$ to Z. Restricted to Z, this map has a lifting of degree 1. The manifold Z is a circle bundle over X_0 which we may assume to be principal (by having passed to a 2-sheeted covering space if necessary). Consequently, the only obstruction to lifting is a class in $H^2(\widetilde{X} - X_0, Z)$, which is zero because (7.50) is a homology equivalence. Hence there exists a map $\widetilde{g} : \widetilde{X} - X_0 \to Z$

$$\begin{tikzcd} & S^1 \arrow[d] \\ Z \subset \widetilde{X} - X_0 \arrow[r, dashed, "\widetilde{g}"] \arrow[dr, "g"'] & Z \arrow[d] \\ & X_0 \end{tikzcd}$$

which when restricted to Z has degree 1. The manifold Z is enlargeable since it is a circle bundle over an enlargeable manifold (see [GL_1]). Therefore, we have shown that the unbounded component X_+ of $\widetilde{X} - X_0$ is a bad end.

Any metric on X satisfies $\mathrm{Ric} \geqq$ constant since X is compact. Hence, Theorem 7.46 applies to complete the proof. ■

Proof of Theorem **7.48**. — This follows directly from 7.47 and the second part of 7.46. ■

Note that any 1-manifold and any 2-manifold other than S^2 and $\mathbf{P}^2(\mathbf{R})$, is enlargeable and of $K(\pi, 1)$-type. Hence, the above results are particularly useful in dimensions 3 and 4.

8. RESULTS FOR 3-MANIFOLDS

In this section we shall study the existence of complete, positive scalar curvature metrics on 3-manifolds. We begin with a discussion of the compact case. (These results were announced in [GL_1].) There are even more striking results, however, for non-compact manifolds. We have already seen in § 6 that there are no complete metrics with positive scalar curvature on $\Sigma \times \mathbf{R}$ for any compact surface Σ with $\chi(\Sigma) \leqq 0$. We shall now prove that there exists no metric of uniformly positive scalar curvature on $S^1 \times \mathbf{R}^2$. It follows that there is no metric of uniformly positive scalar curvature on $\Sigma \times \mathbf{R}$ for any non-compact surface Σ. Results for more general non-compact 3-manifolds are formulated below.

Suppose, to begin, that X is a compact 3-manifold, and by passing to a 2-sheeted covering if necessary, assume that X is oriented. According to Milnor [M_4], X can be decomposed uniquely into *prime* 3-manifolds:

$$X = X_1 \# \dots \# X_\ell \# \underbrace{(S^1 \times S^2) \# \dots \# (S^1 \times S^2)}_{m \text{ factors}} \# K_1 \# \dots \# K_n \tag{8.0}$$

where $\pi_1(X_j)$ is finite for $j = 1, \dots, \ell$ and where each K_j is a $K(\pi, 1)$-manifold.

Consider first the simple case where X is a $K(\pi, 1)$-manifold. Then there exists an embedded circle $X_0 \cong S^1 \subset X$ representing an element of infinite order in $\pi_1(X)$. Theorem 7.47 applies immediately to X. Hence Theorem 7.48 applies immediately to any X of the form (8.0) with $n \geqq 1$. This gives us the following.

THEOREM **8.1**. — *Let* X *be a compact (oriented) 3-manifold which has a* $K(\pi, 1)$*-factor in its prime decomposition* (8.0). *Then* X *carries no metric of positive scalar curvature. In fact any metric with* $\kappa \geqq 0$ *on* X *is flat.*

It was shown in [GL_1] that any 3-manifold of the form

$$X = X_1 \# \dots \# X_\ell \# (S^1 \times S^2) \# \dots \# (S^1 \times S^2)$$

does carry $\kappa > 0$, provided that each X_j is diffeomorphic to S^3/Γ_j for some $\Gamma_j \subset O(4)$ acting standardly on S^3.

We now consider the non-compact case. To state the results concisely we need some definitions.

DEFINITION **8.2**. — An *incompressible surface* in a manifold X is a compact embedded surface $\Sigma \subset X$ such that $|\pi_1 \Sigma| = \infty$ and the induced map $\pi_1 \Sigma \to \pi_1 X$ is injective.

Definition **8.3**. — A *taut surface* in a manifold X is a properly embedded surface $\Sigma \subset X$ such that $|\pi_1 \Sigma| = \infty$ and the induced map $\pi_1 \Sigma \to \pi_1 X$ is injective.

An incompressible surface is always taut. The surface $\Sigma \times \{0\} \subset \Sigma \times \mathbf{R}$ (for Σ compact) is incompressible. The surface $S^1 \times \mathbf{R} \subset S^1 \times \mathbf{R}^2$ is taut.

Theorem **8.4**. — *A* 3-*manifold which contains a taut surface, cannot carry a complete metric of uniformly positive scalar curvature.*

A 3-*manifold which contains an incompressible surface carries no complete, non-flat metrics with scalar curvature* $\geqq 0$.

Remark **8.5**. — It is obvious that in Theorem 8.4 the hypothesis that the manifold contains a taut (or incompressible) surface could be changed to read: " some covering of the manifold contains a taut (resp., incompressible) surface ". It is also evident that if X satisfies either hypothesis of Theorem 8.4, then so does the manifold $X \# Y$ for *any* 3-manifold Y.

The first part of Theorem 8.4 has a stronger (although perhaps equivalent) formulation.

Definition **8.6**. — Let X be a 3-manifold. We say that a smoothly embedded circle $\gamma \subset X$ is *small* if it has infinite order in $H_1 X$ and if the normal circle to γ has infinite order in $H_1(X - \gamma)$.

Theorem **8.7**. — *Let* X *be an open* 3-*manifold with* $H_1(X)$ *finitely generated. If* X *contains a small circle, then* X *carries no complete metric of uniformly positive scalar curvature.*

Note. — The hypothesis on $H_1(X)$ can be dropped, as we shall see in § 10.

Observe that the circle $\gamma = S^1 \times \{\text{pt.}\} \subset S^1 \times \mathbf{R}^2$ is small. In fact this is true of any curve $\gamma \hookrightarrow X^3$ whose inclusion is a homotopy equivalence. Thus we conclude that *any* 3-*manifold which is homotopy equivalent to a circle cannot carry a complete metric of uniformly positive scalar curvature.*

On the other hand, the methods of [GL_2] show that the manifold

$$X_0 \equiv S^1 \times S^2 - \{\text{pt.}\}$$

does carry a metric of uniformly positive scalar curvature (which is the product metric $S^2 \times [0, \infty)$ at infinity). No circle in X_0, or in any covering of X_0, is small.

The hypothesis of smallness persists under connected sums. That is, if $\gamma \subset X$ is small, then γ remains small in the connected sum $X \# Y$ of Y with any 3-manifold Y. On the other hand, the connected sum of 3-manifolds with complete metrics of uniformly positive scalar curvature again carries such a metric.

We remark that Theorem 8.7 implies Theorem 8.1 as a special case. To see this, note first that any compact $K(\pi, 1)$-manifold admits a covering space which is homotopy equivalent to a circle. Thus, no such 3-manifold can carry positive scalar curvature. The full theorem follows easily by taking connected sums.

PROOF OF THEOREM **8.4**. — Suppose X contains a taut surface Σ, which we may assume to be orientable. Let $\gamma \subset \Sigma$ be an embedded circle which is not homotopic to zero in Σ (and therefore in X). Let $\widetilde{X} \to X$ be the covering space corresponding to the subgroup of $\pi_1 X$ generated by γ. Let $\widetilde{\Sigma} \to \Sigma$ be the induced covering. Since $\pi_1 \Sigma \hookrightarrow \pi_1 X$, there is a lifting of γ:

$$\begin{array}{ccccc} \widetilde{\gamma} & \subset & \widetilde{\Sigma} & \subset & \widetilde{X} \\ & & \downarrow & & \downarrow \\ \gamma & \subset & \Sigma & \subset & X \end{array}$$

inducing isomorphisms $\pi_1 \widetilde{\gamma} \cong \pi_1 \widetilde{\Sigma} \cong \pi_1 \widetilde{X}$. Since $\widetilde{\Sigma}$ is orientable and $\pi_1 \widetilde{\Sigma} \cong Z$ we have that Σ is homeomorphic to $S^1 \times \mathbf{R}$. Furthermore, this surface $\widetilde{\Sigma} \approx S^1 \times \mathbf{R}$ is properly embedded in $\widetilde{X}$. Hence, $\widetilde{X}$ is non-compact.

We now claim that $\widetilde{\gamma}$ is a small circle in $\widetilde{X}$. Clearly $\widetilde{\gamma}$ has infinite order in $\widetilde{X}$, so we need only to show that any small normal circle σ about γ has infinite order in $\pi_1(\widetilde{X} - \widetilde{\gamma})$. Note that the surface $\widetilde{\Sigma} - \widetilde{\gamma} \approx S^1 \times (\mathbf{R}^2 - \{0\})$ has two connected components each of which is properly embedded in $\widetilde{X} - \widetilde{\gamma}$. Considering one of these components as an element in $H^1(\widetilde{X} - \widetilde{\gamma}; Z)$ (via intersection) and evaluating on the normal circle σ gives the value 1. Hence, σ must have infinite order in $\pi_1(\widetilde{X} - \widetilde{\gamma})$, and $\widetilde{\gamma}$ is a small circle in $\widetilde{X}$, as claimed.

The first statement of Theorem 8.4 now follows from Theorem 8.7.

Since any compact surface Σ with $|\pi_1 \Sigma| = \infty$ is enlargeable, the second statement follows immediately from Theorem 6.12 (2). ∎

PROOF OF THEOREM **8.7**. — Let γ be a small circle in X and let $Z \approx S^1 \times S^1$ be the boundary of a small tubular neighborhood of γ in X. (If traversing γ reverses orientation, then we replace γ by 2γ.) We observe that the inclusion homomorphism

$$i_* : H_1 Z \to H_1(X - \gamma)$$

is injective. Indeed, the generator of the kernel of the composition

$$H_1 Z \to H_1(X - \gamma) \to H_1 X$$

is exactly the subgroup generated by the normal circle, which, by assumption, has infinite order in $H_1(X - \gamma)$.

Since $H_1(X - \gamma)$ is finitely generated, there exists a homomorphism

$$f_* : H_1(X - \gamma) \to H_1 Z \cong \mathbf{Z} \oplus \mathbf{Z}$$

such that the composition $f_* \circ i_* : H_1 Z \to H_1 Z$ is injective. Since Z is a $K(\pi, 1)$-space, there exists a mapping $f : X - \gamma \to Z$ which induces the homomorphism f_*. Since the homomorphism $f_* \circ i_*$ is injective, the composition

$$Z \overset{i}{\hookrightarrow} X - \gamma \overset{f}{\to} Z$$

is of non-zero degree. Consequently, X and Z satisfy the hypotheses of Theorem 7.44.

We conclude that X contains transversals to γ of finite area. In fact, there must exist a regular, properly embedded, non-compact minimal surface $\Sigma \subset X$ which has *finite area* and is *stable*. (By " stable " we mean that the second variation of area with respect to compactly supported variations of Σ is $\geqq 0$.)

The existence of this minimal surface Σ is a consequence of the work of Federer, Fleming and Almgren. We see this as follows. We first note that in codimension-one it suffices to test on *integral* currents when applying the Hahn-Banach techniques of § 7. More specifically, by [F_2] we know that the function $e(a)$ defined in (7.30) remains unchanged if in this definition we restrict T to the integral 2-currents. By standard compactness theorems (cf. [FF]), we can solve the relative Plateau problem in $(X_{[0,a]}, \partial X_{[0,a]})$ for each a. We then extract a sequence of minimizing 2-currents Σ_{a_n} (where $a_n \to \infty$) which converges on each compact subset to a minimizing integral 2-current Σ without boundary. The homology condition insures that every Σ_{a_n} must meet $\tilde{\gamma}$, and standard monotonicity results then imply that the limit must be non-zero. The minimizing property of the Σ_{a_n}'s implies that Σ is homologically minimizing. The regularity theory for minimizing surfaces in riemannian 3-manifolds, which follows from Fleming [Fl] and Almgren [Al], proves that the limit Σ is a regularly embedded smooth orientable minimal surface. This surface is non-compact and stable. Furthermore, by lower semi-continuity of mass and the uniform bound on $\text{Area}(\Sigma_{a_n})$, we know that $\text{Area}(\Sigma) < \infty$.

We conclude the proof Theorem 8.7 by showing that in contradiction to the above one must have $\text{Area}(\Sigma) = \infty$. This will follow immediately from the central result, Theorem 10.1 of the tenth section. ■

We offer below a more delicate area estimate in the case of 3-manifolds with $\kappa > 0$. In what follows all surfaces are assumed orientable.

THEOREM **8.8**. — *Let* X *be a riemannian* 3*-manifold of positive scalar curvature. Then any complete stable minimal surface of finite area in* X *is homeomorphic to* S^2. *(In particular, no such surface can be non-compact.)*

PROOF. — Let $\Sigma \subset X$ be a complete stable minimal surface. The stability of Σ is equivalent to the fact that for all $\varphi \in C_0^\infty(\Sigma)$ we have the inequality

$$\textbf{(8.9)} \qquad \int_\Sigma \left\{ |\nabla\varphi|^2 + K\varphi^2 - \frac{1}{2}(\kappa + ||A||^2)\varphi^2 \right\} \geqq 0,$$

where κ denotes the scalar curvature function of X and where K and A denote the Gaussian curvature and the second fundamental form of Σ respectively (cf. [SY$_1$]). Note in particular that for $\kappa > 0$, stability implies that

$$\int_\Sigma \{|\nabla\varphi|^2 + K\varphi^2\} > 0 \tag{8.10}$$

for all $\varphi \in C_0^\infty(\Sigma)$. If Σ is compact, we can choose $\varphi \equiv 1$ and conclude that $\Sigma \approx S^2$. If Σ is not compact, we apply the following result (in the special case $\alpha = 1$). ■

THEOREM **8.11**. — *Let Σ be a complete non-compact riemannian surface with the property that, for some fixed $\alpha > \frac{1}{2}$,*

$$I_\alpha(\varphi) \equiv \int_\Sigma \{|\nabla\varphi|^2 + \alpha K\varphi^2\} \geqq 0 \tag{8.12}$$

for all $\varphi \in C_0^\infty(\Sigma)$. *Then* $\mathrm{Area}(\Sigma) = \infty$.

PROOF. — We may assume without loss of generality that Σ is connected. We take a compact domain $\Omega \subset \Sigma$ with piecewise smooth boundary $\gamma = \partial\Omega$, and consider the family of " concentric " domains

$$\Omega(s) \equiv \{x \in \Sigma : \mathrm{dist}(x, \Omega) \leqq s\}.$$

These are the sublevel sets of the compact exhaustion function $s(x) = \mathrm{dist}(x,\Omega)$. By C^∞ approximation of the originally given Ω and the metric on Σ, we may assume that the boundaries $\gamma(s) = \partial\Omega(s)$ are piecewise smooth. We then set

$$\begin{aligned} L(s) &\equiv \mathrm{length}(\gamma(s)) \\ K(s) &\equiv \int_{\gamma(s)} K(\sigma)\, d\sigma \qquad (\sigma \equiv \text{arc length on } \gamma(s)) \\ \chi(s) &\equiv \text{Euler characteristic of } \Omega(s) \\ \Gamma(s) &\equiv \text{total geodesic curvature of } \gamma(s) \end{aligned}$$

where by " total geodesic curvature " we mean the integral of geodesic curvature over the regular set of $\gamma(s)$ plus the sum of exterior angles at vertices. By Gauss-Bonnet applied to the region $\Omega(s_2) - \Omega(s_1)$ we have that

$$\int_{s_1}^{s_2} K(s)\, ds = 2\pi(\chi(s_2) - \chi(s_1)) - (\Gamma(s_2) - \Gamma(s_1)). \tag{8.13}$$

That is,

$$K(s) = [2\pi\chi(s) + \Gamma(s)]' \tag{8.14}$$

in the distributional sense.

We shall now apply the inequality (8.12) with a function φ of the form $\varphi(x) = \Phi(s(x))$. (It is clearly permissible to pass to Lipschitz functions φ with compact support on Σ.) We shall choose $\Phi(s)$ so that $\Phi(0) > 0$, Φ decreases to zero on an interval $[0, \rho]$, and $\Phi(s) = 0$ for $s \geqq \rho$. The inequality (8.12) then takes the form:

$$(8.15) \qquad 0 \leqq \int_0^\rho (\Phi'(s))^2 L(s)\,ds + \alpha \int_{\Omega(\rho)} K\varphi^2\, dA.$$

We shall estimate the second term on the right. Using (8.14) and Gauss-Bonnet, we see that

$$(8.16) \qquad \begin{aligned} \int_{\Omega(\rho)} K\varphi^2\,dA &= \int_{\Omega(0)} \varphi^2 K\,dA + \int_{\Omega(\rho)-\Omega(0)} K\varphi^2\,dA \\ &= \Phi^2(0)\{2\pi\chi(0) - \Gamma(0)\} + \int_0^\rho K(s)\,\Phi^2(s)\,ds \\ &= \int_0^\rho \{\Gamma(s) - 2\pi\chi(s)\}(\Phi^2(s))'\,ds. \end{aligned}$$

Now for curves $\gamma(s)$ with positive exterior angles, one has that $L'(s) = \Gamma(s)$. In the general case, where there also exist negative exterior angles (or " concave " vertices), we only have the inequality

$$L'(s) \leqq \Gamma(s).$$

Since $(\Phi^2)' \leqq 0$, we conclude that

$$(8.17) \qquad \begin{aligned} \int_0^\rho \Gamma(s)(\Phi^2(s))'\,ds &\leqq \int_0^\rho L'(s)(\Phi^2(s))'\,ds \\ &= -2L(0)\,\Phi(0)\,\Phi'(0) - \int_0^\rho L(s)(\Phi^2(s))''\,ds. \end{aligned}$$

Combining (8.15), (8.16) and (8.17) we get the following inequality:

Lemma 8.18. — *For all $\Phi = \Phi(s)$ as described above, we have*

$$0 \leqq -2\int_0^\rho \left\{\Phi\Phi'' + \left(1 - \frac{1}{2\alpha}\right)(\Phi')^2\right\} L\,ds - 2L(0)\,\Phi(0)\,\Phi'(0) - 4\pi\int_0^\rho \chi\Phi\Phi'\,ds.$$

We now choose $\Phi(s) = \rho - s$. In this case 8.18 becomes

$$(8.19) \qquad 0 \leqq -\left(1 - \frac{1}{2\alpha}\right)\operatorname{Area}(\Omega(\rho) - \Omega(0)) + \rho L(0) + 2\pi\int_0^\rho \chi(s)(\rho - s)\,ds.$$

If we assume $\chi(s) \leqq 0$ for $s \in [0, \rho]$, then (8.19) implies

$$(8.20) \qquad \operatorname{Area}(\Omega(\rho) - \Omega(0)) \leqq \frac{2\alpha}{(2\alpha - 1)} L(0)\,\rho.$$

Let us choose our domain Ω as follows. Fix a point $x_0 \in \Sigma$ and consider the family of concentric " balls " $B(R) = \{x \in \Sigma : \operatorname{dist}(x, x_0) \leqq R\}$. For a given R, the complement $\Sigma - B(R)$ may contain some components, say $D_1(R)$, $D_2(R)$, ... which are *bounded* in Σ. We set $\widetilde{B}(R) = B(R) \cup \bigcup_m D_m(R)$, and for fixed $R_2 > R_1 > 0$ we define Ω to be the difference

$$\Omega = \widetilde{B}(R_2) - \widetilde{B}(R_1)^0.$$

We now observe that on each component Ω_α of Ω we have that

$$\pi_1(\Omega_\alpha) \hookrightarrow \pi_1(\Sigma - \{x_0\}) \tag{8.21}$$

is injective. Indeed, let γ be a closed curve embedded in Ω_α which bounds a disk $D \subset \Sigma - \{x_0\}$. Then $D \subset \Sigma - B(R_1)^0$, since otherwise there would be an interior point nearest to x_0. Furthermore, we have $D \subset \Sigma - \widetilde{B}(R_1)^0$ since if D meets a bounded component of $\Sigma - B(R_1)$, then D contains the closure of that component contrary to the assumption on γ. Observe now that $D \cap (\Sigma - B(R_2))$ consists of bounded components of $\Sigma - B(R_2)$, that is, we have $D \subset \widetilde{B}(R_2)$. Thus, $D \subset \widetilde{B}(R_2) - \widetilde{B}(R_1) = \Omega$, and by connectedness, $D \subset \Omega_\alpha$ as claimed. This proves (8.21). Note that, clearly, $\pi_1(\Omega_\alpha) \neq 0$ for any α.

For each $s < R_2$, consider now the domain $\Omega(s)$ as above. Each component $\Omega_\alpha(s)$ of $\Omega(s)$ contains a component $\Omega_{\alpha'}$ of Ω, and so from the factoring

$$0 \neq \pi_1(\Omega_{\alpha'}) \to \pi_1(\Omega_\alpha(s)) \to \pi_1(\Sigma - \{x_0\}),$$

and by (8.21), $\pi_1(\Omega_\alpha(s)) \neq 0$, and so $\chi(\Omega_\alpha(s)) \leqq 0$. We conclude that

$$\chi(s) = \chi(\Omega(s)) \leqq 0.$$

Thus, (8.20) holds for all $\rho \leqq R_1$.

Recall now that the term $L(0)$ in (8.20) is

$$\begin{aligned} L(0) &= \operatorname{length}(\partial\Omega) \\ &= \operatorname{length}(\partial\widetilde{B}(R_2)) + \operatorname{length}(\partial\widetilde{B}(R_1)) \\ &\leqq \operatorname{length}(\partial B(R_2)) + \operatorname{length}(\partial B(R_1)). \end{aligned}$$

Assuming that $\operatorname{Area}(\Sigma) < \infty$, we have

$$\liminf_{R_2 \to \infty} \operatorname{length}(\partial B(R_2)) = 0. \tag{8.22}$$

We now let $R_2 \to \infty$ and let $\rho \to R_1$. Then (8.20) becomes

$$\operatorname{Area}(\widetilde{B}(R)) \leqq \frac{2\alpha}{2\alpha - 1} \operatorname{length}(\partial\widetilde{B}(R)) . R.$$

Setting $a(R) \equiv \operatorname{Area}(\widetilde{B}(R))$, this implies that

$$a(R) \leqq \frac{2\alpha}{2\alpha - 1} a'(R)\, R$$

which in turn implies that

$$a(\mathrm{R}) \geqq a(1)\, \mathrm{R}^{1-\frac{1}{2\alpha}}. \tag{8.23}$$

This completes the proof. ■

We note that we have also proved the following.

THEOREM **8.24**. — *Suppose* Σ *satisfies the hypotheses of Theorem* 8.11. *Then for all* $\mathrm{R} > 0$

$$\mathrm{Area}(\widetilde{\mathrm{B}}(\mathrm{R})) \geqq c\mathrm{R}^{1-\frac{1}{2\alpha}}$$

where $c = \mathrm{Area}(\widetilde{\mathrm{B}}(1))$.

9. RESULTS FOR 4-MANIFOLDS

The results of section 7 have interesting applications also to 4-manifolds. This is basically because any compact surface Σ with $\chi(\Sigma) \leqq 0$, is enlargeable. The same is true of any circle bundle over Σ. Thus we have immediately the following result.

Definition 9.1. — We say that a compact embedded submanifold $\Sigma \subset X$ is *incompressible* if $|\pi_1 \Sigma| = \infty$ and if the homomorphism $\pi_1 \Sigma \to \pi_1 X$ is injective.

Theorem 9.2. — *A compact* $K(\pi, 1)$ *4-manifold which carries an incompressible surface cannot carry a metric of positive scalar curvature. In fact any metric with* $\kappa \geqq 0$ *on such a manifold must be flat.*

Proof. — This follows immediately from 7.47. ∎

It seems that one should be able to substantially strengthen this result. One expects the conclusion to hold for any compact $K(\pi, 1)$-manifold, at least in dimension 4.

On the other hand the existence of an incompressible surface is independently quite strong, and we have the following generalization of 9.2 above.

Definition 9.3. — A compact incompressible surface Σ embedded in a 4-manifold X is called *small*, if $|\pi_1 X / \pi_1 \Sigma| = \infty$ and if the normal circle about Σ has infinite order in $H_1(\widetilde{X} - \Sigma)$, where $\widetilde{X}$ is the covering of X with $\pi_1 \widetilde{X} \cong \pi_1 \Sigma$.

Theorem 9.4. — *A compact spin 4-manifold which admits a small incompressible surface cannot carry a metric of positive scalar curvature. In fact, any metric with* $\kappa \geqq 0$ *must be Ricci flat.*

Note. — The hypothesis of Theorem 9.4 persists after taking connected sum with an arbitrary (spin) 4-manifold.

Proof. — The proof of Theorem 7.4.7 carries through.

Since on a compact oriented 4-manifold every class in $H^2(X; \mathbf{Z})$ is represented by an embedded surface, it seems that Theorem 9.4 should be capable of further strengthening.

10. SOME FUNDAMENTAL THEOREMS FOR INCOMPLETE 3-MANIFOLDS

(Applications of the stability inequality)

This section concerns the global geometry of a 3-manifolds X whose scalar curvature satisfies

$$(10.1) \qquad \kappa \geqq \kappa_0$$

for some constant κ_0 (usually assumed to be positive). We shall study X by examining the properties of a stable minimal surface $\Sigma \subset X$. Neither X nor Σ will be assumed to be complete, but both will be assumed to be oriented.

For a compact set $\Omega \subset \Sigma$ and $\rho \geqq 0$, we define as in § 8

$$\Omega(\rho) \equiv \{x \in \Sigma : \mathrm{dist}_\Sigma(x, \Omega) \leqq \rho\}.$$

Our first main result is the following.

THEOREM 10.2. — *Let Σ be a compact stable minimal surface in a 3-manifold with scalar curvature $\geqq \kappa_0 > 0$. Let $\Omega \subset \Sigma$ be a compact connected domain, and let $\rho > 0$ be a number such that*

(1) *$\Omega(\rho)$ does not meet $\partial\Sigma$.*

(2) $\mathrm{Image}[H_1(\Omega) \to H_1(\Omega(\rho))] \neq 0$.

Then

$$(10.3) \qquad \rho < \frac{\pi}{\sqrt{\kappa_0}}.$$

Proofs of theorems will be given at the end of the section.

Note that hypothesis (2) always holds if the inclusion $H_1(\Omega) \to H_1(\Sigma)$ is $\neq 0$.

REMARK 10.4. — This result also holds when Ω is not connected, provided that hypothesis (2) is changed to read: " $\mathrm{Image}[H_1(\Omega_\alpha) \to H_1(\Omega(\rho))] \neq 0$ for each component Ω_α of Ω. "

REMARK 10.5. — For (not necessarily connected) Ω, hypothesis (2) can also be replaced by the statement:

$$\text{Euler characteristic}(\Omega(s)) \leqq 0$$

for all $s \in [0, \rho]$.

REMARK **10.6.** — The inequality (10.3) is nearly sharp. To see this, consider the stable minimal surface $\Sigma \equiv \{\text{pt.}\} \times S^2 \subset S^1 \times S^2$, where $S^1 \times S^2$ is a riemannian product of standard unit spheres. Here $\kappa_0 = 2$. Hence, by letting Ω be a thin neighborhood of the equator in Σ, we see that the hypotheses of the theorem can be fulfilled for all values of $\rho < \pi/2$. Hence, the best possible conclusion for Theorem 10.2 would be: $\rho < \pi/\sqrt{2\kappa_0}$.

As a major consequence we have the following.

THEOREM **10.7.** — *Let* X *be a compact* 3*-manifold with a (possibly empty) boundary, and suppose* X *is equipped with a metric of scalar curvature* $\geqq 1$. *Then any closed curve* $\gamma \subset X$ *such that*

(1) $[\gamma] = 0$ *in* $H_1(X, \partial X)$,
(2) $\text{dist}(\gamma, \partial X) > 2\pi$

must already bound in its 2π*-neighborhood,* $U_{2\pi}(\gamma) \equiv \{x \in X : \text{dist}(x, \gamma) \leqq 2\pi\}$. *That is,* $[\gamma] = 0$ *in* $H_1(U_{2\pi}(\gamma))$ *(and in particular, in* $H_1(X)$*).*

If X (as in (10.7)) is also complete and contractible, then every γ bounds in its 2π-neighborhood. In the language of [G],

$$\text{Fill Radius}(\gamma \subset X) \leqq 2\pi.$$

It follows that X contains no " line ", *i.e.* doubly infinite, minimizing geodesic.

Theorem 10.7 is a powerful geometric estimate. It gives the following immediate corollaries.

COROLLARY **10.8.** — *Let* X *be a* 3*-manifold which is diffeomorphic to the interior of a compact* 3*-manifold* $\overline{X}$ *with* $H_1(\partial\overline{X}) \neq 0$. *Then* X *carries no complete metric of uniformly positive scalar curvature.*

PROOF. — Take a curve $\gamma \subset \partial\overline{X}$ such that $[\gamma] \neq 0$ in $H_1(\overline{X})$, and push γ into X. ∎

An example of course is $X = S^1 \times \mathbf{R}^2$. We shall easily retrieve Theorems 8.8 and, in fact, 8.1 as Corollaries. We also have the following

COROLLARY **10.9.** — *A complete* 3*-manifold of uniformly positive scalar curvature and with finitely generated fundamental group is simply-connected at infinity.*

PROOF. — Let X be a complete 3-manifold with $\kappa \geqq 1$ and choose an exhaustion $K_1 \subset K_2 \subset \ldots \subset X$ of X by compact domains with smooth boundaries. Set $\Omega_i = X - K_i$, so that $\Omega_1 \supset \Omega_2 \supset \ldots$ is a neighborhood system of infinity. We may assume that $\text{dist}(\partial\Omega_i, \Omega_{i+1}) > 2\pi$ for each i.

Consider now a circle $\gamma \subset \Omega_i - \Omega_j$ for some $j > i$, and suppose γ is homotopic to a curve $\gamma' \subset \Omega_{i'} - \Omega_{j'}$ where $i' > j$. We now alter the metric far away from the curves $\Gamma = \gamma \cup \gamma'$ in the region Ω_{i-2} so that we can apply the classical theorem of C. B. Morrey [Mor]. This theorem asserts that if $[\gamma] \neq 0$ in $\pi_1(\Omega_{i-2})$, then there is a minimizing annulus A in Ω_{i-2} joining γ to γ'. Let $\Sigma = A \cap X_\Gamma$ where $X_\Gamma \equiv \Omega_{i-1} - \Omega_{j+1}$ carries the original (unaltered) metric of scalar curvature $\kappa \geqq 1$. Then Σ is a stable minimal surface in X_Γ with $\partial\Sigma \equiv \Gamma \pmod{\partial X}$. Since the distance from each component of Γ (γ and γ'), to the other and to ∂X_Γ, is $> 2\pi$, we conclude as in the proof of 10.7 that Σ cannot exist. Hence $[\gamma] = 0$ in $\pi_1(\Omega_{i-2})$. This completes the proof. ■

Corollary 10.10. — *Let* X *be a compact* 3*-manifold with boundary and let* γ *be a circle in* ∂X *such that* $[\gamma] \neq 0$ *in* $H_1(X)$. *Then for any riemannian metric with scalar curvature* $\geqq 1$ *on* X *(there exist many!) and any curve* $\gamma' \subset X$ *which is homologous to* γ, *we have that*

$$\operatorname{dist}(\gamma', \partial X) \leqq 2\pi.$$

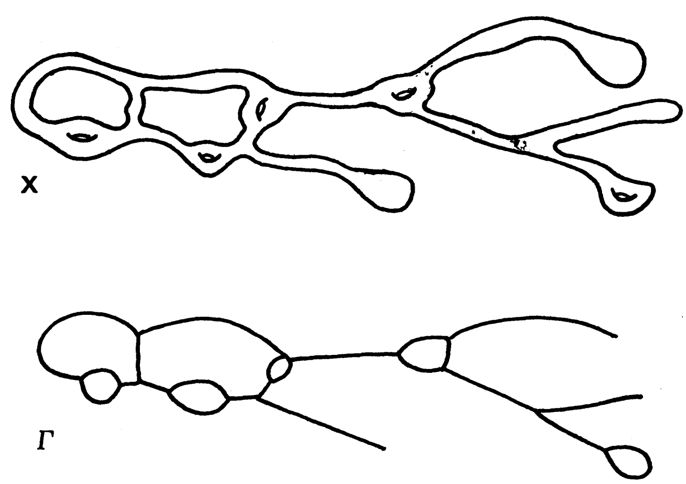
Proof. — This is obvious.

The following Corollary is a consequence of arguments in $[G_1]$.

Corollary 10.11. — *Let* X *be a closed* 3*-manifold of scalar curvature* $\geqq 1$. *Then there exists a distance decreasing map* $f: X \to \Gamma$ *onto a metric graph (a linear graph in* $\mathbf{R}^N$, *say), so that, for each* $p \in \Gamma$,

$$\operatorname{diameter}(f^{-1}(p)) \leqq 12\pi.$$

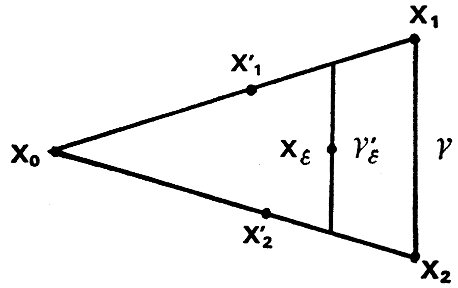
Note. — This means that 3-manifolds with $\kappa \geqq 1$ and diameter $\gg 1$ are " long and thin ". Of course, arbitrarily long and complicated manifolds of this type can be constructed by taking connected sums (as in $[GL_2]$) of copies of $S^1 \times S^2$ and S^3.



Proof. — We may assume X to be connected. We fix a point $x_0 \in X$ and define the function $d(x) = \operatorname{dist}(x, x_0)$. Consider now the quotient $\Gamma = X/\sim$ where the relation " $\sim$ " is defined by setting $x \sim x'$ if x and x' lie in the same arc component of a level set of d. Then (at least after approximation of d), Γ will be a finite metric graph and the projection map $f: X \to \Gamma$ will be distance-non-increasing.

Suppose now that x_1 and x_2 are points with $f(x_1) = f(x_2)$. We shall show that $\mathrm{dist}(x_1, x_2) \leqq 12\pi$. To begin we join x_1 to x_2 by a curve γ on which d is constant. We then join x_1 and x_2 to x_0 by minimizing geodesics γ_1 and γ_2 respectively. These geodesics may be chosen so that the triangle $T = \gamma_1\gamma\gamma_2$ is null-homotopic in X. By Theorem 10.7, the closed curve T bounds a disk D in its 2π-neighborhood.

Note that if $d(x_1)(= d(x_2)) < 6\pi$, we are done. Hence, we assume $d(x_1) \geqq 6\pi$ and consider the curves $\gamma'_\varepsilon \equiv \{x \in D : d(x) = d(x_1) - 2\pi - \varepsilon\}$ joining γ_1 to γ_2. Each such curve, for $\varepsilon > 0$ small, has a point x_ε which lies at a distance $\leqq 2\pi$ from both γ_1 and γ_2. That is there are points $x'_i \in \gamma_i$ with $\mathrm{dist}(x_\varepsilon, x'_i) \leqq 2\pi$. Of course, $|d(x_\varepsilon) - d(x'_i)| \leqq 2\pi$, and so $\mathrm{dist}(x_i, x'_i) \leqq 4\pi + \varepsilon$ for each i. Hence, $\mathrm{dist}(x_1, x_2) \leqq 12\pi$ as claimed. ■



Corollary 10.12. — *A complete stable minimal surface in a 3-manifold of uniformly positive scalar curvature is a 2-sphere.*

Recall that all manifolds are assumed to be orientable.

Proof. — Let Σ be the stable minimal surface. If Σ is compact, the stability inequality applied to $f \equiv 1$ shows that $\chi(\Sigma) > 0$ and so $\Sigma \approx S^2$. If Σ is not compact, fix a point $x_0 \in \Sigma$ consider the curve $\gamma_\rho \equiv \{x \in \Sigma : \widetilde{\mathrm{dist}}_\Sigma(x, x_0) = \rho\}$ where $\widetilde{\mathrm{dist}}$ is a smooth approximation to the distance function in Σ and where $\rho > 0$ is a regular value. Some component $\hat{\gamma}_\rho$ of γ_ρ does not bound in $\Sigma - \{x_0\}$. Let Ω be a thin tubular neighborhood of $\hat{\gamma}_\rho$ in Σ. Then $\mathbf{Z} \cong H_1(\Omega) \to H_1(\Sigma - \{x_0\})$ is injective, and so $H_1(\Omega) \to H_1(\Omega(s))$ is non-zero for all $s < \rho$. Choosing ρ sufficiently large $(> \pi/\sqrt{\kappa_0})$ we get a contradiction to Theorem 10.2. ■

Recall from § 8 that a circle γ embedded in a 3-manifold X is called *small* if γ has infinite order in $H_1(X)$ and if the normal circle to γ has infinite order in $H_1(X - \gamma)$.

Corollary 10.13. — *An open 3-manifold which contains a small circle cannot carry a complete metric of uniformly positive scalar curvature.*

PROOF. — Suppose X carries such a metric, and let $\gamma \subset X$ be a small circle. For each $\rho > 0$, set $X_\rho \equiv \{x \in X : \delta_\gamma(x) \leqq \rho\}$ where $\delta_\gamma(\cdot)$ is a smooth approximation to $\mathrm{dist}_X(\cdot, \gamma)$. Assume ρ is a regular value of δ_γ. Since $[\gamma]$ has infinite order in $H_1(X_\rho)$, we see that there exists a class $u \in H_2(X_\rho, \partial X_\rho) \cong H^1(X_\rho)$ so that $(u, \gamma) \neq 0$. Let $(\Sigma_\rho, \partial\Sigma_\rho) \subset (X_\rho, \partial X_\rho)$ be a surface of least area in u. (Σ_ρ is a regular embedded stable minimal surface.) By a slight deformation we make γ transversal to Σ_ρ.

We now observe that any component of Σ_ρ which meets γ with non-zero intersection number must also meet ∂X_ρ (such a component exists since $\gamma . \Sigma \neq 0$), for otherwise some non-zero multiple of the normal circle would be homologous to zero in $X - \gamma$. Choosing ρ sufficiently large and applying Theorem 10.2 as we did in the previous corollary, leads to a contradiction. (Alternatively, one could take a limit $\Sigma_\rho \to \Sigma$ as $\rho \to \infty$ and apply Corollary 10.12 directly.) ∎

Suppose now that γ is a non-contractible curve in a closed $K(\pi, 1)$ 3-manifold X. Let $\tilde{X} \to X$ be the covering space corresponding to the cyclic subgroup of $\pi_1 X$ generated by γ. Then any $\tilde{\gamma} \subset \tilde{X}$, obtained by lifting γ, is small. This observation remains true if we take the connected sum of X with any other 3-manifold. Hence we conclude (cf. Theorem 8.1):

COROLLARY **10.14**. — *A compact 3-manifold (without boundary) which contains a* $K(\pi, 1)$*-factor in its prime decomposition, cannot carry a metric of positive scalar curvature.*

COROLLARY **10.15**. — *Suppose* $S^1 \times \mathbf{R}^2$ *is equipped with a complete riemannian metric. Fix a circle* $\gamma = S^1 \times \{0\}$ *and for each* $R > 0$, *set*

$$\kappa(R) \equiv \inf\{\kappa(x) : x \in U_R(\gamma)\}.$$

Then
$$\kappa(R) \leqq \frac{4\pi^2}{R^2}.$$

PROOF. — Choose a circle $\sigma \subset X - U_{2R}(\gamma)$ which links γ once. Let Σ be the classical Douglas-Morrey solution to the Plateau problem for σ. Then Σ is an immersed disk which must meet γ in some point, say x_0. Fix R_1 and R_2 with $0 < R_1 < R_2 < R$, and let Ω be an appropriate component of the region $\{x \in \Sigma : R_1 \leq \mathrm{dist}_\Sigma(x, x_0) \leqq R_2\}$. Assuming $\kappa(R_1 + R_2) > 0$, we conclude from Theorem 10.2 that

$$\kappa(R_1 + R_2) \leqq \frac{\pi^2}{R_1^2}.$$

Letting R_1 and R_2 approach $R/2$ gives the result. ∎

THEOREM **10.16**. — *Let* X *be a complete 3-manifold with*

$$\kappa \geqq -1.$$

Let $\Sigma \subset X$ *be a complete (connected) stable minimal surface. Fix* $x_0 \in \Sigma$ *and let*

$$B(R) \equiv \{x \in \Sigma : \mathrm{dist}_\Sigma(x, x_0) \leqq R\}.$$

For each $R > 0$ *let*

$$\alpha(R) \equiv \operatorname{rank}\{\operatorname{image}(H_1(B(R)) \to H_1(\Sigma))\} - 1.$$

Then
$$\operatorname{Area}(B(R)) \geqq \frac{4\pi}{1 + (\pi/R)^2}\,\alpha(R).$$

In particular, if $\chi(\Sigma) \leqq 0$, *then*

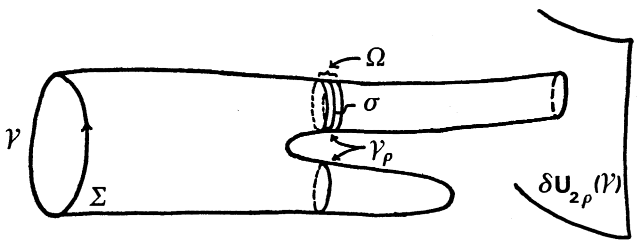
$$\operatorname{Area}(\Sigma) \geqq 4\pi\,|\chi(\Sigma)|.$$

Proof that **10.2** implies **10.7**. — Let Σ be the surface of least area which spans γ modulo ∂X. (This is given by taking the integral current of least mass.) This surface may have boundary points on ∂X, but in $X - \partial X$ it is a regular, oriented stable minimal surface with boundary γ. (This follows from the basic regularity results of Almgren [Al], Fleming [Fl], and Hardt-Simon [HS].)

Suppose Σ is not contained in the 2π-neighborhood of γ. Then it is not contained in the 2ρ-neighborhood for some $\rho > \pi$. Basically, the idea now is this. Let γ_ρ be the ρ-level curve in Σ, *i.e.*,

$$\gamma_\rho = \{x \in \Sigma : \operatorname{dist}_\Sigma(x, \gamma) = \rho\}.$$

Since γ is homologous to γ_ρ and γ does not bound in $U_{2\rho}(\gamma)$ (its 2ρ-neighborhood in X), we know that some component σ of γ_ρ does not bound in $U_{2\rho}(\gamma)$. In particular, σ does not bound in $\sigma(\rho)$, its ρ-neighborhood *on* Σ.



We now let Ω be a very thin tubular neighborhood of σ. (To avoid mess, we assume $\Omega = \sigma$.) By the above, we have that $H_1(\Omega) \to H_1(\Omega(\rho))$ is injective. Furthermore, if we choose ρ (which is $> \pi$) very close to π, then $\Omega(\rho)^0$ will not meet $\partial\Sigma$. To see this note that: $\operatorname{dist}_\Sigma(\Omega, \partial\Sigma) \cong \operatorname{dist}_\Sigma(\sigma, \partial\Sigma) = \min\{\rho, \operatorname{dist}_\Sigma(\sigma, \Sigma \cap \partial X)\}$. However, $\operatorname{dist}_\Sigma(\sigma, \Sigma \cap \partial X) \geqq \operatorname{dist}_\Sigma(\gamma, \Sigma \cap \partial X) - \rho \geqq \operatorname{dist}_X(\gamma, \partial X) - \rho$, and we have assumed that $\operatorname{dist}_X(\gamma, \partial X) > 2\pi$. Consequently, Ω satisfies the hypotheses of Theorem 10.3.

We conclude that $2\rho \leqq 2\pi$, contrary to assumption.

To make the above argument rigorous, we need only to approximate the function $\operatorname{dist}_\Sigma(x, \gamma)$ by a smooth function $\delta(x)$. The curve σ is then an appropriate connected component of $\delta^{-1}(\rho)$ where ρ is a regular value of δ with ρ as above. This completes the proof of 10.7. ■

PROOF OF THEOREM **10.2**. — The stability of the minimal surface $\Sigma \subset X$ implies that for any $f \in C_0^\infty(\Sigma) \equiv \{$smooth functions with compact support in $\Sigma - \partial\Sigma\}$, we have

$$\int_\Sigma \left\{ |\nabla f|^2 + Kf^2 - \frac{1}{2}\kappa_0 f^2 \right\} dA \geqq 0. \tag{10.17}$$

(See (8.9).) Here K is the Gaussian curvature of Σ and κ_0 is a lower bound for the scalar curvature of X. This inequality continues to hold for f in the $L^{1,2}$-closure of $C_0^\infty(\Sigma)$.

We apply this inequality as we did in 8.11. Let $s(x) = \mathrm{dist}_\Sigma(x, \Omega)$ and assume by analytic approximation of Ω and the metric that $s(x)$ has piecewise smooth level sets up to the value ρ (cf. [Fia] (1)). For each constant s_0 we see that $\Omega(s_0) = \{x : s(x) \leqq s_0\}$. We then set

$$\gamma(s) = \partial\Omega(s)$$

$$L(s) = \mathrm{length}(\gamma(s))$$

$$K(s) = \int_{\gamma(s)} K(\sigma)\, d\sigma \qquad (\sigma = \text{arc length on } \gamma(s))$$

and we write

$$\int_{\Omega(s)} K\, dA = 2\pi\chi(s) - \Gamma(s) = 2\pi\chi(0) - \Gamma(0) + \int_0^s K(s')\, ds'$$

where $\chi(s) \equiv$ The Euler characteristic of $\Omega(s)$,

$\Gamma(s) \equiv$ The total geodesic curvature of $\gamma(s)$.

(Recall that $\Gamma(s)$ includes " angle " contributions.) Again we note that in the distribution sense

$$K(s) = \frac{d}{ds}[2\pi\chi(s) - \Gamma(s)]. \tag{10.18}$$

We now choose a positive decreasing function $\varphi(s)$ on $[o, \rho]$ with $\varphi(\rho) = 0$, and define a function

$$f(s) \equiv \begin{cases} 0 & \text{if } x \notin \Omega(\rho) \\ \varphi(s(x)) & \text{if } x \in \Omega(\rho) - \Omega \\ \varphi(0) & \text{if } x \in \Omega. \end{cases}$$

Then, after setting

$$A(s) = \mathrm{Area}(\Omega(s))$$

(1) For a detailed analysis of the functions s, L and L′ in the analytic case, see the paper of Fiala [Fia]. Relevant discussion can also be found in [BZ, §§ 2 and 3 of Ch. I], [Ha], [O], [CF], and [I].

the stability inequality (10.17) on f becomes

(10.19)
$$\int_0^\rho (\varphi'(s))^2 L(s)\, ds + \varphi^2(o)\,[2\pi\chi(o) - \Gamma(o)] + \int_0^\rho K(s)\, \varphi^2(s)\, ds \geqq \frac{1}{2}\kappa_0 \left\{ \varphi^2(o)\, A(o) + \int_0^\rho \varphi^2(s)\, L(s)\, ds \right\}.$$

From (10.18) we have that

(10.20)
$$\varphi^2(o)\,[2\pi\chi(o) - \Gamma(o)] + \int_0^\rho K(s)\, \varphi^2(s)\, ds = -\int_0^\rho [2\pi\chi(s) - \Gamma(s)]\,(\varphi^2(s))'\, ds.$$

As pointed out in § 8 we have the basic inequality (cf. [Fia])

$$L'(s) \leqq \Gamma(s).$$

(In the absence of "concave" corners the equality holds.) This last inequality, together with the fact that $(\varphi^2)' \leqq o$, implies

$$\int_0^\rho \Gamma(s)\,(\varphi^2(s))'\, ds \leqq \int_0^\rho L'(s)\,(\varphi^2(s))'\, ds = -2L(o)\,\varphi(o)\,\varphi'(o) - \int_0^\rho L(s)\,(\varphi^2(s))''\, ds.$$

Combining with (10.19) and (10.20) gives the following fundamental result.

INEQUALITY **10.21.**

$$-4L(o)\,\varphi(o)\,\varphi'(o) - 4\int_0^\rho L(s)\,\varphi(s)\,\varphi''(s)\, ds - 8\pi \int_0^\rho \chi(s)\,\varphi(s)\,\varphi'(s)\, ds \geqq \kappa_0 A(o)\,\varphi^2(o) + \kappa_0 \int_0^\rho \varphi^2(s)\, L(s)\, ds + 2\int_0^\rho (\varphi'(s))^2 L(s)\, ds.$$

COROLLARY **10.22.** — *Setting* $\varphi(s) = \cos(\lambda s)$ *where*

$$\lambda = \frac{\pi}{2\rho}$$

gives

(10.23)
$$\boxed{(4\lambda^2 - \kappa_0)\int_0^\rho \cos^2(\lambda s)\, L(s)\, ds + 4\pi\lambda \int_0^\rho \chi(s)\,\sin(2\lambda s)\, ds \geqq \kappa_0 A(o).}$$

Our second hypothesis guarantees that $\chi(s) \leqq o$ for all s. To see this note that since Ω is connected, $\Omega(s)$ is also connected for each s. The map $H_1(\Omega) \to H_1(\Omega(\rho))$ factors through $H_1(\Omega(s))$. Consequently, $H_1(\Omega(s)) \neq o$, and so $\chi(\Omega(s)) \leqq o$, for all $s \in [o, \rho]$. (Recall that everything is oriented.)

We conclude immediately from (10.23) that

(10.24)
$$(4\lambda^2 - \kappa_0)\int_0^\rho \cos^2(\lambda s)\, L(s)\, ds \geqq \kappa_0 A(o) > o$$

and in particular, that $\lambda^2 > 1/4\kappa_0$. This completes the proof. ■

Returning to (10.21) we also obtain:

COROLLARY **10.25**. — *Setting* $\varphi(s) = \rho - s$ *in* (10.21) *gives*

$$4L(0)\,\rho + 8\pi \int_0^\rho \chi(s)(\rho - s)\,ds \geq \kappa_0 A(0)\,\rho^2 + \kappa_0 \int_0^\rho (\rho - s)^2 L(s)\,ds + 2(A(\rho) - A(0)). \tag{10.26}$$

In particular, if $\kappa_0 = 1$ *and* $\chi(s) \leq 0$, *then*

$$4\rho L(0) \geq \int_{\Omega(\rho)} (\rho - s(x))^2\,dx \tag{10.27}$$

and, in particular,

$$4L(0) \geq A(0)\,\rho. \tag{10.28}$$

REMARK **10.29**. — It should be noted that (10.21), (10.22) and (10.26) hold also for $\kappa_0 \leq 0$. We shall now make use of this fact.

Suppose that Σ is a connected non-compact minimal surface in a 3-manifold X with $\kappa \geq -1$. Let Ω be a compact connected domain in $\Sigma - \partial\Sigma$, and fix a number $\rho > 0$. Then for all $s \in [0, \rho]$, the Euler characteristic of $\Omega(s)$ is written as $\chi(s) = 1 - b_1(s)$ where

$$b_1(s) \geq \operatorname{rank}[H_1(\Omega) \to H_1(\Omega(\rho))] \equiv 1 + \alpha(\rho).$$

It follows that

$$-\chi(s) \geq \alpha(\rho) \quad \text{for all } s \in [0, \rho]. \tag{10.30}$$

We now assume that $\Omega(\rho)$ is compactly contained in $\Sigma - \partial\Sigma$. Then the stability inequality (10.23) applies (with $\kappa_0 = -1$), and we conclude by using (10.30), that

$$\begin{aligned}(4\lambda^2 + 1)(A(\rho) - A(0)) + A(0) &\geq -4\pi\lambda \int_0^\rho \chi(s)\sin(2\lambda s)\,ds \\ &\geq 4\pi\alpha(\rho).\end{aligned}$$

In particular, we have

$$A(\rho) \geq \frac{4\pi}{1 + (\pi/\rho)^2}\,\alpha(\rho) \underset{\rho \to \infty}{\sim} 4\pi\alpha(\rho). \tag{10.31}$$

PROOF OF THEOREM **10.16**. — The argument above proves the first statement. The second follows from the fact that if $\chi(\Sigma) \leq 0$, then $\lim_{R} \alpha(R) \geq |\chi(\Sigma)|$. ∎

11. APPLICATIONS TO THE TOPOLOGY OF MINIMAL HYPERSURFACES

Any theorem concerning the topology of complete manifolds of positive scalar curvature has several immediate applications to the theory of minimal varieties. These applications concern both the topology of the variety itself, and the placement of the variety in the ambient manifold. There are also applications to the structure of singularities.

The first main result is the following.

THEOREM **11.1**. — *Let* M *be a complete stable minimal hypersurface in a manifold with scalar curvature* $\geqq \kappa_0 \geqq 0$. *Let* g *and* A *denote the first and second fundamental forms of* M *respectively. Then there exists a positive function* f *on* M *so that the (complete) warped-product metric* $\widetilde{g} = g + f^2 d\theta^2$ *on* $M \times S^1$ *has scalar curvature*

$$\widetilde{\kappa} \geqq \kappa_0 + \|A\|^2. \tag{11.2}$$

This means that stable (minimal) hypersurfaces in spaces with $\kappa \geqq \kappa_0 \geqq 0$, are " stably " also of $\kappa \geqq \kappa_0 \geqq 0$. The results of §§ 6, 7, 8, ... above can then be applied. For example, we have the following (cf. [SY$_4$]).

COROLLARY **11.3**. — *A complete stable hypersurface in* $\mathbf{R}^n$ *cannot be* Λ^2*-enlargeable. In particular, it cannot be homeomorphic to* $X \times \mathbf{R}$ *where* X *is a compact manifold of non-positive curvature (e.g. a torus* T^{n-2}*).*

The same result holds with $\mathbf{R}^n$ *replaced by any manifold of non-negative scalar curvature.*

THEOREM **11.4**. — *Let* $F : \mathbf{R}^n \to \mathbf{R}$ *be a global solution to the minimal surface equation* $(n \geqq 8)$, *and let* A *be the second fundamental form of the graph of* f, $\Gamma_F \subset \mathbf{R}^{n+1}$. *Set*

$$a(R) = \inf_{\|x\| \leqq R} \|A\|^2.$$

Then there is a constant c_n *depending only on dimension so that*

$$a(R) < \frac{c_n}{R^2}.$$

Our next result applies to the structure of singularities on any codimension-one minimizing current in a riemannian manifold. This was observed previously by R. Schoen [S].

THEOREM 11.5. — *Let* $M \subset S^n$ *be a compact (minimal) hypersurface such that the cone*

$$C(M) \equiv \{tx \in \mathbf{R}^{n+1} : x \in M \text{ and } t \geqq 0\}$$

is a stable variety in $\mathbf{R}^{n+1}$. *(That is, the second variation of area is* > 0 *on compact subdomains of* $C(M) - \{0\}$.*) Then* M *carries a metric of positive scalar curvature.*

This gives the first known restrictions on the topology of stable cones in dimensions $\geqq 8$. For example, note that there are infinitely many isotopy classes of embeddings of T^{n-1} into S^n for any $n > 2$.

COROLLARY 11.6. — *If the cone on* $M^{n-1} \subset S^n$ *is stable in* $\mathbf{R}^{n+1}$, *then* M^{n-1} *cannot be enlargeable. In particular, it cannot be homeomorphic to a torus (or any other manifold admitting a metric of sectional curvature* $\leqq 0$.*)*

We now take up the " placement " question for (not necessarily stable) minimal hypersurfaces.

THEOREM 11.7. — *Let* M *be a compact minimal hypersurface in a compact manifold* X *of positive scalar curvature. Suppose the normal bundle of* M *is trivial (*i.e. M *is " two-sided "), and let* $\hat{X}$ *be the compact manifold with boundary formed by " separating "* X *along* M. *(Note that* $\hat{X}$ *may or may not be connected.) Then the double of each component of* $\hat{X}$ *carries a metric of positive scalar curvature.*

This has very strong implications for the possible placement of M in X. For example, we have the following generalization of results in [L_2].

COROLLARY 11.8. — *Let* $\Sigma \subset S^3$ *be a compact minimal surface for some metric on* S^3 *having positive scalar curvature. Then* Σ *is isotopic to a standard embedding (as the boundary of a " pretzel ").*

To get the flavor of this result note that taking the boundary of a tubular neighborhood of a knot $S^1 \hookrightarrow S^3$, gives a " knotted " embedding $T^2 \hookrightarrow S^3$. Now $(S^3 - T^2)^\wedge$ has two components, one of which is $S^1 \times D^2$. Here the double is $S^1 \times S^2$ which carries $\kappa > 0$. The double of the other component is easily seen to contain a small circle, and therefore cannot carry $\kappa > 0$.

Corollary 11.8 is also found in a series of beautiful results recently proved by Meeks, Simon and Yau [MSY].

The implications in higher dimensions are currently being examined by Sebastião de Carneiro Almeida.

The doubling trick above can be replaced by the following. Let $M \subset X$ be as in Theorem 11.7. Then since M has mean curvature $= 0$, it acts as a barrier for solving the Plateau problem (in codimension-one) in each component of $\hat{X}$. This gives, for example, the following.

THEOREM 11.9. — *Let* X *be a compact riemannian n-manifold* $(n \leqq 7)$ *with* $\kappa > 0$ *and with boundary* ∂X *of mean curvature* $\geqq 0$ *with respect to the interior. Suppose* ∂X *is an enlargeable* $K(\pi, 1)$*-manifold or, more generally, suppose that* ∂X *admits a map of non-zero degree onto such a manifold. Then*

$$\pi_1(\partial X) \to \pi_1(X)$$

is not injective.

For example, let $n = 3$ and suppose ∂X is connected and of genus > 0. Then the hypotheses are automatically satisfied, and we retrieve the unknottedness results above.

Another example comes by considering the " interesting " component of a knotted T^{n-2} in S^n.

Theorem 11.9 can be generalized. For example, suppose K is the enlargeable $K(\pi, 1)$-manifold such that ∂X maps onto K with non-zero degree. Then it is not possible that the map $\pi_1(\partial X) \to \pi_1(K)$ factor through $\pi_1(X)$.

Our methods here are based on the following three basic results. We say that an elliptic differential operator L on an open manifold X is *positive* $(L > 0)$, if for every compact domain $\Omega \subset\subset X$, the first eigenvalue of L on Ω is $\geqq 0$. This means that

$$\lambda_1(\Omega) = \inf_f \frac{\int_\Omega f L(f)}{\int_\Omega f^2} > 0$$

where f ranges over smooth functions on Ω with $f = 0$ on $\partial\Omega$. The following is a direct consequence of the second variational formula (cf. [SY_2]).

PROPOSITION 11.10. — *Let* X *be a stable minimal hypersurface in a manifold with scalar curvature* $\geqq \kappa_0$. *Then*

$$L \equiv -\nabla^2 + \frac{1}{2}(\kappa - \kappa_0 - ||A||^2) > 0, \tag{11.11}$$

where A *and* κ *denote the second fundamental form and the scalar curvature of* X *respectively.*

If $L > 0$, then on any compact domain $\Omega \subset\subset X$, there exists $f > 0$ with $L(f) = 0$. It is proved in [FC-S] that if $L > 0$ and X is complete, then there exists $f > 0$ with $L(f) = 0$ on X.

PROPOSITION 11.12 (cf. [KW]). — *Let* (X, ds^2) *be a riemannian n-manifold with scalar curvature* κ. *Then for any* $f > 0$, *the scalar curvature of the metric*

$$d\hat{s}^2 = f^{4/(n-2)}\, ds^2$$

is given by

$$\hat{\kappa} = f^{(n+2)/(2-n)}\left[-\frac{4(n-1)}{(n-2)}\nabla^2 f + \kappa f\right]. \tag{11.13}$$

In particular, if X *is compact and if* $-t\nabla^2 + \kappa > 0$ *for some* $t \leqq 4(n-1)/(n-2)$, *then* X *carries a metric of positive scalar curvature.*

PROPOSITION **11.14**. — (See 7.33 above.) *Let* (X, ds^2) *be a riemannian n-manifold with scalar curvature* κ. *Then for any* $f > 0$, *the metric*

$$d\hat{s}^2 = ds^2 + f^2 \, d\theta^2$$

on $X \times S^1$ *has scalar curvature*

$$\hat{\kappa} = \kappa - \frac{2}{f} \nabla^2 f. \tag{11.15}$$

In particular, if X *is complete and if* $-t\nabla^2 + \kappa \geqq \kappa_0$ *for some* $t \leqq 2$, *then* $X \times S^1$ *carries a complete metric with scalar curvature* $\geqq \kappa_0$.

We can now enter into the proofs of the theorems above.

PROOF OF THEOREM **11.1**. — By formula (11.11) and completeness we have a positive function f with $-\nabla^2 f + \frac{1}{2}\kappa f = \frac{1}{2}(\kappa_0 + \|A\|^2) f$. The result now follows immediately from Proposition 11.14. ∎

PROOF OF THEOREM **11.4**. — It is a well known fact (cf. [F_1], [L_1]) that any codimension-one minimal graph $\Gamma_F \subset \mathbf{R}^{n+1}$ is stable. (It is actually area-minimizing.) Clearly the projection map $\pi : \mathbf{R}^{n+1} \to \mathbf{R}^n$, chosen so that $\pi(x, F(x)) = x$, gives a distance-decreasing homeomorphism

$$\Gamma_F \xrightarrow{\pi} \mathbf{R}^n.$$

By Theorem 11.1, there is a warped product metric on $\Gamma_F \times S^1$ with scalar curvature

$$\hat{\kappa} \geqq \frac{1}{2} \|A\|^2. \tag{11.16}$$

We now return to our primary methods involving the Dirac operator. Fix a degree-one map $\xi : \mathbf{R}^n \to S^n(1)$ which is constant outside the unit disk. This map can be chosen to be 2π-bounded. Then for each $R > 0$ consider the composition:

$$\Gamma_F \underset{\approx}{\xrightarrow{\pi}} \mathbf{R}^n \xrightarrow{\frac{1}{R}} \mathbf{R}^n \xrightarrow{\xi} S^n(1).$$

This map is $2\pi R^{-1}$ contracting. Passing to a sufficiently high-order covering of $\Gamma_F \times S^1$, we can now construct a $4\pi R^{-1}$-contracting map $\Gamma_F \times S^1 \to S^{n+1} = S^n \wedge S^1$ which is constant outside $\Omega_R \equiv \pi^{-1}(D^n(R)) \times S^1$ and of degree 1. Applying the methods of § 5 ff. we see that there is a constant c_n, depending only on dimension, such that

$$\inf_{\Omega_R} \hat{\kappa} < c_n R^{-2}. \tag{11.17}$$

(Otherwise, the vanishing theorem applies, and one uses the Relative Index Theorem.) Combining (11.16) and (11.17) completes the proof. ∎

Proof of Theorem 11.5. — It is a direct consequence of the calculations in Simons [Si], that if the cone on M^p is stable $(p = n - 1)$, then

$$-\nabla^2 - (p(p-1) - \kappa) \geqq -\left(\frac{p-1}{2}\right)^2.$$

It follows that

$$-\nabla^2 + \kappa \geqq \frac{1}{4}(p-1)(3p+1). \tag{11.18}$$

In particular, Proposition 11.12 applies. ■

Proof of Theorem 11.7. — This follows immediately from the result (see Almeida, Ph.D. Thesis, Stony Brook, 1982) that if a compact manifold X carries a metric with $\kappa \geqq 1$ and such that the mean curvature of ∂X, with respect to the interior normal, is $\geqq 0$, then the double $\mathbf{D}(X)$ carries a metric with $\kappa > 0$. ■

Proof of Theorem 11.9. — Pass to the covering $\widetilde{X} \to X$ corresponding to the subgroup $i_* \pi_1(\partial X) \subset \pi_1(X)$. Then there is an embedding $\tilde{i} : \partial X \subset \widetilde{X}$ covering the boundary below. Let K be the enlargeable $K(\pi, 1)$-manifold for which there is a map $F : \partial X \to K$ of non-zero degree. Since $\tilde{i}$ induces an isomorphism of fundamental groups, this map F extends to $\widetilde{X}$, *i.e.* we have a commutative diagram:

$$\begin{tikzcd}
\partial X \arrow[r, hook, "\tilde{i}"] \arrow[dr, "F"'] & \widetilde{X} \arrow[d, "\widetilde{F}"] \\
& K
\end{tikzcd}$$

It follows that $[\partial X] \neq 0$ in $H_{n-1}(\widetilde{X})$. Since the boundary of $\widetilde{X}$ has non-negative mean curvature, and since $\widetilde{X}$ is a riemannian covering of a compact space, we can solve for a current of least mass in this homology class $[\partial X]$. By regularity theory, the support of this current is a regular stable hypersurface $\Sigma \subset X$. Since $\widetilde{F}_*[\partial X] \neq 0$, one easily sees that $\widetilde{F}$ must map some component of Σ onto K with non-zero degree. This component is therefore enlargeable. However, stability (and $\kappa > 0$ on $\widetilde{X}$) implies that this component carries positive scalar curvature. This contradicts our basic results above (§ 5). ■

12. THEOREMS FOR INCOMPLETE MANIFOLDS OF DIMENSION $\leqq 7$

By combining minimal surface techniques with certain warped-product constructions we are able to extend some of the results of section 10 to all dimensions $\leqq 7$. The methods here were inspired in part by the beautiful results in [FCS] and [$SY_{2,3}$]. The main result is as follows. Set

$$\varepsilon_n = \sqrt{\frac{n}{n-1}}\,\frac{1}{2^n \pi}.$$

THEOREM 12.1. — *Let* X *be a compact oriented riemannian n-manifold,* $n \leqq 7$, *with scalar curvature* $\geqq 1$. *(Here* X *need not be spin.) Then there exists no* ε_n*-contracting map* $(X, \partial X) \to (S^n, *)$, *of non-zero degree.*

NOTE. — The scalar curvature κ in this theorem is, as always, unnormalized. If we assume the normalized scalar curvature $\hat{\kappa} = (1/n(n-1))\,\kappa \geqq 1$, then ε_n will be replaced in theorem 12.1 by

$$\hat{\varepsilon}_n = \frac{n}{2^n \pi}.$$

NOTE. — Schoen and Yau have recently succeeded in establishing similar results in general dimensions by handling the difficulties posed by singularities on minimal hypersurfaces.

It is an immediate consequence of Theorem 12.1 that the results of our previous paper [GL_1] hold without the spin assumption in dimensions $\leqq 7$. For example, it is a corollary that there exists no positive scalar curvature metric on the connected sum $T^4 \# P^2(\mathbf{C})$.

PROOF. — Fix $\varepsilon > 0$ and suppose $f\colon (X, \partial X) \to (S^n, *)$ is an ε-contracting map of non-zero degree. We shall show that ε must be greater than ε_n.

To begin, choose a geodesic subsphere $S^{n-2} \subset S^n$ and a geodesic hemisphere D^{n-1} such that $\partial D^{n-1} = S^{n-2}$ and $* \notin D^{n-1}$. By a close C^1 approximation rel(∂X), make f transversal to S^{n-2} and to D^{n-1}. Set $B^{n-2} \equiv f^{-1}(S^{n-2})$, and note that B^{n-2} is the boundary of the submanifold $\Sigma_0^{n-1} \equiv f^{-1}(D^{n-1})$ in X.

We now solve the oriented Plateau problem for B^{n-2} in $(X, \partial X)$. That is, we let Σ^{n-1} be the current of least mass among all integral currents which are homologous to Σ_0^{n-1} modulo ∂X (*i.e.* Σ^{n-1} minimizes mass among all integral currents Σ_1^{n-1} such that $\Sigma_1^{n-1} - \Sigma_0^{n-1}$ is a boundary in $(X, \partial X)$). The existence of Σ^{n-1} is guaranteed by general compactness theorems [FF]. Since $n \leqq 7$, the regularity theory for minimal hypersurfaces asserts that Σ^{n-1} is a regular embedded submanifold with boundary B^{n-2} (mod. ∂X). Furthermore, by the minimizing property we see that Σ^{n-1} is stable.

The solution Σ^{n-1} may not be regular at points of $\Sigma^{n-1} \cap \partial X$. However, by moving the boundary ∂X in slightly, we can overcome this difficulty. By " moving in " we mean the following. Choose a small regular value t of the function $\mathrm{dist}_X(., \partial X)$ restricted to Σ^{n-1}. Then shave off the t-collar of ∂X.

Consider now the spherical " crushing " map $p : S^n \to D^{n-1}$ given by setting

$$p(x_1, \ldots, x_{n+1}) = \left(x_1, \ldots, x_{n-1}, \sqrt{x_n^2 + x_{n+1}^2}\right)$$

for points $x = (x_1, \ldots, x_{n+1})$ with $\|x\| = 1$. Let $U \subset D^{n-1}$ be a " finger-like " set which connects $p(*)$ to $\partial D^{n-1} = S^{n-2}$, and let $c : D^{n-1} \to S^{n-1}$ be a map defined by collapsing $U \cup \partial D^{n-1}$ to a point, which we again denote by $*$. It is not difficult to see that the composition $c \circ p : S^n \to S^{n-1}$ can be chosen to be approximately 2-contracting, that is, it can be chosen to be $(2 + \delta)$-contracting for any given $\delta > 0$. (To accomplish this, $S^{n-2} \subset S^n$ must be chosen close to $*$.)

Consider now the composition $f_1 = c \circ p \circ f$. We make the following claim: *The map*

$$f_1 : (\Sigma^{n-1}, \partial\Sigma^{n-1}) \to (S^{n-1}, *)$$

is approximately 2ε*-contracting and of non-zero degree.* The first point is clear since f is approximately ε-contracting and $c \circ p$ is approximately 2-contracting. To prove that f_1 has non-zero degree, it suffices to show that $f_1\big|_{\Sigma_0^{n-1}}$ has non-zero degree, since Σ_0^{n-1} is homologous to Σ^{n-1} in $(X, \partial X \cup \partial\Sigma)$, and since the map f_1 is constant on $\partial X \cup \partial\Sigma$. Choose now a regular value of $f_1 : \Sigma_0^{n-1} \to S^{n-1}$. This can be considered as a regular value of the map $f : \Sigma_0^{n-1} \to D^{n-1} \subset S^n$, and, as such, it is in fact a regular value of $f : X \to S^n$. (Recall that $\Sigma_0^{n-1} = f^{-1}(D^{n-1})$ and f is transversal to D^{n-1}.) Summing over the pre-image points of such a regular value shows that these two maps have the same degree. That is, $\deg(f) = \mathrm{def}(f_1) \neq 0$, as claimed.

We now apply the stability condition, and from Propositions 11.10 and 11.14 we obtain a function $\varphi > 0$ on $\Sigma^{n-1} - \partial\Sigma^{n-1}$ so that the warped-product metric

$$ds^2 + \varphi^2 d\theta^2 \tag{12.2}$$

on $\Sigma^{n-1} \times S^1$ has $\kappa \geqq 1$. By shaving off a collar of $\partial\Sigma$, we can assume $\varphi > 0$ on (all of) Σ^{n-1}.

We have now replaced our original manifold X^n with $\kappa \geqq 1$, by a warped-product $\Sigma^{n-1} \times S^1$ also with $\kappa \geqq 1$. The ε-contracting map $f : (X^n, \partial X^n) \to (S^n, *)$ has given rise to an approximately 2ε-contracting map $f_1 : (\Sigma^{n-1}, \partial\Sigma^{n-1}) \to (S^{n-1}, *)$. Note that rotations of the S^1-factor are isometries of the warped-product metric (12.2).

We now continue this process as follows. Fix a geodesic sphere

$$S^{n-3} = \partial D^{n-2} \subset S^{n-1},$$

and construct a boundary $B^{n-3} = f^{-1}(S^{n-3})$ as before. We now solve the Plateau problem, as we did above, for the boundary

$$B^{n-3} \times S^1 \subset \Sigma^{n-1} \times S^1.$$

We claim that the solution must be SO_2-invariant (where SO_2 acts by rotations of the S^1-factor). Indeed, it is a basic result of Gao [Gao] that for any solution T and any $g \in SO_2$, either $gT = T$, or gT and T are disjoint away from the boundary. If the latter case occurs for any g, the vector field generating the action must be everywhere transversal to T (at interior points). In particular, the inner product of this vector field with the field of unit normals to T cannot change sign. It follows that the intersection number of T with any orbit of the action is non-zero. That is, if we consider T as a class in $H^1(\Sigma \times S^1, \partial\Sigma \times S^1 \cup B \times S^1)$, then the intersection pairing $T \cdot S^1 \neq 0$. However, T was chosen in the homology class of $\Sigma_0^{n-2} \times S^1$ where $\Sigma_0^{n-2} \subset \Sigma^{n-1}$ and $\partial\Sigma_0^{n-2} = B^{n-3}$. It is easy to see that $T \cdot S^1$ must be 0. Consequently, *the solution* T *must be* SO_2-*invariant*, that is, it must be of the form

$$\Sigma^{n-2} \times S^1$$

with the induced warped-product metric. As before, the first eigen-function of the stability operator can be used to construct a warped-product metric on $(\Sigma^{n-2} \times S^1) \times S^1$ with $\kappa \geqq 1$. Furthermore, this function can be assumed to be SO_2-invariant, since if it is not, it can be averaged over the group. (The stability operator is invariant and the function is positive.) Consequently, the induced metric on $\Sigma^{n-2} \times S^1 \times S^1$ is doubly warped over Σ^{n-2}.

The same argument as before shows that there exists an approximately 4ε-contracting map

$$f_2 : (\Sigma^{n-2}, \partial\Sigma^{n-2}) \to (S^{n-2}, *)$$

of non-zero degree.

Continuing inductively, we eventually construct a warped-product metric on a manifold $\Sigma^1 \times T^{n-1}$ with $\kappa \geqq 1$ and with the property that there exists an approximately $2^{n-1}\varepsilon$-contracting map

$$f_{n-1} : (\Sigma^1, \partial\Sigma^1) \to (S^1, *)$$

of non-zero degree. This means, in particular, that we have a warped-product metric of the form

$$ds^2 = dt^2 + \varphi_2^2(t)\, d\theta_2^2 + \ldots + \varphi_n^2(t)\, d\theta_n^2 \tag{12.3}$$

on the manifold $[0, 2^{-n+1}\varepsilon^{-1}] \times T^{n-1}$ (where dt denotes the usual arc-length on the interval $[0, 2^{-n+1}\varepsilon^{-1}]$) with $\kappa \geqq 1$. By a straightforward computation we find that the scalar curvature of the metric (12.3) is given by the formula

$$\begin{aligned}
(12.4) \qquad \kappa &= -2\sum_i \frac{\varphi_i''}{\varphi_i} - 2\sum_{i<j} \frac{\varphi_i'}{\varphi_i}\frac{\varphi_j'}{\varphi_j} \\
&= -2\sum_i \frac{\varphi_i''}{\varphi_i} + \sum_i \left(\frac{\varphi_i'}{\varphi_i}\right)^2 - \left(\sum_i \frac{\varphi_i'}{\varphi_i}\right)^2 \\
&= -2\sum_i \left(\frac{\varphi_i''\varphi_i - (\varphi_i')^2}{\varphi_i^2}\right) - \sum_i \left(\frac{\varphi_i'}{\varphi_i}\right)^2 - \left(\sum_i \frac{\varphi_i'}{\varphi_i}\right)^2.
\end{aligned}$$

Setting $F = \log(\varphi_2 \dots \varphi_n)$, we have that

$$(12.5) \qquad \kappa = -2F'' - (F')^2 - \sum_i \left(\frac{\varphi_i'}{\varphi_i}\right)^2.$$

From the inequality $(\Sigma\varphi_i'/\varphi_i)^2 \leqq (n-1)\,\Sigma(\varphi_i'/\varphi_i)^2$ and the fact that $\kappa \geqq 1$, we conclude that

$$(12.6) \qquad -2F'' - \frac{n}{n-1}(F')^2 \geqq 1.$$

Setting

$$u = \sqrt{\frac{n}{n-1}}\, F'$$

we can rewrite the inequality (12.6) as

$$(12.7) \qquad \frac{u'}{1+u^2} \leqq -\frac{1}{2}\sqrt{\frac{n}{n-1}}$$

which integrates to give

$$(12.8) \qquad \tan^{-1}(u(t)) - \tan^{-1}(u(0)) \leqq -\frac{1}{2}\sqrt{\frac{n}{n-1}}\, t.$$

This immediately implies that

$$-\pi \leqq -\frac{1}{2}\sqrt{\frac{n}{n-1}}\, t$$

for all t in the interval, and in particular that

$$\pi \geqq \sqrt{\frac{n}{n-1}}\,\frac{1}{2^n\varepsilon}$$

as claimed. ■

13. MANIFOLDS WHICH REPRESENT HOMOLOGY CLASSES IN COMPACT SPACES OF NON-POSITIVE SECTIONAL CURVATURE

This chapter is based on the idea that a manifold which represents a non-trivial homology class in a compact $K(\pi, 1)$-manifold, should not carry positive scalar curvature. If the $K(\pi, 1)$-manifold is a torus, this is true (since any such submanifold is easily seen to be enlargeable). In this section we show that the statement is, in fact, true for any $K(\pi, 1)$-manifold of non-positive curvature. We shall present two arguments. The first argument uses the Relative Index Theorem for Families, proved in the thesis of Zhiyong Gao at Stony Brook. The second argument, given at the end of this section, uses only the Relative Index Theorem proved here. A comparison of the two arguments is illuminating.

The fundamental construction is as follows. Let K be a compact riemannian manifold of non-positive sectional curvature, and consider a compact oriented n-manifold X which represents a non-zero class in $H_n(K; \mathbf{Q})$. That is, we assume there is a (smooth) map $f: X \to K$ so that $f_*[X] \neq 0$ in $H_n(K; \mathbf{Q})$. By taking a product of K with a torus of sufficiently high dimension, we can assume $f: X \hookrightarrow K$ is an embedding.

We now pass to the universal covering $\pi: \widetilde{K} \to K$ and set $\widetilde{X} \equiv \pi^{-1}(X)$. Note that $\widetilde{X}$ is a properly embedded submanifold of $\widetilde{K}$ which is invariant under the deck group $\Gamma \cong \pi_1(K)$.

Since K carries non-positive curvature, the exponential map $\exp_y: T_y\widetilde{K} \to \widetilde{K}$ is a diffeomorphism at each point $y \in \widetilde{K}$. Furthermore, $\exp_{\gamma(y)} \circ \gamma_* = \gamma \circ \exp_y$ for all $\gamma \in \Gamma$, since Γ acts by isometries. We now construct a family of maps $\widetilde{K} \to S^N$ (where $N = \dim K$) as follows. To begin, we shall assume that there exists a Γ-invariant framing σ of $T\widetilde{K}$. At each point $y \in \widetilde{K}$, the frame σ_y can be considered as an isomorphism $\sigma_y: T_y\widetilde{K} \to \mathbf{R}^N$. We now fix a degree-one map

$$\Phi: \mathbf{R}^N \to S^N$$

which is constant outside the unit disk and, say, 2π-bounded, and we fix $\varepsilon > 0$. Then for each $y \in \widetilde{K}$ we consider the map

$$\varphi_y: \widetilde{K} \to S^N$$

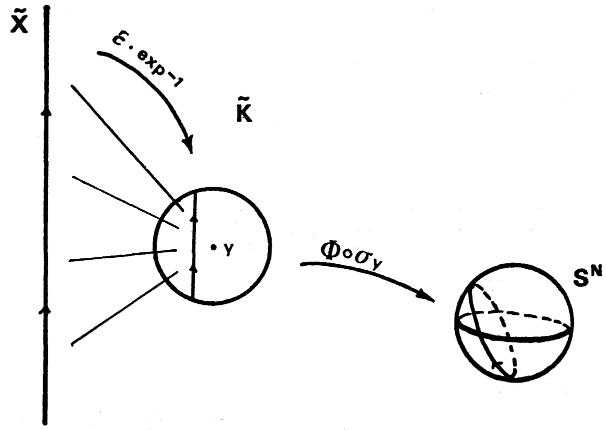
given by setting

$$\varphi_y(x) \equiv \Phi(\sigma_y(\varepsilon . \exp_y^{-1}(x))). \tag{13.1}$$

Observe that for any $\gamma \in \Gamma$ we have

$$\begin{aligned} \varphi_{\gamma y}(\gamma x) &= \Phi(\sigma_{\gamma y}(\varepsilon . \exp_{\gamma y}^{-1}(\gamma x))) \\ &= \Phi(\sigma_{\gamma y}(\gamma_* \varepsilon \exp_y^{-1}(x))) \\ &= \Phi(\sigma_y(\varepsilon \exp_y^{-1}(x))) \\ &= \varphi_y(x). \end{aligned}$$

Hence the family of maps $\varphi : \widetilde{K} \times \widetilde{K} \to S^N$ is invariant under the diagonal action of Γ.



We now restrict this family of maps to the invariant subset $\widetilde{K} \times \widetilde{X} \subset \widetilde{K} \times \widetilde{K}$ and divide by Γ. The resulting space

(13.2) $\qquad Y \equiv \widetilde{K} \times_\Gamma \widetilde{X}$

is the total space of a bundle

(13.3) $\qquad p : Y \to K$

over $\widetilde{K}/\Gamma \cong K$, whose fibre is $\widetilde{X}$. The above construction gives a map

(13.4) $\qquad \varphi : Y \to S^N$

which has compact support on each fibre. This map has the following basic property whose proof we postpone.

LEMMA 13.5. — *Let* $\omega \equiv \varphi^*(\mathrm{vol}) \in H^N_{\mathrm{cpt}}(Y, \mathbf{R})$ *be the compactly supported cohomology class obtained by pulling back the fundamental cohomology class of* S^N *via* φ. *Let*

$$p_* : H^N_{\mathrm{cpt}}(Y; \mathbf{R}) \to H^{N-n}(K; \mathbf{R})$$

denote " integration over the fibre ". Then

$$p_* \omega \neq 0.$$

NOTE. — This cohomology class $p_* \omega$ is essentially (a multiple of) the Poincaré dual of $[X] \in H_n(K; \mathbf{R})$.

We now suppose that X carries a metric of positive scalar curvature, and we lift this metric to a (Γ-invariant) metric on $\widetilde{\mathrm{X}}$. We observe now that *there is a constant* $c > 0$ *(independent of* ε*) so that for each* $y \in \widetilde{\mathrm{K}}$ *the restricted map* $\varphi_y : \widetilde{\mathrm{X}} \to \mathrm{S}^{\mathrm{N}}$ *is* εc*-contracting with respect to this new metric.* Indeed, it was $2\varepsilon\pi$-contracting with respect to the old (induced) metric; and the two metrics are finitely related since they are both lifted from metrics on X.

We now assume N even (as before) and fix a complex vector bundle E_0 over S^{N} with $c_{\mathrm{N}}(\mathrm{E}_0) \neq 0$. We fix also a unitary connection in E_0. We then lift E_0 to the family Y via the map φ; *i.e.* we set $\mathrm{E} = \varphi^* \mathrm{E}_0$ with the induced connection.

This gives us a family of vector bundles on the fibres of the family $\mathrm{Y} \to \mathrm{K}$. Each vector bundle has curvature whose norm is uniformly $\leq c'\varepsilon$ for some constant c'. Furthermore, since φ is constant at infinity, the bundle E is trivialized outside a compact subset.

We now consider two families of Dirac operators. Let D be the family of canonical Dirac operators on spinors in the fibres ($\approx \widetilde{\mathrm{X}}$) of the family. Set

$$\mathrm{D}_{\mathbf{C}^k} \equiv \mathrm{D} \otimes \mathbf{C}^k$$
$$\mathrm{D}_{\mathrm{E}} \equiv \mathrm{D} \otimes \mathrm{E}$$

where $k = \dim_{\mathbf{C}} \mathrm{E}$ (cf. § 5).

For ε sufficiently small, both operators will be pointwise strictly positive and the analytic indices of $\mathrm{D}_{\mathbf{C}^k}$ and D_{E} will be zero. However, by the Relative Index Theorem for Families ([Gao]), the relative topological index $\mathrm{ind}_t(\mathrm{D}_{\mathbf{C}^k}, \mathrm{D}_{\mathrm{E}})$ must be zero, and this index is given by the following formula:

$$\text{(13.6)} \qquad \mathrm{ind}_t(\mathrm{D}_{\mathbf{C}^k}, \mathrm{D}_{\mathrm{E}}) = p_*\{\widehat{\mathrm{ch}}\,\mathrm{E}.\hat{\mathbf{A}}(\mathrm{Y}/\mathrm{K})\} \in \mathrm{H}^*(\mathrm{K}, \mathbf{Q})$$

where $\hat{\mathbf{A}}(\mathrm{Y}/\mathrm{K})$ denotes the total $\hat{\mathrm{A}}$-class of the tangent bundle to the fibres of $p : \mathrm{Y} \to \mathrm{K}$, $\widehat{\mathrm{ch}}\,\mathrm{E} \equiv \mathrm{ch}\,\mathrm{E} - k$ denotes the reduced Chern character, and $p_* : \mathrm{H}^*_{\mathrm{cpt}}(\mathrm{Y}; \mathbf{Q}) \to \mathrm{H}^*(\mathrm{K}; \mathbf{Q})$ denotes integration over the fibres.

Since $\mathrm{E} = \varphi^*\mathrm{E}_0$ and since $c_{\mathrm{N}}(\mathrm{E}_0) \neq 0$ in $\mathrm{H}^{\mathrm{N}}(\mathrm{S}^{\mathrm{N}})$, we see that

$$\begin{aligned} \widehat{\mathrm{ch}}\,\mathrm{E} &= \varphi^*(\widehat{\mathrm{ch}}\,\mathrm{E}_0) = \varphi^*\left[\frac{1}{(\mathrm{N}-1)!}\, c_{\mathrm{N}}(\mathrm{E}_0)\right] \\ &= \varphi^*(k\,\mathrm{vol}) = k\omega \end{aligned}$$

for some non-zero number k. Since $\hat{\mathbf{A}}(\mathrm{Y}/\mathrm{K}) = 1 + \ldots$, we see that the non-zero term of lowest degree in formula (13.6) is $kp_*(\omega)$, which is not zero by Lemma 13.5. This is a contradiction to the existence of positive scalar curvature on X.

In the argument above we assumed the existence of a Γ-invariant framing of $\mathrm{T}\widetilde{\mathrm{K}}$. This amounts to assuming K is parallelizable, or just stably parallelizable since we can multiply K by S^1. This assumption can be avoided by enhancing the family Y to include all oriented tangent frames to K. That is, at each point $y \in \widetilde{\mathrm{K}}$, and for each oriented orthonormal tangent frame σ at y, we define the map $\varphi = \Phi(\sigma(\varepsilon \cdot \exp^{-1}(\cdot)))$ as in (13.1) above. The group Γ acts naturally on the oriented frame bundle $P(\widetilde{K})$ and φ is invariant under the diagonal action on $P(\widetilde{K}) \times \widetilde{K}$. Restricting to $P(\widetilde{K}) \times \widetilde{X}$ and taking a quotient by Γ gives a family

$$p: Z \to K \tag{13.7}$$

where $Z = P(\widetilde{K}) \times_\Gamma \widetilde{X}$ and where the fibre at $y \in K$ is

$$\widetilde{X} \times \{o.o.n.\text{frames at } y\} \cong \widetilde{X} \times SO_N.$$

The map φ again descends to a map $\varphi: Z \to S^N$, and Lemma 13.5 continues to hold. This gives us the following result.

THEOREM **13.8**. — *Let* X *be a compact spin manifold which represents a non-trivial rational homology class in some compact manifold of non-positive sectional curvature. Then* X *cannot carry a metric of positive scalar curvature.*

Note that as a special case (when $\dim X = \dim K$) we retrieve the result that compact manifolds which admit maps of non-zero degree onto manifolds of non-positive curvature cannot carry $\kappa > 0$.

REMARK **13.9**. — When $\pi_1 X$ (or, in fact, $\pi_1 K$) is residually finite, the above theorem can be proved by using the usual Atiyah-Singer Index Theorem for families.

PROOF OF LEMMA 13.5. — By Poincaré duality and the work of Thom we know that there exists a compact oriented manifold X^* of dimension $N - n$, and a map $X^* \to K$ so that the homology intersection

$$X^* \cdot X \neq 0.$$

We can assume the map is smooth and transversal to X.

Consider the covering space $\widetilde{X}^* \to X^*$ induced by the universal covering $\widetilde{K} \to K$. There is a natural lifting $\widetilde{X}^* \to \widetilde{K}$, and we consider the Γ-equivariant map: $\widetilde{X}^* \times \widetilde{X} \to \widetilde{K} \times \widetilde{X}$. Dividing by this (diagonal) action of Γ gives us the commutative diagram of maps:

$$\begin{array}{ccccc} \widetilde{X}^* \times_\Gamma \widetilde{X} & \longrightarrow & \widetilde{K} \times_\Gamma \widetilde{X} & \xrightarrow{\varphi} & S^N \\ \Big\downarrow p & & \Big\downarrow & & \\ X^* & \longrightarrow & K & & \end{array} \tag{13.10}$$

The family $\widetilde{X}^* \times_\Gamma X \to X^*$ is just the $\widetilde{X}$-bundle over X^* induced from the family $Y = \widetilde{K} \times_\Gamma \widetilde{X} \to K$. The map φ has compact support in each fibre.

Recall that by its construction, φ is a map $\varphi: \widetilde{K} \times \widetilde{K} \to \mathbf{R}^N/(\mathbf{R}^N - D^N) = S^N$, and the preimage of 0 under φ is the diagonal in $\widetilde{K} \times \widetilde{K}$. Furthermore, note that φ, when restricted to $\widetilde{X}^* \times \widetilde{X}$, is a map between oriented N-dimensional manifolds, and the set $\varphi^{-1}(0)$ is just the set of intersection points of $\widetilde{X}^*$ with $\widetilde{X}$, *i.e.* it is the set $\pi^{-1}(X^* \cap X) \subset \widetilde{K}$. Since these intersections are all transversal, 0 is a regular value of φ, and this remains true when we push φ down to the quotient $\widetilde{X}^* \times_\Gamma X$. On this quotient we see that $\varphi^{-1}(0) \cong \pi^{-1}(X^* \cap X)/\Gamma \cong X^* \cap X$, and we conclude that

$$\deg(\varphi) = X^* \cdot X.$$

Furthermore, if $\omega = \varphi^* \omega_N \in H^*_{\mathrm{cpt}}(\widetilde{X}^* \times_\Gamma \widetilde{X})$ denotes the pull-back of the fundamental cohomology class of S^N, then $\omega = \deg(\varphi)\, \omega_0$ where ω_0 is the fundamental class of $\widetilde{X}^* \times_\Gamma \widetilde{X}$. It follows immediately that

$$p_*(\omega) = (X^* \cdot X)\, \omega^*$$

where ω^* is the fundamental cohomology class of X^*. In particular $p_*(\omega)[X^*] \neq 0$, and so $p_*(\omega) \neq 0$ in $H^{N-n}(K)$ as asserted. ∎

REMARK 3.11. — This Lemma also holds in the general case where K is replaced by the oriented frame bundle P(K). Essentially the same argument applies; one must integrate over the family of frames at the appropriate time.

Using the constructions in the proof of this Lemma, it is possible to give a proof of Theorem 13.8 without using the Index Theorem for Families. Again for simplicity we remain in the case where there exists an invariant framing.

SECOND PROOF OF THEOREM 13.8. — Let $X^* \to K$ be the dual manifold considered in the proof of Lemma 13.5, and let $W \equiv \widetilde{X}^* \times_\Gamma \widetilde{X}$ be the family constructed there. Recall that

$$W \xrightarrow{p} X^*$$

is a fibre bundle with fibre $\widetilde{X}$, and that for each $\varepsilon > 0$,

$$\varphi : W \to S^N$$

is a map which is constant at infinity and of non-zero degree.

Suppose now that X carries a metric with $\kappa \geqq 1$. Choose an arbitrary metric g^* on X^* and lift the product metric to $\widetilde{X}^* \times \widetilde{X}$. This lifted metric is Γ-invariant and descends to a metric on W.

Assume that $\varepsilon > 0$ is given. Replace g^* by cg^* where $c > 1$ is a constant chosen sufficiently large, so that the product metric on $X^* \times X$ has scalar curvature $\geqq \frac{1}{2}$, and the map φ is $2\varepsilon\pi$-contracting. For ε sufficiently small we get a contradiction by applying the Relative Index Theorem as usual. ∎

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