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ON THE OBSTRUCTION GROUP IN HOMOLOGY SURGERY

by PIERRE VOGEL

0. Introduction

The theory of homology surgery has been introduced by Cappell and Shaneson [1]. This theory plays an important role in the theory of knots and codimension 2 embeddings.

Let $(X, \partial X)$ be a pair of finite complexes and f be a normal map from the normal bundle of a (Top, PL or Diff)-manifold V to a (Top, PL or Diff)-bundle over X and let M be a $\mathbf{Z}[\pi_1 X]$ -module. The problem of homology surgery is to determine the obstruction to the existence of a normal cobordism, constant over ∂X , from f to an M -homology equivalence. Clearly we must suppose that f induces an M -homology equivalence from ∂V to ∂X and that the cap-product by $f_*[V]$ is an isomorphism from $H^*(X, \partial X; M)$ to $H_{n-*}(X; M^w)$, w being the first Stiefel-Whitney class of the bundle over X .

If $M = A$ is a quotient ring with involution of $\mathbf{Z}[\pi_1 X] = \mathbf{Z}\pi$, Cappell and Shaneson have solved the problem and have constructed an obstruction group $\Gamma_n(\mathbf{Z}\pi \rightarrow A)$ defined in terms of algebraic L-theory.

In many cases, this group was known to be the L_n -group of some ring Λ . For example, if there exists a classical localization $S^{-1}\mathbf{Z}\pi$ of $\mathbf{Z}\pi$, where S is the multiplicative subset $1 + \ker(\mathbf{Z}\pi \rightarrow A)$, Smith [7] has proved that $\Gamma_n(\mathbf{Z}\pi \rightarrow A)$ is the group $L_n(S^{-1}\mathbf{Z}\pi)$. An other example is given by Hausmann [3] who proves that $\Gamma_n(\mathbf{Z}\pi \rightarrow \mathbf{Z}[\pi/N])$ is the group $L_n(\mathbf{Z}[\pi/N])$ if N is a locally perfect normal subgroup of π .

My purpose is to show that the homology surgery is possible in a more general situation and that the obstruction group is always the L_n -group of a ring with involution Λ endowed with a subgroup of $\tilde{K}_1(\Lambda)$.

For example, suppose that $\mathbf{Z}\pi \rightarrow A$ is a morphism of rings with involution (the involution of $\mathbf{Z}\pi$ is induced by w). Then we have a diagram of rings with involution

$$\begin{array}{ccc} & \Lambda & \\ \nearrow & & \searrow \\ \mathbf{Z}\pi & \longrightarrow & A \end{array}$$

well defined by the following properties:

- i) For any matrix u with entries in $\mathbf{Z}\pi$, if $u \otimes \Lambda$ is invertible then $u \otimes \Lambda$ is invertible too;
- ii) Λ is universal with respect to the property i).

Theorem. — Suppose the morphism $\Lambda \rightarrow \mathbf{A}$ is onto. Then any normal map f over a n -dimensional $\mathbf{A}$ -Poincaré complex X which is an $\mathbf{A}$ -homology equivalence over ∂X determines an element $\sigma(f) \in L_n^h(\Lambda)$, and, if $n \geq 5$, f is normally cobordant to an $\mathbf{A}$ -homology equivalence if and only if $\sigma(f)$ vanishes.

Corollary. — If $\mathbf{A}$ is a quotient ring with involution of $\mathbf{Z}\pi$, the group $\Gamma_n^h(\mathbf{Z}\pi \rightarrow \mathbf{A})$ is isomorphic to $L_n^h(\Lambda)$.

Theorem. — Let D_{2n} be the dihedral group of order $2n$ (n odd) and $D_{2n} \rightarrow \mathbf{Z}/2$ be the non zero homomorphism. Then we have the following isomorphism:

$$\Gamma_*(\mathbf{Z}D_{2n} \rightarrow \mathbf{Z}) \xrightarrow{\sim} \Gamma_*(\mathbf{Z}[\mathbf{Z}/2] \rightarrow \mathbf{Z}) \simeq L_*^h(\Lambda),$$

where Λ is the pull back of rings:

$$\begin{array}{ccc} \Lambda & \longrightarrow & \mathbf{Z} \\ \downarrow & & \downarrow \\ \mathbf{Z}_{(2)}[\mathbf{Z}/2] & \longrightarrow & \mathbf{Z}_{(2)} \end{array}$$

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1. Statement of the main results

(1.1) Let $\mathbf{A}$ be a ring with involution $a \mapsto \bar{a}$. If M is a left $\mathbf{A}$ -module, it can be given a right $\mathbf{A}$ -module structure, by setting

$$ma = \bar{a}m, \quad \forall a \in \mathbf{A}, \quad \forall m \in M.$$

Conversely any right A -module is a left A -module. From now on an A -module will mean a left or right A -module.

Denote by $\mathcal{C}(A)$ the category of $\mathbf{Z}$ -graded complexes

$$\dots \rightarrow C_{n+1} \rightarrow C_n \rightarrow C_{n-1} \rightarrow \dots$$

such that each C_n is a finitely generated free A -module with fixed (unordered) basis and $\bigoplus_n C_n$ is finitely generated. These complexes will be called finite A -complexes.

We say that a sequence of finite A -complexes $0 \rightarrow C \rightarrow C' \rightarrow C'' \rightarrow 0$ is s -exact if, for any n , the complex $0 \rightarrow C_n \rightarrow C'_n \rightarrow C''_n \rightarrow 0$ is acyclic with torsion 0 in $\tilde{K}_1(A)$; see [4] and [9].

Definition (1.2). — A class $\mathcal{W} \subset \mathcal{C}(A)$ is *exact* if $\mathcal{W}$ contains any acyclic finite A -complex with torsion 0 , and if, for any s -exact sequence in $\mathcal{C}(A)$

$$0 \rightarrow C \rightarrow C' \rightarrow C'' \rightarrow 0,$$

one has the following property:

If two of the complexes C , C' , C'' lie in $\mathcal{W}$, then the third lies in $\mathcal{W}$ too.

Let C be a finite A -complex. Denote by $\hat{C}_n$ the dual module $\text{Hom}(C_n, A)$ endowed with the dual basis, and choose on $\hat{C}$ the differential so that the evaluation from $\hat{C} \otimes C$ to A is a cocycle. So we get a new finite A -complex $\hat{C}$.

Definition (1.3). — An exact class $\mathcal{W} \subset \mathcal{C}(A)$ is called *symmetric* if, for any $C \in \mathcal{W}$, $\hat{C}$ lies in $\mathcal{W}$.

Definition (1.4). — Let $\mathcal{W}$ be an exact class in $\mathcal{C}(A)$. A morphism f in $\mathcal{C}(A)$ is a $\mathcal{W}$ -*equivalence* if the mapping cone of f is in $\mathcal{W}$.

Let f be a map from a finite CW-complex X to a finite connected CW-complex Y , with fundamental group π , and let $\mathcal{W}$ be an exact class in $\mathcal{C}(\mathbf{Z}\pi)$ containing any acyclic finite $\mathbf{Z}\pi$ -complex with torsion in the image of $\pi \rightarrow \tilde{K}_1(\mathbf{Z}\pi)$. Then f is a $\mathcal{W}$ -*equivalence* if the chain map $C_*(X, \mathbf{Z}\pi) \rightarrow C_*(Y, \mathbf{Z}\pi)$ is a $\mathcal{W}$ -equivalence.

Example (1.5). — Let $A \rightarrow B$ be a ring homomorphism and β be a subgroup of $\tilde{K}_1(B)$. Let $\mathcal{W}$ be the class of finite A -complexes C such that $C \otimes_A B$ is acyclic with torsion in β . Then $\mathcal{W}$ is exact and the $\mathcal{W}$ -equivalences are the B -homology equivalences with torsion in β .

If, in addition, $A \rightarrow B$ is a morphism of rings with involution and β is stable under the involution, $\mathcal{W}$ is symmetric.

Example (1.6). — Let M be an A -module. Then the class $\mathcal{W}$ of finite A -complexes C such that $H_*(C, M)$ (resp. $H^*(C, M)$) vanishes, is an exact class and the $\mathcal{W}$ -equivalences are the M -homology (resp. M -cohomology) equivalences.

Notation (1.7). — Let $\mathcal{W}$ be an exact class in $\mathcal{C}(A)$. We denote by Σ the set of matrices u such that the direct sum of the complex $\dots \rightarrow 0 \rightarrow A^p \xrightarrow{u} A^q \rightarrow 0 \rightarrow \dots$ and its suspension is in $\mathcal{W}$.

In example (1.5), Σ is the set of matrices u with entries in A such that $u \otimes B$ is invertible.

Proposition (1.8). — Let $\mathcal{W}$ be an exact class in $\mathcal{C}(A)$. Then there exists a ring homomorphism $A \rightarrow \Lambda$ unique up to isomorphism, which is universal with respect to the following property: For any matrix $u \in \Sigma$, $u \otimes \Lambda$ is invertible.

If $\mathcal{W}$ is symmetric, $A \rightarrow \Lambda$ is a morphism of rings with involution.

Actually, the ring Λ is an inversive localization of A in the sense of Cohn [2].

Definition (1.9). — Let α be the subgroup of $\tilde{K}_1(\Lambda)$ generated by the torsion of all complexes $C \otimes \Lambda$, such that $C \in \mathcal{W}$ and $C \otimes \Lambda$ is acyclic. The pair (Λ, α) will be called the $\mathcal{W}$ -localization of A .

Let f be a normal map from the normal bundle of a compact n -dimensional (Top, PL or Diff)-manifold V to a (Top, PL or Diff)-bundle ξ over a pair $(X, \partial X)$ of finite complexes. Suppose X is connected. The first Stiefel-Whitney class of ξ induces an involution on the ring $A = \mathbb{Z}[\pi_1 X]$.

Let $\mathcal{W}$ be an exact symmetric class in $\mathcal{C}(A)$ containing any acyclic complex with torsion in the image of $\pi_1 X \rightarrow \tilde{K}_1(A)$.

Suppose we have the following properties:

- i) $(X, \partial X)$ is a $\mathcal{W}$ -Poincaré complex; i.e. the cap-product by $f_*[V]$ is a $\mathcal{W}$ -equivalence from $C^*(X; A)$ to $C_*(X, \partial X; A)$.
- ii) The restricted map $f: \partial V \rightarrow \partial X$ is a $\mathcal{W}$ -equivalence.

Theorem (1.10). — Let (Λ, α) be the $\mathcal{W}$ -localization of A . Suppose that any complex in $\mathcal{W}$ is Λ -acyclic. Then, the normal map f determines a well-defined element $\sigma(f) \in L_n^\alpha(\Lambda)$. And, if $n \geq 5$, f is normally cobordant, rel the boundary, to a $\mathcal{W}$ -equivalence if and only if $\sigma(f)$ vanishes.

Theorem (1.11). — With the same hypothesis as above, if $n \geq 6$, and X is a product $M \times I$, M being a (Top, PL or Diff)-manifold, any element of $L_n^\alpha(\Lambda)$ is the obstruction $\sigma(f)$ of a normal map f restricting to an isomorphism over $M \times 0 \cup \partial M \times I$.

Remark (1.12). — The condition of Λ -acyclicity of complexes in $\mathcal{W}$ is a very crucial point because, in the situation of (1.10), $\sigma(f)$ can be defined only if this condition is satisfied, or, more precisely, if the Poincaré duality on $(X, \partial X)$ is a Λ -homology equivalence and f restricts to a Λ -homology equivalence on the boundaries.

On the other hand, this condition is not always satisfied. For example, if $\mathcal{W}$

is the class of finite $\mathbf{Z}[t, t^{-1}]$ -complex with finite homology, the ring Λ is $\mathbf{Z}[t, t^{-1}]$ and there exist many complexes in $\mathcal{W}$ which are not acyclic.

If the condition of Λ -acyclicity of complexes in $\mathcal{W}$ is not satisfied, denote by $\mathcal{W}'$ the class of Λ -acyclic complexes in $\mathcal{W}$. Then theorems (1.10) and (1.11) hold for the class $\mathcal{W}'$. Now, the last problem is to compare the surgery problems corresponding to classes $\mathcal{W}$ and $\mathcal{W}'$. But this question seems to be very difficult.

Let $A \rightarrow B$ be a ring homomorphism. Let Λ be the inversive localization of A in the sense of Cohn [2] obtained by formal inversion of the matrices u with entries in A such that $u \otimes B$ is invertible. The ring homomorphism $A \rightarrow \Lambda$ will be called the localization of $A \rightarrow B$.

Theorem (1.13). — *Let $A \rightarrow B$ be a ring homomorphism and β be a subgroup of $\tilde{K}_1(B)$. Denote by $\mathcal{W}$ the class of finite A -complexes which are B -acyclic with torsion in β , and by (Λ, α) the $\mathcal{W}$ -localization of A .*

Then $A \rightarrow \Lambda$ is the localization of $A \rightarrow B$ and α is the inverse image of β under the canonical morphism $\varepsilon: \Lambda \rightarrow B$.

Moreover, if ε is onto, any complex in $\mathcal{W}$ is Λ -acyclic.

Remark (1.14). — The ring Λ and the group $L_n^a(\Lambda)$ are difficult to compute, but we have some interesting results.

Let $S \subset A$ be the set of elements in A invertible in B . Then, if there exists a classical localization $S^{-1}A$, Λ is the ring $S^{-1}A$. This holds, for example, if A is commutative or if $A \rightarrow B$ is the ring homomorphism $\mathbf{Z}\pi \rightarrow \mathbf{Z}\pi'$ induced by a group homomorphism $\pi \rightarrow \pi'$ with finitely generated nilpotent kernel onto a finite extension of a polycyclic group [7].

Another example is the following (see theorem (9.7)): Let $\pi \rightarrow G$ be a group-epimorphism with locally perfect kernel. Then the localization of $\mathbf{Z}\pi \rightarrow \mathbf{Z}G$ is $\mathbf{Z}\pi \rightarrow \mathbf{Z}G$ itself.

Anyway, the theorems (1.10), (1.11), (1.13) imply that the obstruction groups $\Gamma_n(A \rightarrow B)$ of Cappell and Shaneson [1] are always the L_n -groups of Λ (endowed with a subgroup of $\tilde{K}_1(\Lambda)$), at least when the theory of Cappell and Shaneson holds, i.e. when $A \rightarrow B$ is locally epic. This was already proved in some particular cases by Cappell and Shaneson [1], Smith [7], Hausmann [3] and the author [8].

Nevertheless the condition of surjectivity of $\Lambda \rightarrow B$ holds in many other cases.

Proposition (1.15). — *Let $A \rightarrow B$ be a ring homomorphism and $A \rightarrow \Lambda$ be the localization of $A \rightarrow B$. Let $B_0 \subset B_1 \subset B_2 \subset \dots$ be subrings of B defined by:*

- i) B_0 is the image of $A \rightarrow B$;
- ii) For any $n \geq 0$, B_{n+1} is generated by B_n and the inverses of the units of B contained in B_n .

Then, the image of $\Lambda \rightarrow B$ contains all the rings B_n . Therefore, if B is the union of the rings B_n , the morphism $\Lambda \rightarrow B$ is onto and the theorems (1.10), (1.11), (1.13) hold.

In fact, the image of $\Lambda \rightarrow B$ can be strictly greater than the union of the rings B_n .

Example (1.16). — Let F be the free group with p generators, $p > 1$, and let A be the group ring $\mathbf{Z}[F]$. Let $\mathcal{W}$ be the class of finite A -complexes C such that $H_*(C)$ is finitely generated over $\mathbf{Z}$ and let (Λ, α) be the $\mathcal{W}$ -localization of A . Then the localization of $A \rightarrow \Lambda$ is $A \rightarrow \Lambda$ and the morphism $\Lambda \rightarrow \Lambda$ is the identity. One can prove that any square matrix with entries in A which is invertible in Λ , is invertible in A ; hence $B_n = A$ for all n , but $A \rightarrow \Lambda$ is not surjective!

Remark (1.17). — Let $A \rightarrow B$ be a ring homomorphism and β be a subgroup of $\tilde{K}_1(B)$. Denote by $\mathcal{W}$ the class of finite A -complexes which are B -acyclic with torsion in β and by (Λ, α) the $\mathcal{W}$ -localization of A .

If the morphism $\Lambda \rightarrow B$ is not onto, the condition of Λ -acyclicity of complexes in $\mathcal{W}$ is not always satisfied.

For example, this condition holds if $A \rightarrow B$ is the ring homomorphism $\mathbf{Z} \rightarrow \mathbf{R}$, but it does not hold if A is the ring $\mathbf{Z}[t, t^{-1}]$ and B is the product of the localizations of A with respect to the non zero principal prime ideals.

2. A first homology surgery obstruction group

In a first step, we will construct a surgery obstruction group $\Gamma_n(A, \mathcal{W})$ which looks like the group $\Gamma_n(A \rightarrow B)$ constructed by Ranicki [5], but from a dual point of view.

Throughout sections 2 and 3 we assume that A is a ring with involution and that $\mathcal{W}$ is an exact symmetric class in $\mathcal{C}(A)$ (see (1.2) and (1.3)).

If C and C' are finite A -complexes, we denote by $\text{Hom}(C, C')$ the set of A -homomorphisms from C to C' ; $\text{Hom}(C, C')$ can be given a graded differential $\mathbf{Z}$ -module structure by setting:

$$\begin{aligned} \partial^0 f(x) &= \partial^0 f + \partial^0 x, \quad \text{for any } f \in \text{Hom}(C, C'), x \in C \\ d(f(x)) &= (df)(x) + (-1)^{\partial^0 f} f(dx), \quad \text{for any } f \in \text{Hom}(C, C'), x \in C. \end{aligned}$$

Moreover, by setting

$$\hat{f}(u) = (-1)^{\partial^0 f \partial^0 u} u \circ f, \quad \text{for any } f \in \text{Hom}(C, C'), u \in \hat{C}',$$

we get a morphism $f \rightarrow \hat{f}$ from $\text{Hom}(C, C')$ to $\text{Hom}(\hat{C}', \hat{C})$ which respects the degrees and the differentials.

Notation (2.1). — If C is a finite A -complex, we denote by $B(C)$ the graded differential $\mathbf{Z}$ -module $\text{Hom}(C, \hat{C})$. The composite map:

$$\text{Hom}(C, \hat{C}) \rightarrow \text{Hom}(\hat{\hat{C}}, \hat{C}) \xrightarrow{\sim} \text{Hom}(C, \hat{C})$$

is an involution on $B(C)$ and $B(C)$ is a graded differential $\mathbf{Z}[\mathbf{Z}/2]$ -module.

Definition (2.2). — Let C be a finite A -complex. We use $Q_n(C)$ to denote the group $H_n(\mathbf{Z}/2, B(C))$. By a *quadratic n -form* over C , we mean an element of $Q_n(C)$ and by a *quadratic n -complex* we mean a pair (C, q) , $q \in Q_n(C)$.

Let $C \rightarrow C'$ be an epimorphism of degree 0 between two finite A -complexes. We use $Q_n(C \rightarrow C')$ to denote the group $H_n(\mathbf{Z}/2, B(C)/B(C'))$. By a *quadratic n -form* over $C \rightarrow C'$, we mean an element of $Q_n(C \rightarrow C')$ and by a *quadratic n -pair*, we mean a pair $(C \rightarrow C', q)$, $q \in Q_n(C \rightarrow C')$.

Definition (2.3). — Let (C, q) be a quadratic n -complex. We will say that q or (C, q) is $\mathcal{W}$ -non singular if the image of q by the composite map

$$H_n(\mathbf{Z}/2, B(C)) \xrightarrow{\text{transfer}} H_n(1, B(C)) \simeq H_n(B(C))$$

is represented by a $\mathcal{W}$ -equivalence from C to $\hat{C}$.

Let $(C \rightarrow C', q)$ be a quadratic n -pair. Let K be the kernel of $C \rightarrow C'$. We will say that q or $(C \rightarrow C', q)$ is $\mathcal{W}$ -non singular if the image of q by the composite map

$$H_n(\mathbf{Z}/2, B(C)/B(C')) \xrightarrow{\text{transfer}} H_n(B(C)/B(C')) \rightarrow H_n(\text{Hom}(K, \hat{C}))$$

is represented by a $\mathcal{W}$ -equivalence from K to $\hat{C}$.

Remark (2.4). — If C is zero except in dimension $-p$, a quadratic $2p$ -form over C is exactly a $(-1)^p$ -quadratic form over C_{-p} in the sense of Wall [11].

Remark (2.5). — If $\mathcal{W}$ is the class of acyclic complexes with zero torsion, a $\mathcal{W}$ -non singular quadratic n -form q over a finite A -complex C is an n -dimensional quadratic Poincaré structure on $\hat{C}$, in the sense of Ranicki [5], at least if $\hat{C}$ is (-1) -connected.

Definition (2.6). — We will denote by $\Gamma_n(A, \mathcal{W})$ the set of $\mathcal{W}$ -non singular quadratic n -complexes subject to the following cobordism relation: (C, q) is cobordant to (C', q') if there exists a $\mathcal{W}$ -non singular quadratic $(n+1)$ -pair $(\Sigma \rightarrow C \oplus C', u)$ such that $\partial u = q \oplus -q'$.

Let W be the standard free resolution of the $\mathbf{Z}[\mathbf{Z}/2]$ -module $\mathbf{Z}$:

$$\mathbf{Z}[\mathbf{Z}/2]e_0 \xleftarrow{1-t} \mathbf{Z}[\mathbf{Z}/2]e_1 \xleftarrow{1+t} \mathbf{Z}[\mathbf{Z}/2]e_2 \xleftarrow{1-t} \dots$$

Then $Q_n(C)$ is the n -th homology group of $W \otimes_{\mathbf{Z}/2} B(C)$.

Lemma (2.7). — Two $\mathcal{W}$ -non singular quadratic n -complexes (C, q) and (C', q') are cobordant if and only if there exist two s -exact sequences

$$\begin{aligned} 0 \rightarrow K \rightarrow \Sigma \xrightarrow{\alpha} C \rightarrow 0 \\ 0 \rightarrow K' \rightarrow \Sigma \xrightarrow{\alpha'} C' \rightarrow 0 \end{aligned}$$

and an element $e_0 \otimes \psi_0 + e_1 \otimes \psi_1 + \dots$ in $W \otimes_{\mathbb{Z}/2} B(\Sigma)$ such that:

i) If q and q' are the homology classes of φ and φ' , we have

$$d(\Sigma e_i \otimes \psi_i) = \alpha^*(\varphi) - \alpha'^*(\varphi');$$

ii) $\psi_0 + \hat{\psi}_0$ induces a $\mathcal{W}$ -equivalence from K to $\hat{K}'$.

Proof. — Suppose that q and q' are represented by $\varphi \in W \otimes_{\mathbb{Z}/2} B(C)$ and $\varphi' \in W \otimes_{\mathbb{Z}/2} B(C')$. If (C, q) and (C', q') are cobordant, there exists a s -exact sequence

$$0 \rightarrow \Sigma' \rightarrow \Sigma \xrightarrow{\alpha \oplus \alpha'} C \oplus C' \rightarrow 0$$

together with an element $\Sigma e_i \otimes \psi_i \in W \otimes B(\Sigma)$ such that:

(i) $d(\Sigma e_i \otimes \psi_i) = \alpha^*(\varphi) - \alpha'^*(\varphi')$;

(ii) $\psi_0 + \hat{\psi}_0$ induces a $\mathcal{W}$ -equivalence from Σ' to $\hat{\Sigma}$.

Let K (respectively K') be the kernel of α (respectively α'). We have a homotopy commutative diagram

$$(I) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \Sigma' & \longrightarrow & K & \longrightarrow & C' \longrightarrow 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c \\ 0 & \longrightarrow & \hat{\Sigma} & \longrightarrow & \hat{K}' & \longrightarrow & \hat{C}' \longrightarrow 0 \end{array}$$

where the lines are homotopy s -exact and a and b are induced by $\psi_0 + \hat{\psi}_0$ and c is induced by the transfer of φ' .

Since a and c are $\mathcal{W}$ -equivalences, b is a $\mathcal{W}$ -equivalence too and the first part of the lemma is proved.

Conversely, suppose we have two s -exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & \Sigma & \xrightarrow{\alpha} & C \longrightarrow 0 \\ 0 & \longrightarrow & K' & \longrightarrow & \Sigma & \xrightarrow{\alpha'} & C' \longrightarrow 0 \end{array}$$

and an element $\Sigma e_i \otimes \psi_i$ satisfying the conditions (i) and (ii) of the lemma. Up to simple homotopy type, we may suppose that $\alpha \oplus \alpha'$ is onto with kernel $\Sigma' \in \mathcal{C}(A)$. Then we have the homotopy commutative diagram (I) where b and c are $\mathcal{W}$ -equivalences and $\psi_0 + \hat{\psi}_0$ induces a $\mathcal{W}$ -equivalence from Σ' to $\hat{\Sigma}$. Hence (C, q) and (C', q') are cobordant. ■

Lemma (2.8). — *Let (C, q) be a $\mathcal{W}$ -non singular quadratic n -complex and $f: C' \rightarrow C$ be a $\mathcal{W}$ -equivalence. Then $(C', f^*(q))$ is a $\mathcal{W}$ -non singular quadratic n -complex cobordant to (C, q) .*

Proof. — We may suppose that f is epic with kernel $K \in \mathcal{C}(A)$. Then we have the s -exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & C' & \xrightarrow{f} & C \longrightarrow 0 \\ 0 & \longrightarrow & 0 & \longrightarrow & C' & \xrightarrow{1} & C' \longrightarrow 0 \end{array}$$

and the result is an easy consequence of lemma (2.7). ■

3. Algebraic surgery

In order to kill the homology of a $\mathcal{W}$ -non singular quadratic n -complex, in low dimension, we need the following:

Lemma (3.1). — *Let $0 \rightarrow I \xrightarrow{\alpha} C \xrightarrow{\beta} J \rightarrow 0$ be an s -exact sequence of finite A -complexes. Let q be a $\mathcal{W}$ -non singular quadratic n -form over C such that $\alpha^*q = 0$. Then, q is represented by a cycle $\Sigma e_i \otimes f_i \beta$.*

Moreover if q is represented by such a cycle, (C, q) is cobordant to a $\mathcal{W}$ -non singular quadratic n -complex (C', q') where C' is the mapping cone of $\hat{\alpha}f_0$ (the grading of C' is chosen so that the map $C' \rightarrow J$ has degree 0).

Proof. — Consider the following exact sequences of graded differential $\mathbf{Z}[\mathbf{Z}/2]$ -modules:

$$\begin{aligned} 0 \rightarrow B \rightarrow B(C) \xrightarrow{\alpha^*} B(I) \rightarrow 0 \\ \text{Hom}(C, \hat{J}) \oplus \text{Hom}(J, \hat{C}) \rightarrow B \rightarrow 0. \end{aligned}$$

If α^*q is zero, q is represented by a cycle in $W \otimes_{\mathbf{Z}/2} B$, and there exist morphisms f'_i and f''_i in $\text{Hom}(J, \hat{C})$ such that q is represented by

$$\Sigma e_i \otimes (f'_i \beta + \hat{\beta} f''_i).$$

Now we have

$$d(e_{i+1} \otimes f''_i \beta) = e_i \otimes f''_i \beta + (-1)^{i+1} e_i \otimes \hat{\beta} f''_i + (-1)^{i+1} e_{i+1} \otimes df''_i \beta.$$

Then there exist morphisms $f_i \in \text{Hom}(J, \hat{C})$ such that q is represented by $\Sigma e_i \otimes f_i \beta$. Since $\Sigma e_i \otimes f_i \beta$ is a cycle, we have

$$\forall i \geq 0, \quad (-1)^i df_i \beta + f_{i+1} \beta + (-1)^{i+1} \hat{\beta} f_{i+1} = 0,$$

whence $d(\hat{\alpha}f_0) = 0$, $\hat{\alpha}f_i = 0$, for any $i > 0$.

Let C' be the mapping cone of $\hat{\alpha}f_0$. We have a split exact sequence

$$0 \rightarrow \hat{I} \xrightleftharpoons[r']{\alpha'} C' \xrightarrow{\beta'} J \rightarrow 0$$

such that

$$\partial^0 \alpha' = -n - 1, \quad \partial^0 \beta' = 0, \quad dr' = \hat{\alpha}f_0 \beta', \quad r' \alpha' = 1$$

and

$$0 \rightarrow S^{-n-1} \hat{I} \rightarrow C' \rightarrow J \rightarrow 0$$

is s -exact.

Let Σ be the pull-back of C and C' over J :

$$\begin{array}{ccc} \Sigma & \xrightarrow{\gamma} & C \\ \gamma' \downarrow & & \downarrow \beta \\ C' & \xrightarrow{\beta'} & J \end{array}$$

Let r be a retraction of α and let u be the element $e_0 \otimes \hat{\gamma} \hat{r} r' \gamma' \in W \otimes_{\mathbf{Z}/2} B(\Sigma)$. We have

$$du = e_0 \otimes \hat{\gamma} d\hat{r} r' \gamma' + e_0 \otimes \hat{\gamma} f_0 \beta \gamma + e_0 \otimes \hat{\gamma} (\hat{r} \alpha - 1) f_0 \beta' \gamma'$$

and it is easy to see that $\gamma^*(\Sigma e_i \otimes f_i \beta) - du$ has the form $\gamma'^*(\Sigma e_i \otimes \varphi'_i)$, $\varphi'_i \in B(C')$.

On the other hand, $\hat{\gamma} \hat{r} r' \gamma' + \hat{\gamma}' \hat{r}' r \gamma$ induces the identity from the kernel of γ' to the dual of the kernel of γ . Then $\Sigma e_i \otimes \varphi'_i$ represents a $\mathcal{W}$ -non singular quadratic n -form q' over C' and, by (2.7), (C, q) and (C', q') are cobordant. ■

Corollary (3.2). — Any $\mathcal{W}$ -non singular quadratic n -complex is cobordant to a $\mathcal{W}$ -non singular quadratic n -complex (C, q) such that C is $\left(\left[\frac{-n}{2}\right] - 1\right)$ -connected.

Proof. — Just apply lemma (3.1), I being the $\left(\left[\frac{-n}{2}\right] - 1\right)$ -skeleton of the complex. ■

Lemma (3.3). — Let $0 \rightarrow I \xrightarrow{\alpha} C \xrightarrow{\beta} J \rightarrow 0$ be an s -exact sequence of finite A -complexes and $\gamma: J \rightarrow K$ be an epimorphism of degree 0 which respects the differentials. Let q be a $\mathcal{W}$ -non singular quadratic n -form over $C \rightarrow K$ such that $\alpha^* q = 0$. Then q is represented by $\Sigma e_i \otimes f_i \beta$.

Moreover, if C' is the mapping cone of $\hat{\alpha} f_0$ (the grading being chosen as in lemma (3.1)), there exists a $\mathcal{W}$ -non singular quadratic n -form q' over $C' \rightarrow K$ such that ∂q and $\partial q'$ coincide in $Q_{n-1}(K)$.

Proof. — We have the following exact sequences of graded differential $\mathbf{Z}[\mathbf{Z}/2]$ -modules:

$$\begin{aligned} 0 \rightarrow B \rightarrow B(C)/B(K) \xrightarrow{\alpha^*} B(I) \rightarrow 0 \\ \text{Hom}(C, \hat{J}) \oplus \text{Hom}(J, \hat{C}) \rightarrow B \rightarrow 0. \end{aligned}$$

Then, as in lemma (3.1), we show that q is represented by an element $\Sigma e_i \otimes f_i \beta$ and we have

$$d(\hat{\alpha} f_0) = 0, \quad \hat{\alpha} f_i = 0, \quad \text{for any } i > 0.$$

Consider, as above, the diagram: $0 \rightarrow \hat{I} \xrightarrow{\alpha'} C' \xrightarrow{\beta'} J \rightarrow 0$ and let s be a section of β . We have

$$ds = \alpha \delta, \quad \delta \in \text{Hom}(J, I).$$

It is not difficult to see that the element

$$u = e_0 \otimes \hat{\beta}' \hat{\delta} r' + \Sigma e_i \otimes \hat{\beta}' \hat{s} f_i \beta'$$

represents a quadratic n -form q' over $C' \rightarrow K$ and that ∂q and $\partial q'$ coincide in $Q_{n-1}(K)$. Moreover, the transfer $\tilde{u}$ of u is:

$$\tilde{u} = \hat{\beta}' \hat{\delta} r' + (-1)^{n+1} \hat{r}' \delta \beta' + \hat{\beta}' \hat{s} f_0 \beta' + \hat{\beta}' \hat{s} f_0 s \beta'$$

and we have

$$\tilde{u}\alpha' = \hat{\beta}'\hat{\delta} \quad \text{and} \quad \hat{\alpha}'\tilde{u} = \delta\beta'.$$

Denote by $\bar{C}$, $\bar{J}$, $\bar{C}'$ the kernels of the morphisms $C \rightarrow K$, $J \rightarrow K$ and $C' \rightarrow K$. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hat{I} & \xrightarrow{\alpha'} & \bar{C}' & \xrightarrow{\beta'} & \bar{J} & \longrightarrow & 0 \\ & & \downarrow \hat{\delta} & & \downarrow \tilde{u} & & \downarrow \delta & & \\ 0 & \longrightarrow & \hat{J} & \xrightarrow{\hat{\beta}'} & \hat{C}' & \xrightarrow{\hat{\alpha}'} & I & \longrightarrow & 0, \end{array}$$

and we obtain a s -exact sequence between the mapping cone of $\hat{\delta}$, $\tilde{u}$ and δ . Now the boundary of this s -exact sequence is homotopic to the morphism $(-1)^{n+1}(f_0\beta + \hat{\beta}\hat{f}_0)$ from $\bar{C}$ to $\hat{C}$, which is a $\mathcal{W}$ -equivalence. Then the mapping cone of $\tilde{u}: \bar{C}' \rightarrow \hat{C}'$ is in $\mathcal{W}$ and q' is $\mathcal{W}$ -non singular. ■

Corollary (3.4). — *Let (C, q) be a $\mathcal{W}$ -non singular quadratic n -complex cobordant to zero. Then there exists a $\mathcal{W}$ -non singular quadratic $(n+1)$ -pair $(\Sigma \rightarrow C, u)$ such that q is the boundary of u and the kernel of $\Sigma \rightarrow C$ is $\left(\left[\frac{-n-1}{2}\right] - 1\right)$ -connected.*

Proof. — If (C, q) is cobordant to zero, there exists a $\mathcal{W}$ -non singular quadratic $(n+1)$ -pair $(\Sigma' \rightarrow C, u')$ such that q is the boundary of u' . Then apply lemma (3.3), I being the $\left(\left[\frac{-n-1}{2}\right] - 1\right)$ -skeleton of the kernel of $\Sigma' \rightarrow C$. ■

Now, if we want to kill the homology of a $\mathcal{W}$ -non singular quadratic n -form beyond the middle dimension, we must suppose that $\mathcal{W}$ satisfies some other properties. Actually, it is useful to consider the new class $\mathcal{W}'$ of all Λ -acyclic finite A -complexes.

Splitting lemma (3.5). — *Let C be a complex in $\mathcal{W}'$ and let n be an integer. Then, there exist two finite A -complexes L and L' concentrated in dimension n and a $\mathcal{W}'$ -equivalence from L to the complex*

$$L' \oplus (\dots \rightarrow C_{n+1} \rightarrow C_n \rightarrow 0 \rightarrow \dots).$$

This lemma will be proved in § 7.

Lemma (3.6). — *Any $\mathcal{W}'$ -non singular quadratic n -complex is cobordant to a $\mathcal{W}'$ -non singular quadratic n -complex (C, q) where C vanishes except in dimension $\left[\frac{-n}{2}\right]$ (and $\left[\frac{-n}{2}\right] + 1$ if n is odd).*

Proof. — Let (C, q) be a $\mathcal{W}'$ -non singular quadratic n -complex. By corollary (3.2), we may as well suppose that C_i vanishes for $i < \left[\frac{-n}{2}\right]$.

Suppose $n = -2p$. Since (C, q) is $\mathcal{W}'$ -non singular, we have the following complex in $\mathcal{W}'$:

$$\dots \rightarrow C_{p+1} \rightarrow C_p \rightarrow \hat{C}_p \rightarrow \hat{C}_{p+1} \rightarrow \dots$$

and, by splitting lemma (3.5), there exist two complexes L and L' concentrated in dimension p and a $\mathcal{W}'$ -equivalence

$$f: L \rightarrow C \oplus L'.$$

Up to stabilization, we may suppose that L'_p is even dimensional. Let $q' \in Q_n(L')$ be a standard hyperbolic structure on L'_p .

Then (C, q) is cobordant to $(C \oplus L', q \oplus q')$ and by lemma (2.8), (C, q) is cobordant to $(L, f^*(q \oplus q'))$.

Suppose $n = -2p - 1$. Since (C, q) is $\mathcal{W}'$ -non singular, we have the following complex in $\mathcal{W}'$:

$$\dots \rightarrow C_{p+1} \rightarrow C_p \oplus \hat{C}_p \rightarrow \hat{C}_{p+1} \rightarrow \dots,$$

and, by the splitting lemma (3.5), there exist two complexes L and L' concentrated in dimension $p + 1$ and a $\mathcal{W}'$ -equivalence

$$L \rightarrow L' \oplus (\dots \rightarrow C_{p+2} \rightarrow C_{p+1} \rightarrow 0 \dots).$$

We deduce a $\mathcal{W}'$ -equivalence

$$f: (\dots \rightarrow 0 \rightarrow L_{p+1} \rightarrow C_p \oplus L'_{p+1} \rightarrow 0 \dots) \rightarrow C$$

and (C, q) is cobordant to $(\dots \rightarrow 0 \rightarrow L_{p+1} \rightarrow C_p \oplus L'_{p+1} \rightarrow 0 \dots, f^*q)$. ■

Lemma (3.7). — *Let (C, q) be a $\mathcal{W}'$ -non singular quadratic $(-2p)$ -complex such that C_i vanishes for $i \neq p$. Then (C, q) is cobordant to zero if and only if there exists a $\mathcal{W}'$ -non singular quadratic $(-2p + 1)$ -pair $(\Sigma \rightarrow C, u)$ such that q is the boundary of u and Σ_i vanishes for $i \neq p, p - 1$.*

Proof. — Suppose (C, q) is cobordant to zero. By corollary (3.4), there exists a $\mathcal{W}'$ -non singular quadratic $(-2p + 1)$ -pair $(\Sigma' \rightarrow C, u')$ such that q is the boundary of u' and Σ'_i vanishes for $i < p - 1$. Let K' be the kernel of $\Sigma \rightarrow C$.

Since u' is $\mathcal{W}'$ -non singular, we have the following complex in $\mathcal{W}'$:

$$\dots \rightarrow K'_{p+1} \rightarrow K'_p \rightarrow K'_{p-1} \oplus \hat{\Sigma}'_{p-1} \rightarrow \hat{\Sigma}'_p \rightarrow \hat{\Sigma}'_{p+1} \rightarrow \dots$$

and, by the splitting lemma (3.5), there exist two complexes $L, L' \in \mathcal{C}(A)$ concentrated in dimension p and a $\mathcal{W}'$ -equivalence

$$(\dots \rightarrow 0 \rightarrow L_p \rightarrow K'_{p-1} \oplus L'_p \rightarrow 0 \rightarrow \dots) \rightarrow K'.$$

Let K be the complex $\dots \rightarrow L_p \rightarrow K'_{p-1} \oplus L'_p \rightarrow 0 \rightarrow \dots$. Since the $\mathcal{W}'$ -equivalence $K \rightarrow K'$ is $(p-1)$ -connected, the boundary $C \rightarrow K'$ lifts through K and we get a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & \Sigma & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow & & \downarrow f & & \parallel \\ 0 & \longrightarrow & K' & \longrightarrow & \Sigma' & \longrightarrow & C \longrightarrow 0 \end{array}$$

where the lines are s -exact.

Then $(\Sigma \rightarrow C, f^*u')$ is the desired quadratic pair. ■

Lemma (3.8). — *Let (C, q) be a $\mathcal{W}'$ -non singular quadratic $(-2p-1)$ -form such that C_i vanishes for $i \neq p, p+1$. Then (C, q) is cobordant to zero if and only if there exists a $\mathcal{W}'$ -non singular quadratic $(-2p)$ -pair $(\Sigma \rightarrow C, u)$ such that q is the boundary of u and $\Sigma_i \rightarrow C_i$ is a simple isomorphism for $i \neq p$.*

Proof. — Suppose (C, q) is cobordant to zero. By corollary (3.4), there exists a $\mathcal{W}'$ -non singular quadratic $(-2p)$ -pair $(\Sigma' \rightarrow C, u')$ such that q is the boundary of u' and Σ'_i vanishes for $i < p$.

Let K' be the kernel of $\Sigma' \rightarrow C$. We have a complex in $\mathcal{W}'$

$$\dots \rightarrow \Sigma'_{p+1} \rightarrow \Sigma'_p \rightarrow \hat{K}'_p \rightarrow \hat{K}'_{p+1} \rightarrow \dots$$

and, by the splitting lemma (3.5), there exist two complexes L and $L' \in \mathcal{C}(A)$ concentrated in dimension p and a $\mathcal{W}'$ -equivalence

$$f: L \rightarrow \Sigma' \oplus L'.$$

Up to stabilization, we may suppose that L'_p is even dimensional. Let $v \in Q_{-2p}(L')$ be a standard hyperbolic structure on L'_p .

Let X be an acyclic finite A -complex with torsion zero concentrated in dimension $p-1, p, p+1$ and $X \rightarrow C$ be an epimorphism with kernel in $\mathcal{C}(A)$ such that $X_{p+1} \rightarrow C_{p+1}$ is an isomorphism. Let $(\Sigma'' \rightarrow C, u'')$ be the quadratic $(-2p)$ -pair defined by $\Sigma'' = L \oplus X$, $u'' = f^*(u' \oplus v) \oplus 0$.

It is easy to see that u'' is $\mathcal{W}'$ -non singular and that $\partial u'' = q$. Moreover the kernel K'' of $\Sigma'' \rightarrow C$ is concentrated in dimension $p-1$ and p .

Now, by lemma (3.3), we can kill the $p-1$ skeleton of K'' by surgery and we get a $\mathcal{W}'$ -non singular $(-2p)$ -pair $(\Sigma \rightarrow C, u)$ such that $\partial u = q$ and the kernel of $\Sigma \rightarrow C$ vanishes except in dimension p . ■

Now, with the above lemmas, it is possible to give an interpretation of $\Gamma_n(A, \mathcal{W}')$ in term of special forms in the sense of Wall [10] and Cappell and Shaneson [1].

Definition (3.9). — Let $\eta = \pm 1$ and $I_\eta = \{a - \eta \bar{a}, a \in A\}$. A $\mathcal{W}'$ -special η -form is a triple (H, λ, μ) where H is a finitely generated free A -module, λ a $\mathbb{Z}$ -bilinear map from $H \otimes_{\mathbb{Z}} H$ to A and μ a map from H to A/I_η , and satisfying the following conditions:

- Q_1 $\lambda(ax, yb) = a\lambda(x, y)b, \quad \forall x, y \in H, \forall a, b \in A$
- Q_2 $\lambda(x, y) = \overline{\eta\lambda(y, x)}, \quad \forall x, y \in H$
- Q_3 $\mu(x) + \eta\mu(y) = \lambda(x, y), \quad \forall x, y \in H$
- Q_4 $\mu(x + y) \equiv \mu(x) + \mu(y) + \lambda(x, y) \bmod I_\eta, \quad \forall x, y \in H$
- Q_5 $\mu(xa) = \bar{a}\mu(x)a, \quad \forall x \in H, \forall a \in A$
- Q_6 the morphism $\tilde{\lambda}$ induced by λ is a Λ -isomorphism (i.e. $\tilde{\lambda} \otimes \Lambda$ is an isomorphism).

Definition (3.10). — Let (H, λ, μ) be a $\mathcal{W}'$ -special η -form. A $\mathcal{W}'$ -subkernel of (H, λ, μ) is a free A -module K endowed with a morphism $f: K \rightarrow H$ satisfying the following conditions:

- S_1 $f^*\lambda = 0, \quad f^*\mu = 0$
- S_2 the following complex lies in $\mathcal{W}'$: $0 \rightarrow K \xrightarrow{f} H \xrightarrow{\tilde{\lambda}} \hat{K} \rightarrow 0$.

(3.11) Let $\eta = (-1)^p$ and let (H, λ, μ) be a $\mathcal{W}'$ -special η -form. Since H is free, there exists a map $\varphi_0: H \rightarrow \hat{H}$ such that

$$\begin{aligned} \lambda(x, y) &= \varphi_0(x)(y) + \overline{\eta\varphi_0(y)(x)}, \quad \forall x, y \in H \\ \mu(x) &\equiv \varphi_0(x)(x) \bmod I_\eta, \quad \forall x \in H. \end{aligned}$$

And, if φ_0 and φ'_0 are such two maps, $\varphi_0 - \varphi'_0$ has the form $\psi - \eta\hat{\psi}$.

Choose a basis for H and denote by H_* the finite A -complex defined by

$$H_i = \begin{cases} H, & i = -p \\ 0, & i \neq -p \end{cases}$$

Then $e_0 \otimes \varphi_0$ represents a $\mathcal{W}'$ -non singular quadratic $2p$ -form q over H_* and the cobordism class of (H_*, q) is a well defined element $\omega(H, \lambda, \mu) \in \Gamma_{2p}(A, \mathcal{W}')$.

(3.12) Let $\eta = (-1)^p$ and let $f: K \rightarrow B \oplus \hat{B}$ be a $\mathcal{W}'$ -subkernel of a standard η -kernel $B \oplus \hat{B}$ (B is a finitely generated free A -module). The map f is induced by maps $d: K \rightarrow B$ and $\varphi_0: K \rightarrow \hat{B}$. Since the quadratic form is trivial over K , there exists a map $\varphi_1: K \rightarrow \hat{K}$ such that $\hat{\varphi}_0 \circ d = \varphi_1 - (-1)^p \hat{\varphi}_1$. Choose basis for K and B . Let C be the $-p$ -dimensional complex

$$\dots \rightarrow 0 \rightarrow K \xrightarrow{d} B \rightarrow 0 \rightarrow \dots$$

Let $\varphi_0|B = 0$. We get two bilinear forms φ_0 and φ_1 on C , and we have

$$d\varphi_0 = \varphi_1 - \hat{\varphi}_1.$$

Then, $e_0 \otimes \varphi_0 - e_1 \otimes \varphi_1$ is a cycle in $W \otimes_{\mathbb{Z}/2} B(C)$ inducing a quadratic $(2p+1)$ -form q over C .

It is easy to see that q is $\mathcal{W}'$ -non singular. We denote by $\omega(f) \in \Gamma_{2p+1}(A, \mathcal{W}')$ the cobordism class of (C, q) . This element depends *a priori* on the choice of φ_1 .

On the other hand, the tensorization by Λ induces a map from $\Gamma_n(A, \mathcal{W}')$ to $\Gamma_n(\Lambda, \mathcal{W}_1)$ where $\mathcal{W}_1$ is the class of finite acyclic Λ -complexes. But the group $\Gamma_n(\Lambda, \mathcal{W}_1)$ is isomorphic to $L_n^h(\Lambda)$. Then we get a morphism ε from $\Gamma_n(A, \mathcal{W}')$ to $L_n^h(\Lambda)$ and $\varepsilon\omega(f)$ is the class of $f \otimes \Lambda$ in $L_n^h(\Lambda)$. We deduce that $\varepsilon\omega(f)$ does not depend on the choice of φ_1 . But it will be proved in § 8 that ε is an isomorphism. Therefore $\omega(f)$ is well defined.

Proposition (3.13). — *Any element of $\Gamma_{2p}(A, \mathcal{W}')$ has the form $\omega(H, \lambda, \mu)$ for some $\mathcal{W}'$ -special $(-1)^p$ -form (H, λ, μ) and any element of $\Gamma_{2p+1}(A, \mathcal{W}')$ has the form $\omega(f)$ for some $\mathcal{W}'$ -subkernel $f: K \rightarrow B \oplus \hat{B}$ of a standard $(-1)^p$ -kernel $B \oplus \hat{B}$.*

Proof. — In the even dimensional case, this is a trivial consequence of lemma (3.6).

In the odd dimensional case, we know by lemma (3.6) that any element of $\Gamma_{2p+1}(A, \mathcal{W}')$ is the cobordism class of a $\mathcal{W}'$ -non singular $2p+1$ -complexes (C, q) where C_i vanishes for $i \neq -p, -p-1$. It is not difficult to see that q is represented by $e_0 \otimes \varphi_0 + e_1 \otimes \varphi_1$, where the morphism φ_0 is trivial over C_{-p-1} . Then the cobordism class of (C, q) is $\omega(f)$ where f is the map $d \oplus \varphi_0: C_{-p} \rightarrow C_{-p-1} \oplus \hat{C}_{-p-1}$. ■

Proposition (3.14). — *Let (H, λ, μ) be a $\mathcal{W}'$ -special $(-1)^p$ -form. Then $\omega(H, \lambda, \mu)$ is zero if and only if the direct sum of (H, λ, μ) and a standard kernel has a $\mathcal{W}'$ -subkernel.*

Proof. — Suppose that (H, λ, μ) has a $\mathcal{W}'$ -subkernel $f: K \rightarrow H$. Consider the quadratic $2p$ -complex (H_*, q) constructed in (3.11). Choose a basis for K and denote by $K_* \in \mathcal{C}(A)$ the complex defined by

$$K_i = \begin{cases} K, & i = -p \\ 0, & i \neq -p. \end{cases}$$

Let $K_* \rightarrow H'_* \xrightarrow{g} H_*$ be a factorization of f such that g is a simple homotopy equivalence and $K_* \rightarrow H'_*$ is a monomorphism with free cokernel. After doing an algebraic surgery along $K_* \rightarrow H'_*$, we show that (H'_*, g^*q) is cobordant to (H'_*, q'') where H'_* has the simple homotopy type of

$$\dots \rightarrow 0 \rightarrow K \rightarrow H \rightarrow \hat{K} \rightarrow 0 \rightarrow \dots$$

The complex H'_* is thus Λ -acyclic and (H_*, q) is cobordant to zero.

Now suppose that the direct sum of (H, λ, μ) and a standard kernel H' has a $\mathcal{W}'$ -subkernel. We have

$$\omega(H, \lambda, \mu) = \omega(H, \lambda, \mu) + \omega(H') = 0.$$

Conversely suppose that $\omega(H, \lambda, \mu)$ vanishes. By lemma (3.7), there exists a $\mathcal{W}'$ -non singular quadratic $(2p+1)$ -pair $(\Sigma \xrightarrow{\alpha} H_*, u)$ such that q is the boundary of u and Σ_i vanishes for $i \neq -p, -p-1$.

The form u can be represented by $e_0 \otimes \psi_0 + e_1 \otimes \psi_1$, ψ_0 vanishing on Σ_{-p-1} . Let K be the kernel of $\Sigma_{-p} \rightarrow H$.

Since u is $\mathcal{W}'$ -non singular, the following complex is Λ -acyclic:

$$0 \rightarrow \Sigma_{-p} \xrightarrow{d \oplus (-1)^p \psi_0} \Sigma_{-p-1} \oplus \hat{\Sigma}_{-p-1} \xrightarrow{\hat{\psi}_0 + \hat{d}} \hat{K} \rightarrow 0,$$

and since $\tilde{\lambda}: H \rightarrow \hat{H}$ is a Λ -isomorphism, we deduce that

$$\alpha \oplus d \oplus (-1)^p \psi_0: \Sigma_{-p} \rightarrow H \oplus \Sigma_{-p-1} \oplus \hat{\Sigma}_{-p-1}$$

is a $\mathcal{W}'$ -subkernel of the direct sum of (H, λ, μ) and the standard kernel $\Sigma_{-p-1} \oplus \hat{\Sigma}_{-p-1}$.

Proposition (3.15). — *Let $f: K \rightarrow B \oplus \hat{B}$ be a $\mathcal{W}'$ -subkernel of the standard $(-1)^p$ -kernel $B \oplus \hat{B}$. Then $\omega(f)$ is zero if and only if there exist a kernel $C \oplus \hat{C}$ endowed with its standard subkernel $g: C \rightarrow C \oplus \hat{C}$ and an isometry h of $B \oplus \hat{B} \oplus C \oplus \hat{C}$ leaving each element of $B \oplus \hat{C}$ fixed, such that the composite map*

$$K \oplus C \xrightarrow{h \circ (f \oplus g)} B \oplus \hat{B} \oplus C \oplus \hat{C} \rightarrow B \oplus \hat{C}$$

is a Λ -isomorphism.

Proof. — Consider the “if” part first. If g is the standard subkernel of $C \oplus \hat{C}$, the complex associated to g (see (3.12)) is acyclic and then $\omega(g)$ vanishes.

The complex associated to $f \oplus g$ is

$$0 \rightarrow K \oplus C \rightarrow B \oplus C \rightarrow 0 \rightarrow \dots$$

If we perform a surgery along B , we get a new complex

$$\dots \rightarrow K \oplus C \rightarrow \hat{B} \oplus C \rightarrow 0 \rightarrow \dots$$

and $\omega(f)$ is equal to $\omega(f')$, f' being the new $\mathcal{W}'$ -subkernel

$$K \oplus C \xrightarrow{f \oplus g} \hat{B} \oplus C \oplus (\widehat{\hat{B} \oplus C}).$$

It is easy to show that, for any isometry h of $\hat{B} \oplus C \oplus B \oplus \hat{C}$ leaving each element of $B \oplus \hat{C}$ fixed ($h \in \text{UU}_r(A)$ with the notations of [10]), the two $\mathcal{W}'$ -subkernels f' and $h \circ f'$ represent the same quadratic $(2p+1)$ -form over the same complex.

It suffices now to perform a surgery along $\hat{B} \oplus C$ to get a Λ -acyclic complex and $\omega(f)$ is zero.

Conversely, suppose $\omega(f)$ is zero. Let (C_*, q) be the quadratic complex associated

to f (see (3.12)). By lemma (3.8), there exists a $\mathcal{W}'$ -non singular quadratic $(2p+2)$ -pair $(\Sigma_* \rightarrow C_*, u)$ such that q is the boundary of u and $\Sigma_i \rightarrow C_i$ is a simple isomorphism for $i \neq -p-1$.

The map $\Sigma_* \rightarrow C_*$ has the form

$$\begin{array}{ccccccc} & & 0 & \longrightarrow & K & \xrightarrow{1} & K \\ & & \downarrow & & \downarrow d' & & \downarrow d \\ 0 & \longrightarrow & X & \xrightarrow{\alpha} & \Sigma & \xrightarrow{\beta} & B \longrightarrow 0 \end{array}$$

where $K \xrightarrow{d'} \Sigma$ is the complex Σ_* .

If u is represented by $\Sigma e_i \otimes \psi_i$, ψ_0 is a homomorphism from Σ to $\hat{\Sigma}$ satisfying

$$\tilde{\psi} \circ d' + \hat{\beta} \circ \varphi_0 = 0 \quad \text{with } \tilde{\psi} = \psi_0 - (-1)^p \hat{\psi}_0$$

and the following complex is Λ -acyclic:

$$0 \longrightarrow K \xrightarrow{d'} \Sigma \xrightarrow{\hat{\alpha} \circ \tilde{\psi}} \hat{X} \longrightarrow 0.$$

By the splitting lemma (3.5), there exist two finitely generated free Λ -modules C and I and a homomorphism $\gamma: C \rightarrow \Sigma \oplus I$ such that $(\gamma \oplus d') \otimes \Lambda$ is an isomorphism. After adding a kernel to Σ_* , we may suppose that I is zero and γ is a homomorphism from C to Σ .

Then the morphism $\hat{\alpha} \circ \tilde{\psi} \circ \gamma: C \rightarrow \hat{X}$ is a Λ -isomorphism, and the morphism $\tilde{\psi} \circ \gamma \oplus \hat{\beta}: C \oplus \hat{B} \rightarrow \hat{\Sigma}$ is also a Λ -isomorphism. That implies that the composite map from $C \oplus K$ to $\hat{C} \oplus B$

$$(\hat{\gamma} \circ \hat{\tilde{\psi}} \oplus \beta) \circ (\gamma \oplus d') = -(-1)^p \hat{\gamma} \circ \tilde{\psi} \circ \gamma \oplus (-1)^p \hat{\gamma} \circ \hat{\beta} \circ \varphi_0 \oplus \beta \circ \gamma \oplus d$$

is a Λ -isomorphism.

Let h be the homomorphism from $B \oplus \hat{B} \oplus C \oplus \hat{C}$ to itself defined by

$$h = 1 \oplus (-1)^p \hat{\gamma} \circ \hat{\beta} \oplus (-1)^{p+1} \beta \circ \gamma \oplus (-1)^{p+1} \hat{\gamma} \circ \tilde{\psi} \circ \gamma.$$

It is easy to check that h is an isometry leaving each element of $B \oplus \hat{C}$ fixed and that the composite map

$$K \oplus C \xrightarrow{h \circ (f \oplus g)} B \oplus \hat{B} \oplus C \oplus \hat{C} \longrightarrow B \oplus \hat{C}$$

is a Λ -isomorphism. ■

4. Geometric surgery

Throughout this section, we will suppose that Λ is the group ring $\mathbb{Z}\pi$ with an involution induced by a morphism $w: \pi \rightarrow \pm 1$, and that $\mathcal{W}$ is an exact symmetric class in $\mathcal{C}(\Lambda)$ containing any acyclic complex with torsion in the image of $\pi \rightarrow \tilde{K}_1(\Lambda)$.

We denote by (Λ, α) the $\mathcal{W}$ -localization of Λ (1.9) and by $\mathcal{W}'$ the class of Λ -acyclic

complexes in $\mathcal{C}(\Lambda)$. The class $\mathcal{W}'$ is exact and symmetric and the $\mathcal{W}'$ -localization of Λ is $(\Lambda, \tilde{K}_1(\Lambda))$. The fact that any element in $\tilde{K}_1(\Lambda)$ is the torsion of a complex $C \otimes \Lambda$, $C \in \mathcal{W}'$, will be proved in § 7.

Let f be a degree one normal map from the normal bundle of a compact n -dimensional (Top, PL or Diff)-manifold V to a (Top, PL or Diff)-bundle ξ over a connected $\mathcal{W}$ -Poincaré complex with fundamental group π , such that the first Stiefel-Whitney class of ξ is w . We assume that f induces a $\mathcal{W}$ -equivalence on the boundaries.

Suppose that any complex in $\mathcal{W}$ is Λ -acyclic. Then f induces a Λ -homology equivalence with torsion in α between the boundaries. Then we can use Wall's technique [10] in order to define $\sigma(f) \in L_n^\alpha(\Lambda)$ and $\sigma(f)$ depends only on the normal cobordism class (relative the boundary) of f , and vanishes if f is normally cobordant to a $\mathcal{W}$ -equivalence.

(4.1) *Proof of theorem (1.10) in the case $\mathcal{W} = \mathcal{W}'$*

Suppose $n = 2p$ or $2p + 1 \geq 5$ and $\sigma(f) = 0$. After performing surgeries, we may suppose that the normal map $f: V \rightarrow X$ is p -connected.

Denote by C_* the complex $\Sigma^{-1}C_*(X, V; \mathbf{Z}\pi)$. If g is a homotopy inverse of the cap product $C^*(V; \mathbf{Z}\pi) \rightarrow C_*(V, \partial V; \mathbf{Z}\pi)$, the composite map

$$C_* \rightarrow C_*(V; \mathbf{Z}\pi) \rightarrow C_*(V, \partial V; \mathbf{Z}\pi) \xrightarrow{g} C^*(V; \mathbf{Z}\pi) \rightarrow \hat{C}_*$$

is a $\mathcal{W}'$ -equivalence.

a) *The even dimensional case*

If $n = 2p$, we have a complex in $\mathcal{W}'$

$$\dots \rightarrow C_{p+1} \rightarrow C_p \rightarrow \hat{C}_p \rightarrow \hat{C}_{p+1} \rightarrow \dots$$

and by the splitting lemma (3.5), there exist two complexes L and L' concentrated in dimension p and a $\mathcal{W}'$ -equivalence $L \rightarrow C_* \oplus L'$.

After performing trivial surgeries, we may suppose that L' is zero. Then the intersection and self-intersection forms on $H_{p+1}(X, V; \mathbf{Z}\pi)$ induce forms λ and μ on L_p and (L_p, λ, μ) is a $\mathcal{W}'$ -special $(-1)^p$ -form. Clearly, $\omega(L_p, \lambda, \mu)$ is sent to $\sigma(f)$ by the canonical map: $\varepsilon: \Gamma_n(\mathbf{Z}\pi, \mathcal{W}') \rightarrow L_n^h(\Lambda)$.

But ε is an isomorphism. This will be proved in § 8.

Then $\omega(L_p, \lambda, \mu)$ is zero and by proposition (3.14), the direct sum of (L_p, λ, μ) and a $(-1)^p$ -kernel has a $\mathcal{W}'$ -subkernel. We can realize the direct sum by trivial surgeries. So we may as well suppose that (L_p, λ, μ) has a $\mathcal{W}'$ -subkernel $K \rightarrow L_p$. Now it suffices to perform surgeries along a basis of K , via the map $K \rightarrow L_p \rightarrow C_p \rightarrow H_{p+1}(X, V; \mathbf{Z}\pi)$, to get a $\mathcal{W}'$ -equivalence.

b) *The odd dimensional case*

If $n = 2p + 1$, we have a complex in $\mathcal{W}'$

$$\dots \rightarrow C_{p+2} \rightarrow C_{p+1} \rightarrow C_p \oplus \hat{C}_p \rightarrow \hat{C}_{p+1} \rightarrow \hat{C}_{p+2} \rightarrow \dots$$

and by the splitting lemma (3.5), there exist two complexes L and L' concentrated in dimension $p+1$ and a $\mathscr{W}'$ -equivalence

$$L \rightarrow (\dots \rightarrow C_{p+2} \rightarrow C_{p+1} \rightarrow 0 \rightarrow \dots) \oplus L'.$$

So we get a $\mathscr{W}'$ -equivalence $(\dots \rightarrow 0 \rightarrow L_{p+1} \rightarrow C_p \oplus L'_{p+1} \rightarrow 0 \rightarrow \dots) \rightarrow C$.

Denote by $K \xrightarrow{d} B$ the map $L_{p+1} \rightarrow C_p \oplus L'_{p+1}$, and consider the composite map $B \rightarrow C_p \rightarrow \pi_{p+1}(X, V)$. The basis of B induces maps from S^p to V homotopic to zero in X . These maps are covered by fibered maps and we get immersions $\alpha_i: S^p \rightarrow V$, which we can suppose to be disjoint embeddings. Let U be a regular neighborhood of the images of these embeddings, connectified with 1-handles. The group $H_{p+1}(\text{pt}, \partial U; \mathbf{Z}\pi)$ endowed with intersection and self-intersection forms is the standard $(-1)^p$ -kernel $B \oplus \hat{B}$.

The morphisms $K \rightarrow B$ and $K \rightarrow C_{p+1}$ induce a morphism from K to the relative homology group

$$H_{p+2} \left(\begin{array}{ccc} U & \longrightarrow & \text{pt} \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array} ; \mathbf{Z}\pi \right) = H_{p+2} \left(\begin{array}{ccc} \partial U & \longrightarrow & \text{pt} \\ \downarrow & & \downarrow \\ V - \dot{U} & \longrightarrow & X \end{array} ; \mathbf{Z}\pi \right)$$

and we get, upon composing with the boundary, a morphism h from K to

$$H_{p+1}(\partial U \rightarrow \text{pt}; \mathbf{Z}\pi) = H_{p+1}(\text{pt}, \partial U; \mathbf{Z}\pi) = B \oplus \hat{B}.$$

It is not difficult to see that the image under h of the basis of K can be represented by spheres immersed in ∂U with zero intersections and self-intersections. To prove that h is a $\mathscr{W}'$ -subkernel, it suffices to show that the complex $\dots \rightarrow 0 \rightarrow K \rightarrow B \oplus \hat{B} \rightarrow \hat{K} \rightarrow 0 \rightarrow \dots$ lies in $\mathscr{W}'$; and this follows from the $\mathscr{W}'$ -equivalences

$$(\dots \rightarrow 0 \rightarrow K \rightarrow B \rightarrow \dots) \rightarrow C_* \rightarrow \hat{C}_* \rightarrow (\dots \rightarrow 0 \rightarrow \hat{B} \rightarrow \hat{K} \rightarrow 0 \rightarrow \dots).$$

Then we get a $\mathscr{W}'$ -subkernel h and an invariant $\omega(h) \in \Gamma_n(\mathbf{Z}\pi, \mathscr{W}')$. By construction, $\omega(h)$ is sent to $\sigma(f)$ by the isomorphism $\varepsilon: \Gamma_n(\mathbf{Z}\pi, \mathscr{W}') \rightarrow L_n^h(\Lambda)$. Hence $\omega(h)$ is zero. By proposition (3.15), there exist a standard $(-1)^p$ -kernel $C \oplus \hat{C}$ endowed with its standard subkernel $g: C \rightarrow C \oplus \hat{C}$ and an automorphism φ on $B \oplus \hat{B} \oplus C \oplus \hat{C}$ leaving each element of $B \oplus \hat{C}$ fixed, such that the composite map

$$K \oplus C \xrightarrow{\varphi \circ (h \oplus g)} B \oplus \hat{B} \oplus C \oplus \hat{C} \longrightarrow B \oplus \hat{C}$$

is a Λ -isomorphism.

If we add trivial disjoint embeddings β_j , from S^p to V , corresponding to the basis of C , the new $\mathscr{W}'$ -subkernel is $h \oplus g$. If we perform surgeries along the spheres α_i , the $\mathscr{W}'$ -subkernel $h \oplus g$ is replaced by $T \circ (h \oplus g)$, where T exchanges the factors B and $\hat{B}$. The new embedded spheres are the duals $\bar{\alpha}_i$ of α_i and β_j .

Now we can choose a regular homotopy depending on φ (see [10]) to get new disjoint embeddings α'_i and β'_j and the $\mathscr{W}'$ -subkernel $T \circ \varphi \circ (h \oplus g)$.

If we perform surgeries along the spheres α'_i and β'_j , we get the $\mathcal{W}'$ -subkernel $T' \circ \varphi \circ (h \oplus g)$ where T' exchanges the factors C and $\hat{C}$.

So we obtain a new normal map $f' : V' \rightarrow X$ normally cobordant to f and a $\mathcal{W}'$ -equivalence

$$(\dots \rightarrow 0 \rightarrow K \oplus C \rightarrow B \oplus \hat{C} \rightarrow 0 \rightarrow \dots) \rightarrow \Sigma^{-1}C_*(X, V'; \mathbb{Z}\pi).$$

Therefore f' is a $\mathcal{W}'$ -equivalence. ■

(4.2) *Proof of theorem (1.11) in the case $\mathcal{W} = \mathcal{W}'$*

a) *The even dimensional case*

Suppose $n = 2p \geq 6$ and let $\sigma \in L_n^h(\Lambda)$. Since the morphism

$$\varepsilon : \Gamma_n(\mathbb{Z}\pi, \mathcal{W}') \rightarrow L_n^h(\Lambda)$$

is an isomorphism, σ is represented by a $\mathcal{W}'$ -special $(-1)^p$ -form (H, λ, μ) (3.13). Then we construct a normal map $f : W \rightarrow M \times I$ exactly as in ([10], p. 53). This normal map is an isomorphism over $M \times 0 \cup \partial M \times I$ and a $\mathcal{W}'$ -equivalence over $M \times 1$ because λ is $\mathcal{W}'$ -non singular. By construction, σ is the surgery invariant of f .

b) *The odd dimensional case*

Suppose $n = 2p + 1 \geq 7$ and let $\sigma \in L_n^h(\Lambda)$. We can represent σ by a trivial $(-1)^p$ -kernel $B \oplus \hat{B}$ endowed with a $\mathcal{W}'$ -subkernel $g : K \rightarrow B \oplus \hat{B}$ ((3.14)). After adding p -handles to $M \times I$ corresponding to the basis of B , we get a normal map $f_0 : W_0 \rightarrow M \times \left[0, \frac{1}{2}\right]$ which restricts to an isomorphism over $M \times 0 \cup \partial M \times \left[0, \frac{1}{2}\right]$. The inverse image M' of $M \times \frac{1}{2}$ is the connected sum of M and copies of $S^p \times S^p$ and the group $\pi_{p+1}\left(M \times \frac{1}{2}, M'\right)$ is the kernel $B \oplus \hat{B}$. Then we can perform surgeries along the image under g of the basis of K and we get a normal map

$$f_1 : W_1 \rightarrow M \times \left[\frac{1}{2}, 1\right].$$

These two normal maps induce a normal map $f : W \rightarrow M \times I$. It is easy to see that f restricts to an isomorphism over $M \times 0 \cup \partial M \times I$ and a $\mathcal{W}'$ -equivalence over $M \times 1$. Moreover σ is the surgery obstruction $\sigma(f)$. ■

Actually this proof is almost identical with [10], p. 66.

Lemma (4.3). — Let $\tau \in \tilde{K}_1(\Lambda)$. Then there exist two matrices u and v with entries in Λ such that $u \otimes \Lambda$ and $v \otimes \Lambda$ are invertible and $\tau = \tau(u \otimes \Lambda) - \tau(v \otimes \Lambda)$.

This lemma will be proved in § 7.

Lemma (4.4). — Let M be a connected compact (Top, PL or Diff)-manifold, $\dim M \geq 5$. Let φ be an epimorphism from $\pi_1 M$ to π and τ be an element of $\tilde{K}_1(\Lambda)$. Then, there exists a normal

map $f: V \rightarrow M \times I$ restricting to an isomorphism over $M \times 0 \cup \partial M \times I$ and such that f is a Λ -homology equivalence with torsion τ .

Proof. — By lemma (4.3), there exist two matrices

$$u: \mathbb{Z}\pi^p \rightarrow \mathbb{Z}\pi^q \quad \text{and} \quad v: \mathbb{Z}\pi^r \rightarrow \mathbb{Z}\pi^s$$

such that $u \otimes \Lambda$ and $v \otimes \Lambda$ are invertible and

$$\tau = \tau(u \otimes \Lambda) - \tau(v \otimes \Lambda).$$

After adding q 1-handles to $M \times I$, we get a normal map $f_1: V_1 \rightarrow M \times I$ which is trivial on the handles. Now we add p 2-handles on V_1 along u and we get a normal map $f_2: V_2 \rightarrow M \times I$ restricting to an isomorphism over $M \times 0 \cup \partial M \times I$ and such that: $\tau(f_2) = \tau(u \otimes \Lambda) \in \tilde{K}_1(\Lambda)$.

Let M' be the manifold $f_2^{-1}(M \times 1)$. After adding s trivial 2-handles and r 3-handles along v , we construct a normal map $f_3': V_3' \rightarrow M' \times I$ which restricts to an isomorphism over $M' \times 0 \cup \partial M' \times I$, and f_3' is a Λ -homology equivalence with torsion $-\tau(v \otimes \Lambda)$.

Then after gluing f_2 and f_3' together, we get a normal map $f: V \rightarrow M \times I$ which has the desired property. ■

(4.5) *Proof of theorem (1.10) in the general case*

Consider the Ranicki-Rothenberg exact sequence

$$L_{n+1}^h(\Lambda) \xrightarrow{\partial} H^n(\mathbb{Z}/2, \tilde{K}_1(\Lambda)/\alpha) \rightarrow L_n^\alpha(\Lambda) \rightarrow L_n^h(\Lambda).$$

Suppose that $\sigma(f)$ vanishes in $L_n^\alpha(\Lambda)$. Then the surgery invariant of f is zero in $L_n^h(\Lambda)$ and f is normally cobordant (relative the boundary) to a normal map $f_1: V_1 \rightarrow X$ which is a $\mathcal{W}'$ -equivalence. Moreover f_1 is $\left[\frac{n}{2}\right]$ -connected.

Let $\tau \in \tilde{K}_1(\Lambda)$ be the torsion of f_1 . Since $\sigma(f)$ is zero, there exists an element $u \in L_{n+1}^h(\Lambda)$ such that ∂u is represented by τ . But f_1 is 2-connected and $\pi_1 V_1 = \pi$. Then, by theorem (1.11) (proved in the case $\mathcal{W} = \mathcal{W}'$, $M = V_1$), there exists a normal map $g_1: W_1 \rightarrow V_1 \times I$ restricting to an isomorphism over $V_1 \times 0 \cup \partial V_1 \times I$ and such that $\sigma(g) = u$. This normal map induces a normal cobordism (relative the boundary) from f_1 to a normal map $f_2: V_2 \rightarrow X$ which is a $\mathcal{W}'$ -equivalence. Moreover the torsion of f_2 is zero in $H^n(\mathbb{Z}/2, \tilde{K}_1(\Lambda)/\alpha)$.

Then, there exists $\tau' \in \tilde{K}_1(\Lambda)$ such that: $\tau(f_2) \equiv \tau' + (-1)^n \bar{\tau}' \pmod{\alpha}$.

By lemma (4.4), there exists a normal map $g_2: W_2 \rightarrow V_2 \times I$ restricting to an isomorphism over $V_2 \times 0 \cup \partial V_2 \times I$ such that g_2 is a $\mathcal{W}'$ -equivalence with torsion $-\tau'$. This normal map induces a normal cobordism from f_2 to $f_3: V_3 \rightarrow X$ and f_3 is a $\mathcal{W}'$ -equivalence with torsion in $\alpha \subset \tilde{K}_1(\Lambda)$. Thus, theorem (1.10) is a trivial consequence of the following lemma (proved in § 7):

Lemma (4.6). — *Any finite Λ -complex which is Λ -acyclic with torsion in α lies in $\mathcal{W}$.*

(4.7) Proof of theorem (1.11) in the general case

Consider again the Ranicki-Rothenberg exact sequence

$$H^n(\mathbf{Z}/2, \tilde{K}_1(\Lambda)/\alpha) \rightarrow L_n^\alpha(\Lambda) \rightarrow L_n^h(\Lambda) \rightarrow H^{n-1}(\mathbf{Z}/2, \tilde{K}_1(\Lambda)/\alpha).$$

Let σ be an element of $L_n^\alpha(\Lambda)$ and σ' be the image of σ in $L_n^h(\Lambda)$. By theorem (1.11) (proved in the case $\mathcal{W} = \mathcal{W}'$) there exists a normal map $f_1: W_1 \rightarrow M \times I$ restricting to an isomorphism over $M \times 0 \cup \partial M \times I$ and such that the surgery obstruction of f_1 is σ' in $L_n^h(\Lambda)$. Let V_1 be the inverse image of $M \times 1$. Since σ' is sent to zero in $H^{n-1}(\mathbf{Z}/2, \tilde{K}_1(\Lambda)/\alpha)$ the torsion of $f_1: V_1 \rightarrow M$ is congruent to $\tau - (-1)^n \bar{\tau} \pmod{\alpha}$ for some $\tau \in \tilde{K}_1(\Lambda)$.

Then, by lemma (4.4), we can glue together f_1 and a normal map $f'_1: W'_1 \rightarrow M \times I$ in order to construct a new normal map $f_2: W_2 \rightarrow M \times I$ such that

- (i) f_1 and f_2 have the same invariant in $L_n^h(\Lambda)$;
- (ii) f_2 restricts over $M \times 1$ to a $\mathcal{W}'$ -equivalence with torsion in α .

By construction, $\sigma(f_2) - \sigma$ is the image of an element of $H^n(\mathbf{Z}/2, \tilde{K}_1(\Lambda)/\alpha)$ represented by $\tau' \in \tilde{K}_1(\Lambda)$. By lemma (4.4), there exists a normal map

$$f'_2: W'_2 \rightarrow f_2^{-1}(M \times 1) \times I$$

restricting to an isomorphism over $f_2^{-1}(M \times 1) \times 0 \cup \partial f_2^{-1}(M \times 1) \times I$ and such that f'_2 is a $\mathcal{W}'$ -equivalence with torsion $-\tau'$. Then, after gluing f_2 and f'_2 together, we get a normal map $f: W \rightarrow M \times I$ with surgery obstruction σ . ■

5. Localization in the category of graded differential modules

Consider now the general case: A is a ring and $\mathcal{W}$ is an exact class in $\mathcal{C}(A)$. The $\mathcal{W}$ -localization of A is (Λ, α) .

Definition (5.1). — A complex $C \in \mathcal{W}$ will be called $\mathcal{W}$ -splittable if there exist, for any n , an n -dimensional complex $C' \in \mathcal{W}$ and an $(n-1)$ -connected morphism from C' to C .

The class of $\mathcal{W}$ -splittable complexes of $\mathcal{W}$ will be called $\mathcal{W}^s$.

Lemma (5.2). — The class $\mathcal{W}^s$ is exact.

Proof. — The class $\mathcal{W}^s$ is clearly stable under simple homotopy equivalence and under any suspension.

Now let $0 \rightarrow C \rightarrow C' \rightarrow C'' \rightarrow 0$ be a s -exact sequence of finite A -complexes. Suppose that C and C' are $\mathcal{W}$ -splittable.

Let n be an integer. There exists a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C & \longrightarrow & C' & \longrightarrow & C'' \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \\ & & \bar{C} & & \bar{C}' & & \end{array}$$

such that $\bar{C}$ (respectively $\bar{C}'$) is an $(n-1)$ -dimensional (respectively n -dimensional) complex in $\mathcal{W}$ and the morphism $\bar{C} \rightarrow C$ (respectively $\bar{C}' \rightarrow C'$) is $(n-2)$ -connected (respectively $(n-1)$ -connected). The obstructions to factoring the morphism $\bar{C} \rightarrow C'$ through $\bar{C}'$ are in the groups $H^p(\bar{C}, H_p(C', \bar{C}'))$ which are all trivial. So we get a morphism $\bar{C} \rightarrow \bar{C}'$. It is easy to see that the mapping cone $\bar{C}''$ of $\bar{C} \rightarrow \bar{C}'$ is an n -dimensional complex in $\mathcal{W}$ and the induced morphism from $\bar{C}''$ to C'' is $(n-1)$ -connected.

Then C'' is $\mathcal{W}$ -splittable and, since $\mathcal{W}^s$ is stable under simple homotopy equivalence and suspension, it is easy to prove that $\mathcal{W}^s$ is exact. ■

Lemma (5.3). — $\mathcal{W}^{ss} = \mathcal{W}^s$.

Proof. — The proof is by induction on the length of the complex. Clearly any complex in $\mathcal{W}^s$ of length two is $\mathcal{W}^s$ -splittable. Suppose any complex in $\mathcal{W}^s$ of length $< p$ is $\mathcal{W}^s$ -splittable, and let $C \in \mathcal{W}^s$ be a $\mathcal{W}$ -splittable complex of length p . The complex C is n -dimensional and $(n-p)$ -connected. Since C is $\mathcal{W}$ -splittable, there exist an $(n-p+2)$ -dimensional complex $C' \in \mathcal{W}$ and an $(n-p+1)$ -connected morphism $C' \rightarrow C$.

The length of C' is 2 and C' lies in $\mathcal{W}^{ss}$. Then the mapping cone of $C' \rightarrow C$ is a complex in $\mathcal{W}^s$ of length $p-1$. By induction the mapping cone of $C' \rightarrow C$ lies in $\mathcal{W}^{ss}$ and $C \in \mathcal{W}^{ss}$. ■

We will work out a theory of localization in the category of graded differential modules. Unfortunately, the category $\mathcal{C}(A)$ is too small to do that and we must consider the category $\bar{\mathcal{C}}(A)$ of graded differential free A -modules bounded from below.

Notations (5.4). — Denote by $\mathcal{W}_0$ the exact class of finite A -complexes C such that $C \oplus \Sigma C$ lies in $\mathcal{W}$ and by $\mathcal{W}_0^s$ the class $(\mathcal{W}_0)^s$. We use $\bar{\mathcal{W}}$ to denote the class of complexes $C \in \bar{\mathcal{C}}(A)$ such that any morphism from a finite A -complex to C factorizes through a complex in $\mathcal{W}_0^s$.

A morphism f in $\bar{\mathcal{C}}(A)$ is a $\bar{\mathcal{W}}$ -equivalence if the mapping cone of f lies in $\bar{\mathcal{W}}$.

Definition (5.5). — A complex $C \in \bar{\mathcal{C}}(A)$ will be called *local* if any morphism from a complex $C' \in \bar{\mathcal{W}}$ to C is null homotopic.

A morphism $f: C \rightarrow C'$ is a *localization* of C if f is a $\bar{\mathcal{W}}$ -equivalence and C' is local. Clearly, if C has a localization, this localization is unique up to homotopy.

Proposition (5.6). — Any complex in $\overline{\mathcal{C}}(A)$ has a localization.

Proof. — Let $C \in \overline{\mathcal{C}}(A)$. Suppose C is $(n-1)$ -connected. Let $\mathcal{A}$ be the set of morphisms $K \rightarrow C$ such that K is a $(n-2)$ -connected complex in $\mathcal{W}_0^s$. Let $\Phi(C)$ be the mapping cone of the morphism $\bigoplus_{\mathcal{A}} K \rightarrow C$.

Clearly $\Phi(C)$ is $(n-1)$ -connected and we can carry on this process:

$$C \rightarrow \Phi(C) \rightarrow \Phi^2(C) \rightarrow \Phi^3(C) \rightarrow \dots$$

Denote by $E(C)$ the limit of this system.

The complex $\Phi^{p+1}(C)/\Phi^p(C)$ is a direct sum of complexes in $\mathcal{W}_0^s$. Then, by induction, it is easy to show that $\Phi^p(C)/C$ lies in $\overline{\mathcal{W}}$. But, by construction, $E(C)$ is $(n-1)$ -connected and $E(C) \in \overline{\mathcal{C}}(A)$. Moreover $E(C)/C$ lies in $\overline{\mathcal{W}}$ and $C \rightarrow E(C)$ is a $\overline{\mathcal{W}}$ -equivalence.

Now, let $\mathcal{C}$ be the class of complexes $C' \in \overline{\mathcal{C}}(A)$ such that any morphism from C' to $E(C)$ is null homotopic. The class $\mathcal{C}$ is stable under homotopy equivalence and extension. The last problem is to prove that $\mathcal{C}$ contains $\overline{\mathcal{W}}$.

Let $K \in \mathcal{W}_0^s$. Since any complex in $\mathcal{W}_0^s$ is $\mathcal{W}_0^s$ -splittable ((5.3)), there exists a homotopy s -exact sequence $0 \rightarrow K' \rightarrow K \rightarrow K'' \rightarrow 0$ such that K' is a $n-1$ -dimensional complex in $\mathcal{W}_0^s$ and K'' an $(n-2)$ -connected complex in $\mathcal{W}_0^s$. Clearly $K' \in \mathcal{C}$. Let f be a morphism from K'' to $E(C)$. Since K'' is finitely generated, the image of f is contained in some $\Phi^p(C)$ and f is homotopic to zero in $\Phi^{p+1}(C)$. Hence $K'' \in \mathcal{C}$ and $K \in \mathcal{C}$ too. Then $\mathcal{C}$ contains the class $\mathcal{W}_0^s$.

If $K \in \overline{\mathcal{C}}(A)$, denote by $\mathcal{H}^i(K)$ the group $[\Sigma^{-i}K, E(C)]$ of homotopy classes of morphisms from $\Sigma^{-i}K$ to $E(C)$. The group $\mathcal{H}^i(K)$ vanishes for any $K \in \mathcal{W}_0^s$ and any $i \in \mathbb{Z}$, and we must prove that $\mathcal{H}^0(K)$ is zero for any $K \in \overline{\mathcal{W}}$.

If $K \in \overline{\mathcal{W}}$, K has the homotopy type of the limit of a directed system K_i , $K_i \in \mathcal{W}_0^s$, and we have a spectral sequence with the following E_2 term:

$$E_2^{pq} = \varprojlim^p \mathcal{H}^q(K_i).$$

The E_2 term is trivial and the spectral sequence converges to $\mathcal{H}^*(K)$. Then $\mathcal{H}^0(K)$ vanishes and $C \rightarrow E(C)$ is a localization of C . ■

The localization plays an important role in view of the following propositions:

Proposition (5.7). — Let C and C' be two complexes in $\mathcal{C}(A)$, with $\dim C = n$. Let $C' \xrightarrow{\varepsilon} E(C')$ be a localization of C' . Then, for any morphism $f: C \rightarrow E(C')$, there exist an n -dimensional complex $\overline{C} \in \mathcal{C}(A)$ and a homotopy commutative diagram

$$\begin{array}{ccc} \overline{C} & \longrightarrow & C' \\ \downarrow & & \downarrow \varepsilon \\ C & \xrightarrow{f} & E(C') \end{array}$$

such that $\overline{C} \rightarrow C$ is a $\mathcal{W}_0^s$ -equivalence.

Proposition (5.8). — Let C and C' be two complexes in $\mathcal{C}(A)$ with $\dim C = n$. Let $C' \xrightarrow{\varepsilon} E(C')$ be a localization of C' . Let $f: C \rightarrow C'$ be a map such that $\varepsilon \circ f$ is null homotopic. Then, there exists a $\mathcal{W}_0^s$ -equivalence $\bar{C} \rightarrow C$ such that $\bar{C} \in \mathcal{C}(A)$ is n -dimensional and the composite map $\bar{C} \rightarrow C \xrightarrow{f} C'$ is null homotopic.

Proof of (5.7). — Suppose ε is monic with free cokernel. We have an exact sequence

$$0 \rightarrow C' \rightarrow E(C') \rightarrow K' \rightarrow 0, \quad K' \in \overline{\mathcal{W}}.$$

Let us construct the homotopy commutative diagram

$$\begin{array}{ccc} \bar{C} & \longrightarrow & C' \\ \downarrow & & \downarrow \\ C & \longrightarrow & E(C') \\ \downarrow & \searrow & \downarrow \\ K & \longrightarrow L \longrightarrow & K' \end{array}$$

in the following way: Since C is finitely generated, the map $C \rightarrow K'$ factorizes through a complex $L \in \mathcal{W}_0^s$ and by (5.3), there exist an $(n+1)$ -dimensional complex $K \in \mathcal{W}_{0s}$ and an n -connected map $K \rightarrow L$. Then there is no obstruction to factorize the map $C \rightarrow L$ through K .

Let $\bar{C}$ be the homotopy kernel of $C \rightarrow K$. It is easy to check that $\bar{C}$ is n -dimensional and that the map $\bar{C} \rightarrow E(C')$ factorizes through C' . ■

Proof of (5.8). — Suppose ε is epic with kernel $K' \in \overline{\mathcal{W}}$. Since the composite map $C \xrightarrow{f} C' \xrightarrow{\varepsilon} E(C')$ is null homotopic, f is homotopic to a map $f': C \rightarrow K'$. Then f' factorizes through a complex $L \in \mathcal{W}_0^s$. By (5.3), there exist an $(n+1)$ -dimensional complex $K \in \mathcal{W}_0^s$ and an n -connected map $K \rightarrow L$. As before the map $C \rightarrow L$ retracts in K and the homotopy kernel of $C \rightarrow K$ has the desired properties. ■

6. The ring Λ

In this section, we will compute the homology groups of the localization of a complex $C \in \mathcal{C}(A)$ in terms of the ring Λ defined in (1.8).

Let M be a (right) A -module. This module will be said local if any $q \times p$ matrix in Σ induces an isomorphism $\text{Hom}(A^q, M) \rightarrow \text{Hom}(A^p, M)$.

Lemma (6.1). — A module M is local if and only if $H^n(C, M)$ vanishes for any $n \in \mathbb{Z}$ and any $C \in \overline{\mathcal{W}}$.

Proof. — Suppose that $H^n(C, M)$ vanishes for any $n \in \mathbf{Z}$ and any $C \in \overline{\mathcal{W}}$. If u is a matrix in Σ , denote by C the 1-dimensional complex

$$\dots \rightarrow 0 \rightarrow A^p \xrightarrow{u} A^q \rightarrow 0 \rightarrow \dots$$

Then $C \oplus \Sigma C$ lies in $\mathcal{W}$ (see (1.7)) and C is a complex of $\mathcal{W}_0^s \subset \overline{\mathcal{W}}$. Hence $H^*(C, M)$ vanishes and M is local.

Conversely, suppose M is local and denote by $\mathcal{C}$ the class of complexes $C \in \overline{\mathcal{C}}(A)$ such that $H^*(C, M) = 0$.

If C is a complex of length two in $\mathcal{W}_0^s$, C lies in $\mathcal{C}$ by definition.

If C is a complex in $\mathcal{W}_0^s$ of length $p > 2$, there exists a homotopy s -exact sequence

$$0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$$

such that C' and C'' are complexes in $\mathcal{W}_0^s$ of length $< p$.

By induction, C is in $\mathcal{C}$ and $\mathcal{C}$ contains the class $\mathcal{W}_0^s$.

If $C \in \overline{\mathcal{W}}$, C is the limit of a directed system $C_i \in \mathcal{W}_0^s$ and we have a spectral sequence with E_2 term $E_2^{pq} = \varprojlim^p H^q(C_i, M)$. The E_2 term is zero and the spectral sequence converges to $H^*(C, M)$. Hence $H^*(C, M)$ vanishes and the lemma is proved.

Corollary (6.2). — *A complex $C \in \overline{\mathcal{C}}(A)$ is local if and only if $H_n(C)$ is local for any $n \in \mathbf{Z}$.*

Proof. — If K is a complex, denote by $\mathcal{H}^i(K)$ the group of homotopy classes of maps $\Sigma^{-i}K \rightarrow C$. We have a spectral sequence with E_2 term

$$E_2^{pq} = H^p(K, H_{-q}(C))$$

and this spectral sequence usually converges to $\mathcal{H}^*(K)$.

Suppose C is local and let $K \in \mathcal{W}_0^s$ be a complex of length 2 defined by a matrix $u \in \Sigma$. Then the above spectral sequence collapses to exact sequences

$$0 \rightarrow H^n(K, H_{-i}(C)) \rightarrow \mathcal{H}^{n+i}(K) \rightarrow H^{n-1}(K, H_{-i-1}(C)) \rightarrow 0 \quad (n = \dim K).$$

Then all the groups $H^*(K, H_i(C))$ vanish and $H_i(C)$ is local for any $i \in \mathbf{Z}$.

Conversely suppose $H_*(C)$ is local. Then for any $K \in \overline{\mathcal{W}}$, the E_2 term of the above spectral sequence vanishes and the spectral sequence converges to $\mathcal{H}^*(K)$. Hence this last group vanishes and C is local. ■

Lemma (6.3). — *Localization respects exact sequences.*

Proof. — Let $0 \rightarrow C \rightarrow C' \rightarrow C'' \rightarrow 0$ be a short exact sequence in $\overline{\mathcal{C}}(A)$. Take localizations $C \rightarrow E(C)$ and $C' \rightarrow E(C')$ of C and C' . We get a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C & \longrightarrow & C' & \longrightarrow & C'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & E(C) & \longrightarrow & E(C') & & \end{array}$$

Let $E(C'')$ be the mapping cone of $E(C) \rightarrow E(C')$. We have a homotopy commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C & \longrightarrow & C' & \longrightarrow & C'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E(C) & \longrightarrow & E(C') & \longrightarrow & E(C'') \longrightarrow 0 \end{array}$$

Clearly $E(C'')$ is local and the map $C'' \rightarrow E(C'')$ is a $\overline{\mathcal{W}}$ -equivalence. Then $C'' \rightarrow E(C'')$ is a localization of C'' and the result follows.

Lemma (6.4). — *Localization respects direct sums.*

Proof. — Let $C_i \in \overline{\mathcal{C}}(A)$ be a class of complexes. Suppose that C_i is $(n-1)$ -connected for any i , and take localizations $C_i \rightarrow E(C_i)$.

Clearly the mapping cone of $\bigoplus_i C_i \rightarrow \bigoplus_i E(C_i)$ lies in $\overline{\mathcal{W}}$ and, by (6.2), the sum $\bigoplus_i E(C_i)$ is local. Then the map $\bigoplus_i C_i \rightarrow \bigoplus_i E(C_i)$ is a localization of $\bigoplus_i C_i$. ■

Now if C is a complex in $\overline{\mathcal{C}}(A)$, denote by $\Phi_n(C)$ the group $H_n(E(C))$ where $C \rightarrow E(C)$ is a localization of C .

If M is a (right) A -module, we will also denote by $\Phi_n(M)$ the group $\Phi_n(C)$ where C is a free resolution of M . The Φ_n 's are functors and we have a natural transformation $\eta : M \rightarrow \Phi_0(M)$.

Clearly, if M is local, a resolution of M is local ((6.2)). So η is bijective and $\Phi_i(M)$ vanishes for $i \neq 0$.

Lemma (6.5). — *Let M be an A -module. Then, there is a natural homomorphism*

$$\varepsilon' : M \otimes_{\mathbf{Z}} \Phi_0(A) \rightarrow \Phi_0(M),$$

such that the following diagram commutes:

$$\begin{array}{ccc} M \otimes_{\mathbf{Z}} A & \longrightarrow & M \\ \downarrow 1 \otimes \eta & & \downarrow \eta \\ M \otimes_{\mathbf{Z}} \Phi_0(A) & \xrightarrow{\varepsilon'} & \Phi_0(M) \end{array}$$

Proof. — Let $m \in M$. Denote by $\varphi : A \rightarrow M$ the homomorphism $a \mapsto ma$. By setting $\varepsilon'(m, x) = \Phi_0(\varphi)(x)$, for any $x \in \Phi_0(A)$, we get a map $\varepsilon' : M \times \Phi_0(A) \rightarrow \Phi_0(M)$. Clearly, $\varepsilon'(m, x)$ is $\mathbf{Z}$ -linear on x and, since Φ_0 respects direct sums, it is easy to see that $\varepsilon'(m, x)$ is $\mathbf{Z}$ -linear on m . ■

Lemma (6.6). — *The module $\Phi_0(A)$ is a ring and ε' induces a homomorphism*

$$\varepsilon : M \otimes_A \Phi_0(A) \rightarrow \Phi_0(M).$$

Proof. — Let $m \in M$ and $x, y \in \Phi(A)$. Denote by $\varphi : A \rightarrow M$ the map $a \mapsto ma$ and by $\psi : A \rightarrow \Phi_0(A)$ the map $a \rightarrow xa$.

We have a commutative diagram

$$\begin{array}{ccccc} & & A & \xrightarrow{\varphi} & M \\ & & \downarrow & & \downarrow \\ A & \xrightarrow{\psi} & \Phi_0(A) & \xrightarrow{\Phi_0(\varphi)} & \Phi_0(M) \\ \downarrow & & \sim \downarrow \eta & & \sim \downarrow \eta \\ \Phi_0(A) & \xrightarrow{\Phi_0(\psi)} & \Phi_0^2(A) & \xrightarrow{\Phi_0^2(\varphi)} & \Phi_0^2(M) \end{array}$$

and the following formulas:

$$\Phi_0^2(\varphi) \circ \Phi_0(\psi)(y) = \Phi_0^2(\varphi)(\varepsilon'(x, y)) = \eta \varepsilon'(m, \eta^{-1} \varepsilon'(x, y))$$

$$\Phi_0[\Phi_0(\varphi) \circ \psi](y) = \varepsilon'(\varepsilon'(m, x), y)$$

whence
$$\eta \varepsilon'(m, \eta^{-1} \varepsilon'(x, y)) = \varepsilon'(\varepsilon'(m, x), y).$$

Then the map $\eta^{-1} \varepsilon'$ from $\Phi_0(A) \otimes_{\mathbf{Z}} \Phi_0(A)$ to $\Phi_0(A)$ induces a ring structure on $\Phi_0(A)$ and η is a ring homomorphism from A to $\Phi_0(A)$. Moreover ε' induces a homomorphism $\varepsilon : M \otimes_A \Phi_0(A) \rightarrow \Phi_0(M)$. ■

Lemma (6.7). — *The ring homomorphism $A \rightarrow \Phi_0(A)$ is isomorphic to the homomorphism $A \rightarrow \Lambda$.*

Proof. — Let $A \rightarrow B$ be a ring homomorphism. The A -module B is local if and only if any $q \times p$ matrix $u \in \Sigma$ induces an isomorphism $u^* : \text{Hom}(A^q, B) \rightarrow \text{Hom}(A^p, B)$. But the matrix of u^* is the transpose of $u \otimes B$. Then, B is local if and only if, for any $u \in \Sigma$, $u \otimes B$ is invertible.

Hence, for any matrix $u \in \Sigma$, $u \otimes \Phi_0(A)$ is invertible and we will prove that $\Phi_0(A)$ is universal with respect to this property.

Let $A \rightarrow B$ be a ring homomorphism such that $u \otimes B$ is invertible for any $u \in \Sigma$. Let us choose free resolutions A_* and B_* of A and B and a localization $A_* \rightarrow E(A_*)$ of A_* . Since B is local, there exists an extension $E(A_*) \rightarrow B_*$ unique up to homotopy. Then there exists a unique extension $\Phi_0(A) \rightarrow B$ of $A \rightarrow B$.

Consider the following diagram:

$$\begin{array}{ccc} A & \longrightarrow & \Phi_0(A) \\ \downarrow & & \downarrow \\ B & \xrightarrow{\sim} & \Phi_0(B) \end{array}$$

All the morphisms of this diagram are ring homomorphisms and $B \xrightarrow{\sim} \Phi_0(B)$ is an isomorphism. Then the extension $\Phi_0(A) \rightarrow B$ is a ring homomorphism. So $A \rightarrow \Phi_0(A)$ satisfies the universal property of Λ and $A \rightarrow \Phi_0(A)$ is isomorphic to $A \rightarrow \Lambda$.

Lemma (6.8). — *For any module M , the morphism $\varepsilon : M \otimes \Lambda \rightarrow \Phi_0(M)$ is an isomorphism.*

Proof. — By lemma (6.4), the functor Φ_0 respects direct sums and ε is an isomorphism if M is free. Moreover, by lemma (6.5), Φ_0 is right exact and ε is an isomorphism for any M . ■

Corollary (6.9). — *If M is local, the canonical map $M \rightarrow M \otimes \Lambda$ is an isomorphism.*

Lemma (6.10). — *If M is local, $\text{Tor}_1(M, \Lambda)$ is trivial.*

Proof. — Choose a free module L and an exact sequence

$$0 \rightarrow N \rightarrow L \rightarrow M \rightarrow 0.$$

By lemma (6.4), we have an exact sequence

$$\Phi_1(M) \rightarrow \Phi_0(N) \rightarrow \Phi_0(L) \rightarrow \Phi_0(M) \rightarrow 0.$$

If M is local, $\Phi_1(M)$ is zero and $\Phi_0(N) \rightarrow \Phi_0(L)$ is monic. But this map is isomorphic to the map $N \otimes \Lambda \rightarrow L \otimes \Lambda$ and its kernel is $\text{Tor}_1(M, \Lambda)$. ■

Corollary (6.11). — *Let $C \in \overline{\mathcal{C}}(A)$ be an $(n-1)$ -connected local complex. Then the canonical map $H_i(C) \rightarrow H_i(C \otimes \Lambda)$ is an isomorphism for $i \leq n$ and an epimorphism for $i = n+1$.*

Proof. — We have a spectral sequence with E^2 term $E_{pq}^2 = \text{Tor}_p(H_q(C), \Lambda)$ which converges to $H_*(C \otimes \Lambda)$. Since C is local, $H_*(C)$ is local and, by (6.9) and (6.10), we have

$$E_{0q}^2 = \text{Tor}_0(H_q(C), \Lambda) = H_q(C),$$

$$E_{1q}^2 = \text{Tor}_1(H_q(C), \Lambda) = 0.$$

The result follows.

Theorem (6.12). — *Let C and C' be two finite A -complexes and suppose that $C' \otimes \Lambda$ is $(n-1)$ -connected. Then we have the following properties:*

(i) *If $H^i(C, \Lambda)$ vanishes for $i > n+1$ and f is a morphism from $C \otimes \Lambda$ to $C' \otimes \Lambda$, there exist a $\mathcal{W}_0^s$ -equivalence $\varepsilon: \bar{C} \rightarrow C$ with $\dim \bar{C} = \dim C$ and a morphism $g: \bar{C} \rightarrow C'$ such that $g \otimes \Lambda$ is homotopic to $f \circ (\varepsilon \otimes \Lambda)$.*

(ii) *If $H^i(C, \Lambda)$ vanishes for $i > n$ and f is a morphism from C to C' such that $f \otimes \Lambda$ is null homotopic, there exists a $\mathcal{W}_0^s$ -equivalence $\varepsilon: \bar{C} \rightarrow C$, with $\dim \bar{C} = \dim C$ such that $f \circ \varepsilon$ is null homotopic.*

Proof. — Let $C' \rightarrow E(C')$ be a localization of C' and consider the following diagram:

$$\begin{array}{ccc} C' & \longrightarrow & C' \otimes \Lambda \\ \downarrow & & \downarrow \\ E(C') & \longrightarrow & E(C') \otimes \Lambda \end{array}$$

If f is a morphism from $C \otimes \Lambda$ to $C' \otimes \Lambda$, f is defined by an A -homomorphism $f': C \rightarrow C' \otimes \Lambda$.

The obstructions to lift the composite map $f'': C \rightarrow C' \otimes \Lambda \rightarrow E(C') \otimes \Lambda$ through $E(C')$ lie in the groups $H^p(C, H_p(E(C') \otimes \Lambda, E(C')))$. Let H_p be the module $H_p(E(C') \otimes \Lambda, E(C'))$. Since $E(C')$ is local, H_p is a Λ -module and is trivial for $p \leq n+1$, by (6.11). But $H^i(C, \Lambda)$ vanishes for $i > n+1$ and the localization $E(\hat{C})$ of $\hat{C}$ is $(-n-2)$ -connected. Then we have, for $p > n+1$,

$$H^p(C, H_p) = H_{-p}(\hat{C}, H_p) = H_{-p}(E(\hat{C}), H_p) = 0.$$

Then f'' lifts through $E(C')$ and, by (5.7), there exist a complex $\bar{C} \in \mathcal{C}(A)$ with $\dim \bar{C} = \dim C$, a $\mathcal{W}_0^s$ -equivalence $\varepsilon: \bar{C} \rightarrow C$ and a morphism $g: \bar{C} \rightarrow C'$ such that the following diagram is homotopy commutative:

$$\begin{array}{ccc} \bar{C} & \xrightarrow{g} & C' \\ \varepsilon \downarrow & & \downarrow \\ C & \xrightarrow{f''} & E(C') \otimes \Lambda \end{array}$$

On the other hand, any complex in $\mathcal{W}_0^s$ of length two is Λ -acyclic and, by induction, any complex in $\mathcal{W}_0^s$ is Λ -acyclic. This implies that any complex in $\overline{\mathcal{W}}$ is Λ -acyclic and $C' \otimes \Lambda \rightarrow E(C') \otimes \Lambda$ is a homotopy equivalence.

Then the following diagram commutes up to homotopy:

$$\begin{array}{ccc} \bar{C} & \xrightarrow{g} & C' \\ \varepsilon \downarrow & & \downarrow \\ C & \xrightarrow{f'} & C' \otimes \Lambda \end{array}$$

and part (i) of the theorem is proved.

Suppose now f is a morphism from C to C' with $\dim C = n$. If $f \otimes \Lambda$ is null homotopic, the composite map $C \rightarrow C' \rightarrow E(C') \otimes \Lambda$ is null homotopic and, by obstruction, the map $C \rightarrow E(C')$ is null homotopic. Then we may apply (5.8) and the theorem is proved.

7. The structure of $\mathcal{W}$

Lemma (7.1). — *The class $\mathcal{W}_0^s$ is the class $\mathcal{W}'$ of Λ -acyclic complexes in $\mathcal{C}(\Lambda)$.*

Proof. — If C is a complex in $\mathcal{W}_0^s$ of length two, it is Λ -acyclic by definition of Λ . Then, by induction, any complex in $\mathcal{W}_0^s$ is Λ -acyclic.

Conversely, let $C \in \mathcal{C}(\Lambda)$ be a Λ -acyclic complex and $C \rightarrow E(C)$ be a localization of C . Since C is Λ -acyclic, $E(C)$ is Λ -acyclic too. Suppose $E(C)$ is not acyclic and let H_n be the first non trivial homology group of $E(C)$. The module H_n is local and

$$H_n \simeq H_n \otimes \Lambda \simeq H_n(E(C) \otimes \Lambda) = 0.$$

Hence $E(C)$ is acyclic and $C \in \overline{\mathcal{W}}$. Since C is finite, the identity $C \rightarrow C$ factorizes through a complex $K \in \mathcal{W}_0^s$ and we get a split exact sequence

$$0 \rightarrow C' \rightarrow K \rightarrow C \rightarrow 0.$$

This implies that $C \oplus C'$ has the simple homotopy type of K and $C \oplus C'$ lies in $\mathcal{W}_0^s$.

On the other hand, ΣK has the simple homotopy type of the mapping cone of the zero map $C' \rightarrow \Sigma C$ and $C' \rightarrow \Sigma C$ is a $\mathcal{W}_0^s$ -equivalence. Then $C \oplus \Sigma C$ lies in $\mathcal{W}_0^s$.

Now we will prove that C is in $\mathcal{W}_0^s$ by induction on the length of C .

If the length of C is two, $C \oplus \Sigma C$ is contained in $\mathcal{W}_0$ and $C \oplus \Sigma C \oplus \Sigma C \oplus \Sigma^2 C$ lies in $\mathcal{W}$. But $\Sigma(C \oplus \Sigma C \oplus \Sigma C \oplus \Sigma^2 C)$ is the mapping cone of the zero map $\Sigma C \oplus \Sigma C \oplus \Sigma^2 C \rightarrow \Sigma C$ which is a $\mathcal{W}$ -equivalence. Then $C \oplus \Sigma C$ lies in $\mathcal{W}$ and C lies in $\mathcal{W}_0$. Since the length of C is two, C lies in $\mathcal{W}_0^s$.

If the length of C is $p > 2$, C is n -dimensional and $(n-p)$ -connected. Since $C \oplus \Sigma C$ is $\mathcal{W}_0^s$ -splittable, there exist an $(n-p+2)$ -dimensional complex $K \in \mathcal{W}_0^s$ and an $(n-p+1)$ -connected morphism $f \oplus g$ from K to $C \oplus \Sigma C$.

The morphism $f \oplus 0$ is clearly $(n-p+1)$ -connected. Let M be the mapping cone of f . The complex $M \oplus \Sigma M$ is the mapping cone of $f \oplus \Sigma f$ and lies in $\mathcal{W}_0^s$. But the length of M is $p-1$. By induction, M lies in $\mathcal{W}_0^s$ and C lies in $\mathcal{W}_0^s$ too. ■

(7.2) Proof of the splitting lemma (3.5)

Let C be a complex in $\mathcal{W}'$ and let n be an integer. Since $\mathcal{W}' = \mathcal{W}_0^s$, C is $\mathcal{W}'$ -splittable and there exist an n -dimensional complex $C' \in \mathcal{W}'$ and an $(n-1)$ -connected morphism $C' \rightarrow C$.

Up to simple homotopy type, we may suppose that the map $C'_i \rightarrow C_i$ is bijective for $i < n-1$ and is epic with free kernel L'_n for $i = n-1$. Then we have the following complex in $\mathcal{W}'$:

$$\dots \rightarrow C_{n+2} \rightarrow C_{n+1} \oplus C'_n \rightarrow C_n \oplus L'_n \rightarrow 0 \rightarrow \dots$$

Now by setting

$$L = (\dots \rightarrow 0 \rightarrow C'_n \rightarrow 0 \rightarrow \dots)$$

$$L' = (\dots \rightarrow 0 \rightarrow L'_n \rightarrow 0 \rightarrow \dots),$$

we get a $\mathcal{W}'$ -equivalence

$$L \rightarrow L' \oplus (\dots \rightarrow C_{n+1} \rightarrow C_n \rightarrow 0 \rightarrow \dots). \quad \blacksquare$$

Lemma (7.3). — *For any complex $C \in \mathcal{W}'$, the complex $C \oplus \Sigma C$ lies in $\mathcal{W}$.*

Proof. — If C is Λ -acyclic, C lies in $\mathcal{W}_0^s \subset \mathcal{W}_0$ and then $C \oplus \Sigma C \in \mathcal{W}$. $\blacksquare$

(7.4) We use $K(\mathcal{W})$ to denote the class of complexes $C \in \mathcal{W}'$ fulfilling the following relation:

$$C \sim C' \Leftrightarrow C \oplus \Sigma C' \in \mathcal{W}.$$

By (7.3), this relation is an equivalence relation and $K(\mathcal{W})$ is a well defined set. Moreover the direct sum of complexes induces an abelian group structure on $K(\mathcal{W})$.

If C is a Λ -acyclic complex in $\mathcal{C}(A)$, the class of C in $K(\mathcal{W})$ will be denoted by $\theta(C)$.

Lemma (7.5). — *Let $0 \rightarrow C \rightarrow C' \rightarrow C'' \rightarrow 0$ be an s -exact sequence of Λ -acyclic complexes in $\mathcal{C}(A)$. Then $\theta(C') = \theta(C) + \theta(C'')$.*

Proof. — We have an s -exact sequence

$$0 \rightarrow C \oplus \Sigma C \rightarrow C' \oplus \Sigma C \oplus \Sigma C'' \rightarrow C'' \oplus \Sigma C'' \rightarrow 0$$

and, by lemma (7.3), $C' \oplus \Sigma C \oplus \Sigma C''$ is in $\mathcal{W}$. That proves the lemma.

Now if f is a Λ -homology equivalence between two finite Λ -complexes, we will define $\theta(f)$ as the class of the mapping cone of f in $K(\mathcal{W})$.

Lemma (7.6). — *Let $f: C \rightarrow C'$ and $g: C' \rightarrow C''$ be two Λ -homology equivalences between finite Λ -complexes. Then $\theta(g \circ f) = \theta(f) + \theta(g)$.*

Proof. — We have a short s -exact sequence between the mapping cones of f , g , $g \circ f \oplus 1_{C'}$. Then the result follows from (7.5).

(7.7) Let $f: \Lambda^p \rightarrow \Lambda^q$ be an isomorphism. Denote also by A the 0-dimensional complex $\dots \rightarrow 0 \rightarrow A \rightarrow 0 \rightarrow \dots$. Then f is a morphism from $A^p \otimes \Lambda$ to $A^q \otimes \Lambda$, and, by (6.12), there exist a $\mathcal{W}'$ -equivalence $\varepsilon: \bar{C} \rightarrow A^p$ and a map $g: \bar{C} \rightarrow A^q$ such that $f \circ (\varepsilon \otimes \Lambda)$ is homotopic to $g \otimes \Lambda$.

Since f is an isomorphism, g is a $\mathcal{W}'$ -equivalence.

Then we define $\theta(f)$ as $\theta(g) - \theta(\varepsilon)$. By (6.12), it is easy to show that $\theta(f)$ does not depend on the choices.

Lemma (7.8). — Let $f: \Lambda^p \rightarrow \Lambda^q$ and $g: \Lambda^q \rightarrow \Lambda^r$ be two isomorphisms. Then we have

$$\theta(g \circ f) = \theta(f) + \theta(g).$$

Proof. — By theorem (6.12), there exists a homotopy commutative diagram in $\mathcal{C}(A)$

$$\begin{array}{ccccc} \bar{\bar{C}} & & & & \\ \downarrow \bar{\varepsilon} & \searrow \bar{h} & & & \\ \bar{C} & & \bar{C}' & & \\ \downarrow \varepsilon & \searrow h & \downarrow \varepsilon' & \searrow h' & \\ A^p & & A^q & & A^r \end{array}$$

such that the morphisms are Λ -homology equivalences and $h \otimes \Lambda$ and $h' \otimes \Lambda$ are homotopic to $f \circ (\varepsilon \otimes \Lambda)$ and $g \circ (\varepsilon' \otimes \Lambda)$. Then we have

$$\theta(g \circ f) = \theta(h' \circ \bar{h}) - \theta(\varepsilon \circ \bar{\varepsilon}) = \theta(h') + \theta(\bar{h}) - \theta(\varepsilon) - \theta(\bar{\varepsilon})$$

whence $\theta(g \circ f) = \theta(h') - \theta(\varepsilon) + \theta(h) - \theta(\varepsilon') = \theta(f) + \theta(g)$. ■

Theorem (7.9). — The torsion homomorphism $\varepsilon: K(\mathcal{W}) \rightarrow \tilde{K}_1(\Lambda)/\alpha$ is an isomorphism.

Proof. — If $x \in \tilde{K}_1(\Lambda)/\alpha$ is represented by an isomorphism $f: \Lambda^p \rightarrow \Lambda^q$, we have

$$\varepsilon(\theta(f)) \equiv \tau(f) \bmod \alpha \Rightarrow x = \varepsilon(\theta(f))$$

and ε is surjective.

Now let θ be an element of $\text{Ker } \varepsilon$, represented by a complex $C \in \mathcal{W}'$. Since $\varepsilon(\theta)$ vanishes, $\tau(C \otimes \Lambda)$ is in α and $\tau(C \otimes \Lambda)$ is the torsion of a complex $C' \otimes \Lambda$ where C' is a Λ -acyclic complex in $\mathcal{W}$. Then θ is represented by $C \oplus \Sigma C'$ and the torsion of $(C \oplus \Sigma C') \otimes \Lambda$ vanishes. Since $\mathcal{W}'$ is splittable, we can “split” $C \oplus \Sigma C'$ into complexes $C_i \in \mathcal{W}'$ of length 2. And we have

$$\theta = \Sigma \theta(C_i) \quad \text{and} \quad \Sigma \tau(C_i \otimes \Lambda) = 0.$$

On the other hand, the suspension Σ^2 does not change the invariants θ and τ . So we may as well suppose that the complexes C_i are 1 or 2-dimensional.

Then there exist two 1-dimensional complexes in $\mathcal{W}'$

$$X = (\dots \rightarrow 0 \rightarrow A^p \xrightarrow{f} A^q \rightarrow 0 \rightarrow \dots)$$

$$Y = (\dots \rightarrow 0 \rightarrow A^{p'} \xrightarrow{g} A^{q'} \rightarrow 0 \rightarrow \dots)$$

such that $\theta = \theta(X) - \theta(Y)$ and $\tau(X \otimes \Lambda) = \tau(Y \otimes \Lambda)$.

But the image of $\tau(X \otimes \Lambda) = \tau(f \otimes \Lambda)$ under the boundary $\tilde{K}_1(\Lambda) \xrightarrow{\partial} K_0(\mathbf{Z})$ is $q - p$ [9]. Then, after stabilization on X and Y , we may suppose

$$p = p' \quad \text{and} \quad q = q'.$$

Let $\varphi \in GL_q(\Lambda)$ be the map for $(f \otimes \Lambda) \circ (g \otimes \Lambda)^{-1}$. Since $\tau(f \otimes \Lambda) - \tau(g \otimes \Lambda)$ is zero, the class of φ in $K_1(\Lambda)$ is in the image of $K_1(\mathbf{Z}) \rightarrow K_1(\Lambda)$. Then, after a permutation on the basis of A^q (in X) and after stabilization on X and Y , we may suppose that φ lies in the commutator subgroup of $GL_q(\Lambda)$:

$$\varphi = \prod_i [\varphi_i, \psi_i].$$

And we have

$$\theta = \theta(X) - \theta(Y) = \theta(f) - \theta(g) = \theta(f \otimes \Lambda) - \theta(g \otimes \Lambda) = \theta(\varphi)$$

whence $\theta = \Sigma(\theta(\varphi_i) + \theta(\psi_i) - \theta(\varphi_i) - \theta(\psi_i)) = 0$.

This completes the proof.

Corollary (7.10). — *The class of Λ -acyclic complexes in $\mathcal{W}$ is the class of Λ -acyclic complexes C such that the torsion of $C \otimes \Lambda$ is in α .*

Now we prove lemmas (4.3) and (4.6).

Lemma (4.6) is actually the corollary (7.10).

Let $\tau \in \tilde{K}_1(\Lambda)$. By theorem (7.9), there exists a complex $C \in \mathcal{W}'$ such that τ is the torsion of $C \otimes \Lambda$. Since C is splittable ((7.1)), we can split C into Λ -acyclic complexes C_i of length two and we have $\tau = \Sigma \tau(C_i \otimes \Lambda)$. If C_i is $(n_i + 1)$ -dimensional and the differential of C_i is u_i , we have:

$$\tau = \Sigma(-1)^{n_i} \tau(u_i \otimes \Lambda)$$

and lemma (4.3) follows.

8. The isomorphism theorem

Suppose now that A is a ring with involution and $\mathcal{W}$ is an exact symmetric class in $\mathcal{C}(A)$. The $\mathcal{W}$ -localization of A is (Λ, α) and $A \rightarrow \Lambda$ is a morphism of rings with involution.

The class of Λ -acyclic complexes in $\mathcal{C}(A)$ is denoted by $\mathcal{W}'$ and the class of acyclic complexes in $\mathcal{C}(\Lambda)$ is denoted by $\mathcal{W}_\Lambda$.

We have a canonical map

$$\varepsilon: \Gamma_n(A, \mathcal{W}') \rightarrow \Gamma_n(\Lambda, \mathcal{W}_\Lambda) \simeq L_n^h(\Lambda).$$

In this section, we will prove that ε is an isomorphism.

Lemma (8.1). — *Let C (respectively Σ) be a p -dimensional and $(p-2)$ -connected complex in $\mathcal{C}(A)$ (respectively $\mathcal{C}(\Lambda)$) and $f: \Sigma \rightarrow C \otimes \Lambda$ be a map. Then there exist a p -dimensional complex $\Sigma' \in \mathcal{C}(A)$, a homotopy equivalence $\varepsilon: \Sigma' \otimes \Lambda \rightarrow \Sigma$ and a map $g: \Sigma' \rightarrow C$ such that $f \circ \varepsilon$ is homotopic to $g \otimes \Lambda$.*

Proof. — Let us consider the modules Σ_p, Σ_{p-1} as p -dimensional complexes $C'_p \otimes \Lambda, C'_{p-1} \otimes \Lambda$. The differential d on Σ is a map from $C'_p \otimes \Lambda$ to $C'_{p-1} \otimes \Lambda$. Then, by theorem (6.12), there exist a p -dimensional complex $\bar{C} \in \mathcal{C}(A)$, a $\mathcal{W}'$ -equivalence $\bar{\varepsilon}: \bar{C} \rightarrow C'_p$ and a morphism $g: \bar{C} \rightarrow C'_{p-1}$ such that $g \otimes \Lambda$ is homotopic to $d \circ (\bar{\varepsilon} \otimes \Lambda)$.

Let M be the mapping cone of g . The $\mathcal{W}'$ -equivalence $\bar{\varepsilon}$ induces a homotopy equivalence $\varepsilon': M \otimes \Lambda \rightarrow \Sigma$. Moreover M is p -dimensional and $C \otimes \Lambda$ is $(p-2)$ -connected. Then by (6.12), there exist a p -dimensional complex $\Sigma' \in \mathcal{C}(A)$, a $\mathcal{W}'$ -equivalence $\varepsilon'': \Sigma' \rightarrow M$ and a morphism $g: \Sigma' \rightarrow C$ such that $f \circ \varepsilon' \circ (\varepsilon'' \otimes \Lambda)$ is homotopic to $g \otimes \Lambda$. The result follows.

Lemma (8.2). — *Let C be a finite A -complex such that $H^i(C, \Lambda)$ vanishes for $i > p$ and let $\varphi \in B(C \otimes \Lambda)$ be a bilinear form such that*

$$\partial^0 \varphi \leq -2p + 1, \quad d\varphi = 0.$$

Then there exist a complex $C' \in \mathcal{C}(A)$ with $\dim C' = \dim C$, a $\mathcal{W}'$ -equivalence $\varepsilon: C' \rightarrow C$ and a bilinear form $\varphi' \in B(C')$ such that $d\varphi' = 0$ and $\varepsilon^(\varphi) - \varphi' \otimes \Lambda$ is a boundary.*

Proof. — By theorem (6.12), there exist a complex $C' \in \mathcal{C}(A)$ with $\dim C' = \dim C$, a $\mathcal{W}'$ -equivalence $\varepsilon: C' \rightarrow C$ and a morphism $g: C' \rightarrow \hat{C}$ such that $\varphi \circ (\varepsilon \otimes \Lambda)$ is homotopic to $\Lambda \otimes g$. Then $\varphi' = \hat{\varepsilon}g$ is the desired form. ■

Lemma (8.3). — *Let C be a finite A -complex such that $H^i(C, \Lambda)$ vanishes for $i > p$ and let $\varphi \in B(C)$ be a bilinear form such that*

$$\partial^0 \varphi \leq -2p, \quad d\varphi = 0.$$

Then, if $\varphi \otimes \Lambda$ is a boundary, there exist a complex $C' \in \mathcal{C}(A)$ with $\dim C' = \dim C$ and a $\mathcal{W}'$ -equivalence $\varepsilon: C' \rightarrow C$ such that $\varepsilon^(\varphi)$ is a boundary.*

Proof. — If $\varphi \otimes \Lambda$ is a boundary, $\varphi \otimes \Lambda$ is null homotopic and, by (6.12), there exist a complex $C' \in \mathcal{C}(A)$ with $\dim C' = \dim C$ and a $\mathcal{W}'$ -equivalence $\varepsilon: C' \rightarrow C$ such that $\varphi \circ \varepsilon$ is null homotopic. Then $\varepsilon^*(\varphi) = \hat{\varepsilon} \circ \varphi \circ \varepsilon$ is a boundary. ■

Theorem (8.4). — *The morphism $\varepsilon : \Gamma_n(A, \mathcal{W}') \rightarrow L_n^h(\Lambda)$ is an isomorphism.*

Proof. — Suppose $n = -2p$ or $n = -2p + 1$, and let $\sigma \in L_n^h(\Lambda)$.

By lemma (3.6), σ is represented by a $\mathcal{W}_\Lambda$ -non singular quadratic n -complex (C, q) where C is concentrated in dimension p (and $p - 1$ if n is odd).

By lemma (8.1), there exist a p -dimensional complex $C' \in \mathcal{C}(A)$ and a homotopy equivalence from $C' \otimes \Lambda$ to C . Then σ is represented by $(C' \otimes \Lambda, q')$. Since C' is p -dimensional, q' is the class of $e_0 \otimes \varphi_0 + e_1 \otimes \varphi_1$ and we have

$$d\varphi_0 + \varphi_1 - \widehat{\varphi}_1 = 0, \quad d\varphi_1 = 0.$$

By lemma (8.2), we may suppose that φ_1 has the form $\psi_1 \otimes \Lambda$, $\psi_1 \in B(C')$ and $d\psi_1$ is zero. Then $(\psi_1 - \widehat{\psi}_1) \otimes \Lambda$ is a boundary and, by lemma (8.3), we may suppose that $\psi_1 - \widehat{\psi}_1$ is a boundary $d\xi$.

Now, $\varphi_0 + \xi \otimes \Lambda$ is a cycle and, by (8.2), we may suppose that

$$\varphi_0 + \xi \otimes \Lambda = \varphi' \otimes \Lambda + d\eta$$

where φ' is a cycle in $B(C')$ and $\eta \in B(C' \otimes \Lambda)$. Then, we have

$$e_0 \otimes \varphi_0 + e_1 \otimes \varphi_1 = (e_0 \otimes (\varphi' - \xi) + e_1 \otimes \psi_1) \otimes \Lambda + d(e_0 \otimes \eta).$$

Moreover $e_0 \otimes (\varphi' - \xi) + e_1 \otimes \psi_1$ is a cycle and represents a $\mathcal{W}'$ -non singular quadratic n -form over C' . Then the morphism ε is surjective.

Now let $\sigma' \in \Gamma_n(A, \mathcal{W}')$ be an element in $\text{Ker } \varepsilon$. By lemma (3.6), σ' is represented by a $\mathcal{W}'$ -non singular quadratic n -complex (C, q) where C is a complex in $\mathcal{C}(A)$ concentrated in dimension p (and $p - 1$ if n is odd).

Since $\varepsilon\sigma'$ is zero, $(C \otimes \Lambda, q \otimes \Lambda)$ is cobordant to zero and, by lemmas (3.7) and (3.8), there exists a $\mathcal{W}_\Lambda$ -non singular quadratic $(n + 1)$ -pair $(\Sigma \rightarrow C \otimes \Lambda, u)$ such that q is the boundary of u and Σ_i vanishes for $i \neq p, p - 1$.

By lemma (8.1), we may suppose that the morphism $\Sigma \rightarrow C \otimes \Lambda$ is the morphism $g \otimes \Lambda : \Sigma' \otimes \Lambda \rightarrow C \otimes \Lambda$, where Σ' is a p -dimensional complex in $\mathcal{C}(A)$. The quadratic form u is represented by

$$e_0 \otimes \psi_0 + e_1 \otimes \psi_1 + e_2 \otimes \psi_2, \quad \psi_i \in B(\Sigma'),$$

and we have

$$\begin{aligned} d\psi_0 + \psi_1 - \widehat{\psi}_1 &= \widehat{g}\varphi_0 g \otimes \Lambda \\ -d\psi_1 + \psi_2 + \widehat{\psi}_2 &= \widehat{g}\varphi_1 g \otimes \Lambda \\ d\psi_2 &= 0 \end{aligned}$$

where $e_0 \otimes \varphi_0 + e_1 \otimes \varphi_1$ represents q .

By lemma (8.2), we may suppose that

$$\psi_2 = \psi'_2 \otimes \Lambda + d\xi_1, \quad d\psi'_2 = 0$$

and, after adding to $e_0 \otimes \psi_0 + e_1 \otimes \psi_1 + e_2 \otimes \psi_2$ the boundary of $e_2 \otimes \xi_1$, we have

$$\psi_2 = \psi'_2 \otimes \Lambda, \quad d\psi'_2 = 0.$$

Then $(\hat{g}\varphi_1g - \psi'_2 - \hat{\psi}'_2) \otimes \Lambda$ is a boundary and, by lemma (8.3), we may suppose that

$$\hat{g}\varphi_1g = \psi'_2 + \hat{\psi}'_2 + d\eta_1.$$

Since $\psi_1 + \eta_1 \otimes \Lambda$ is a cycle, we may suppose, by lemma (8.2), that

$$\psi_1 + \eta_1 \otimes \Lambda = \psi'_1 \otimes \Lambda + d\xi_0, \quad d\psi'_1 = 0,$$

and, after adding to $e_0 \otimes \psi_0 + e_1 \otimes \psi_1 + e_2 \otimes \psi_2$ the boundary of $-e_1 \otimes \xi_0$, we may suppose that

$$\psi_1 + \eta_1 \otimes \Lambda = \psi'_1 \otimes \Lambda, \quad d\psi'_1 = 0.$$

Then, we have

$$d\psi_0 + (\psi'_1 - \eta_1 - \hat{\psi}'_1 + \hat{\eta}_1) \otimes \Lambda = \hat{g}\varphi_0g \otimes \Lambda.$$

Let ψ be the form $\hat{g}\varphi_0g - \psi'_1 + \eta_1 + \hat{\psi}'_1 - \hat{\eta}_1$. The bilinear form ψ is a cycle of degree n and $\psi \otimes \Lambda$ is a boundary. Moreover, by Poincaré duality, $H^i(\Sigma', \Lambda)$ vanishes for $i > -n - p$. Then lemma (8.3) holds and we may suppose that

$$\hat{g}\varphi_0g - \psi'_1 + \eta_1 + \hat{\psi}'_1 - \hat{\eta}_1 = d\eta_0.$$

So $\psi_0 - \eta_0 \otimes \Lambda$ is a cycle and, by (8.2), we may suppose that

$$\psi_0 - \eta_0 \otimes \Lambda = \psi'_0 \otimes \Lambda + d\xi_{-1}, \quad d\psi'_0 = 0,$$

and, after adding to $e_0 \otimes \psi_0 + e_1 \otimes \psi_1 + e_2 \otimes \psi_2$ the boundary of $e_0 \otimes \xi_{-1}$, we may suppose that

$$\psi_0 - \eta_0 \otimes \Lambda = \psi'_0 \otimes \Lambda.$$

Now it is easy to check that

$$e_0 \otimes \psi_0 + e_1 \otimes \psi_1 + e_2 \otimes \psi_2 = [e_0 \otimes (\eta_0 + \psi'_0) + e_1 \otimes (-\eta_1 + \psi'_1) + e_2 \otimes \psi'_2] \otimes \Lambda$$

and

$$d[e_0 \otimes (\eta_0 + \psi'_0) + e_1 \otimes (-\eta_1 + \psi'_1) + e_2 \otimes \psi'_2] = g^*(e_0 \otimes \varphi_0 + e_1 \otimes \varphi_1).$$

Then $e_0 \otimes (\eta_0 + \psi'_0) + e_1 \otimes (-\eta_1 + \psi'_1) + e_2 \otimes \psi'_2$ represents a $\mathcal{W}'$ -non singular quadratic $(n+1)$ -form v over $\Sigma' \rightarrow \mathbb{C}$ with boundary q . So σ' is zero and ε is injective. ■

9. Some results about Λ and $L_n(\Lambda)$

Throughout this section, we assume that $A \rightarrow B$ is a ring homomorphism and β is a subgroup of $\tilde{K}_1(B)$.

The class of finite A -complexes C such that $C \otimes B$ is acyclic with torsion in β is denoted by $\mathcal{W}^\beta$, and the $\mathcal{W}^\beta$ -localization of A is denoted by (Λ, α) .

Proposition (9.1). — *Let u be a matrix with entries in Λ . Then, if $u \otimes B$ is invertible, u is invertible too.*

Proof. — Let u be a matrix with entries in Λ . If we denote by A the 0-dimensional complex $\dots \rightarrow 0 \rightarrow A \rightarrow 0 \rightarrow \dots$, u is a morphism $A^p \otimes \Lambda \rightarrow A^q \otimes \Lambda$ and, by theo-

rem (6.12), there exist a 0-dimensional complex $\bar{C} \in \mathcal{C}(A)$, a $(\mathcal{W}^\beta)_0^s$ -equivalence $\varepsilon: \bar{C} \rightarrow A^p$ and a morphism $g: \bar{C} \rightarrow A^q$ such that $g \otimes \Lambda$ is homotopic to $u \circ (\varepsilon \otimes \Lambda)$.

Let K be the homotopy kernel of ε . Since K is $\mathcal{W}_0^\beta$ -splittable, there exist a (-1) -dimensional complex $K' \in \mathcal{W}_0^\beta$ and a (-2) -connected morphism $f: K' \rightarrow K$. The composite map $K' \rightarrow K \rightarrow \bar{C}$ is (-2) -connected. Denote by C' its mapping cone. The complex C' lies in $\mathcal{W}_0^\beta$ and has the simple homotopy type of a complex C'' such that C_i'' vanishes for $i \neq 0, -1$. Moreover ε and g factorize through C'' and we get two morphisms $\varepsilon': C'' \rightarrow A^p$ and $g': C'' \rightarrow A^q$ such that $g' \otimes \Lambda$ is homotopic to $u \circ (\varepsilon' \otimes \Lambda)$.

But $u \otimes B$ is invertible, then $g' \otimes B$ is a homotopy equivalence and the mapping cone of g' is B -acyclic and lies in $\mathcal{W}_0^\beta$. Since the length of this mapping cone is 2, g' is a $(\mathcal{W}^\beta)_0^s$ -equivalence. Then, by (7.1), g' is a Λ -homology equivalence, and u is an isomorphism. ■

(9.2) Proof of theorem (1.13)

If u is a matrix with entries in A , denote by $M(u)$ the 1-dimensional complex $\dots \rightarrow 0 \rightarrow A^p \xrightarrow{u} A^q \rightarrow 0 \rightarrow \dots$

The set Σ is the set of matrices u such that $(M(u) \oplus \Sigma M(u)) \otimes B$ is acyclic with torsion in β . But $M(u) \oplus \Sigma M(u)$ is B -acyclic if and only if $M(u)$ is B -acyclic. Moreover if $M(u)$ is B -acyclic, we have

$$\tau[M(u) \otimes B \oplus \Sigma M(u) \otimes B] = 0.$$

Then Σ is the set of matrices u such that $u \otimes B$ is invertible and $A \rightarrow \Lambda$ is the localization of $A \rightarrow B$.

Now let τ be an element of $\tilde{K}_1(\Lambda)$. By lemma (4.3), there exists a finite A -complex C such that $C \otimes \Lambda$ is acyclic with torsion τ . Then, by lemma (7.10), τ lies in α if and only if C lies in $\mathcal{W}^\beta$. But the torsion of $C \otimes B$ is the image of τ by the morphism $\varepsilon: \Lambda \rightarrow B$. Hence α is the inverse image of β under ε .

Now suppose ε is onto, and let $C \in \mathcal{W}^\beta$. The complex $C \otimes B$ is acyclic and the identity is a homotopy: $1 = d \circ k + k \circ d$.

But $C \otimes \Lambda \rightarrow C \otimes B$ is onto and we can lift k in a map k' from $C \otimes \Lambda$ to itself. The morphism $d \circ k' + k' \circ d$ is invertible after tensorization by B . Then, by (9.1), $d \circ k' + k' \circ d$ is an isomorphism and $C \otimes \Lambda$ is acyclic. ■

(9.3) Proof of Proposition (1.15)

Let $B_0 \subset B_1 \subset B_2 \subset \dots$ be subrings of B defined by:

- (i) B_0 is the image of $A \rightarrow B$;
- (ii) for any $n \geq 0$, B_{n+1} is generated by B_n and the inverses of the units of B contained in B_n .

Denote by B' the image of $\Lambda \rightarrow B$. The subring B' contains A and, by (9.1), any unit of B contained in B' is a unit of B' . Then B' contains all the rings B_n . ■

As a corollary of (9.1), we have:

Lemma (9.4). — *If $\Lambda \rightarrow B$ is onto, $\tilde{K}_1(\Lambda) \rightarrow \tilde{K}_1(B)$ is onto.*

From now on, we will suppose that $\Lambda \rightarrow B$ is a morphism of rings with involution and that β is stable under the involution. Then $\mathcal{W}^\beta$ is symmetric and Λ has an involution. We suppose also that $\Lambda \rightarrow B$ is onto.

Theorem (9.5). — *If n is even, the morphism $L_n^\alpha(\Lambda) \rightarrow L_n^\beta(B)$ is epic. If n is odd, this morphism is monic.*

Proof. — By lemma (9.4), the relative group $L_n^{\alpha, \beta}(\Lambda \rightarrow B)$ does not depend on β . Then it suffices to prove the theorem in the case $\beta = \tilde{K}_1(B)$.

Let $n = 2p$. An element $u \in L_{2p}^h(B)$ is represented by a hermitian $(-1)^p$ -form (H, λ, μ) such that the induced map $\tilde{\lambda}: H \rightarrow \hat{H}$ is an isomorphism. Since H is free over B and $\Lambda \rightarrow B$ is epic, there exists a hermitian $(-1)^p$ -form (H', λ', μ') such that

$$\begin{aligned} H' &\text{ is free over } \Lambda, \\ H' \otimes B &= H, \quad \lambda' \otimes B = \lambda, \quad \mu' \otimes B = \mu. \end{aligned}$$

Then, by lemma (9.1), λ' induces an isomorphism from H' to $\hat{H}'$ and (H', λ', μ') represents an element $v \in L_{2p}^h(\Lambda)$ such that $\varepsilon_*(v) = u$.

Let now $n = 2p + 1$. An element $v \in L_{2p+1}^h(\Lambda)$ is represented by an isometry between two standard kernel K and K' . If v is sent to zero in $L_{2p+1}^h(B)$, $K = K'$ and $g \otimes B$ is an element of $RU^h(B)$ (with the notations of [10]).

Consider the following diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & UU(\Lambda) & \longrightarrow & TU^h(\Lambda) & \longrightarrow & GL(\Lambda) \longrightarrow 1 \\ & & \downarrow a & & \downarrow b & & \downarrow c \\ 1 & \longrightarrow & UU(B) & \longrightarrow & TU^h(B) & \longrightarrow & GL(B) \longrightarrow 1 \end{array}$$

By lemma (9.1), a and c are surjective. Then b is epic and the morphism $RU^h(\Lambda) \rightarrow RU^h(B)$ is epic too. Hence v can be represented by an isometry f such that $f \otimes B$ is the identity map.

Let $H \oplus \hat{H}$ be the standard kernel K . The isometry f is defined by

$$f(x, y) = (x + a(x) + b(y), y + c(x) + d(y)), \quad \forall x \in H, y \in \hat{H}$$

and $a \otimes B, b \otimes B, c \otimes B, d \otimes B$ vanish. By (9.1), $1 + a$ is invertible and, after composing f with an element of $GL(\Lambda)$, we may as well suppose that a is zero.

Since f is an isometry, it is easy to see that the map g defined by

$$g(x, y) = (x, y - c(x))$$

is an isometry leaving each element of $\hat{H}$ fixed and g lies in $RU^h(\Lambda)$. We have

$$g \circ f(x, y) = (x + b(y), y + d(y) - c \circ b(y)).$$

But $1 + d - c \circ d$ is invertible and there is an isometry $h \in RU^h(\Lambda)$ such that

$$h \circ g \circ f(x, y) = (x + a'(x) + b'(y), y).$$

It is easy to see that a' is zero and $h \circ g \circ f$ lies in $RU^h(\Lambda)$. Therefore V is zero. ■

Theorem (9.6). — *The relative group $L_{2p+1}^h(\Lambda \rightarrow B)$ is the group of equivalence classes of pairs (H, K) where H is a hermitian $(-1)^p$ -form over Λ and K a subkernel of $H \otimes B$, subject to the following relation:*

(H, K) is equivalent to (H', K') if there exist two Λ -kernels H_0 and H'_0 with subkernels S_0 and S'_0 and an isometry $\varphi : H \oplus H_0 \rightarrow H' \oplus H'_0$ such that

$$\varphi(K \oplus S_0 \otimes B) = K' \oplus S'_0 \otimes B.$$

Proof. — By Wall ([10], p. 72), $L_{2p+1}^h(\Lambda \rightarrow B)$ is generated by such pairs. Moreover (H, K) and (H', K') represent the same element in $L_{2p+1}^h(\Lambda \rightarrow B)$ if there exist two kernels $\bar{H}_0$ and H'_0 with subkernels $\bar{S}_0$ and S'_0 and an isometry

$$\bar{\varphi} : H \oplus \bar{H}_0 \oplus -H' \rightarrow H'_0$$

such that any automorphism $\bar{\psi}$ taking $\bar{S}'_0 \otimes B$ to $\bar{\varphi}(K \oplus \bar{S}_0 \otimes B \oplus K')$ lies in $RU^h(B)$. But the map $RU^h(\Lambda) \rightarrow RU^h(B)$ is epic (see the proof of (9.5)). Hence we can lift $\bar{\psi}$ to an automorphism ψ on H'_0 .

Let S'_0 be the subkernel $\psi(\bar{S}'_0)$. We have an isometry

$$\varphi : H \oplus \bar{H}_0 \oplus -H' \oplus H' \rightarrow H' \oplus H'_0$$

taking $K \oplus \bar{S}_0 \otimes B \oplus K' \oplus K'$ to $K' \oplus S'_0 \otimes B$.

On the other hand, the diagonal $\bar{K}$ is a subkernel of $-H' \oplus H'$ and there exists an automorphism in $RU^h(B)$ taking $\bar{K} \otimes B$ to $K' \oplus K'$. By lifting this automorphism in $RU^h(\Lambda)$ we get an automorphism f and $f(\bar{K})$ is a subkernel of $-H' \oplus H'$ such that $f(\bar{K}) \otimes B = K' \oplus K'$. Let H_0 be the kernel $\bar{H}_0 \oplus -H' \oplus H'$ with subkernel $S_0 = \bar{S}_0 \oplus f(\bar{K})$. Then φ is an isometry taking $K \oplus S_0 \otimes B$ to $K' \oplus S'_0 \otimes B$. ■

Now, consider the following question: Under what conditions is the map $\varepsilon : \Lambda \rightarrow B$ an isomorphism? To study this problem, it is convenient to use the following definitions:

An A -module M is called *B-perfect* if $M \otimes B$ is zero; it is called *locally B-perfect* if any element in M is contained in a finitely generated B -perfect submodule.

Theorem (9.7). — *Suppose the kernel of $A \rightarrow B$ is locally B-perfect and B is the localization of $\text{Im}(A \rightarrow B)$ with respect to a multiplicative subset of the center. Then the morphism $\varepsilon : \Lambda \rightarrow B$ is an isomorphism.*

Proof. — Let $a \in \text{Ker}(A \rightarrow B)$ and suppose that a is contained in a finitely generated B -perfect submodule I . Let us choose a free resolution of I

$$C \xrightarrow{f} A^n \rightarrow I \rightarrow 0.$$

Since I is B -perfect, $f \otimes B$ is epic and has a section s . But $A \rightarrow B$ is epic and we can lift s to a morphism $g: A^n \rightarrow C \otimes A$. By (9.1), $f \otimes A \circ g$ is an isomorphism and $f \otimes A$ is epic. Hence I is A -perfect and the composite map $I \rightarrow A \rightarrow \Lambda$ is zero. Then $A \rightarrow B$ and $A \rightarrow \Lambda$ have the same kernel K .

Now it is easy to see that the maps $A/K \rightarrow B$ and $A/K \rightarrow \Lambda$ have the same universal property and $\varepsilon: \Lambda \rightarrow B$ is an isomorphism. ■

This theorem is in fact a generalization of a theorem of Hausmann [3] proved also in [6] and [8], theorem (1.4).

Finally, we will give an example of computation.

Let D_{2n} be the dihedral group of order $2n$ (n odd) and let $\mathbf{Z}D_{2n} \rightarrow \mathbf{Z}$ be the evaluation map. The group D_{2n} is not perfect and not nilpotent, then we cannot use the techniques of Hausmann or Smith in order to compute the group $\Gamma_*(\mathbf{Z}D_{2n} \rightarrow \mathbf{Z})$.

Theorem (9.8). — We have the isomorphisms

$$\Gamma_*(\mathbf{Z}D_{2n} \rightarrow \mathbf{Z}) \simeq \Gamma_*(\mathbf{Z}[\mathbf{Z}/2] \rightarrow \mathbf{Z}) \simeq L_*^h(\Lambda)$$

where Λ is the pull back of rings

$$\begin{array}{ccc} \Lambda & \longrightarrow & \mathbf{Z} \\ \downarrow & & \downarrow \\ \mathbf{Z}_{(2)}[\mathbf{Z}/2] & \longrightarrow & \mathbf{Z}_{(2)} \end{array}$$

Proof. — The group D_{2n} is generated by t and τ with the following relations:

$$t^n = 1, \quad \tau^2 = 1, \quad \tau t = t^{-1} \tau.$$

Let $\mathbf{Z}D_{2n} \rightarrow \Lambda$ be the localization of $\mathbf{Z}D_{2n} \rightarrow \mathbf{Z}$ and let x and y be the images of t and τ in Λ . We have

$$\left[\frac{1-n}{2} (1 + \tau) + 1 + t + \dots + t^{n-1} \right] (1 - \tau)(1 - t) = 0.$$

But $\frac{1-n}{2} (1 + \tau) + 1 + t + \dots + t^{n-1}$ is sent to 1 in $\mathbf{Z}$ and

$$\frac{1-n}{2} (1 + y) + 1 + x + \dots + x^{n-1}$$

is invertible. This implies that

$$(1 - y)(1 - x) = 0.$$

On the other hand, $\mathbf{ZD}_{2n} \rightarrow \Lambda$ is a morphism of rings with involution. So we have:

$$(1-y)(1-x) = (1-x^{-1})(1-y) = 0 \Rightarrow (1-x)(1-y) = 0.$$

And x and y commute. Then:

$$yx = x^{-1}y = xy \Rightarrow x = 1.$$

Hence t is sent to 1 in Λ and Λ is the localization of $\mathbf{Z}[\mathbf{Z}/2] \rightarrow \mathbf{Z}$. But $\mathbf{Z}[\mathbf{Z}/2]$ is commutative and Λ is the localization $S^{-1}\mathbf{Z}[\mathbf{Z}/2]$ where S is the set of elements $a + b\tau \in \mathbf{Z}[\mathbf{Z}/2]$ with $a + b = 1$. Then it is easy to see that Λ is the subring of $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$ defined by

$$\Lambda = \{a + b\tau, a, b \in \mathbf{Z}_{(2)} \text{ and } a + b \in \mathbf{Z}\}.$$

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