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EQUIVALENCE OF DIFFERENTIABLE MAPPINGS AND ANALYTIC MAPPINGS

by MASAHIRO SHIOTA

INTRODUCTION

In this paper, we consider the problem to know when a differentiable function or mapping can be transformed to an ("equivalent") analytic or polynomial one by a diffeomorphism.

In the 1 dimensional case this problem is simple. R. Thom showed in *Topology*, 3 (1965) that a non-constant C^∞ function on $\mathbf{R}$ is equivalent to a polynomial if and only if it is proper, the number of critical points is finite and the derivative is nowhere flat. The author showed in [23] that it is equivalent to an analytic function if and only if the derivative is nowhere flat.

The problem for general dimension consists of a local one and a global one. In chapter I we consider the local problem for functions. Here are some of the main results, which will be proved in Section 2.

Theorem. — If a germ of a C^∞ function in n variables is zero at the origin and of the form $\prod_{i=1}^{k} f_i^{\alpha_i}$ where f_i are C^∞ germs and α_i are integers such that the Taylor expansions of f_i generate distinct prime ideals in the formal power series ring, then the germ is equivalent to a polynomial in two variables with germs of C^∞ functions in $n-2$ variables as coefficients.

Theorem. — Any convergent (or formal) power series in n variables is equivalent to some polynomial in two variables with convergent (or formal) power series in $n-2$ variables as coefficients.

N. Levinson [10] and J. C. Tougeron [31] showed the equivalence to a polynomial in one variable with C^∞ or analytic germs in $n-1$ variables as coefficients, and H. Whitney gave an example of a convergent series in three variables which is not equivalent to any polynomial [34].

The theorem above and a result in [31] imply that, if f is a complex convergent series in n variables such that $f(0)=0$, if the singular set of $f^{-1}(0)$ has codimension s' in $\mathbf{C}^n$, f is equivalent to a polynomial in s' variables with convergent series in $n-s'$ variables as coefficients. This is a generalization of the well-known fact (see [31]) that if f has its singular set of codimension s, f is equivalent to a polynomial in s variables with convergent series in $n-s$ variables as coefficients.

The case $n=2$ will be studied in more detail in § 3. Section 4 describes other properties of the equivalence and the C^r equivalence for $0 \leqslant r < \infty$.

Theorem. — If two real convergent power series vanish at the origin and have the same sign at each point near the origin, then they are topologically equivalent.

Corollary. — Let f_1, f_2 be analytic functions on a compact real analytic manifold, and let S_i be the set of singular values of f_i. Assume $f_1^{-1}(S_1 \cup S_2) = f_2^{-1}(S_1 \cup S_2) = M$ and $f_1|_M = f_2|_M$. Then f_1 and f_2 are globally C^0 equivalent.

We know by [31] that if a real convergent series f in n variables has a singular set of codimension s in $\mathbf{R}^n$, f is C^r equivalent to a polynomial in s variables with convergent series in $n-s$ variables as coefficients for any $r<\infty$. We generalize this as follows.

Theorem. — Assume $f(0)=0$. Let S' be the intersection of $\mathbf{R}^n$ with the singular set of $f^{-1}(0)$ in $\mathbf{C}^n$ around 0, and let s' be the codimension of S' in $\mathbf{R}^n$. Then f is C^r equivalent to a polynomial in s' variables with convergent series in $n-s'$ variables as coefficients for any $r<\infty$.

Next we show the existence of a homomorphism from the formal power series ring in one variable to the ring of germs of C^∞ functions in one variable whose composition with the Taylor expansion at the origin is the identity, and which commutes with the derivation (§ 5).

In chapter II we treat the global C^∞ problem.

Conjecture I. — Let f be a C^∞ mapping between analytic manifolds. Suppose that the germ of f at each point is equivalent to a germ of an analytic mapping. Then f is equivalent to an analytic mapping.

Conjecture II. — Any two C^∞ equivalent analytic mappings are analytically equivalent, that is, the diffeomorphism can be chosen analytic.

In Section 6 we prove a fundamental lemma on the equivalence, the local case of which is due to J. C. Tougeron [31]. We also give a lemma which glues analytic function germs. In Sections 7 and 8 we work on functions and prove the conjectures I and II respectively for functions taking locally one of the following forms, except at a discrete set:

(i) regular,

(ii) the sum or the difference of a constant and a power of a regular function,

(iii) $\pm x_1^2 \pm \ldots \pm x_k^2 + \text{const.}$ for a suitable coordinate system $(x_1, \ldots, x_k, \ldots, x_n)$.

In general an analytic function takes locally the form (i) or (ii) except on an analytic set of codimension 2, and a coherent analytic set is the vanishing set of some analytic function locally of the form (i) or (iii) except on the singularities. Consequently, the conjectures hold for functions if the dimension of the manifold is 1 or 2. If a closed subset of an analytic manifold is locally equivalent to germs of coherent analytic sets with isolated singularities, then the set is equivalent to an analytic set. As a corollary corresponding to the conjecture II, two C^∞ equivalent coherent analytic sets with isolated singularities are analytically equivalent. Moreover, we prove this fact in a slightly more general form, that is, we admit vanishing sets of type (iii) and we weaken the condition of coherence. We also consider the following question: Suppose that in I and II some analytic submanifolds are invariant under the given local or global diffeomorphism. Then, can we choose the resulting diffeomorphism so that those submanifolds remain invariant? We shall give a partial result in that direction.

In Section 9 we introduce local canonical forms (i)′, (ii)′ and (iii)′ of *mappings* corresponding to the types of *functions* (i), (ii) and (iii) respectively. We show that any analytic mapping takes locally the form (i)′ or (ii)′ except on an analytic set of codimension two in the source manifold. Furthermore, the condition for being of the form (iii) involves only the first partial derivatives. Hence we use the determinants of Jacobian submatrices to generalize (iii) to (iii)′. Section 10 gives a proof of the conjecture I for mappings locally of the form (i)′, (ii)′ or (iii)′ except on a discrete set. We also prove the conjecture II in a special case. In particular, the conjectures are valid for any one- or two-dimensional source manifold.

Section 11 deals with other equivalence relations of mappings by adding diffeomorphisms of the target manifold. There we give some counter-examples to the corresponding conjectures.

In § 12 we consider the case of a Nash function, and we obtain results similar to the above. We also prove that a factorization of a non-zero analytic function on a connected analytic manifold into C^∞ functions is a factorization into analytic functions.

The fundamental tools of the proofs in this chapter except in the last section are M. Artin's theorem [1] and the fundamental theorems A and B on Stein manifold of H. Cartan.

In chapter III we study other global problems, of a topological nature. In Section 13, we prove theorems about the topological equivalence of a continuous function f on a C^∞ manifold of dimension $\neq 4, 5$ to a C^∞ function.

Theorem. — If f has only isolated topological singularities, f is C^0 equivalent to a C^∞ function.

Theorem. — If the set of topologically singular values has no inner point in $\mathbf{R}$, *then f is right-left C^0 equivalent to a C^∞ function.*

The problem studied in section 14 is when a $\mathbf{C}$-valued function on $\mathbf{R}^2 = \mathbf{C}$ can be transformed into a $\mathbf{C}$-polynomial. We prove:

Theorem. — If a **C**-*valued* C^∞ *function on* $\mathbf{R}^2$ *is proper and is locally equivalent to a germ of a* **C**-*polynomial, then it is equivalent to a* **C**-*polynomial.*

By Stoïlow's theorem this means that any **C**-valued light open proper continuous function on $\mathbf{R}^2$ is C^0 equivalent to a **C**-polynomial.

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CONTENTS

I. — LOCAL EQUIVALENCE

1. Preparation

Let $\mathscr{E}_n$ denote the ring of germs of C^∞ functions at 0 in $\mathbf{R}^n$, $\mathcal{O}_n$ the ring of convergent power series $\mathbf{R}\{x_1, \ldots, x_n\}$ and $\mathscr{F}_n$ the ring of formal power series $\mathbf{R}[[x_1, \ldots, x_n]]$. We sometimes write $\mathscr{E}(x_1, \ldots, x_n)$ for $\mathscr{E}_n$ in order to make clear the coordinate system. Let A be one of the rings above and f be an element of A. Then $\mathfrak{m}(A)$ and I_f mean the maximal ideal of A and the ideal generated by all the first derivatives of f in A respectively. We sometimes write simply $\mathfrak{m}$ for $\mathfrak{m}(\mathscr{E}_n)$. Let T denote the Taylor expansion at 0. An element f of $\mathscr{E}_n$ is called *flat* if $Tf=0$. Let $[:]$ and ht denote the quotient ideal and the height of an ideal respectively.

Elements f and g of $\mathscr{E}_n$ are called *equivalent* if there exists a local C^∞ diffeomorphism τ of $\mathbf{R}^n$ around 0 such that $f\circ\tau=g$. If the local diffeomorphism is only of class C^r with $0 \leq r \leq \infty$, we say that f and g are C^r *equivalent*. In the same way we can define an equivalence relation in $\mathcal{O}_n$, to be called C^ω *equivalence*. We remark that elements f and g of $\mathcal{O}_n$ are C^ω equivalent if and only if there exists an automorphism of $\mathcal{O}_n$ mapping g onto f. Similarly, we can consider an equivalence relation in $\mathscr{F}_n$, two elements f and g of $\mathscr{F}_n$ being called *equivalent* if there exists an automorphism of $\mathscr{F}_n$ mapping g onto f.

In this chapter we consider all functions in a sufficiently small neighborhood of the origin. We do not always distinguish germs from functions.

The next lemma is essentially stated in Mather [15], p. 33. See [25] for the proof.

Lemma (**1.1**). — *Let f, g be in $\mathscr{E}_n$ (resp. $\mathcal{O}_n$) and let $a_i(x, t)$ $(i=1, \ldots, n)$ be germs at $0\times[0, 1]$ in $\mathbf{R}^n\times\mathbf{R}$ of C^∞ functions (resp. analytic functions). Assume that*

$$f(x)-g(x)=\sum_{i=1}^{n} a_i(x, t)\left(\frac{\partial f(x)}{\partial x_i}t+\frac{\partial g(x)}{\partial x_i}(1-t)\right),$$

$$a_i(0, t)=0 \quad \text{for} \quad i=1, \ldots, n.$$

Then f and g are equivalent (resp. C^ω equivalent).

As a corollary, we obtain the next lemma which is due to Tougeron [31] (see also [25]).

Lemma (**1.2**). — *Let f, g be in $\mathfrak{m}^2(\mathscr{E}_n)$ (resp. $\mathfrak{m}^2(\mathcal{O}_n)$) such that $f-g$ is an element of the ideal generated by $\dfrac{\partial g}{\partial x_i}\dfrac{\partial g}{\partial x}x_k$ with $i, j, k=1, \ldots, n$. Then f and g are equivalent (resp. C^ω equivalent).*

Remark (**1.3**). — Assume moreover that, in the lemma above, $f-g$ is contained in the product of the ideal above and an ideal $\mathfrak{a}$. Then the local diffeomorphism $\tau=(\tau_1, \ldots, \tau_n)$ can be chosen so that $\tau_i - x_i$ are contained in $\mathfrak{a}$. For the proof, see [31].

The next lemma is due to Skoda and Briançon [30]. They proved the case of a complex convergent power series. The real case follows trivially, and from the closedness of any ideal of $\mathscr{F}_n$ in the Krull topology (Krull) we deduce the $\mathscr{F}_n$ case.

Lemma (**1.4**). — *Let f be in $\mathfrak{m}(\mathscr{F}_n)$ or in $\mathfrak{m}(\mathcal{O}_n)$. Then f^n is contained in I_f.*

Remark (**1.5**). — The $\mathscr{E}_n$ case of the lemma (1.4) is not true. For example, $f=\exp(-1/x^2)\sin 1/x$ in $\mathscr{E}_1$ is not contained in I_f. But the following is very probable; in fact, we easily prove it for $n\leq 2$ from the corollary (2.4).

Let q be in $\mathfrak{m}(\mathscr{F}_n)$. Then there exists f in $\mathscr{E}_n$ such that $\mathrm{T}f=q$ and that f^n is contained in I_f.

2. A canonical form for certain differentiable germs of functions

If an element f of $\mathfrak{m}(\mathscr{E}_n)$ can be factorized in $\mathfrak{m}(\mathscr{E}_n)$ as

$$f=\prod_{i=1}^{k} f_i^{\alpha_i}$$

in such a way that the $\mathrm{T}f_i$ generate distinct prime ideals, we call *f factorizable.* We know by the theorem of Zariski-Nagata that any germ of a C^∞ function equivalent to an analytic germ is factorizable.

The next theorem is a generalization of a theorem in Shiota [26].

Theorem (**2.1**). — *Let an element f in $\mathscr{E}_n$ be factorizable. Then f is equivalent to a polynomial in two variables with coefficients in $\mathscr{E}_{n-2}$.*

Proof. — The case $n=1$ is trivial, hence we assume $n\geq 2$. By the hypothesis there are elements f_i for $i=1, \ldots, k$ in $\mathfrak{m}$ such that

$$f=\prod_{i=1}^{k} f_i^{\alpha_i}$$

and that $\mathrm{T}f_i$ are different and prime. Put

$$f'=\prod_{i=1}^{k} f_i, \quad f''=f/f', \quad \mathfrak{p}=\mathfrak{m}\mathrm{I}_{f'}[\mathrm{I}_f : f].$$

Then we shall see that

$$\mathrm{ht}\ \mathrm{T}\mathfrak{p}=\mathrm{ht}\{\mathrm{T}g \mid g\in\mathfrak{p}\}\geq 2.$$

For the proof it is sufficient to show:

$$\mathrm{ht}\ \mathrm{T}\mathrm{I}_{f'}\geq 2, \quad \mathrm{ht}\ \mathrm{T}[\mathrm{I}_f : f]\geq 2.$$

The first inequality was observed by Tougeron [31].

From the factoriality assumption on f, the $\frac{\partial f}{\partial x_i}$ are divisible by f'' for $i=1, \ldots, n$. Let the quotients be g_i. Then we have

$$[\mathrm{I}_f : f''] = (g_1, \ldots, g_n), \qquad [\mathrm{I}_{\mathrm{T}f} : \mathrm{T}f''] = (\mathrm{T}g_1, \ldots, \mathrm{T}g_n),$$

where $(g_1, \ldots, g_n)$ denotes the ideal generated by g_i. Hence

$$\mathrm{T}[\mathrm{I}_f : f''] = [\mathrm{I}_{\mathrm{T}f} : \mathrm{T}f''].$$

Since it is trivial that

$$[\mathrm{I}_f : f''] \subset [\mathrm{I}_f : f],$$

the next lemma is sufficient to establish the needed inequality.

Lemma (**2.2**). — *Let p be in $\mathfrak{m}(\mathscr{F}_n)$ or in $\mathfrak{m}(\mathcal{O}_n)$. Let*

$$p = \prod_{i=1}^{k} p_i^{\alpha_i}, \qquad p' = \prod_{i=1}^{k} p_i, \qquad p'' = p/p',$$

where p_i generate distinct prime ideals. Then we have

$$\mathrm{I}_{p'} \supset [\mathrm{I}_p : p'']^{2n}, \qquad [\mathrm{I}_p : p''] \supset \mathrm{I}_{p'}^{2n}.$$

Proof. — We only prove the $\mathscr{F}_n$ case, the $\mathcal{O}_n$ case being entirely the same. We put

$$\mathrm{J} = \sum_{i=1}^{k} \mathrm{I}_{p_i} \prod_{j \neq i} p_j.$$

We want to prove that $\mathrm{J} \supset \mathrm{I}_{p'} \supset \mathrm{J}^{2n}$. It is trivial that $\mathrm{J} \supset \mathrm{I}_{p'}$. Now J^2 is generated by

$$q_{ij\ell m} = \frac{\partial p_j}{\partial x_i} \prod_{s \neq j} p_s \frac{\partial p_m}{\partial x_\ell} \prod_{t \neq m} p_t$$

for $i, l = 1, \ldots, n$ and $j, m = 1, \ldots, k$. If $j \neq m$, $q_{ij\ell m} \in p' \mathscr{F}_n$. If $j = m$

$$q_{ij\ell m} = \frac{\partial p_j}{\partial x_i} \prod_{s \neq j} p_s \left(\frac{\partial p'}{\partial x_\ell} - \sum_{u \neq m} \frac{\partial p_u}{\partial x_\ell} \prod_{t \neq u} p_t \right) \in p' \mathscr{F}_n + \frac{\partial p'}{\partial x_\ell} \mathscr{F}_n.$$

By Lemma (1.4) we have $p'^n \in \mathrm{I}_{p'}$. Hence

$$(p' \mathscr{F}_n + \mathrm{I}_{p'})^n \subset \mathrm{I}_{p'}.$$

This implies that $\mathrm{J}^{2n} \subset \mathrm{I}_{p'}$. In the same way we prove that $\mathrm{J} \supset [\mathrm{I}_p : p''] \supset \mathrm{J}^{2n}$. Here we use the fact that $p'^n \in [\mathrm{I}_p : p'']$, for whose proof it is sufficient to read the proof of Lemma (1.4). Therefore we see:

$$\mathrm{I}_{p'} \supset \mathrm{J}^{2n} \supset [\mathrm{I}_p : p'']^{2n}, \qquad [\mathrm{I}_p : p''] \supset \mathrm{J}^{2n} \supset \mathrm{I}_{p'}^{2n}.$$

Thus the lemma is proved.

Now we continue the proof of the theorem. From the normalization theorem for rings of formal power series (e.g. Nagata [20]), we have a coordinates system $(x_1, \ldots, x_n)$ such that the sum of $\mathrm{T}p$ and the ideal generated by $x_1, \ldots, x_{n-2}$ in $\mathscr{F}_n$ contains $\mathfrak{m}^p(\mathscr{F}_n)$

for sufficiently large p. Then, by Nakayama's lemma, the sum of $\mathfrak{p}$ and the ideal generated by $x_1, \ldots, x_{n-2}$ in $\mathscr{E}_n$ contains $\mathfrak{m}^p$. Hence the images $\bar{x}_{n-1}$ and $\bar{x}_n$ of x_{n-1} and x_n by projection in $\mathscr{E}_n/\mathfrak{p}$ are integral over $\mathscr{E}(x_1, \ldots, x_{n-2})$. This property remains valid after a small change of the coordinates system in the Krull topology. Let φ be a monic polynomial in one variable with coefficients in $\mathscr{E}(x_1, \ldots, x_{n-2})$ such that $\varphi(x_{n-1})$ and $\varphi(x_n)$ are contained in the product of a sufficiently large number of copies of $\mathfrak{p}$. By Malgrange's preparation theorem, there exist φ_1 and φ_1' in $\mathscr{E}_n$ and F_1 a (x_{n-1}, x_n)-polynomial with coefficients in $\mathscr{E}(x_1, \ldots, x_{n-2})$, such that

$$f_1 = F_1 + \varphi(x_{n-1})\,\varphi_1 + \varphi(x_n)\,\varphi_1'.$$

Hence we have

$$f' = F_1 \prod_{i\neq 1} f_i + \varphi(x_{n-1})\,\varphi_1 \prod_{i\neq 1} f_i + \varphi(x_n)\,\varphi_1' \prod_{i\neq 1} f_i.$$

By Lemma (1.2), f' and $F_1 \prod_{i\neq 1} f_i$ are equivalent. Moreover, by our remark (1.3), the local diffeomorphism has the following form:

$$\tau = (x_1 + \theta_1 \prod_{i\neq 1} f_i, \ldots, x_n + \theta_n \prod_{i\neq 1} f_i),$$

where θ_i are elements of the product of a sufficiently large number of copies of $\mathfrak{p}$. Let $y = (y_1, \ldots, y_n)$ be another coordinates system. For each $j \neq 1$ there exist $\beta_{j1}, \ldots, \beta_{jn}$ in $\mathscr{E}(x, y)$ such that

$$f_j(x+y) = f_j(x) + \sum_{i=1}^{n} \beta_{ji} y_i.$$

In particular,

$$f_j \circ \tau = f_j(x + \theta \prod_{i\neq 1} f_i) = f_j(x) + \sum_{i=1}^{n} \beta_{ji}(x, \theta \prod_{l\neq 1} f_l)\, \theta_i \prod_{l\neq 1} f_l.$$

Hence we have $h_2, \ldots, h_n$ in a product of sufficiently many copies of $\mathfrak{p}$ such that

$$f_j \circ \tau = f_j(x)(1 + h_j(x)) \qquad \text{for} \qquad j = 2, \ldots, n.$$

As $\prod_{i=1}^{k} f_i \circ \tau = F_1 \prod_{i\neq 1} f_i$, we have $f_1 \circ \tau = F_1 / \prod_{i\neq 1} (1 + h_i)$. Therefore

$$f \circ \tau = c F_1^{\alpha_1} \prod_{i\neq 1} f_i^{\alpha_i}$$

where c is the sum of 1 and an element of a product of a sufficiently large number of copies of $\mathfrak{p}$. Moreover the lemma below shows the existence of a local diffeomorphism τ' such that

$$f \circ \tau \circ \tau' = F_1^{\alpha_1} \prod_{i\neq 1} f_i^{\alpha_i},$$

and that each component of τ' minus the identity is an element of the product of sufficiently many copies of $\mathfrak{m}$.

We consider the case when f is of the form $(*)$, and we want to modify f_2 in $(*)$ into an (x_{n-1}, x_n)-polynomial F_2 with coefficients in $\mathscr{E}(x_1, \ldots, x_{n-2})$. We need the following facts. The integrality of $\bar{x}_{n-1}$ and $\bar{x}_n$ in $\mathscr{E}_n/\mathfrak{p}$ over $\mathscr{E}(x_1, \ldots, x_{n-2})$ is preserved by a small change of the coordinate system. If we define $\mathfrak{p}^*$ by the factorization $(*)$ in the same way as $\mathfrak{p}$, we have

$$\mathfrak{p} \supset \mathfrak{p}^{*n}, \qquad \mathfrak{p}^* \supset \mathfrak{p}^n \bmod \bigcap_{\ell=1}^{\infty} \mathfrak{m}^\ell$$

by Lemma (1.4). These inclusions imply that $\bar{x}_{n-1}$ and $\bar{x}_n$ in $\mathscr{E}_n/\mathfrak{p}^*$ are integral over $\mathscr{E}(x_1, \ldots, x_{n-2})$. Hence we can see in the same way as above that f is equivalent to $F_1^{\alpha_1} F_2^{\alpha_2} \prod_{i \neq 1,2} f_i^{\alpha_i}$. Repeating the process, we obtain (x_{n-1}, x_n)-polynomials $F_1, \ldots, F_n$ with coefficients in $\mathscr{E}(x_1, \ldots, x_{n-2})$ such that f and $\prod_{i=1}^{k} F_i^{\alpha_i}$ are equivalent. Thus the theorem is proved.

Lemma **(2.3)**. — *Let* A *be one of the rings* $\mathscr{E}_n$, $\mathscr{F}_n$ *and* $\mathcal{O}_n$. *Let* f *and* g *be elements of* $\mathfrak{m}(A)$ *such that*

$$g \in \mathfrak{m}(A)^2 [I_f : f]^2.$$

Then $f + gf$ *and* f *are equivalent.*

Proof. — We consider the $\mathscr{E}_n$ case. The other cases can be proved in the same way. By Lemma (1.1), it is enough to find germs $a_i(x, t)$, for $i = 1, \ldots, n$, which satisfy the following conditions:

$$gf = \sum_{i=1}^{n} a_i(x, t) \left(\frac{\partial gf}{\partial x_i} t + \frac{\partial f}{\partial x_i} \right) \qquad (*)$$

$$a_i(0, t) = 0 \qquad (**)$$

By hypothesis there exist a finite number of elements b_{ij}, c_i and d_j in $\mathscr{E}_n$ such that

$$c_i, d_j \in [I_f : f], \qquad b_{ij} \in \mathfrak{m}^2 \qquad \text{and} \qquad g = \sum_{i,j} b_{ij} c_i d_j.$$

Hence $\dfrac{\partial g}{\partial x_i}$ are contained in $\mathfrak{m}[I_f : f]$, therefore there are elements α_i and β_{jk} in $\mathfrak{m}$, for $i, j, k = 1, \ldots, n$, such that

$$gf = \sum_{i=1}^{n} \alpha_i \frac{\partial f}{\partial x_i}, \qquad \frac{\partial g}{\partial x_i} f = \sum_{j=1}^{n} \beta_{ij} \frac{\partial f}{\partial x_j}.$$

Substituting in $(*)$, we have:

$$\sum_{i=1}^{n} \alpha_i \frac{\partial f}{\partial x_i} = \sum_{i=1}^{n} a_i \left[\sum_{j=1}^{n} \beta_{ij} \frac{\partial f}{\partial x_j} t + (1 + gt) \frac{\partial f}{\partial x_i} \right] = \sum_{i=1}^{n} \left[\sum_{j=1}^{n} a_j \beta_{ji} t + a_i (1 + gt) \right] \frac{\partial f}{\partial x_i}.$$

Hence it is sufficient to solve (in a_i) the equations

$$\alpha_i = \sum_{j=1}^{n} a_j \beta_{ji} t + a_i (1 + gt).$$

Let B be the matrix whose (i,j)-component is $\beta_{ji}t$ and let I be the unit matrix. Then the equations above become

$$\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix} = (\mathrm{B} + (1+gt)\,\mathrm{I})\begin{pmatrix}a_1\\ \vdots \\ a_n\end{pmatrix}.$$

Since β_{ij}, $g \in \mathfrak{m}$, the determinant of $\mathrm{B}+(1+gt)\,\mathrm{I}$ does not vanish at any point of $0\times[0,1]$. Hence there is an inverse matrix C of $\mathrm{B}+(1+gt)\,\mathrm{I}$ whose components are germs at $0\times[0,1]$ of C^∞ functions. Therefore we have the solution

$$\begin{pmatrix}a_1\\ \vdots \\ a_n\end{pmatrix} = \mathrm{C}\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix}.$$

Since $\alpha_i(0)=0$, all a_i satisfy $(**)$. The lemma is proved.

We can prove just in the same way as above the next corollary which is a generalization of results in Levinson [10] and [11]. We omit the proof.

Corollary **(2.4)**. — *Any element of $\mathcal{O}_n$ (resp. of $\mathcal{F}_n$) is* C^ω *equivalent (resp. equivalent) to a polynomial in two variables with coefficients in $\mathcal{O}_{n-2}$ (resp. $\mathcal{F}_{n-2}$).*

Remark **(2.5)**. — The corollary above is valid for the complex case. We omit the proof which is entirely similar to that of the theorem.

The proof of the theorem shows:

Corollary **(2.6)**. — *Let $f_1, \ldots, f_m$ be elements of $\mathcal{E}_n$ such that $\prod_{i=1}^{m} f_i$ is factorizable. Then there exist $g_1, \ldots, g_m$ in $\mathcal{E}_n$ and τ, a local* C^∞ *diffeomorphism, such that $g_i(0)\neq 0$ for $i=1, \ldots, m$ and that all $g_i \times f_i \circ \tau$ are (x_{n-1}, x_n)-polynomials with coefficients in $\mathcal{E}(x_1, \ldots, x_{n-2})$.*

The same result holds in the $\mathcal{O}_n$ or $\mathcal{F}_n$ case.

Since $\mathcal{O}_n$ and $\mathcal{F}_n$ are Noetherian rings, we have the immediate

Corollary **(2.7)**. — *Let $\mathfrak{p}$ be an ideal of $\mathcal{O}_n$ or $\mathcal{F}_n$. Then $\mathfrak{p}$ is generated by 2-polynomials with coefficients in $\mathcal{O}_{n-2}$ or $\mathcal{F}_{n-2}$ in a suitable coordinate system.*

We remark that, by the normalization theorem and Nakayama's lemma, if $\operatorname{ht}\mathfrak{p}=h$, then $\mathfrak{p}$ is generated by h-polynomials with coefficients in $\mathcal{O}_{n-h}$ or $\mathcal{F}_{n-h}$ in a suitable coordinate system.

The condition " factorizable " in the theorem is also a necessary condition for $n=2$. We shall consider that case in detail in the next section.

Problem. — Let an element f of $\mathcal{O}_3$ (or $\mathcal{F}_3$) be prime. Then, is f equivalent to a polynomial?

Without the primeness condition, the problem has a negative answer. See Whitney's example [34]. The author has given an example of a formal power series in 3-variables which is not equivalent to any convergent power series [27].

3. Differentiable germs of functions in two variables

In this section we study some properties of elements of $\mathscr{E}_2$. We saw the following already in the previous section.

(3.1) Let f be an element of $\mathfrak{m}(\mathcal{O}_2)$ or $\mathfrak{m}(\mathscr{F}_2)$. Then there exists a positive integer k such that for any element g of $\mathfrak{m}^k(\mathcal{O}_2)$ (resp. $\mathfrak{m}^k(\mathscr{F}_2)$), $f+gf$ is C^ω equivalent (resp. equivalent) to f.

(3.2) Any element of $\mathcal{O}_2$ (resp. $\mathscr{F}_2$) is C^ω equivalent (resp. equivalent) to a polynomial.

There are non-flat and non-factorizable elements in $\mathscr{E}_2$, for example

$$x^2+\exp(-1/y^2).$$

But any non-flat element is " *semi-factorizable* ":

Proposition **(3.3)**. — *Let f be a non-flat element of $\mathscr{E}_2$. Then there exist a finite number of elements $g_1, \ldots, g_n$ of $\mathscr{E}_2$ such that $f=\prod_{i=1}^{n} g_i$, that any two of $\mathrm{T}g_i$ are relatively prime and that, for each i, $\mathrm{T}g_i$ is a power of a prime formal power series. Moreover this " semi-factorization " is unique, that is, if $f=\prod g_i'$ is another semi-factorization, then for each i there exist a unique j and an invertible element α_i of $\mathscr{E}_2$ such that $g_i=\alpha_i g_j'$.*

Proof. — The first half is proved in Risler [23]. We prove the uniqueness. Let $\prod g_i'$ be another semi-factorization. Let

$$\mathrm{T}f=\prod_{i=1}^{n} f_i^{\alpha_i}$$

be a unique factorization into prime elements of $\mathscr{F}_2$. We assume that each f_i is convergent and that $f_i^{\alpha_i}=\mathrm{T}g_i=\mathrm{T}g_i'$. We only have to prove that g_1'/g_1 is well defined as an element of $\mathscr{E}_2$. If $f_1^{-1}(0)=\{0\}$, then there exists a positive integer α such that

$$|f_1(x)|\geq|x|^\alpha \quad \text{hence} \quad |g_1(x)|\geq|x|^{\alpha+1}, \quad |g_1'(x)|\geq|x|^{\alpha+1}.$$

Since $g_1'(x)-g_1(x)$ is flat, $g_1'/g_1=1+(g_1'-g_1)/g_1$ is defined as an element of $\mathscr{E}_2$. If $\mathrm{X}=f_1^{-1}(0)$ is not $\{0\}$, there exists an open neighborhood Y of $\mathrm{X}-\{0\}$ in which $\prod_{i\neq 1} f_i$ does not vanish and whose boundary is a union of analytic sets. The existence is showed as follows. We consider the zero set of $\prod_{i\neq 1}(f_1^2-f_i^2)$; it separates $\mathrm{X}-\{0\}$ from $\bigcup_{i\neq 1} f_i^{-1}(0)-\{0\}$ and we take as Y a union of connected components of the complement of that set. Let Z be the complement of $Y \cup \{0\}$. By the inequality of Łojasiewicz, there exists a constant $\alpha > 0$ such that

$$|f_1(x)| \geq \text{dist}\,(x, X)^\alpha.$$

Since any two analytic sets are regularly situated (see [12]), we have

$$\text{dist}\,(x, X) \geq |x|^\beta \text{ for } x \text{ in } Z,$$

for some constant β. Hence there exists a constant $\gamma > 0$ such that

$$|f_1(x)| \geq |x|^\gamma \text{ for } x \text{ in } Z.$$

We easily see that g_1'/g_1 is well defined on Z as a C^∞ function, that it can be extended to the closure of Z and that its Taylor expansion at the origin is 1. As

$$g_1'/g_1 = \prod_{i \neq 1} g_i / \prod_{i \neq 1} g_i',$$

we have a definition of g_1'/g_1 on Y, extending to $\overline{Y}$, whose Taylor expansion at the origin is 1. We can glue together the definitions of g_1'/g_1 on $\overline{Z}$ and $\overline{Y}$ in view of the regular situation of $\overline{Z} \cap \overline{Y}$ [12]. Thus the proposition is proved.

The proposition above is not true in higher dimensions, and, on the other hand, irreducible non-flat elements of $\mathscr{E}_2$ are not always prime.

Example 1. — $f(x, y, z) = xy + \exp(-1/z^2)$ is not semi-factorizable.

Example 2. — We put

$$f(y) = \exp(-1/y^2)\sin^2 1/y,$$

$$g(y) = \exp(-1/y^2)\cos^2 1/y,$$

$$h_1(y) = \begin{cases} f(y) & \text{if } y \geq 0 \\ g(y) & \text{if } y \leq 0, \end{cases}$$

$$h_2(y) = \begin{cases} g(y) & \text{if } y \geq 0 \\ f(y) & \text{if } y \leq 0. \end{cases}$$

Then $$(x^2 + f(y))(x^2 + g(y)) = (x^2 + h_1(y))(x^2 + h_2(y)),$$

two different factorizations into irreducible elements of $\mathscr{E}_2$.

It may be possible to generalize the uniqueness in the proposition above for higher dimensions:

Problem. — Let f, f_i, g_i, with $i = 1, 2$, be elements of $\mathscr{E}_n$ such that $f = f_1 g_1 = f_2 g_2$, and that Tf_i and Tg_i are relatively prime for each i. Then, do there exist h_i, $i = 1, \ldots, 4$ in $\mathscr{E}_n$ such that

$$f = \prod_{i=1}^{4} h_i, \quad f_1 = h_1 h_2, \quad f_2 = h_1 h_3,$$

$$g_1 = h_3 h_4, \quad g_2 = h_2 h_4?$$

We generalize the theorem (2.1) in the two dimensional case.

Theorem **(3.4)**. — *Let* $f \neq 0$ *be an element of* $\mathfrak{m}(\mathscr{E}_2)$. *The following conditions are equivalent*:

1) f *is equivalent to a polynomial*;

2) f *is equivalent to an element of* $\mathcal{O}_2$;

3) f *is* C^r *equivalent to an element of* $\mathcal{O}_2$ *for* $0 < r < \infty$;

4) f *is factorizable*;

5) $f^{-1}(0)$ *is the union of the images of a finite number of* C^∞ *mappings from* $[0, 1]$ *to* $\mathbf{R}^2$ *such that the images do not intersect each other except at the origin, and the* $F_i = \sum_{|\beta| \leq i} |D^\beta f|$ *satisfy the Łojasiewicz' inequality for* $i = 0, 1, \ldots$.

Proof. — We already proved $4) \Leftrightarrow 1)$. It is trivial that $1) \Rightarrow 2)$ and $2) \Rightarrow 3)$. The implication $2) \Rightarrow 5)$ is easy, and we omit it. We prove the implications $3) \Rightarrow 4)$ and $5) \Rightarrow 4)$.

Proof of $3) \Rightarrow 4)$. — We can assume that $r = 1$. Let τ be a C^1 local diffeomorphism around 0 such that $g = f \circ \tau$ is a convergent power series. Let $\prod_{i=1}^{m} g_i^{\alpha_i}$ be the unique factorization of g into prime elements of $\mathcal{O}_2$. Let $\prod_{i=1}^{p} f_i$ be a semi-factorization of f, after Proposition (3.3). We only have to prove that each f_i is factorizable. We remark that if an element χ of $\mathfrak{m}(\mathscr{E}_n)$ satisfies

$$|\chi(x)| \geq |x|^\alpha \text{ for some constant } \alpha, \qquad (*)$$

then χ is factorizable. The reason is the following. From the inequality $(*)$, χ is C^0 sufficient (see for example [2]). Hence χ is equivalent to any realization of $T\chi$, especially to a realization which is factorizable. We assume that f_i, g_j satisfy $(*)$ for $i \geq k+1$, $j \geq \ell + 1$, and not for $i \leqslant k$, $j \leq \ell$. We assume that each Tf_i is the α'_i power of a prime convergent power series h_i. We have

$$f^{-1}(0) = \tau(g^{-1}(0)) = \tau\Big(\bigcup_{i=1}^{m} g_i^{-1}(0)\Big) = \bigcup_{i=1}^{m} \tau(g_i^{-1}(0)).$$

Since any two $g_i^{-1}(0)$ are regularly situated, $f^{-1}(0)$ consists of mutually regularly situated curves. We see also that any two $f_i^{-1}(0)$ are regularly situated. This means that for each $i \leq \ell$ there exists a unique $j \leq k$ such that $f_j^{-1}(0) = \tau(g_i^{-1}(0))$. We can assume that $j = i$. We see that $\ell = k$ as follows. If $\ell < k$, $f(x)$ converges to zero with infinite order when $x \to 0$ along the curve $h_k^{-1}(0)$. Hence $g(x)$ converges to zero with infinite order when $x \to 0$ along the curve $(h_k \circ \tau)^{-1}(0)$. But this curve is regularly situated with respect to $g^{-1}(0)$. This contradicts the second property of 5) in the theorem. (It is sufficient that the curve is of class C^1.) Hence we only have to prove that f_1 is factorizable. Let U be a small neighborhood of $g_1^{-1}(0) - \{0\}$. Put

$$\rho(x) = g(x) / \operatorname{dist}(x, g_1^{-1}(0))^{\alpha_1}$$

on $U-g_1^{-1}(0)$. Then $\rho(x)$ can be extended to the analytic curve $g_1^{-1}(0)-\{0\}$, and the restriction of the function to the curve does not take the value zero and converges to zero with finite order when $x\to 0$. This follows from the Łojasiewicz's inequality if we consider the case where $g=g_1$ and $\alpha_1=1$. Put

$$\rho'(x)=g\circ\tau^{-1}(x)/\mathrm{dist}(x,f_1^{-1}(0))^{\alpha_1}$$

on $\tau(U)-f^{-1}(0)$. Then we have

$$K\ \mathrm{dist}(\tau^{-1}(x),g^{-1}(0))\leq \mathrm{dist}(x,f_1^{-1}(0))\leq K'\ \mathrm{dist}(\tau^{-1}(x),g_1^{-1}(0))$$

for $x\in\tau(U)-f_1^{-1}(0)$ here K and K' are positive constants. Hence

$$L\rho\circ\tau^{-1}(x)\leq\rho'(x)\leq L'\rho\circ\tau^{-1}(x)$$

for $x\in\tau(U)-f_1^{-1}(0)$, where L, L' are positive constants. Therefore ρ' has the same property as ρ. We see easily that

$$|\rho'(x)|\leq c\sum_{|\beta|=\alpha_1}|D^\beta f(x)| \text{ for some constant } c \qquad (**)$$

Since $f_1-h_1^{\alpha'_1}$ is flat, $h_1^{\alpha'_1}$ converges to zero with an infinite order when $x\to 0$ along the curve $f_1^{-1}(0)-\{0\}$, hence so does h_1. This means that $h_1^{-1}(0)\neq\{0\}$. Moreover it follows that the curves $f_1^{-1}(0)$ and $h_1^{-1}(0)$ have a contact of infinite order at the origin. Now we show that $\alpha_1=\alpha'_1$. If $\alpha_1<\alpha'_1$, then for any β such that $|\beta|=\alpha_1$, $D^\beta h_1^{\alpha'_1}(x)$ converges to zero with infinite order when $x\to 0$ along the curve $f_1^{-1}(0)-\{0\}$, hence so do $D^\beta f_1$ and ρ' for $|\beta|=\alpha_1$ by (**). This contradicts the property of ρ'. If $\alpha_1>\alpha'_1$, we have

$$\{x\mid D^\gamma f_1(x)=0 \text{ for all } |\gamma|\leq\alpha'_1\}\supset f_1^{-1}(0).$$

On the other hand the ideal of $\mathcal{O}_2$ generated by $TD^\gamma f_1$ for all $|\gamma|\leq\alpha'_1$ is of height 2. Hence the set of common zeroes of $D^\gamma f_1$ $|\gamma|\leq\alpha'_1$ is $\{0\}$. This is a contradiction. Thus we proved that $\alpha_1=\alpha'_1$. Let $\mathfrak{p}$ be the ideal of $\mathscr{F}_2$ generated by all $TD^\gamma f_1$ with $|\gamma|\leq\alpha'_1$. Then we see that the radical of $[\mathfrak{p}:h_1]$ is $\mathfrak{m}_2$. Hence there exist elements χ_γ in $\mathscr{E}_2$ with $|\gamma|<\alpha'_1$ such that

$$T\Big(\sum_{|\gamma|<\alpha_1}\chi_\gamma D^\gamma f_1\Big)=h_1(x_1^{2N}+x_2^{2N})$$

where $x=(x_1,x_2)$ and N is sufficiently large. We put

$$\varphi=\sum_{|\gamma|<\alpha_1}\chi_\gamma D^\gamma f_1.$$

By Proposition (3.3) there exist elements φ_1 and φ_2 in $\mathscr{E}_2$ such that

$$\varphi=\varphi_1\varphi_2,\quad T\varphi_1=h_1 \quad\text{and}\quad T\varphi_2=x_1^{2N}+x_2^{2N}.$$

Since $T\varphi_1$ is prime, φ_1 is equivalent to a convergent power series. Obviously, for any point a near 0, T_af_1 is divisible by $T_a\varphi_1$ where T_a denotes the Taylor expansion at a.

Hence, in view of a theorem in Malgrange [12], f_1 is divisible by φ_1. Repeating this argument, we see that there exist elements $\varphi_{11}, \ldots, \varphi_{1\alpha_1}$ such that

$$f_1 = \prod_{i=1}^{\alpha_1} \varphi_{1i}, \qquad T\varphi_{1i} = h_1 \text{ for all } i,$$

$$\varphi_{1i}^{-1}(0) = \varphi_{1j}^{-1}(0) \text{ for all } i, j.$$

Applying once more the theorem of Malgrange, we may assume that $\varphi_{11} = \ldots = \varphi_{1\alpha_1}$. Hence f_1 is factorizable. Thus we have proved $3) \Rightarrow 4)$.

Proof of $5) \Rightarrow 4)$. — Since f satisfies the Łojasiewicz' inequality, f is not flat [3] Let $\prod_{i=1}^{p} g_i$ be the semi-factorization of f in Proposition (3.3). Let Tg_i be the α_i-th power of a prime formal power series h_i, which we may assume to be convergent. Then we only have to prove that g_1 is factorizable. From the proof above we can assume $h_1^{-1}(0) \neq \{0\}$. By the remark below, $h_1^{-1}(0) - \{0\}$ is an analytic curve with two connected components A_1 and A_2, and the theorem of Bruhat-Whitney shows that the closure of each component is the image of an analytic mapping defined on $[0, 1]$. Then the composition of the analytic mapping with f is flat. Let

$$\varphi_i : [0, 1] \to \mathbf{R}^2 \quad \text{for} \quad i = 1, \ldots, n (\geqslant 2)$$

be the mappings in condition 5) such that $\varphi_i(0) = 0$ and each element of $\{\text{image } \varphi_i\}$ is not regularly situated with respect to $h_1^{-1}(0)$. We will see that $n = 2$. Assume that the elements of $\{\text{image } \varphi_i\}_{i=1,\ldots,n'}$ are not regularly situated with respect to A_1. We choose a coordinate system (x_1, x_2) so that A_1 is tangent to the positive x_1 axis. Then, for sufficiently small $t > 0$, there exist real numbers a_i for $i = 1, \ldots, n'$ such that $s_i = (t, a_i) \in \text{image } \varphi_i$. Consider the restriction of f to the segment jointing s_i, then the q_i flatness of f at s_i for $i = 1, \ldots, n'$ implies the existence of a point b on the segment where the $\sum_{i=1}^{n'} (q_i + 1) - 1$-th derivative of the restriction is zero. This means that $\Sigma(q_i + 1) \leqslant \alpha_1$. By the way, from the assumption, it follows that $\dfrac{\partial^{\alpha_1} f}{\partial x_2^{\alpha_1}}$ never vanishes on any sequence of points which has a contact with $h_1^{-1}(0) - \{0\}$. Hence $F_{\alpha_1 - 1} \circ \varphi_i$ (for $i = 1, \ldots, n'$) does not vanish except at 0, if $n' \geqslant 2$. But those functions are flat at 0, and any sequence of zeroes of $F_{\alpha_1 - 1}$ which has a contact of infinite order with A_1 is contained in $\bigcup \text{image } \varphi_i$. This contradicts the assumption that $F_{\alpha_1 - 1}$ satisfies Łojasiewicz' inequality. Thus we proved that $n' = 1$, hence $n = 2$. This argument also shows that, for small $t > 0$, the line $\{x_1 = t\}$ intersects the image of φ_1 at only a point. Moreover we see that $\dfrac{\partial g_1}{\partial x_1^{\alpha_1 - 1}}$ vanishes and is regular on the image of φ_1. Now, since the $h_i \circ \varphi_j$ are not flat at 0 for $i = 2, 3, \ldots$ and $j = 1, 2$, neither are the $g_i \circ \varphi_j$. Hence $g_1 \circ \varphi_i = 0$ for $i = 1, 2$.

We apply the proposition (3.3) to $\frac{\partial g_1}{\partial x_1^{\alpha_1-1}}$. Then we have the semi-factorization $\prod_{i=1}^{p'} g'_i$ of the function. Here we can assume $Tg'_1 = h_1$ and $g_1'^{-1}(0) = g_1^{-1}(0)$. Hence the image of φ_1 is transformed to a semi-analytic set by a C^∞ local diffeomorphism around 0. Choosing a suitable coordinate system, we can assume that

$$\varphi_1(t) = (t^n, a(t)) = (x_1, x_2)$$

where $a(t)$ is C^∞ on $[0, 1]$. Put

$$H_1 = h_1 \circ \varphi_1 \quad \text{and} \quad \psi_1(x_1, x_2) = \begin{cases} H_1(x_1^{1/n}) & \text{if} \quad x_1 \geq 0 \\ 0 & \text{if} \quad x_1 < 0. \end{cases}$$

Then, since H_1 is flat, ψ_1 is a flat element of $\mathscr{E}_2$. Let $\bar{\psi}_1$ be the Whitney function defined on the image of φ_1, which is the restriction of ψ_1 (see [12]). In the same way we construct a Whitney function $\bar{\psi}_2$ on the image of φ_2. Because the images are regularly situated, we can extend $\bar{\psi}_1$ and $\bar{\psi}_2$ to the set $g_1^{-1}(0)$ and hence to a neighborhood of the origin. Let ψ be the extended germ. Then ψ is flat and satisfies

$$\psi \circ \varphi_i = h_1 \circ \varphi_i \quad \text{for} \quad i = 1, 2.$$

Put $h'_1 = h_1 - \psi$. Then h'_1 is equivalent to a convergent power series and satisfies $h_1'^{-1}(0) = g_1^{-1}(0)$. Applying Malgrange's theorem in the same way as in the proof of $3) \Rightarrow 4)$, we prove that g_1 is divisible by h'_1 and even by $h_1'^{\alpha_1}$ from the assumption. Thus we have proved that $5) \Rightarrow 4)$.

We used in the proof above the case $n=2$ of the next remark.

Remark **(3.5)**. — *Let* X *be the zero set of a prime ideal of* $\mathcal{O}_n$ *of height* $n-1$. *Then* $X - \{0\}$ *is empty or has two connected components.*

Proof. — We will only prove the case $n=2$, the other cases being proved in just the same way. Let f be a prime element of $\mathfrak{m}(\mathcal{O}_2)$. Then we want to show that $f^{-1}(0) - \{0\}$ is empty or has two connected components. We assume that $f^{-1}(0) \neq \{0\}$. Let X_1 be the closure of a connected component of $f^{-1}(0) - \{0\}$. Then, by the theorem of Bruhat-Whitney, there exists an analytic mapping φ from $[0, 1]$ to $\mathbf{R}^2$ such that the image is X_1, $\varphi(0) = 0$, the set of singular points of φ is $\{0\}$ and φ is injective. Let $\varphi(t) = (\varphi_1(t), \varphi_2(t))$. We may suppose through a change of the coordinate system that

$$\varphi_1(t) = t^k, \quad \varphi_2(t) = \sum_{\alpha > k} a_\alpha t^\alpha, \quad a_\alpha \in \mathbf{R},$$

where the greatest common divisor of all the numbers in $A = \{\alpha \in \mathbf{Z} \mid a_\alpha \neq 0\}$ and of k is 1. Then we can extend analytically φ to $[-\varepsilon, 1]$ with $\varepsilon > 0$ so that the extension is injective and that its set of singular points is $\{0\}$. We use the same notation φ for the extension. We may assume $\varepsilon = 1$. It is trivial that $f^{-1}(0) \supset \varphi([-1, 1])$. Let $\varphi_{\mathbf{C}}$ be the complexification of φ defined on a small open neighborhood U of $[-1, 1]$ in $\mathbf{C}$. We see easily that $\varphi_{\mathbf{C}}$ is injective and its set of singuler points is $\{0\}$. Hence the image of $U-\{0\}$ by $\varphi_{\mathbf{C}}$ is a one-dimensional complex manifold and is contained in a complexification of $f^{-1}(0)$. By a theorem on p. 101 of Hervé [4], $\varphi_{\mathbf{C}}(U)$ is an analytic set in an open set of $\mathbf{C}^2$. Hence the intersection of the image of $\varphi_{\mathbf{C}}$ and $\mathbf{R}^2$ is an analytic set in an open set of $\mathbf{R}^2$. It is sufficient to prove that

$$\text{image } \varphi_{\mathbf{C}} \cap \mathbf{R}^2 = \text{image } \varphi \qquad (*)$$

Let $t \neq 0$ be a point of U such that $\varphi_{\mathbf{C}}(t) \in \mathbf{R}^2$. Since $\varphi_{1\mathbf{C}}(t) = t^k$, we see that

$$t = s \exp \ell\pi i/k \quad \text{with} \quad \ell = 0, \ldots, 2k-1, \quad s \text{ real.}$$

Then the set of points of U whose images by $\varphi_{2\mathbf{C}}$ are real is a real analytic set in U of dimension one if we regard $\mathbf{C}$ as $\mathbf{R}^2$, and $\varphi_{2\mathbf{C}}$ as a real analytic mapping from U to $\mathbf{R}^2$. Hence the set of points of U whose images by $\varphi_{\mathbf{C}}$ are real is a real analytic set of dimension one and is a sum of lines $\{s \exp \ell\pi i/k \mid s \text{ real}\}$ for $\ell = 0, \ldots, 2k-1$. Let $\{s \exp \ell\pi i/k\}$ be such a line. Then

$$\varphi_{\mathbf{C}}(s \exp \ell\pi i/k) = \sum_{\alpha \in A} a_\alpha (s \exp \ell\pi i/k)^\alpha$$

is real for any real s near 0. Let α_0 be the smallest element of A. Then $a_{\alpha_0}(s \exp \ell\pi i/k)^{\alpha_0}$ is real for real s. The reason is the following. If it were not so, the numbers

$$\sum_{\alpha > \alpha_0} a_\alpha (\exp \ell\pi i/k)^\alpha s^{\alpha - \alpha_0} \quad \text{and} \quad a_{\alpha_0}(\exp \ell\pi i/k)^{\alpha_0}$$

would not be real for some real s arbitrary close to 0. But their sum is real, the first converges to zero when $s \to 0$ and the second is a constant. This is a contradiction. In the same way we see that the $a_\alpha(s \exp \ell\pi i/k)^\alpha$ for $\alpha = k+1, \ldots$ are real. This means that $\ell\alpha \equiv 0 \bmod k$ for $\alpha \in A$. Because the greatest common divisor of A and k is 1, ℓ is equal to zero or k. Hence we have $(*)$ which proves the remark.

4. Other results about differentiable germs of functions

The author showed in [25] a sort of inverse of Lemma (2.3). But the proof was faulty, so we again prove it here, in a generalized form.

Proposition **(4.1)**. — *Let f be an element of $\mathfrak{m}(\mathcal{O}_n)$. The ideal $[I_f : f]$ contains $\mathfrak{m}^q(\mathcal{O}_n)$ for $q \geq 0$ if and only if there exists $r \geq 0$ such that for any g in $\mathfrak{m}^r(\mathcal{O}_n)$, $fg + f$ is C^ω equivalent to f. Here we assume $g(0) > -1$ if $r = 0$, and the correspondences $q \mapsto r$ and $r \mapsto q$ are given by $r = q+2$, $q = r$ respectively. The same result holds for $\mathcal{F}_n$.*

Proof. — " Only if " follows from the proof of Lemma (2.3). For " if ", we detail the case $r = 0$ as follows. For the other cases, the proof is the same, and we omit it.

" If tf is C^ω equivalent to f for any constant $t > 0$, then f is contained in $\mathfrak{m}(\mathcal{O}_n) I_f$ ".

Let a natural number k be fixed. Let Adiff_n be the set of local analytic diffeomorphisms around o in $\mathbf{R}^n$. Let Adiff_n^k denote the set of k-jets of elements of Adiff_n. Then Adiff_n^k is contained in a Euclidean space. We put

$$\mathrm{A}=\{(t,\tau)\in\mathbf{R}\times\mathrm{Adiff}_n^k \mid tf\equiv f\circ\tau \bmod \mathfrak{m}^{k+1}(\mathcal{O}_n)\},$$
$$\mathrm{B}=\{\tau\in\mathrm{Adiff}_n^k \mid \exists t\in\mathbf{R} \text{ s.t. } tf=f\circ\tau \bmod \mathfrak{m}^{k+1}(\mathcal{O}_n)\},$$
$$\mathrm{C}=\{\tau\in\mathrm{Adiff}_n^k \mid f=f\circ\tau \bmod \mathfrak{m}^{k+1}(\mathcal{O}_n)\}.$$

Since A and C are algebraic sets and groups, they are Lie groups. Because B is a projection of A, it is a semi-algebraic set. Hence B is also a Lie group, and C is a Lie subgroup of B. From the assumption, there exists τ_t in Adiff_n for any $t\in(0,1]$, such that $tf=f\circ\tau_t$. The cardinality of the set of τ_t whose Jacobians are greater that $1/\mathrm{N}$ at the origin for a sufficiently large N is the continuum. Hence there exists a sequence $t_0, t_1, \dots$ converging to s in $(0,1]$ such that $\{\tau_{t_i}\}$ converges to τ_s in Adiff_n^k. Take subsequences of $\{t_i/t_j\}$ and $\{\tau_{t_i}\circ\tau_{t_j}^{-1}\}$ if necessary, and assume that $s=1$ and $\tau_s=$ identity. This shows that C is of codimension one in B. Consider an element of Adiff_n^k as n polynomials. Then there exists an analytic mapping $\varphi(t,x)=(\varphi_1,\dots,\varphi_n)$ from $(-\varepsilon,\varepsilon)\times\mathbf{R}^n$ to $\mathbf{R}^n$ with $\varepsilon>0$ such that for each $t\in(-\varepsilon,\varepsilon)$

$$(1+t)f(x)=f\circ\varphi(t,x) \bmod \mathfrak{m}^{k+1}(\mathcal{O}_n),$$
$$\varphi(t,x)\in\mathrm{Adiff}_n^k, \quad \text{and} \quad \varphi(0,x)=\text{identity}.$$

We put $\mathrm{F}(t,x)=f\circ\varphi(t,x)$. Then there exists $g(t,x)\in\mathcal{O}(t,x)$ such that

$$\mathrm{F}(t,x)=\mathrm{F}(0,x)+t\frac{\partial\mathrm{F}}{\partial t}(0,x)+t^2g(t,x).$$

Hence we have for each $t\in[-\varepsilon,\varepsilon]$

$$f(x)\equiv(f\circ\varphi(t,x)-f\circ\varphi(0,x))/t \bmod \mathfrak{m}^{k+1}(\mathcal{O}_n)$$
$$=\sum_{i=1}^n\frac{\partial f}{\partial x_i}(x)\,\frac{\partial\varphi_i}{\partial t}(0,x)+tg(t,x).$$

This implies

$$f(x)\equiv\sum_{i=1}^n\frac{\partial f}{\partial x_i}(x)\,\frac{\partial\varphi_i}{\partial t}(0,x) \bmod \mathfrak{m}^{k+1}(\mathcal{O}_n).$$

The relation $\varphi(t,0)\equiv 0$ means that $\frac{\partial\varphi_i}{\partial t}(0,x)\in\mathfrak{m}(\mathcal{O})$. Hence, from the closedness of any ideal of $\mathcal{O}_n$ in the Krull topology (Krull), we deduce $f\in\mathfrak{m}(\mathcal{O}_n)\mathrm{I}_f$.

Here we used a consequence of the theorem of Zariski-Nagata, namely the fact that an element of $\mathcal{O}_n$ is prime in $\mathcal{O}_n$ if and only if it is prime in $\mathcal{F}_n$. This is not generally true in the $\mathcal{E}_n$ case. But

Proposition **(4.2)**. — *Let f be an element of $\mathfrak{m}(\mathcal{O}_n)$. Then a factorization of f in irreducible elements of $\mathcal{E}_n$ is the unique factorization in prime elements of $\mathcal{O}_n$.*

We shall generalize this to the global situation in the next chapter.

Proof. — We need a result of Tougeron [31]. Let $g_1, \ldots, g_k$ be C^∞ functions defined in the neighborhood of $0 \in \mathbf{R}^n$. Let $J_a(g_1, \ldots, g_k)$ denote the ideal of $\mathscr{F}_n$ generated by $T_a g_1, \ldots, T_a g_k$ where T_a is the Taylor expansion at a. Then the function $\operatorname{ht} J_a(g_1, \ldots, g_k)$ in a is lower semi-continuous. We only have to prove the following:

Let g_1 and g_2 be elements of $\mathscr{E}_n$ such that $g_1 g_2 = f$: then there exist elements φ_1 and φ_2 in $\mathscr{E}_n$ such that $\varphi_1 g_1$ and $\varphi_2 g_2$ are analytic and that $\varphi_1 \varphi_2 = 1$.

Let $f = \prod_{i=1}^{k} f_i^{\alpha_i}$ be the unique factorization in prime elements of $\mathcal{O}_n$. We can assume that

$$Tg_1 = \prod_{i=1}^{k} f_i^{\alpha_i'}, \qquad Tg_2 = \prod_{i=1}^{k} f_i^{\alpha_i - \alpha_i'}.$$

We prove our assertion by induction on k. The case $k=0$ is trivial. We suppose the assertion true $k-1$. Let i satisfy $\alpha_i - \alpha_i' \geq \alpha_j - \alpha_j'$ for any j. Then

$$\operatorname{ht} J_0(\{D^\beta g_2\}_{|\beta| \leq \alpha_i - \alpha_i'}) \geq 2.$$

Hence by Tougeron's result

$$\operatorname{ht} J_a(\{D^\beta g_2\}_{|\beta| \leq \alpha_i - \alpha_i'}) \geq 2 \text{ for } a \text{ near } 0.$$

If $f_i(a) \neq 0$, obviously $T_a g_1$ is divisible by $T_a f_i^{\alpha_i'}$. If $f_i(a) = 0$, $T_a g_2$ is not divisible by $T_a f_i^{\alpha_i - \alpha_i' + 1}$. Hence $T_a g_1$ is divisible by $T_a f_i^{\alpha_i'}$ for any a. From a theorem of Malgrange on p. 82 of [12], g_1 is divisible by $f_i^{\alpha_i'}$. Hence we may assume that $\alpha_i' = 0$. Then

$$\operatorname{ht} J_0(g_1, f_i) \geq 2,$$

consequently

$$\operatorname{ht} J_a(g_1, f_i) \geq 2 \text{ for } a \text{ near } 0.$$

In the same way as above we see that g_2 is divisible by $f_i^{\alpha_i - \alpha_i'}$. By the induction assumption on k, this completes the proof.

We now have a result about the C^0 equivalence.

Theorem **(4.3)**. — *Let f, g be elements of $\mathfrak{m}(\mathcal{O}_n)$ such that f and g have the same sign at each point. Then f and g are C^0 equivalent.*

Proof. — We define vector fields near the origin by

$$X_f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) \quad \text{and} \quad X_g = \left(\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n}\right).$$

Then X_f and X_g are non-singular on $\{x \mid f(x) \neq 0\}$. Using them, we will construct a local homeomorphism τ. We first define τ except on $f^{-1}(0)$. Let A be the set of x where

$kX_f = \ell X_g$ for some $k \geq 0$ and $\ell \leq 0$ with $k^2 + \ell^2 > 0$. Then A is a semi-analytic set, contained in $f^{-1}(0)$. The reason is the following. We have

$$A = \left\{ x \,\middle|\, \sqrt{\left(\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2\right)\left(\sum_{i=1}^{n}\left(\frac{\partial g}{\partial x_i}\right)^2\right)} = -\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i} \text{ at } x \right\}.$$

Hence A is a semi-analytic set. Assume that A is not contained in $f^{-1}(0)$. Then, from the theorem of Bruhat-Whitney it follows that there exists an analytic curve in $A - f^{-1}(0)$ one of whose ends is the origin. By the assumption we have $fg > 0$ on the curve. We consider the restrictions of f and g to the curve. They are monotone. If one of them is monotone increasing, then the other is monotone decreasing. As they are zero at the origin, we see that $fg \leqslant 0$ on the curve. This is a contradiction. Hence A is contained in $f^{-1}(0)$. We put

$$E = \left\{ (x, t) \in \mathbf{R}^n \times \mathbf{R} \,\middle|\, \sqrt{\sum_{i=1}^{n}\left(\frac{\partial f(x)}{\partial x_i}\right)^2 \sum_{i=1}^{n}\left(\frac{\partial g(x)}{\partial x_i}\right)^2} = (t-1)\sum_{i=1}^{n}\frac{\partial f(x)}{\partial x_i}\frac{\partial g(x)}{\partial x_i} \right\}.$$

Then E is a semi-analytic set. Let $\eta(x)$ be the function on $\mathbf{R}^n$ such that the graph of $t = \eta(x)$ on $\mathbf{R}^n - f^{-1}(0)$ is $E \cap (\mathbf{R}^n - f^{-1}(0)) \times \mathbf{R}$, and that $\eta(x) = 0$ on $f^{-1}(0)$. Then, by a property of semi-analytic sets, η satisfies the Łojasiewicz' inequality. Moreover, if f is analytically parameterized, the inequality is satisfied on the product space of $\mathbf{R}^n$ and the domain of the parameter. This means that, also in the parameter case, $f^{-1}(0)$ contains the intersection of A with a neighborhood of $f^{-1}(0)$. Let $r > 0$ be a constant. We put

$$S_r = \{x \in \mathbf{R}^n \mid |x| = r\}, \quad f\big|_{S_r} = f_r \quad \text{and} \quad g\big|_{S_r} = g_r.$$

Let X_{f_r}, X_{g_r} be the tangent components of $X_f\big|_{S_r}$, $X_g\big|_{S_r}$ respectively. Then

$$X_{f_r} = X_f - \langle X_f, \overrightarrow{Ox} \rangle \frac{\overrightarrow{Ox}}{|x|^2}, \qquad X_{g_r} = X_g - \langle X_g, \overrightarrow{Ox} \rangle \frac{\overrightarrow{Ox}}{|x|^2}.$$

We define A_r by X_{f_r} and X_{g_r} in the same way as A. We consider that f_r, g_r are parameterized by $r > 0$. Then $f^{-1}(0) - \{0\}$ contains the intersection of $\bigcup_{r>0} A_r$ with a neighborhood of $f^{-1}(0) - \{0\}$. We put

$$A'_r = A_r - f^{-1}(0), \qquad B = \left(\bigcup_{r>0} A'_r\right) \cup \{0\}.$$

Then we see that B is a closed semi-analytic set. Let φ be a C^∞ function on $\mathbf{R}^n - \{0\}$ such that $0 \leqslant \varphi \leqslant 1$, the values in a neighborhood of $f^{-1}(0) - \{0\}$ are 0 and the values in a neighborhood of $B - \{0\}$ are 1. Let Y be a vector field on $\mathbf{R}^n - f^{-1}(0)$ defined by

$$Y = X_f / |X_f| + X_g / |X_g|.$$

Then Y has no singular points. There are vector fields Y_1 and Y_2 on $\mathbf{R}^n - f^{-1}(0)$ such that Y_1 is the $\overrightarrow{Ox}$ component of Y at each point x, and Y_2 is the orthogonal component.

We write $Y' = \varphi Y_1 + Y_2$. Then Y' has no singular points and satisfies $Y'f > 0$, $Y'g > 0$. Let a point x be sufficiently close to the origin and such that $f(x) \neq 0$. Then there exists a unique y on the integral curve of Y' passing through x such that $f(y) = g(x)$, and the mapping $\tau : x \to y$ is continuous. The reason is the following. Let $r > 0$ be fixed. The vector field Y' is tangent to S_r at any point of S_r in a neighborhood of $f^{-1}(0)$. Hence all integral curves passing through points of the interior of S_r never intersect with S_r at any point close to $f^{-1}(0)$. Let u be the minimum of $|f_r|$ outside of the neighborhood of $f^{-1}(0)$. Then we have the solution $y = \tau(x)$ for x such that $|x| < r$ and $|g(x)| < u$. Hence τ is well defined in a neighborhood of 0. The continuity is trivial. This shows also that τ can be extended to $(\mathbf{R}^n - f^{-1}(0)) \cup \{0\}$. We replaced Y by Y' in order to extend τ to the origin. For the extension to the whole space, we shall moreover modify Y.

Let $\{C_i; i = 1, \ldots, m\}$ be the stratification of $f^{-1}(0)$ in Mather [14]. That is, the C_i are analytic manifolds such that

$$\bigcup_{i=1}^{m} C_i = f^{-1}(0),$$

$$\bar{C}_i \supset C_j \quad \text{if} \quad \bar{C}_i \cap C_j \neq \varnothing,$$

and there exist a tubular neighborhood U_i of C_i and a diffeomorphism $\pi_i : C_i \times D_i \to U_i$ for each i, where D_i is a closed disc centered at 0 with radius 1, such that

1) $\pi_i \big|_{C_i \times 0}$ is the identity and

2) if $C_j \subset \bar{C}_i$, for any $x \in C_i \cap U_j$ we can choose uniquely $y \in C_j$ so that

$$\pi_i(x \times D_i) \subset \pi_j(y \times D_j).$$

We consider analytic approximations of the maps π_i, $i = 1, \ldots, m$ which satisfy the condition 1). We use the same notations π_i, U_i. Then they do not generally satisfy the condition 2). We shall modify π_i so that 2) is satisfied. Let C_i, C_j be strata such that $\bar{C}_i \supset C_j$. We define a modification for (C_i, C_j). Let p_i, q_i be the projections of $C_i \times D_i$ to the first component and to the second, respectively. Define

$$\rho_{it} : C_i \times D_i \to C_i \times D_i \quad \text{by} \quad \rho_{it}(x, \xi) = (x, t\xi)$$

for $0 \leq t \leq 1$. We put

$$\pi_{ijt}(z) = \pi_j(p_j \circ \pi_j^{-1} \circ \pi_i \circ \rho_{it} \circ \pi_i^{-1}(z),\ q_j \circ \pi_j^{-1}(z))$$

for $z \in U_i \cap V_j$, where $V_j = \pi_j(C_j \times \{|\xi| \leq 9/10\})$. Then, if the approximations are strong enough, π_{ijt} is well defined and is a diffeomorphism from $U_i \cap V_j$ into U_j. Let $\chi : \mathbf{R} \to \mathbf{R}$ be a C^∞ function such that

$$\chi(x) = \begin{cases} 0 & \text{if} \quad x \leq 1/3, \\ 1 & \text{if} \quad x \geq 2/3, \end{cases}$$

and $\chi(x)$ is analytic and monotone on $(1/3, 2/3)$. We put

$$\tau_{ij}(z)=\begin{cases}\tau_{ij\chi(|\rho_j\circ\pi_j^{-1}(z)|^2)}(z) & \text{if} \quad z\in U_i\cap V_j\\ \tau_i(z) & \text{if} \quad z\in U_i-V_j,\end{cases}$$

$$U_i'=\tau_{ij}(U_i), \qquad U_j'=\tau_j(A_j\times\{|\xi|\leq 1/2\}),$$

$$\pi_i'=\tau_{ij}\circ\pi_i, \qquad \text{and} \qquad \pi_j'=\pi_j\circ\rho_{j1/2}.$$

We assume the approximations sufficiently strong. Then

$$\pi_i': C_i\times D_i\rightarrow U_i' \qquad \text{and} \qquad \pi_j': C_j\times D_j\rightarrow U_j'$$

are diffeomorphisms, satisfy 1) and 2), and are analytic except on a semi-analytic subset of $C_i\times D_i$. Here the semi-analytic set is a sum of fibres of the projection p_i. Let C_i, C_j, C_k be strata such that $C_k\subset\overline{C}_i$, $C_k\subset\overline{C}_j$. Then the same modifications of π_i, π_j, π_k as above can be done at once for (C_k, C_i) and (C_k, C_j). Hence we first modify all the pairs (π_i, π_j) such that $\dim C_i=\dim f^{-1}(0)=n'$, $\dim C_j=n'-1$ and then all the pairs (π_k, π_ℓ) such that $\dim C_k=n'$ or $n'-1$, $\dim C_\ell=n'-2$. We proceed with the modifications in this way. Then we have diffeomorphism $\pi_i': C_i\times D_i\rightarrow U_i'$, for $i=1, \ldots, m$ which satisfy 1) and 2) where each π_i' is analytic except on a semi-analytic set. Here the semi-analytic set is a sum of fibres of the projection p_i, and the restriction of π_i' to the semi-analytic set is analytic except on a semi-analytic subset of smaller dimension and so on. We order C_1, C_2, ... so that

$$0=\dim C_1\leq\ldots\leq\dim C_m.$$

Since we consider functions in a small neighborhood of the origin, we have $\dim C_i=0$ only if $i=1$. We write $Y_1=Y$. Let a fibre of $U_2'\rightarrow C_2$ be fixed. Let x be a point of the fibre. Let Y_{2x} be the component of Y_{1x} tangent to the fibre. Then the " quasi " analytic property of π_i' shows in the same way as for Y that the vector field Y_2 is non-singular in a neighborhood V of C_2. Let φ_2 be a C^∞ function defined on $\mathbf{R}^n-\overline{C}_2$ so that $0\leqslant\varphi_2\leqslant 1$ and

$$\varphi_2=\begin{cases}1 \text{ outside of V}\\ 0 \text{ in a small neighborhood of } C_2.\end{cases}$$

We put

$$Y_2'=\varphi_2Y_1+(1-\varphi_2)Y_2.$$

In the same way we define Y_3' for C_3 and Y_2'. Repeating this process, we get a vector field Y_m' and a mapping $\tau': x\mapsto y$ such that $f(y)=g(x)$ and x, y are contained in the same integral curve of Y_m'. We saw that τ defined by Y' could be extended to $(\mathbf{R}^n-f^{-1}(0))\cup\{0\}$. This method shows that τ' can be extended to $f^{-1}(0)$. Hence $f\circ\tau'=g$. It is trivial that the extension is a local homeomorphism around 0. The proof is complete.

Remark. — It is natural to ask the following question. Let f, g be elements of $\mathfrak{m}(\mathcal{O}_n)$. If there exists a local homeomorphism τ such that $f\circ\tau$ and g have the same sign at each point. Then, are f and g C^0 equivalent?

This is wrong. There is a counterexample in King [7].

Remark. — As a special case of the theorem above, we see that f and fh are C^0 equivalent for any f in $\mathfrak{m}(\mathcal{O}_n)$ and h in $\mathcal{O}_n$ such that $h(0)>0$. Moreover, this statement is true for any germ h of a C^1 function whose value at 0 is positive.

Proof. — In the above proof, we modified the vector field $Y=X_f/|X_f|+X_g/|X_g|$ on $\mathbf{R}^n-f^{-1}(0)$ into Y'_m in order that

1) the diameter of the subset of each integral curve of Y'_m of points whose distance from $f^{-1}(0)$ is smaller than $\varepsilon>0$ converge uniformly to 0 when ε decreases to 0, and that
2) neither $Y'_m f$ nor $Y'_m g$ vanish anywhere on $\mathbf{R}^n-f^{-1}(0)$.

Since $g=fh$ is not analytic, it is not acceptable to consider the vector field Y. Hence we modify X_f on $\mathbf{R}^n-f^{-1}(0)$ to X'_m in place of Y. By the same argument as in the proof of the theorem, it is sufficient that the modification X'_m satisfy 1),

2)′ $X'_m f$ vanishes nowhere on $\mathbf{R}^n-f^{-1}(0)$ and
2)″ $X'_m fh$ vanishes nowhere on $\mathbf{R}^n-f^{-1}(0)$.

The conditions 1) and 2)′ are easy to be satisfied. For 2)″, we use the idea of the method of majorant as follows. We saw already the C^0 equivalence of f and af for any constant $a>0$. Hence we may assume that $h(0)=1$. Let $y=(y_1, \dots, y_n)$ be other variables. We set

$$\bar{g}(x,y)=f(x)\,(1+\sum_{i=1}^{n} y_i x_i), \quad \bar{f}(x,y)=f(x)$$

and $$\bar{X}=(X_f, 0).$$

We regard $\bar{X}$ as a vector field on $(\mathbf{R}^n-f^{-1}(0))\times\mathbf{R}^n$. Let K be a compact subset of the $y_1, \dots, y_n$ space. By the way, we easily see that

$$|f(x)|\leqslant a' \sum_{i=1}^{n}\left| x_i \frac{\partial f(x)}{\partial x_i}\right|$$

in a neighborhood of 0, where a' is a positive constant (e.g. [30]). This implies that

$$\bar{X}\bar{g}=\sum_{i=1}^{n}\frac{\partial f^2}{\partial x_i}(1+\sum_{j=1}^{n} y_j x_j)+\sum_{i=1}^{n} f\frac{\partial f}{\partial x_i} y_i>0$$

on the intersection of a neighborhood of $\{0\}\times K$ and $(\mathbf{R}^n-f^{-1}(0))\times\mathbf{R}^n$. By the implicit function theorem, there exist C^0 functions $y_i=y_i(x)$ for $i=1, \dots, n$ such that the gradient of $g=fh$ is the multiple of $\left(\frac{\partial \bar{g}}{\partial x_1}(x,y(x)), \dots, \frac{\partial \bar{g}}{\partial x_n}(x,y(x))\right)$ by a positive valued

C^0 function $a''(x)$. Let K be the image of a small closed neighborhood of the origin by $y(x)$. Then

$$X_f fh(x) = a''(x)\,\overline{X}\bar{g}(x, y(x)) > 0$$

on the intersection of the neighborhood of 0 and $\mathbf{R}^n - f^{-1}(0)$. As in the proof of the theorem, we can preserve this inequality when modifying X_f to $X'_2, \ldots, X'_m$. Therefore the condition 2)'' is satisfied. Thus we have proved the remark.

The proof of the theorem above works also in the global case. Hence we have, for example,

Corollary **(4.4)**. — *Let f_1, f_2 be analytic functions on a compact real analytic manifold, and S_i be the set of singular values of f_i. Assume $M = f_1^{-1}(S_1 \cup S_2) = f_2^{-1}(S_1 \cup S_2)$ and $f_1\big|_M = f_2\big|_M$. Then f_1 and f_2 are C^0 equivalent.*

From the remark (2.5) and a result in [31], we have the following. Let f be a convergent power series in n variables over $\mathbf{C}$ such that $f(0) = 0$, let s be the codimension of S, the singular set of $f^{-1}(0)$ around 0 in $\mathbf{C}^n$. Then f is equivalent to a $\mathbf{C}$-polynomial in s variables with convergent $(n-k)$-series as coefficients. Here s is 2 except when f is a power of a prime element. But the codimension s' of $S' = S \cap \mathbf{R}^n$ in $\mathbf{R}^n$ is not always 2 even if f has coefficients in $\mathbf{R}$ and is not a power of a prime element. Hence the next theorem is not implied immediately by the theorem (2.4). We remark that S' is the subset of $f^{-1}(0)$ of real points around which f is not a power of a regular function.

Theorem **(4.5)**. — *Let f be an element of $\mathfrak{m}(\mathcal{O}_n)$. Let r be a non-negative integer. Then f is C^r equivalent to a polynomial in s' variables with convergent $(n-s')$-series as coefficients.*

For the proof we need the following lemmas which are the C^r cases of the Lemmas (1.1) and (2.3), and a refinement of (1.2). The proofs are the same and we omit them.

Lemma **(4.6)**. — *In Lemma (1.1), we assume the C^r differentiability of $a_i(x, t)$ if $r \geq 1$ and Lipschitz' condition if $r = 0$ in place of the C^∞ differentiability. Then f and g are C^r equivalent.*

Lemma **(4.7)**. — *Let $f(x)$ be in $\mathfrak{m}^2(\mathcal{E}_n)$ (resp. $\mathfrak{m}^2(\mathcal{O}_n)$). Let y_{ij}, for $i, j = 1, \ldots, n$, be variables. Then there exist elements $b_{ij}(x, y)$ in $\mathfrak{m}(\mathcal{E}_{n+n^2})$ (resp. $\mathfrak{m}(\mathcal{O}_{n+n^2})$) such that*

$$f\left(\ldots, x_i + \sum_{j=1}^{n} b_{ij}(x, y)\frac{\partial f(x)}{\partial x_j}, \ldots\right) = f(x) + \sum_{i,j=1}^{n} y_{ij}\frac{\partial f(x)}{\partial x_i}\frac{\partial f(x)}{\partial x_j},$$

and $\qquad b_{ij}(x, 0) = 0 \quad$ *for* $\quad i, j = 1, \ldots, n.$

Lemma **(4.8)**. — *In the Lemma* (2.3), *we assume that g is an element of the ideal generated by $\mathfrak{m}^2(A)[I_f : f]^2$ in the ring of germs of C^{r+1} functions for $r \geqslant 1$. Then f and $f+fg$ are C^r equivalent.*

Proof of the Theorem (4.5). — We proceed in the same way as we proved the Theorem (2.1), and we use the same notations f, f_i, k, f' and $\mathfrak{p}$, replacing $\mathscr{E}_n$ by $\mathcal{O}_n$. We assume $k>1$. By Lemma (2.2), S′ is the zero set of $\mathfrak{p}$. Let $\mathfrak{q}$ be the ideal in $\mathcal{O}_n$ of germs vanishing on S′. Then we have ht $\mathfrak{q}=s'$. Hence, from the normalization theorem for the ring of convergent power series, we have a coordinate system $(x_1, \dots, x_n)$ such that the sum of $\mathfrak{q}$ and the ideal generated by $x_1, \dots, x_{n-s'}$ in $\mathcal{O}_n$ contains $\mathfrak{m}^p(\mathcal{O}_n)$ for sufficiently large p. Let φ be a monic polynomial in a variable with coefficients in $\mathcal{O}(x_1, \dots, x_{n-s'})$ such that $\varphi(x_{n-s'+1}), \dots, \varphi(x_n)$ are contained in the product of a sufficiently large number of copies of $\mathfrak{q}$. Then we remark that by the Łojasiewicz' inequality, the $\varphi(x_j)/g(x)$ are germs of C^r functions for $j=n-s'+1, \dots, n$ where $g(x)=\sum_{i=1}^{n}\left(\frac{\partial f'(x)}{\partial x_i}\right)^2$. From Weierstrass' preparation theorem there exist $\varphi_1, \dots, \varphi_{s'}$ in $\mathcal{O}_n$ and F_1, a $(x_{n-s'+1}, \dots, x_n)$-polynomial with coefficients in $\mathcal{O}(x_1, \dots, x_{n-s'})$, such that

$$f_1 = F_1 + \varphi(x_{n-s'+1})\varphi_1 + \dots + \varphi(x_n)\varphi_{s'}.$$

Hence we have

$$f' = F_1 \prod_{i \neq 1} f_i + \varphi(x_{n-s'+1})\varphi_1 \prod_{i\neq 1} f_i + \dots + \varphi(x_n)\varphi_{s'} \prod_{i \neq 1} f_i.$$

Apply Lemma (4.7) to f'. Then replacing y_{ij} by

$$-(\varphi(x_{n-s'+1})\varphi_1 \prod_{\ell \neq 1} f_\ell + \dots)/g(x)$$

if $i=j$, and by 0 if $i \neq j$, we have

$$F_1 \prod_{i \neq 1} f_i = f'\left(\dots, x_i + \sum_{j=1}^{n} b_{ij}[x, y(x)]\frac{\partial f'}{\partial x_j}, \dots\right).$$

We denote the local C^r diffeomorphism by τ. Since $b_{ij}(x, 0)=0$, we have $h_2(x, t), \dots, h_n(x, t)$ in $\mathfrak{m}(\mathcal{O}_{n+1})$ such that

$$f_j \circ \tau = f_j(x)(1 + h_j[x, (\varphi(x_{n-s'+1})\varphi_1 + \dots)/g(x)]),$$
$$h_j(x, 0) = 0 \quad \text{for} \quad j=2, \dots, n.$$

Hence there exists $c(x, t)$ in $\mathcal{O}_{n+1}$ such that $c(x, 0)=1$ and

$$f \circ \tau = c[x, (\varphi(x_{n-s'+1})\varphi_1 + \dots)/g(x)]\, F_1^{\alpha_1} \prod_{i \neq 1} f_i^{\alpha_i}.$$

We put $F = F_1^{\alpha_1} \prod_{i \neq 1} f_i^{\alpha_i}$. Let h be the sum of the squares of generators of $\mathfrak{m}(\mathcal{O}_n)[I_F : F]$. Then the zero set of h is contained in S′, and we may assume from the beginning that the $\varphi(x_j)/gh$ are germs of C^r functions for $j=n-s'+1, \dots, n$. Hence, by Lemma (4.8), F and $f \circ \tau$ are C^r equivalent. Here we remark that the singular set of $F_1 \prod_{i \neq 1} f_i$ is S'. This means that we can use again $\varphi(x_{n-s'+1}), \ldots$ when we transform f_2 into a $(x_{n-s'+1}, \ldots, x_n)$-polynomial F_2 with coefficients in $\mathcal{O}_{n-s'}$. Therefore we see in the same way as above that f and $F_1^{\alpha_1} F_2^{\alpha_2} f_3^{\alpha_3} \ldots$ are C^r equivalent. Repeating this process, we prove the theorem.

Remark. — *Let f, f' be as in the proof of Theorem* (2.1). *Suppose that the height of the ideal* $\left\{ Tg \mid g \in \mathcal{E}_n,\ g \Big/ \sum_{j=1}^{n} \left(\frac{\partial f'}{\partial x_j}\right)^2 \text{ is a well defined continuous function germ} \right\}$ *in* $\mathcal{F}_n$ *be h. Then, for any* $r < \infty$, *f is* C^r *equivalent to a polynomial in h variables with germs of* C^∞ *functions in* $n-h$ *variables as coefficients.*

5. Homomorphism from $\mathcal{F}_1$ to $\mathcal{E}_1$

Reichard [22] and the author [27] showed the existence of a homomorphism from $\mathcal{F}_1$ to $\mathcal{E}_1$ the composition of which with the Taylor expansion is the identity of $\mathcal{F}_1$. Van der Put [32] generalized it (see Theorem (5.4)). Here we show another generalization.

Theorem (**5.1**). — *There exists a homomorphism* S *from* $\mathcal{F}_1$ *to* $\mathcal{E}_1$ *such that* $T \circ S$ *is the identity of* $\mathcal{F}_1$ *and that for any f in* $\mathcal{F}_1$ *we have* $S\left(\frac{df}{dx}\right) = \frac{d(Sf)}{dx}$.

For the proof we need a theorem of Malgrange [13].

Lemma (**5.2**). — *Let* Φ *be a* C^∞ *mapping in* $2m+1$ *variables* $x, Y = (y_1, \ldots, y_m)$ *and* $Z = (z_1, \ldots, z_m)$ *in a neighborhood of* $(0, Y_0, Z_0)$ *with values in* $\mathbf{R}^m$. *Suppose that there exist formal power series* $H \in \mathcal{F}_1^m$ *such that*

$$H(0) = Y_0, \quad \frac{dH(0)}{dx} = Z_0 \quad \text{and} \quad T_{(0, Y_0, Z_0)} \Phi\left(x, H, \frac{dH}{dx}\right) = 0.$$

Suppose that the determinant of the matrix $T_{(0, Y_0, Z_0)} \frac{\partial \Phi}{\partial Z}\left(x, H, \frac{dH}{dx}\right)$ *is not zero. Then there exists* $F \in \mathcal{E}_1^m$ *such that*

$$TF = H, \quad \Phi\left(x, F, \frac{dF}{dx}\right) = 0.$$

Proof of the theorem. — Let A be a subring of $\mathcal{F}_1 = \mathcal{F}(x)$ containing $\mathbf{R}$ such that for any ζ in A, the derivative $\frac{d\zeta}{dx}$ is contained in A. Let φ be a homomorphism from A to $\mathcal{E}_1$ such that $T \circ \varphi$ is the identity and that for any ζ in A we have $\varphi\left(\frac{d\zeta}{dx}\right) = \frac{d(\varphi\zeta)}{dx}$.

Let X denote the ordered set consisting of the such pairs (A, φ). We define the order as follows. Let (A, φ), (B, ψ) be elements of X. Then

$$(A, \varphi) \leq (B, \psi) \quad \text{if} \quad A \subset B \quad \text{and} \quad \psi\Big|_A = \varphi.$$

Apply Zorn's lemma to X. Then X has a maximal element (A, φ). We shall prove that $A = \mathscr{F}_1$. Assume that A is a proper subset of $\mathscr{F}_1$. Let ζ be an element of $\mathscr{F}_1$ not contained in A. Let A_ζ be the ring generated by $\zeta, \frac{d\zeta}{dx}, \ldots$ over A and let $A[t_0, t_1, \ldots]$ be the polynomial ring in $t_0, \ldots$ with coefficients in A. We have a natural homomorphism θ from $A[t_0, \ldots]$ to A_ζ defined by $\theta(t_i) = \frac{d^i\zeta}{dx^i}$ for all i. There are two cases to consider:

1) θ is injective, or
2) θ is not injective.

Case 1. — Let f be a realization of ζ, that is, $f \in \mathscr{E}_1$ and $Tf = \zeta$. Then we have an extension homomorphism φ' of φ from A_ζ to $\mathscr{E}_1$ defined by $\varphi' \frac{d^i\zeta}{dx^i} = \frac{d^if}{dx^i}$. It is trivial that $(A_\zeta, \varphi') \in X$, $(A, \varphi) \lneqq (A_\zeta, \varphi')$. This contradicts to the maximality of (A, φ). Hence this case cannot occur.

Case 2. — Let $\mathfrak{p}$ be the kernel of θ. We set $\mathfrak{p}_n = \mathfrak{p} \cap A[t_0, \ldots, t_n]$. Then $\mathfrak{p} = \bigcup_{n=0}^{\infty} \mathfrak{p}_n$. Let m be the minimum of n such that $\mathfrak{p}_n \neq 0$.

First we shall prove the existence of a homomorphism φ' from $A[t_0, \ldots, t_m]$ to $\mathscr{E}_1$ extending φ and such that

$$(*) \qquad T \circ \varphi'(t_0) = \zeta, \quad \varphi' t_i = \frac{d\varphi' t_0}{dx^i} \quad \text{for} \quad i \leq m, \quad \varphi'\Big|_{\mathfrak{p}_m} = 0.$$

We regard elements of $\mathfrak{p}_m$ as polynomials in t_m with coefficients in $A[t_0, \ldots, t_{m-1}]$. Let p be an element of $\mathfrak{p}_m$ whose degree takes the minimum value in $\mathfrak{p}_m$. For any element q of $\mathfrak{p}_m$, there exist p' in $A[t_0, \ldots, t_m]$ and q' in $A[t_0, \ldots, t_{m-1}] - \{0\}$ such that

$$(**) \qquad pp' = qq'.$$

The reason is the following. Let us divide q by p. Then there exist p', r in $A[t_0, \ldots, t_m]$ and q' in $A[t_0, \ldots, t_{m-1}] - \{0\}$ such that

$$qq' = pp' + r, \quad \text{and} \quad \text{degree } r < \text{degree } p.$$

Since $r \in \mathfrak{p}_m$, we see that $r = 0$. Hence (**). We put

$$p = \sum_\alpha a_\alpha t^\alpha, \quad \text{with} \quad \alpha = (\alpha_0, \ldots, \alpha_m) \in \mathbf{N}^{m+1}, \quad a_\alpha \in A,$$

$$t = (t_0, \ldots, t_m), \quad t^\alpha = \prod_{i=1}^{m} t_i^{\alpha_i}.$$

We set

$$P(t_0, \ldots, t_m) = \sum_{\alpha} \varphi(a_\alpha)\, t^\alpha,$$

and, if $m = 0$

$$p_0(t_0, t_1) = \sum_{\alpha} \frac{da_\alpha}{dx} t_0^\alpha + t_1 \sum_{\alpha} \alpha a_\alpha t_0^{\alpha-1},$$

$$P_0(t_0, t_1) = \sum_{\alpha} \frac{d\varphi(a_\alpha)}{dx} t_0^\alpha + t_1 \sum_{\alpha} \alpha\varphi(a_\alpha)\, t_0^{\alpha-1}.$$

We put $m' = \max(1, m)$. Let $\Phi = (\Phi_1, \ldots, \Phi_{m'})$ be the C^∞ mapping in the variables $x, y_1, \ldots, y_{m'}, z_1, \ldots, z_{m'}$ defined by

$$\text{if} \quad m > 0, \quad \Phi_1 = z_1 - y_2, \ldots, \Phi_{m-1} = z_{m-1} - y_m,$$

$$\Phi_m = P(y_1, z_1, \ldots, z_m),$$

$$\text{if} \quad m = 0, \quad \Phi_1 = P_0(y_1, z_1).$$

Then $H = \left(\zeta, \frac{d\zeta}{dx}, \ldots, \frac{d^{m'-1}\zeta}{dx^{m'-1}}\right)$ is a formal solution of

$$T\Phi\left(x, H, \frac{dH}{dx}\right) = 0.$$

The determinant of the matrix $T\frac{\partial\Phi}{\partial Z}\left(x, H, \frac{dH}{dx}\right)$ is $\frac{\partial p}{\partial t_m}\left(\zeta, \frac{d\zeta}{dx}, \ldots, \frac{d^m\zeta}{dx^m}\right)$. Since $\frac{\partial p}{\partial t_m}(t_0, \ldots, t_m)$ is not zero and of smaller degree than p as polynomial in t_m, the minimality of the degree of p implies that $\frac{\partial p}{\partial t_m}$ is not contained in $\mathfrak{p}_m$, that is, $\frac{\partial p}{\partial t_m}\left(\zeta, \frac{d\zeta}{dx}, \ldots, \frac{d^m\zeta}{dx^m}\right)$ is not zero. Hence we can apply Lemma (5.2). Therefore there exists a solution

$$F = (F_1, \ldots, F_{m'}) \in \mathscr{E}_1^{m'}$$

of $TF = H$ and $\Phi\left(x, F, \frac{dF}{dx}\right) = 0$. Let us define an extension φ' of φ by $\varphi'(t_i) = \frac{d^iF}{dx^i}$. If $m > 0$, $\varphi(p(t_0, \ldots, t_m)) = P\left(F, \frac{dH}{dx}, \ldots, \frac{d^mF}{dx^m}\right) = 0$. Hence $p \in \mathfrak{p}_m$, and we have $\varphi'(q) = 0$ for any element q of $\mathfrak{p}_m$ by virtue of (**). Hence φ' satisfies (*). If $m = 0$, we have $\frac{d}{dx}(p(t_0)) = \frac{dP}{dx}(F) = P_0\left(F, \frac{dF}{dx}\right) = 0$. Hence $\varphi(p(t_0)) = P(F) = 0$. This means that $p(t_0) \in \mathfrak{p}_0$. Therefore we can prove, as the case $m > 0$, that φ' satisfies (*).

Secondly we shall extend φ' to $A[t_0, t_1, \ldots]$ so that, if we use the same φ', then

$$(***) \qquad \varphi' t_i = \frac{d^i\varphi' t_0}{dx^i} \quad \text{and} \quad \varphi'\Big|_{\mathfrak{p}} = 0.$$

We define the homomorphism by $\varphi' t_i = \frac{d^i \varphi' t_0}{dx^i}$ for $i > 0$. Then we only have to prove that $\varphi'\Big|_{\mathfrak{p}} = 0$. Let ρ be the linear transformation of $A[t_0, \ldots]$ defined by

$$\rho(a) = \frac{da}{dx} \quad \text{for} \quad a \in A, \quad \rho(t_i) = t_{i+1},$$

and

$$\rho(fg) = \rho(f)g + f\rho(g) \quad \text{for} \quad f, g \in A[t_0, \ldots].$$

Then we have $\varphi'(\rho f) = \frac{d\varphi' f}{dx}$ for any $f \in A[t_0, \ldots]$. Hence any composition $\rho \circ \ldots \circ \rho(f)$ is mapped to 0 by φ' if f is. Therefore, for the proof of (***) it is sufficient to see that if f is an element of $\mathfrak{p}$ there exists f' in $A[t_0, \ldots] - \mathfrak{p}$ such that ff' is contained in the ideal $\mathfrak{a}$ generated by p, $\rho(p)$, $\rho \circ \rho(p)$, ... We prove this by the induction on the least integer n such that $f \in \mathfrak{p}_n$. If $n = m$, the assertion is already proved. We suppose it true for $n \leq k - 1$, and assume that $f \in \mathfrak{p}_k$, $f \notin \mathfrak{p}_{k-1}$. We easily see that

$$\overbrace{\rho \circ \ldots \circ \rho}^{i}(p) = q t_{m+i} + q'_{m+i},$$

where $q \in A[t_0, \ldots, t_m] - \mathfrak{p}_m$, $q'_{m+i} \in A[t_0, \ldots, t_{i-1}]$ for $i \geq 1$. We regard the elements of $A[t_0, \ldots, t_k]$ as polynomials in t_k with coefficients in $A[t_0, \ldots, t_{k-1}]$. Dividing f by $qt_k + q'_k$, we see that there exist f' in $A[t_0, \ldots, t_m] - \mathfrak{p}_m$, g in $A[t_0, \ldots, t_k]$ and h in $A[t_0, \ldots, t_{k-1}]$ such that

$$ff' = g(qt_k + q'_k) + h.$$

Since $qt_k + q'_k \in \mathfrak{a} \subset \mathfrak{p}$, we have $h \in \mathfrak{p}$. By induction, there exists h' in $A[t_0, \ldots] - \mathfrak{p}$ such that $hh' \in \mathfrak{a}$. Hence $ff'h' \in \mathfrak{a}$. Thus (***) is proved.

Finally, we induce a homomorphism φ'' from A_ζ to $\mathscr{E}_1$ by φ'. The condition (***) implies the existence of φ''. The conditions (*) and (***) show that $(A_\zeta, \varphi'') \in X$. It is trivial that $(A, \varphi) \lneqq (A_\zeta, \varphi'')$. This contradicts the maximality of (A, φ). Hence $A = \mathscr{F}_1$. The theorem is proved.

It seems impossible to generalize the theorem above to arbitrary dimensions (see p. 164 in Hörmander [6]).

Van der Put's result is the following.

Theorem (5.3). — *Let* A *be the image of a* **R**-*homomorphism from* $\mathcal{O}_m$ *to* $\mathscr{F}_n$. *Then there exists a homomorphism* S *from* A *to* $\mathscr{E}_n$ *such that* $T \circ S$ *is the identity of* A.

II. — GLOBAL EQUIVALENCE

6. Preparation

In this chapter, manifolds are assumed to be real, connected, paracompact, analytic and of dimension n, functions and germs of functions are real-valued C^∞, and mappings, diffeomorphisms and their germs are C^∞, unless otherwise specified.

Let f_1 and f_2 be mappings from a manifold M_1 to another M_2. We say that f_1 and f_2 are *equivalent* if there exists a diffeomorphism τ of M_1 such that $f_1 \circ \tau = f_2$. The mappings are R-L (right-left) *equivalent* if $\tau' \circ f_1 \circ \tau = f_2$ for suitable diffeomorphisms τ' of M_2 and τ of M_1. We define in the same way the L *equivalence* by means of a diffeomorphism of M_2 only, the *analytic equivalences* by means of analytic diffeomorphisms, and the *equivalences* of germs by means of local diffeomorphisms. Let A_1 and A_2 be subsets or germs of subsets of a manifold M. A_1 and A_2 are *equivalent* when A_1 is translated to A_2 by a diffeomorphism or a local diffeomorphism of M. If the diffeomorphism is analytic, the equivalence is called analytic. If we talk about the Jacobian matrix of a mapping, local coordinate systems are assumed to be already given.

The notations X, Y, ... stand for C^∞ or analytic vector fields on manifolds, in particular $X_1, X_2, \ldots$ are analytic fields defined as follows. Assume the manifold M to be analytically imbedded in $\mathbf{R}^N$. Let $(x_1, \ldots, x_N)$ be the affine coordinate system of $\mathbf{R}^N$. For each $i = 1, \ldots, N$ we restrict the vector field $\frac{\partial}{\partial x_i}$ to M. Then X_i is defined as the tangential component of the restriction to M. We remark that $X_1, \ldots, X_N$ span the tangent space of M at each point. Let f and A be a function and a set respectively. Then we denote by f_x and A_x the germs of f and A at a point x. We write $\mathcal{O}$ and $\mathcal{F}$ for the sheaves of germs of analytic functions and C^∞ functions on a manifold respectively.

Throughout this chapter the topology put on a set of mappings or on a set of vector fields will always be the strong Whitney topology, defined as follows. Let M be a manifold and let $C^\infty(M)$ be the set of functions on M. Let $\{O_i\}$ be any locally finite open covering of M and let V_i be an open set in $C^\infty(O_i)$ for the topology of uniform convergence of all derivatives. Then an open set of $C^\infty(M)$ consists of functions whose restriction to O_i is contained in V_i for all i. We define the topology on a set of diffeomorphisms, a set of mappings or a set of vector fields in the same way. We recall Whitney's theorem asserting that any function can be approximated by analytic functions in this topology [35].

The fundamental lemma on the equivalence is the following global version of Lemma (1.2). The proof is the same, because the global analogue of Lemma (1.1) holds. See the proof of Lemma (1.2) in [25].

Lemma **(6.1)**. — *Let* M *be a manifold, let* f *be a function on* M, *and let* $Y_1, \ldots, Y_k$ *be vector fields on* M. *Then there exists a continuous correspondence associating to a system of* k^2 *functions* a_{ij} *on* M, *for* $i, j = 1, \ldots, k$, *a vector field* Y *on* M *such that*:

(a) *for* $a_{ij} = 0$, Y *is the zero vector field*;

(b) Y *is a linear combination of* Y_i *with functions as coefficients*;

(c) *if we write* $\tau = \varphi_1$ *where* φ_t *is the one parameter group of transformations "generated by"* Y, *then we have, for small* a_{ij}

$$f \circ \tau = f + \sum_{ij=1}^{k} a_{ij}.Y_i f.Y_j f.$$

Remark **(6.2)**

1) In addition, if f, Y_i and a_{ij} are all analytic in an open subset of M, then Y is analytic in the same subset.

2) We assume that $M = \mathbf{R}^n$, that f, Y_i and a_{ij} are analytic in M and that the a_i are all divisible by an analytic function h. Then the components of the mapping τ minus the identity are divisible by h.

3) In the above lemma, we can replace the condition that a_{ij} are small by the condition that $\sum_{ij=1}^{k} a_{ij} Y_i f Y_j f$ is small if f is analytic.

Proof. — The first assertion immediately follows from the proof in [25]. Regarding a vector field on $\mathbf{R}^n$ as a mapping, we also see that the components of Y are divisible by h. Hence the components of τ minus the identity vanish on $h^{-1}(0)$. This implies that, if h is regular, 2) holds. Moreover we easily settle the case where h is a power of a regular function by the properties of vector fields. We remark here that the correspondence $a_{ij} \mapsto Y$ can be extended to the complex field and that the resulting diffeomorphism is the complexification of τ if the a_{ij} are real valued on $\mathbf{R}^n$.

Now we consider a general h. Since the problem is local, we regard the functions and vector fields as germs at a point of $\mathbf{R}^n$ and we assume that h vanishes at that point. Let $\widetilde{h}$ be the complexification of h and let $\widetilde{h} = \prod_{i=1}^{\ell} \widetilde{h}_i^{\alpha_i}$ be the unique factorization into prime elements in the ring of holomorphic function germs. We already saw that the complexified germs of the components of τ minus the identity are divisible by $\widetilde{h}$ on the regular set of $\prod_{i=1}^{\ell} \widetilde{h}_i$. Hence, by Hilbert Nullstellensatz, the complexified germs are divisible by $\prod_{i=1}^{\ell} \widetilde{h}_i$, because the closure of the regular set of $\prod_{i=1}^{\ell} \widetilde{h}_i$ is the zero set. This implies again that the quotients are divisible by $\prod_{\alpha_i>1} \tilde{h}_i$. Repeating this argument, we obtain the divisibility by the germ of h. Thus 2) is proved.

3) follows from the statement below.

Let $f_1, \ldots, f_m$ be analytic functions on an analytic manifold M *and let g be a linear combination of $f_1, \ldots, f_m$ with functions as coefficients. If g is sufficiently near to the zero function, the coefficients can be chosen near to the zero function. If the given coefficients are analytic, so are the resulting ones.*

Proof. — First we prove this in the topology of uniform convergence of all derivatives on compact subsets of M. Let p be the mapping from $[C^\infty(M)]^m$ to $C^\infty(M)$ defined by

$$p(\varphi_1, \ldots, \varphi_m) = \sum_{i=1}^{m} \varphi_i f_i.$$

We denote the kernel of p by K and the intersection of K and $[C^\omega(M)]^m$ by K'. Because the image of p is closed, the open mapping theorem for Fréchet spaces shows that if $g = p(\varphi_1, \ldots, \varphi_m)$ is sufficiently near to 0, there exists $(\varphi'_1, \ldots, \varphi'_m)$ in K such that the $\varphi_i - \varphi'_i$ are near to 0 for $i = 1, \ldots, m$. We only need that if g is analytic, so are $\varphi_i - \varphi'_i$. By Cartan's theorem we may assume the analyticity of φ_i. Hence it is sufficient to see that the closure of K' is K. Let $\mathscr{K} \subset \mathscr{F}^m$ and $\mathscr{K}' \subset \mathscr{O}^m$ be the sheaves of submodules defined by the stalk

$$\mathscr{K}_a = \{(\varphi_1, \ldots, \varphi_m) \in \mathscr{F}_a^m \mid \sum_{i=1}^{m} \varphi_i f_{ia} = 0\}$$

$$\mathscr{K}'_a = \mathscr{K}_a \cap \mathscr{O}_a^m \quad \text{for} \quad a \in M.$$

Then, by Oka's theorem, $\mathscr{K}'$ is coherent, and Artin's theorem implies that the closure of $T_a\mathscr{K}'_a$ in the Krull topology is $T_a\mathscr{K}_a$ for $a \in M$. Hence $T_a\mathscr{K}'_a\mathscr{F}_a = T_a\mathscr{K}_a$ (Krull) Here T_a means "Taylor expansion at a".

Now let ψ be an element of K. It is a cross section of $\mathscr{K}$. The theorem of Cartan shows that, for $a \in M$, $T_a\psi$ can be approximated by $T_a\xi$ where ξ are cross sections of $\mathscr{K}'$, that is, elements of K'. Therefore, for each a of M there exists ξ in $C^\infty(M)K'$ such that $T_a\psi = T_a\xi$. Then we say that ψ is pointwise in $C^\infty(M)K'$ (see [12]). Whitney proved that any function which is pointwise in a submodule of $[C^\infty(M)]^m$ is contained in the closure of the submodule. Hence ψ is contained in the closure of $C^\infty(M)K'$. Since the closure of K' is that of $C^\infty(M)K'$, ψ is an element of the closure of K'. Thus the statement is proved for the weak topology.

Case of the strong Whitney topology. Using a partition of unity, we find easily $(\varphi'_1, \ldots, \varphi'_m)$ as above. We need to show that the closure of K' in the strong topology is K. For this we apply the method of Lemma 6 in Whitney [35]. We may assume $M = \mathbf{R}^n$. Let ψ be an element of K, let the open set $U \subset \mathbf{C}^n$ be a Stein manifold containing $\mathbf{R}^n$ to which $f_1, \ldots, f_m$ can be extended, and let h be a function on $\mathbf{R}^n$ which is 1 on $\{|x| \geq 1\}$ and 0 on $\{|x| \leq 1/2\}$. Then, by the above result, ψ can be approximated on $\{|x|\leqslant 2\}$ by some Ψ_1 in K'. Cartan's theorem tells us that Ψ_1 can be chosen to be extendible to U. We choose the approximation so strong that $(\psi-\Psi_1)h_1$ is small on $\{|x|\leqslant 2\}$, where $h_1(x)=h(x/2)$. Next we approximate $(\psi-\Psi_1)h_1$ on $\{|x|\leqslant 3\}$ by Ψ_2. Repeating this process, we obtain an approximation $\Psi=\Psi_1+\Psi_2+\ldots$ of ψ. To finish the proof, we only have to see the analyticity of Ψ. Let Ψ_2' be an approximation of $\psi-\Psi_1$ on $\{|x|\leqslant 3\}$ defined on U. We put

$$\Psi_2''(x)=Ck^n\int_{\mathbf{R}^n}h_1(y)\,e^{-k^2|x-y|^2}\,dy$$

where C is a constant and k is a real number. We take k large. Then Ψ_2'' is near to h_1 on $\{|x|\leqslant 3\}$ and to 0 on $\{z\in\mathbf{C}^n \mid |z|\leqslant 1/2\}$. Hence $\Psi_2=\Psi_2'\Psi_2''$ is an approximation of $(\psi-\Psi_1)h_1$ on $\{|x|\leqslant 3\}$ and $\Psi_1+\Psi_2+\ldots$ converges on U. This finishes the proof.

Remark **(6.3)**. — With the same f and $Y_1, \ldots, Y_k$ as in the lemma above, let $g_1, \ldots, g_{k'}$ be linear combinations of the $Y_i f$, for $i=1, \ldots, k$, with functions as coefficients, such that, for $i=1, \ldots, k$ and $j=1, \ldots, k'$, also the $Y_i g_j$ are linear combinations. Then f is equivalent to $f+\sum_{i=1}^{k'} a_i g_j$ for small a_i, and the same properties of the diffeomorphism as above hold. The proof is the same.

The principal idea of the proofs in this chapter is to enlarge the subdomain where functions or mappings are analytic. Up to equivalence, the lemma above allows one to add certain types of functions. For that reason, the next lemma will be used many times.

Lemma **(6.4)**. — *Let ψ_1 and ψ_2 be functions on a manifold* M. *Suppose that ψ_2 is analytic and that ψ_1 is analytic around $\psi_2^{-1}(0)$. Then there exists an arbitrarily small function ψ_3 such that $\psi_1+\psi_2\psi_3$ is analytic.*

Proof. — Let $\mathfrak{p}$ be the sheaf of ideals of $\mathcal{O}$ generated by ψ_2. Then we have the exact sequence of coherent sheaves

$$0\rightarrow\mathfrak{p}\rightarrow\mathcal{O}\rightarrow\mathcal{O}/\mathfrak{p}\rightarrow 0.$$

Cartan's theorem on Stein manifold asserts the surjectivity of the mapping between cross-section spaces $\Gamma(M, \mathcal{O})\rightarrow\Gamma(M, \mathcal{O}/\mathfrak{p})$. Since $\mathcal{O}/\mathfrak{p}$ is zero outside $\psi_2^{-1}(0)$, ψ_1 determines an element of $\Gamma(M, \mathcal{O}/\mathfrak{p})$. Let ψ be an element of $\Gamma(M, \mathcal{O})$ whose image in $\Gamma(M, \mathcal{O}/\mathfrak{p})$ is ψ_1. Then $\psi-\psi_1$ is divisible by ψ_2. Hence $\psi=\psi_1+\psi_2\psi_3'$ for some function ψ_3'. Let ψ_3'' be an analytic approximation of ψ_3'. Then $\psi_3=\psi_3'-\psi_3''$ meets our requirement.

Corollary **(6.5)**. — *Let f be a function on a manifold* M *and let* V *be an open subset of* M. *Suppose that f is regular on* M—V *and analytic on* V. *Then f is equivalent to an analytic function. We can choose the inverse of the diffeomorphism to be analytic in a neighborhood of any given closed set contained in* V.

Proof. — Let $\mathfrak{q}$ be the sheaf of ideals of $\mathcal{O}$ generated by $X_i f$ for $i=1, \ldots, N$ on V and by 1 on M−V. Then, by Cartan's theorem there exist $g_1, \ldots, g_k$ in $\Gamma(M, \mathfrak{q})$ such that at least one of the g_i does not vanish at all regular points of f. Let ψ be the sum of the squares of the g_i. Then ψ is a linear combination of $X_i f . X_j f$ for $i, j=1, \ldots, N$ with functions as coefficients. Since $\psi^{-1}(0)$ is contained in V, f is analytic in a neighborhood of $\psi^{-1}(0)$. Hence we can apply Lemma (6.4). Let ψ_1 be a small function such that $f+\psi_1\psi$ is analytic. Then, by Lemma (6.1), f is equivalent to $f+\psi_1\psi$, because $\psi_1\psi$ is a linear combination of the $X_i f . X_j f$ with small coefficients. The last statement follows from Remark (6.2.1). The corollary is proved.

Corollary **(6.6)**. — *Let f be a function on a manifold* M. *Suppose that the critical points of f form a discrete set and that the germ of f at each point is equivalent to an analytic germ. Then f is equivalent to an analytic function.*

Proof. — Let a be a critical point of f. By the assumption, there exists a local diffeomorphism τ in a neighborhood of a such that $f\circ\tau$ is defined and analytic in a neighborhood of a. Let τ' be a strong analytic approximation of τ^{-1}. Then $\tau\tau'$ can be extended to M so that the extension τ'' is the identity outside a small neighborhood of a. Considering $f\circ\tau''$ instead of f, we may assume that f is analytic in a neighborhood of a. We repeat this process at each critical point. Thus, we can assume that f is analytic on an open set containing all the critical points, and the corollary follows from (6.5).

Remark **(6.7)**. — We can generalize the results above except Lemma (6.4) to the case of a mapping, as follows. In Lemma (6.1), we replace f and a_{ij} by mappings into $\mathbf{R}^m$, $f=(f_1, \ldots, f_m)$ and $a_{\alpha\beta}=(a^1_{\alpha\beta}, \ldots, a^m_{\alpha\beta})$ where $m \leqslant n$ and α, β run through the set of sequences of m integers $(i_1, \ldots, i_m)$ such that $1 \leqslant i_1 < \ldots < i_m \leqslant k$. We denote by g_α the determinant of the $m\times m$-matrix whose (j, ℓ) component is $Y_{i_j} f_\ell$, for $\alpha=(i_1, \ldots, i_m)$. Then we have

$$f_\ell\circ\tau = f_\ell + \sum_{\alpha\beta} a_{\alpha\beta} g_\alpha g_\beta \quad \text{for} \quad \ell=1, \ldots, m.$$

The proof proceeds in the same way, using the method of p. 206 in [31]. The other generalizations are similar.

7. Equivalence to analytic functions

In this section we deal with the conjecture I for functions.

Theorem **(7.1)**. — *Let f be a function on a manifold* M. *Suppose that the germ of f at each point of* M *is equivalent to an analytic germ. In addition suppose that f takes locally the canonical forms* (i), (ii) *or* (iii) *of the introduction except on a discrete set. Then f is equivalent to an analytic function.*

Proof. — Let S be the exceptional discrete set and let S′ be the set of critical points of f outside of S. By Corollaries (6.5), (6.6) and their proofs we can assume that f is analytic in a neighborhood U of S and that the only critical value of f is 0. For any x in S′, f_x has one of the forms $\pm x_1^2 \pm \ldots \pm x_k^2$ or $\pm x_1^\ell$ for some local coordinate system $(x_1, \ldots, x_n)$ and some integers $1 \leq k \leq n$ and $\ell \geq 2$. We assume that ℓ is globally constant on M. The general case follows in the same way. Let S'_k or $S'_{1\ell}$ be the subset of S′ of such points. We put

$$F_\alpha(x) = \begin{cases} (X_{i_1} f(x), \ldots, X_{i_k} f(x)) & \text{for} \quad \alpha = (1; i_1, \ldots, i_k), \quad 1 \leq i_j \leq N, \\ \overbrace{X_i \ldots X_i}^{\ell - 1} f(x) & \text{for} \quad \alpha = (\ell - 1; i), \quad \ell \geq 2 \quad \text{and} \quad 1 \leq i \leq N. \end{cases}$$

Then F_α is a mapping from M to $\mathbf{R}^k$ or to $\mathbf{R}$. Let S'_α be the set of points of S'_k or $S'_{1\ell}$ at which the Jacobian matrix of F_α has the rank k or 1 respectively. We write $S'_\alpha(f)$ to specify f if necessary. It is trivial that the union of all S'_α is S′. Let (V, S''_α), α as above, be a closed covering of $S \cup S'$ such that V and S''_α are contained in U and S'_α respectively.

Let us order the set of all α. Fix one α. Assume inductively that f is analytic in a neighborhood of the union $S^{(3)}_\alpha$ of V and all S''_β such that $\beta < \alpha$. We want to transform f by a sufficiently small diffeomorphism τ so that

1) $f \circ \tau$ is analytic in a neighborhood of $\tau^{-1}(S''_\alpha \cup S^{(3)}_\alpha)$ and that
2) for each γ, $\tau^{-1}(S''_\gamma)$ is contained in $S'_\gamma(f \circ \tau)$.

Suppose this done; then we can assume that f is analytic in a neighborhood of the union of V and S''_β for $\beta \leq \alpha$, and by the induction on α and Corollary (6.5) the theorem ensues.

By definition, F_α is regular on S_α. Hence we can choose a small neighborhood U_α of $S''_\alpha \cup S^{(3)}_\alpha$ so that if $x \in U_\alpha$ is a critical point of f, f_x is analytic or x is an element of S''_α and that F_α is regular or analytic everywhere on U_α. Remark (6.7) applied to the restriction of F_α on U_α implies the existence of a small diffeomorphism τ_α of U_α such that $F_\alpha \circ \tau_\alpha$ is analytic, that τ_α is analytic around $\tau_\alpha^{-1}(S^{(3)}_\alpha)$ and that τ_α is extensible to the identity outside U_α (see the proof of (6.6)). Considering $f \circ \tilde{\tau}_\alpha$ instead of f, where $\tilde{\tau}_\alpha$ is the extension of τ_α, we can assume from the start that $(S \cup S') \cap U_\alpha$ is analytic.

Let $\mathfrak{p}$ be the intersection of $\mathcal{O}$ on U_α and the sheaf of ideals of $\mathcal{F}$ on U_α generated by $X_i f$ for $i = 1, \ldots, N$. Then $\mathfrak{p}$ is coherent and $\mathfrak{p}\mathcal{F}$ contains and is generated by $X_i f$ for $i = 1, \ldots, N$ because of the definition of S''_α and the analyticity of $(S \cup S') \cap U_\alpha$. Let $\mathfrak{p}'$ be the sheaf of ideals defined in the lemma below. Then $\mathfrak{p}'$ is coherent, and for each x of S''_α the stalk $\mathfrak{p}'_x$ is the square of the ideal of germs vanishing on S'_x if $\alpha = (1; i_1, \ldots, i_k)$ and its ℓ-th power if $\alpha = (\ell - 1, i)$. Cartan's theorem implies the existence of cross-sections $g_1, \ldots, g_{k'}$ of $\mathfrak{p}'$ on U_α such that for any x outside S, $g_{1x}, \ldots, g_{k'x}$ generate $\mathfrak{p}'_x$. Here the global finiteness easily follows from the regularity of $\mathfrak{p}'$ outside S. Moreover we can see that, in general, any coherent sheaf of $\mathcal{O}$-modules is generated by a finite number of global cross-sections if all the stalks are generated by a bounded number of germs. The proof, which proceeds by induction on the dimension of the zero set of the sheaf, is easy and will be omitted. Since f is a cross-section of $\mathfrak{p}'\mathscr{F}$ in a neighborhood of S''_α, f determines an element of $\Gamma(U_\alpha, \mathcal{O}/\mathfrak{p}')$. As in the proof of (6.4), we have an analytic function g on U_α such that $f-g$ is a cross-section of $\mathfrak{p}'\mathscr{F}$ on U_α. Using a partition of unity, one shows the existence of k' C^∞ functions h_i for $i=1, \ldots, k'$ on U_α such that $f-g=\sum_{i=1}^{k'} h_i g_i$. Let h'_i be a strong analytic approximation of h_i for each i. Then $f\big|_{U_\alpha}$ and $g+\sum_{i=1}^{k'} h'_i g_i$ satisfy the conditions of Remark (6.3). Hence they are equivalent, and the diffeomorphism can be chosen to be arbitrarily near to the identity and extensible to the identity outside of U_α. These facts imply the conditions 1) and 2). The theorem is proved.

Lemma **(7.2)**. — *Let* $\mathfrak{p}$ *be a coherent sheaf of ideals of* $\mathcal{O}$ *and let* $Y_1, \ldots, Y_k$ *be analytic vector fields. Let* $\mathfrak{p}' \subset \mathfrak{p}$ *be the sheaf of ideals consisting of all germs* g *such that* $Y_i g$ *belongs to* $\mathfrak{p}$ *for all* i. *Then* $\mathfrak{p}'$ *is coherent.*

Proof. — Since the problem is local, we can assume that $\mathfrak{p}$ is generated by analytic functions $g_1, \ldots, g_\ell$. Then the stalk $\mathfrak{p}'_x$ at a point x consists of germs g of the form $\sum_{=1}^{\ell} a_i g_{ix}$ such that $\sum_{i=1}^{\ell} a_i Y_j g_{ix}$ belongs to $\mathfrak{p}_x$ for all j. Hence, if we define a sheaf of modules $\mathfrak{q}$ by

$$\mathfrak{q}_x=\{(a_1, \ldots, a_\ell, b_{11}, \ldots, b_{\ell k}) \in \mathcal{O}_x^{(k+1)\ell} \mid \sum_{i=1}^{\ell} a_i Y_j g_{ix} - \sum_{i=1}^{\ell} b_{ij} g_{ix} = 0 \text{ for all } j\},$$

then $\mathfrak{p}'$ is the image of $\mathfrak{q}$ by the homomorphism ρ from $\mathcal{O}^{(k+1)\ell}$ to $\mathcal{O}$ defined by $\rho(a_1, \ldots, a_\ell, b_{11}, \ldots) = \sum_{i=1}^{\ell} a_i g_i$. Oka's theorem implies that $\mathfrak{q}$ and hence $\mathfrak{p}'$ are coherent. The lemma is proved.

An immediate corollary of the theorem is

Corollary **(7.3)**. — *The conjecture I holds for functions if the dimension of the manifold is* 1 *or* 2.

As corollaries we shall give conditions for a closed subset of a manifold to be equivalent to an analytic set. We call a point of a closed set *regular* if the set is C^∞ smooth in a neighborhood of the point and *singular* otherwise. An analytic set is called *coherent* if the sheaf of germs vanishing on the set is coherent.

Corollary **(7.4)**. — *Let* A *be a closed subset of a manifold. Assume that the germ* A_x *at each point is defined by the germ of a function of one of the types considered in the theorem above. Then* A *is equivalent to an analytic set.*

Proof. — Let M be the manifold, let R be the set of regular points of A of codimension 1, and let S be the exceptional discrete set in the sense of the theorem. Namely, for any point x of A outside S, A_x is the germ of the zero set of a function of the form (i) or (iii). By the proof of Corollary (6.6), we can assume the existence of an open neighborhood U of S such that $A \cap U$ is the zero set of an analytic function g. Here g has the form (i) or (iii) outside S and is regular on R. This regularity is shown as follows. Let x be an adherence point of R. Then A_x is defined by an analytic function germ of the form (i) or (iii) outside x. Let $\prod_{i=1}^{m} h_i^{\alpha_i}$ be the unique factorization of the germ in the stalk $\mathcal{O}_x$. Consider $\prod_{i=1}^{m} h_i$ instead of the germ. Then g is regular on R and satisfies the properties.

Let $\mathcal{G}$ be the subsheaf of $\mathcal{F}$ of germs whose zero set germs are the germs of A, which take the form (i) or (iii) on A outside S, are analytic on U and are regular on R. We introduce an equivalence relation on $\mathcal{G}$ as follows. Two germs at the same point are equivalent if they have the same sign at all points near the given point. Let M′ be the set of equivalence classes. We have a natural mapping p from M′ to M. It is easily seen that p is onto and that the inverse image of each point consists of two points. We give M′ a structure of an analytic manifold such that p is a 2-fold covering. Then, by using a partition of unity, we can find a function f' on M′ such that $f'^{-1}(0) = p^{-1}(A)$ and that f' is analytic on $p^{-1}(U)$ and outside a neighborhood of $p^{-1}(A)$, takes the form (i) or (iii) on $p^{-1}(A)$ outside $p^{-1}(S)$ and is regular on $p^{-1}(R)$. Let u be the non-identity diffeomorphism of M′ such that $p \circ u = p$. Considering $f' - f' \circ u$ instead of f' if necessary, we can assume that $f' = -f' \circ u$. We denote by X'_i the vector field $p^* X_i$ on M′ for $i = 1, \ldots, N$. In view of the theorem applied to f' and X'_i, the function f, is equivalent to an analytic function. Furthermore, the proof tells us that the diffeomorphism commutes with u, hence induces a diffeomorphism of M. It is obvious that the induced diffeomorphism transforms A into an analytic set. Thus the corollary is proved.

Examples. — 1) Let A be a closed set whose germ at each point is equivalent to the germ of a coherent analytic set with isolated singularity. Then A satisfies the condition above. Hence A is equivalent to an analytic set.

2) The above analytic set is coherent. But the coherence is not always necessary. An example of a non-coherent analytic set which satisfies the condition is the umbrella $\{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_3(x_1^2 + x_2^2) = x_1^3\}$.

Corollary (7.5). — *Let* A *be a closed subset of a metric manifold whose germ at each point is equivalent to the germ of an analytic set with isolated singularity. Assume that the connected components of the set of regular points are all unbounded and have the same dimension, and that the intersection of* A *with any sphere of sufficiently small radius centered at a singular point is a disjoint union of spheres. Then* A *is equivalent to an analytic set.*

Proof. — Let M and R be the manifold and the set of regular points of A respectively, and let k be the dimension of R. Let s be a singular point of A and let B be a ball of center s and small radius. Then $R \cap \partial B$ is a union of spheres $S_1, \ldots, S_\ell$. We can transform A by a diffeomorphism so that A is analytic in a neighborhood of B. Let g be the sum of the squares of the generators of the ideal of $\mathcal{O}_s$ consisting of the germs vanishing on S_s. We assume the convergence of g on B and consider g as a function on B. Then g takes the form $x_1^2 + \ldots + x_{n-k}^2$ in some local coordinate system around each point outside an analytic subset of A near B, and the dimension of the subset is equal to or smaller than k. Hence we can assume that the intersection of the subset with S_i is close to a point b_i for each i.

Let $\alpha_i : [0, \infty) \to R - \mathrm{Int}\, B$ be C^∞ proper simple curves from b_i to infinity for $i = 1, \ldots, \ell$, which do not intersect each other. Since the triplet (M, A, one curve) is diffeomorphic to $((-1, \infty) \times \mathbf{R}^{n-1}, (-1, \infty) \times \mathbf{R}^{k-1}, [0, \infty))$ in a neighborhood of the curve, there is a C^∞ imbedding α of M into itself such that, for all i, $\alpha(A)$ is contained in α_i and close to b_i, that $\alpha(A) \cap \alpha_i = \alpha(\alpha_i)$ and that α is the identity outside a neighborhood of the curves. Let β be an approximation of α which is analytic on a neighborhood of B and the curves, and is the identity outside another neighborhood. Then $g \circ \beta$ is analytic near B and the curves, and $\beta^{-1}(A)$ is equivalent to A. From the construction of β, $g \circ \beta$ takes locally the form $x_1^2 + \ldots + x_{n-k}^2$ outside a small neighborhood of B and the curves. We repeat this for each singular point of A. Then, by means of a partition of unity we construct a function f on M which is analytic on an open set containing all the singular points, takes locally the form $x_1^2 + \ldots + x_{n-k}^2$ outside the open set and is such that $f^{-1}(0) = A$. Hence the theorem implies our corollary.

8. Analytic equivalence

This section deals with the conjecture II for functions. We will need the following lemma.

Lemma **(8.1)**. — *Let* A *be a coherent analytic subset of* $\mathbf{R}^n$ *with isolated singularity at* 0. *Then there exists an integer* $m > 0$ *satisfying the following condition. Let* π *be a local diffeomorphism in a neighborhood of* 0 *which maps* A *into itself and whose Taylor expansion at* 0 *is equal to that of the identity up to m-th order. Then there exists a global diffeomorphism* τ *which is equal to* π *on a neighborhood of* 0 *and to the identity outside another such neighborhood, and which maps* A *onto itself. We can choose* τ *arbitrarily close to the identity.*

Proof. — By Hironaka's desingularization theorem, we have an analytic proper mapping p from a smooth manifold $\widetilde{A}$ onto A whose restriction to $p^{-1}(A - \{0\})$ is a diffeomorphism. Let $\widetilde{A}$ be analytically imbedded in $\mathbf{R}^\ell$ and let $T\widetilde{A}$ be the tangent space of A. Let $\| \ \|$ denote the naturally defined length of elements of any Euclidean space. We sometimes regard a tangent vector of $A - \{0\}$ or $\widetilde{A}$ as tangent to $\mathbf{R}^n$ or $\mathbf{R}^\ell$ respectively.

We define two analytic functions $\rho_{\sim}$ and ρ on $T\tilde{A}\times T\tilde{A}$ as follows. Let t and t' be tangent vectors of $\tilde{A}$ at t_1 and t'_1 respectively. We put

$$\rho_{\sim}(t, t') = ||t-t'||^2 ||p(t_1)||^2$$

and

$$\rho(t, t') = ||p_*(t) - p_*(t')||^2.$$

We compare the zero sets of the functions. Since the zero set of $\rho_{\sim}$ contains that of ρ, we have, by Łojasiewicz' inequality

$$\rho_{\sim}(t, t')^r \leq \rho(t, t')$$

for (t, t') in $T\tilde{A}\times T\tilde{A}$ such that t_1 and t'_1 are near to $p^{-1}(0)$, where r is a positive constant.

Let t be in $T\tilde{A}$ such that $p(t_1)$ is near but not equal to 0. We write $t=(t_1, t_2)$ regarding it as an element of $T\mathbf{R}^\ell = \mathbf{R}^\ell \times \mathbf{R}^\ell$ and we put $t' = p_*^{-1}(\pi_* p_* t)$. It is clear that t' is uniquely determined. Then we see that

$$\rho(t, t') = ||\pi_* p_* t - p_* t||^2 \leq C_1(||p(t_1)|| + ||p_* t_2||)^2 ||p(t_1)||^{2m}$$

for some constant $C_1 > 0$. Hence we have

$$||t-t'||^r ||p(t_1)||^r \leq C_1(||p(t_1)|| + ||p_* t_2||)\, ||p(t_1)||^m.$$

If we assume $||t_2|| = 1$ and choose m large, this implies that

$$||t-t'|| \leq C_2 ||p(t_1)||$$

for some constant $C_2 > 0$. Hence, by the trivial inequality

$$||p(t_1)|| \leq C_3 \operatorname{dist}(t_1, p^{-1}(0)),$$

we have

$$||t-t'|| \leq C_4 \operatorname{dist}(t_1, p^{-1}(0))$$

for $||t_2|| = 1$ where C_3 and C_4 are positive constants. This means that the mapping $p^{-1}\circ\pi\circ p$ defined on $\tilde{A} - p^{-1}(0)$ is extensible to $p^{-1}(0)$ and that the extension is of class C^1 and equal to the identity on $p^{-1}(0)$ up to the first partial derivatives. We apply this method to $T\ldots T\tilde{A}$. Then we prove in the same way that for any m_1 the extension is of class m_1 and equal to the identity up to the m_1-th partial derivatives if m is large.

Let $\varphi : \mathbf{R}\to\mathbf{R}$ be a decreasing function which is 1 on $(-\infty, 1/2)$ and 0 on $(2/3, +\infty)$. Let $t_1 \mapsto t'_1$ be the above extension of $p^{-1}\pi p$. We denote by t''_1 a projection of

$$t'_1 \varphi(k||p(t_1)||^2) + t_1(1 - \varphi(k||p(t_1)||^2))$$

into $\tilde{A}$ for some sufficiently large k. Then the C^{m_1} mapping $t_1 \mapsto t''_1$ is equal to the mapping $t_1 \mapsto t'_1$ in a neighborhood of $p^{-1}(0)$ and to the identity outside another. Taking sufficiently large m_1 and k, we can assume that the mapping is close to the identity. By definition, this mapping induces a C^∞ mapping π' of A which is equal to $\pi|_A$ in a neighborhood of o and to the identity outside another. From Łojasiewicz' inequality we see that

$$||p(t_1)||^{r'}||t_2|| \leq ||p_*(t_2)||$$

near $p^{-1}(o)$ for some constant $r'>o$, and moreover we prove in the same way as above that π' is close to the identity. Here we endow the space of mappings of A with the topology of uniform convergence of all derivatives on $A-\{o\}$.

We want to extend π' to a Whitney mapping on A (see the definition in [12]) preserving the properties of π'. The idea of the method is the same, so we only give an outline. We assume that the set of regular points of A has codimension n' everywhere. For the general case, we only have to deal with each irreducible components of A separately. By the conditions on A, there are analytic functions $f_1, \ldots, f_e$ defined near o in $\mathbf{R}^n$ such that the set of common zeroes is A and that the Jacobian matrix of the mapping $(f_1, \ldots, f_e)$ has constant rank n' on $A-\{o\}$. Let G be an analytic non-negative function on $\mathbf{R}^{nn'}$ such that for n' vectors $y_1, \ldots, y_{n'}$ in $\mathbf{R}^n$, $G(y_1, \ldots, y_{n'})$ is o if and only if $y_1, \ldots, y_{n'}$ are linearly dependent. For any $\alpha=(i_1, \ldots, i_{n'})$, $1 \leq i_1 < \ldots < i_{n'} \leq e$ we put

$$G_\alpha = G(\operatorname{grad} f_{i_1}, \ldots, \operatorname{grad} f_{i_{n'}}).$$

Then o is the only common zero point of all G_α. Hence, by Łojasiewicz' inequality we have

$$\max_\alpha G_\alpha(x) > ||X||^{r''}$$

for any x near but not equal to o, where r'' is a positive constant. We put

$$U_\alpha = \{x \mid G_\alpha(x) > ||x||^{r''}\}.$$

The sets U_α are open semi-analytic, and $\{o\} \bigcup_\alpha U_\alpha$ is a neighborhood of o. Let p_α be the mapping from $T\tilde{A} \times \mathbf{R}^{n'}$ to $T\mathbf{R}^n\Big|_A = A \times \mathbf{R}^n$ defined by

$$p_\alpha(t, x) = (p(t_1), p_*(t_2) + \sum_{j=1}^{n'} \operatorname{grad} f_{i_j}(p(t_1))\, x_j)$$

where $t=(t_1, t_2)$ is in $T\tilde{A}$ and $x=(x_1, \ldots, x_{n'})$ in $\mathbf{R}^{n'}$. We consider p_α on $p_\alpha^{-1}(U_\alpha \times \mathbf{R}^n)$. In the same way as for the mapping p_* on $T\tilde{A}$, we show the existence of a C^1 Whitney mapping π'_α on U_α which is equal to the Whitney mapping induced by π in a neighborhood of o and to the identity outside another. Furthermore, π'_α can be chosen to be arbitrarily close to the identity and equal to π' as a C^0 mapping. Hence, using a partition of unity on $\{U_\alpha\}$, one sees that there exists a C^1 Whitney mapping π'' on A which has the same properties as π'_α. It is easy to modify π'' to be C^∞; moreover we can realize π'' by a diffeomorphism τ of $\mathbf{R}^n$ preserving the required properties. The lemma is proved.

Lemma (8.1) is equivalent to the next statement.

Lemma **(8.2)**. — *Let* A_1 *and* A_2 *be coherent analytic subset of* $\mathbf{R}^n$ *with isolated singularity at* o. *Assume given a diffeomorphism* π *of* $\mathbf{R}^n$ *such that* $\pi(A_1)=A_2$. *Then there exists an integer* $m>0$ *with the following property. Let* π' *be a local diffeomorphism in a neighborhood of* o, *which maps* A_1 *into* A_2 *and whose Taylor expansion at* o *is equal to that of* π *up to order* m. *Then there exists a global diffeomorphism* π'' *which is equal to* π' *on a neighborhood of* o *and to* π *outside another, and which maps* A_1 *onto* A_2. *We can choose* π'' *arbitrarily close to* π.

We generalize this as follows.

Lemma **(8.3)**. — *Let* A_1 *and* A_2 *be analytic subsets of* $\mathbf{R}^n$ *whose germs at any point* x *are the zero set germs of functions of the type considered in Theorem* (7.1) *if* $x=0$, *and of functions of the form* (iii) *of the introduction if* $x\neq 0$. *Then the conclusions of Lemma* (8.2) *hold.*

Proof. — Let S_1 or S_2 be the subset of A_1 or A_2 respectively consisting of all singular points and all regular points where the germs of A_1 or A_2 are of dimension $\leq n-2$. By the assumption made on A_2, there is an analytic function h in a neighborhood of o in $\mathbf{R}^n$ such that $h^{-1}(0)=A_2$ near o, that for any non-zero x in S_2 the germ h_x takes locally the form (iii) and that h is regular outside S_2 (see the proof of the remark (6.2)). Let φ be a function on $\mathbf{R}^n$ which is equal to 1 in a neighborhood of o and o outside another. We put

$$h_k=(h\circ\pi'(x))\,\varphi(kx)+(h\circ\pi(x))\,(1-\varphi(kx))$$

for large numbers k. We want to prove that h is transformed into h_k by a diffeomorphism π'' which is equal to π' in a neighborhood of o and to π outside another. That diffeomorphism π'' will map A_1 to A_2, because h_k vanishes on A_1.

We see that S_1 and S_2 are coherent analytic sets with singularity at most at o. Hence, by the lemma above, there exists a global diffeomorphism π''_s which is equal to π' on a neighborhood of o and to π outside another, and which maps S_1 onto S_2. If k is sufficiently large, we have

$$h_k=(h\circ\pi''_s(x))\,\varphi(kx)+(h\circ\pi(x))\,(1-\varphi(kx)).$$

Comparing $h\circ\pi''_s$ and h_k, we see that their difference is small and contained in the ideal of the ring of functions generated by $\dfrac{\partial h\circ\pi''_s}{\partial x_i}\dfrac{\partial h\circ\pi''_s}{\partial x_j}$ for $i,j=1,\ldots,n$, if we take m large and π''_s close to π. Hence the pair $(h, h_k\circ\pi''^{-1}_s)$ satisfies the conditions of Lemma (6.1) and Remark (6.2.3). Therefore h and $h_k\circ\pi''^{-1}_s$ are equivalent by a diffeomorphism which is equal to the identity on a neighborhood of o and outside another. This shows that h is transformed into h_k by a diffeomorphism π'' of the desired form, and the lemma is proved.

Now we establish one of our main results.

Theorem **(8.4)**. — *Let* f_1 *and* f_2 *be analytic functions on a manifold* M. *Assume that they are equivalent and take locally the forms* (i), (ii), *or* (iii) *except on discrete sets. Then* f_1 *and* f_2 *are analytically equivalent.*

Proof. — 1) Assume that the sets of critical points of f_1 and f_2 are discrete. We denote them by S_1 and S_2 respectively. By assumption there is a diffeomorphism π of M such that $f_1 = f_2 \circ \pi$. We first reduce the problem to the case where π is analytic in a neighborhood of S_1. This problem is local. Hence we can assume that $M = \mathbf{R}^n$ and $S_1 = S_2 = \{0\}$. By Artin's theorem there exists an analytic local diffeomorphism π' in a neighborhood of 0 whose Taylor expansion at 0 is equal to that of π up to an arbitrarily large order satisfying $f_1 = f_2 \circ \pi'$ near 0. Let φ be as in the proof of the last lemma. We easily see that

$$f_2 \circ (\pi'(x)\,\varphi(kx) + \pi(x)\,(1 - \varphi(kx)))$$

is equal to f_1 in a neighborhood of 0 and outside another, and is close to f_1 for large k. Hence, by Remark (6.2), this function is equivalent to f_1 by a diffeomorphism which is analytic in a neighborhood of 0. Since $\pi'(x)\,\varphi(kx) + \pi(x)\,(1 - \varphi(kx))$ is a diffeomorphism analytic in a neighborhood of 0, we can assume the analyticity of π in a neighborhood of S_1.

Let F be the sum of the squares of $X_i f_1$ for $i = 1, \ldots, N$. Then the zero set of F is S_1. By Lemma (6.4), there exists an analytic mapping π'' from M to $\mathbf{R}^N$ such that $\pi'' - \pi$ is the product of F and a small mapping. Let $\pi^{(3)}$ be the composition of π'' with an analytic projection of $\mathbf{R}^N$ onto M; it is an analytic diffeomorphism on M and is close to π. Consider the difference $f_2 \circ \pi^{(3)} - f_1 = f_2 \circ \pi^{(3)} - f_2 \circ \pi$ on a neighborhood of each point of S_1. Complexifying F and the difference in the same way as in the proof of Remark (6.2.2), we see that the difference is divisible by F. Since the difference is small, f_1 and $f_2 \circ \pi^{(3)}$ are analytically equivalent by Lemma (6.1) and Remark (6.2). Here the diffeomorphism is near to the identity. Hence f_2 is transformed into f_1 by an analytic diffeomorphism which is arbitrarily close to π.

2) *Remarks.* — From the proof above we have: *(a)* In general, let f_1 and f_2 be analytic functions. If f_2 is transformed into f_1 by a diffeomorphism which is analytic in a neighborhood of the set of critical points of f_1, then f_1 and f_2 are analytically equivalent and the diffeomorphism can be chosen arbitrarily close to the previous one.

By Remark (6.7), we prove in the same way as above and as below: *(b)* Let f_1 and f_2 be equivalent analytic mappings between manifolds. Assume that the rank of the Jacobian matrix of f_1 is constant except on a discrete set. Then f_1 and f_2 are analytically equivalent, and the diffeomorphism satisfies the condition in *(a)*. Moreover the same remark as *(a)* holds for mappings.

For the proof, we only have to reduce the problem to the case where the target manifold is $\mathbf{R}^\ell$ and the constant value of the rank is ℓ. The reduction follows from *(c)* below. The reason is the following. We can assume that the target manifold is $\mathbf{R}^{N'}$

for some N', and that f_1 and f_2 are analytically equivalent on U, a neighborhood of the exceptional set. We write $f_1=(f_{11}, \ldots, f_{1N'})$ where the f_{1i} are component functions. For each $\alpha=(i_1, \ldots, i_\ell)$, $1 \leq i_1 < \ldots < i_\ell \leq N'$, let U_α be the subset of the source manifold consisting of the points where the Jacobian matrix of $f_{1\alpha}=(f_{1i_1}, \ldots, f_{1i_\ell})$ has rank ℓ. Then we prove the analytic equivalence of $f_{1\alpha}\big|_{U \cup U_\alpha}$ and $f_{2\alpha}\big|_{U \cup \tau(U_\alpha)}$ where τ is a diffeomorphism such that $f_2 \circ \tau = f_1$. We repeat this for $(\alpha', U \cup U_\alpha \cup U_{\alpha'})$ and so on. The assertion *(b)* ensues.

(c) Let $f_1=(f_{11}, \ldots, f_{1m})$ and $f_2=(f_{21}, \ldots, f_{2m})$ be analytically equivalent germs of analytic mappings at 0 in $\mathbf{R}^n$. Assume that the maximum of the rank of the Jacobian matrix of f_1 equals that of $f_1'=(f_{11}, \ldots, f_{1\ell})$ for some ℓ. Then any diffeomorphism, which transforms f_1' into $f_2'=(f_{21}, \ldots, f_{2\ell})$ and whose Taylor expansion of large order at 0 is equal to that of the given one, transforms f_1 to f_2.

Proof. — We can assume that $f_1=f_2$. We have a neighborhood U of 0 in $\mathbf{R}^n$ and a proper analytic subset A of U such that the restriction of f_1 to $U-A$ is a submersion onto its image and that the projection of $f_1(U-A)$ onto $f_1'(U-A)$ is a covering. The proof is then easier than that of Lemma (10.2), to which we refer the reader.

3) The general case. Let π be a diffeomorphism such that $f_1=f_2 \circ \pi$. Let S or S_i' be the sets of exceptional critical points or other critical points respectively of f_i for $i=1$ and 2. By *(a)* we only have to modify π into an analytic diffeomorphism in a neighborhood of each connected component of $S_1 \cup S_1'$. Hence we can assume that $S_1 \cup S_1'$ is connected and then the only critical value is 0. We also assume that for any x in S_1', f_{1x} is of the form $\pm x_1^2 + \ldots \pm x_k^2$ for some local coordinate system $(x_1, \ldots, x_n)$ and some integer k, $1 \leq k \leq n$. The other cases follow in the same way as in the proof of Theorem (7.1). We define the notations $F_{(\alpha)}$, $S'_{(\alpha)}$, $S''_{(\alpha)}$ and $S^{(3)}_{(\alpha)}$ for f_2 like $F_\alpha, \ldots$ for f in the proof of Theorem (7.1), and an ordering of the set of all α is fixed. By Artin's theorem, there exists an analytic diffeomorphism π' from a neighborhood of S_1 to a neighborhood of S_2 such that $f_1 \circ \pi' = f_2$ on the neighborhood and that the jets of large degree of π and π' at each point of S_1 are identical. Hence the previous lemma implies the existence of a diffeomorphism π'' of M which is analytic in a neighborhood of S_1, which satisfies $f_2 \circ \pi'' = f_1$ in that neighborhood and such that $\pi''(S_1 \cup S_1') = S_2 \cup S_2'$. Moreover we can choose π'' arbitrarily close to π.

By induction on α, we shall modify π'' slightly in such a way that the analyticity of π'' and the relation $f_2 \circ \pi'' = f_1$ become valid in a neighborhood of $S_1 \cup \pi''^{-1}(S''_{(\alpha)})$, the equality $\pi''(S_1 \cup S_1') = S_2 \cup S_2'$ being preserved. We assume this done for all β such that $\beta < \alpha$. Then, by definitions, $F_{(\alpha)} \circ \pi''$ is analytic in a neighborhood $U'_{(\alpha)}$ of $\pi''^{-1}(S^{(3)}_{(\alpha)})$, has Jacobian rank k on $\pi''^{-1}(S''_{(\alpha)})$ and vanishes on $\pi''^{-1}(S'_{(\alpha)})$. For each i, $(X_i f_2) \circ \pi''$ is a linear combination of $X_1(f_2 \circ \pi''), \ldots, X_N(f_2 \circ \pi'')$ with functions as coefficients. By the induction hypothesis we assume that $f_1 = f_2 \circ \pi''$ hence $X_j f_1 = X_j(f_2 \circ \pi'')$ on $U'_{(\alpha)}$ for all j. On the other hand, the canonicity of f_1 on S_1' means that $X_j(f_2 \circ \pi'')$ is a linear combination of $X_1 f_1, \ldots, X_N f_1$ in a neighborhood of S'_1. Hence we have functions a_{ij}, for $i, j = 1, \ldots, N$, in a small neighborhood $U_{(\alpha)}$ of $\pi''^{-1}(S''_{(\alpha)} \cup S^{(3)}_{(\alpha)})$, such that

$$(X_i f_2) \circ \pi'' = \sum_{j=1}^{N} a_{ij} X_j f_1 \text{ on } U_{(\alpha)}$$

for all i, and that a_{ij} are analytic in a neighborhood of $\pi''^{-1}(S^{(3)}_{(\alpha)})$, by Cartan's theorem. Let F be a non-negative analytic function on $U_{(\alpha)}$ whose zero set is contained in $U'_{(\alpha)}$ and whose restriction to $U'_{(\alpha)}$ is the product of a function by the sum of the squares of all the Jacobian minor determinants of degree k of $F_{(\alpha)} \circ \pi''$. The existence of such function follows from Cartan's theorem. Let $\mathfrak{p}$ be the sheaf of ideals of $\mathcal{O}$ on $U_{(\alpha)}$ of germs 1-flat on $S_1 \cup S'_1$, and let $\mathfrak{p}'$ be the intersection of $\mathcal{O}$ on $U_{(\alpha)}$ with the sheaf of ideals of $\mathscr{F}$ on $U_{(\alpha)}$ generated by $X_1 f_1, \ldots, X_N f_1$. We put $\mathfrak{p}'' = [\mathfrak{p}'^2 : \mathfrak{p}]$, the quotient ideal. Those sheaves are coherent. We have $\mathfrak{p} = \mathfrak{p}'^2$ outside S_1. Hence the zero set of $\mathfrak{p}''$ is contained in S_1. Let G be a cross-section of $\mathfrak{p}''$ such that $G^{-1}(0) \subset S_1$.

By Lemma (6.4) there exists a small function b_{ij} on $U_{(\alpha)}$ for each i and j such that $a_{ij} + FGb_{ij}$ is analytic. Then

$$F'_{(\alpha)} = F_{(\alpha)} \circ \pi'' + \left(\sum_{j=1}^{N} b_{i_1 j} X_j f_1, \ldots, \sum_{j=1}^{N} b_{i_k j} X_j f_1\right) FG$$

is analytic on $U_{(\alpha)}$, and $F_{(\alpha)}$ and $F_{(\alpha)} \circ \pi''$ are equivalent, by (6.7). Hence $F'_{(\alpha)}$ and $F_{(\alpha)}$ are analytically equivalent, by *(b)* in 2). Here the diffeomorphism $\pi^{(3)}$ is close to π'' on $U_{(\alpha)}$ and satisfies $\pi^{(3)}(S'_1 \cap U_{(\alpha)}) \subset S'_2$. Moreover, in the neighborhood of any point x of S_1, $\pi^{(3)}$ is the sum of π'' and the product of G and a mapping for some local coordinate system at $\pi''(x)$ by Remark (6.2.2). Hence we have

$$f_2 \circ \pi^{(3)} = f_2 \circ \pi'' + G \times \text{a function}$$

on $U_{(\alpha)}$. This implies that, near S_1, $f_2 \circ \pi^{(3)} - f_1$ is the product of G and a function. Since G does not vanish on S'_1, $f_2 \circ \pi^{(3)} - f_1$ is divisible by G. The quotient is 1-flat on $(S'_1 \cup S_1) \cap U_{(\alpha)}$ because $f_2 \circ \pi^{(3)}$ and f_1 are 1-flat there. Hence it is a cross-section of $\mathfrak{p}$. Therefore, by the definition of G, $f_2 \circ \pi^{(3)} - f_1$ is a cross-section of $\mathfrak{p}'^2$. It follows that on $U_{(\alpha)}$, f_1 and $f_2 \circ \pi^{(3)}$ satisfy the conditions of Lemma (6.1) and Remark (6.2). Thus we have proved that f_2 is transformed into f_1 on $U_{(\alpha)}$ by an analytic diffeomorphism arbitrarily close to π'' on $U_{(\alpha)}$. Hence, by induction on α, the theorem is proved.

Remark **(8.5)**. — In the theorem above, if the diffeomorphism π which satisfies $f_1 = f_2 \circ \pi$ is analytic on an open set, we may admit any form of f_1 on that set. We can also choose the analytic diffeomorphism arbitrarily close to the previous one.

We now give a condition for two analytic sets to be analytically equivalent.

Corollary **(8.6)**. — *Let* A_1 *and* A_2 *be equivalent coherent analytic subsets of a manifold* M. *Assume that the germs* A_{1x} *and* A_{2x} *at each point are defined by germs of analytic functions of the type considered in Theorem* (8.4). *Then* A_1 *and* A_2 *are analytically equivalent.*

Proof. — For $i=1$ and 2, let R_i be the set of regular points of A_i of codimension 1 and S_i be the exceptional discrete set of A_i. We put $S'_i = A_i - R_i - S_i$ and denote by Z_i the closure of $A_i - \overline{R}_i$. For $i=1, 2$, $\overline{R}_i$ and Z_i are coherent analytic sets. Let $\mathfrak{p}$ be the sheaf of ideals of $\mathcal{O}$ of germs vanishing on Z_1. By Cartan's theorem we find in the same way as in the proof of Theorem (7.1) a finite number of cross-sections of $\mathfrak{p}$ which generate $\mathfrak{p}$ outside $S_1 \cap Z_1$. We let g be the sum of the squares of the cross-sections. It is trivial that $g^{-1}(0) = Z_1$ and that g takes locally the form $x_1^2 + \ldots + x_k^2$ on $Z_1 - S_1$.

Let g' be a function on M whose zero set is A_1, is regular on R_1 and takes locally the form $\pm x_1^2 \pm \ldots \pm x_k^2$ on S_1. Such a function does not always exist. But if we use the two fold covering $p : M' \rightarrow M$ of the proof of Corollary (7.4) and if we consider $p^{-1}(A_1)$ and $p^{-1}(A_2)$ in M' instead of A_1 and A_2, then the existence of g' follows by means of a partition of unity. Of course the diffeomorphisms of M' which will be constructed step by step must commute with the non-identity diffeomorphism u such that $p \circ u = p$ in order to induce diffeomorphisms of M. We assume the existence of g' for the sake of brevity. We modify g' as follows to make it analytic. Let $\mathfrak{p}'$ be the sheaf of ideals of $\mathcal{O}$ of germs vanishing on R_1. It is coherent, since $\overline{R}_1$ is. In the same way as for the construction of g, we have a non-negative analytic function h on M whose zero set is the closure of R_1 and whose germ at each point generates the stalk of $\mathfrak{p}'^2$. Since the stalks of $\mathfrak{p}'$ are principal ideals, the square roots of germs of h are analytic. We put

$$h'(x) = \begin{cases} h^{1/2}(x) & \text{if} \quad g(x) \geq 0 \\ -h^{1/2}(x) & \text{if} \quad g(x) \leq 0 \end{cases}$$

on M. Taking $g' = h'g$, we can from the start assume g' analytic.

Let π be the given diffeomorphism which maps A_2 onto A_1. We can assume that $S_1 = S_2 = S$ and that π is the identity on S and arbitrarily close to the identity on M. We want to transform π slightly so that it becomes analytic in a neighborhood of S. The problem is local, therefore we suppose $M = \mathbf{R}^n$ and $S = \{0\}$. Let $\varphi_1, \ldots, \varphi_\ell$ or $\psi_1, \ldots, \psi_m$ be a system of generators of the ideal of $\mathcal{O}$ of germs vanishing on A_1 or A_2 respectively. There exist formal power series a_{ij} for $i = 1, \ldots, \ell$ and $j = 1, \ldots, m$ such that for each i

$$\varphi_i \circ \pi = \sum_{j=1}^{m} a_{ij} \psi_j$$

(cf. the theorem on page 90 of [12]). By Artin's theorem applied to these equalities, there exist an analytic local diffeomorphism π' in a neighborhood of 0 and analytic functions a'_{ij} for $i = 1, \ldots, \ell$ and $j = 1, \ldots, m$ whose Taylor expansions at 0 of large order equal π and a_{ij} respectively and which satisfy

$$\varphi_i \circ \pi' = \sum_{j=1}^{m} a_{ij} \psi_j$$

for each i. These equalities imply $\pi'(A_{20}) \subset A_{10}$. As π' is a local diffeomorphism, we see that $\pi'(A_{20}) = A_{10}$. By Lemma (8.3) we can connect π' and π preserving the property that A_2 is mapped onto A_1. Hence we can assume that π is analytic near S.

Secondly we will modify π to make it analytic in a neighborhood U of Z_2. Choosing U small and transforming π so that it be close to the identity, we can assume that π is analytic in a neighborhood of $U \cap (S \cup \bar{R}_2)$ and that U contains Z_1. We put $g_2 = g \circ \pi$. Then g_2 is analytic near $U \cap (S \cup \bar{R}_2)$ and takes locally the form $x_1^2 + \ldots + x_k^2$ on $Z_2 - S$. Let $\mathfrak{q}$ be the intersection of $\mathcal{O}$ on U and the sheaf of ideals of $\mathcal{F}$ on U generated by $X_1 g_2, \ldots, X_N g_2$. Then the zero set of $\mathfrak{q}$ is Z_2 and $\mathfrak{q}$ on $Z_2 - S$ is the sheaf of ideals of germs vanishing on $Z_2 - S$. Let μ_i for $i = 1, \ldots, r$ be cross-sections of $\mathfrak{q}$ which generate $\mathfrak{q}$ on $Z_2 - S$. By a method which we used repeatedly, we have an analytic function ν on U and functions ξ_{ij} on U for $i, j = 1, \ldots, r$ such that

$$g_2 - \nu = \sum_{i,j=1}^{r} \xi_{ij} \mu_i \mu_j h_2'$$

where h_2' is defined for R_2 in the same way as h' for R_1. Let ξ_{ij}' be an analytic strong approximation of ξ_{ij} for all i and j. Then g_2 and

$$G_2 = \nu + \sum_{i,j=1}^{r} \xi_{ij}' \mu_i \mu_j h_2'$$

satisfy the conditions of Lemma (6.1). Hence they are equivalent, and the diffeomorphism is analytic near $U \cap (S \cup \bar{R}_2)$ and is the identity on $S \cup \bar{R}_2$. This fact and Remark (8.5) imply that g and G_2 are analytically equivalent and that the diffeomorphism of U can be chosen arbitrarily close to π on U. We easily see that the zero set of G_2 is Z_2. Thus π can be made analytic in U. Henceforth we assume that π is analytic in a neighborhood of $Z_2 \cup S$.

Finally we will modify π to make it analytic globally. Let $\mathfrak{a}$ be the intersection of $\mathcal{O}$ and the sheaf of ideals of $\mathcal{F}$ generated by $g_2' = g' \circ \pi$. For any point x outside Z_2, $\mathfrak{a}_x$ is the ideal of germs vanishing on A_{2x}. Hence, by Cartan's theorem there exist a finite number of cross-sections of $\mathfrak{a}$ whose germs generate the stalk of $\mathfrak{a}$ at each point. Let σ be the sum of the squares of those cross-sections. We put

$$\sigma' = \begin{cases} \sigma^{1/2} & \text{if} \quad g_2' \geq 0, \\ -\sigma^{1/2} & \text{if} \quad g_2' \leq 0. \end{cases}$$

Then σ' is divisible by g_2' and the quotient α is positive. Considering the sheaf of ideals of $\mathcal{O}$ whose stalk at $x \in M$ is the quotient ideal [the ideal generated by $(X_i g_2' X_j g_2')_x$ for $i, j = 1, \ldots, N : \mathfrak{a}$], we construct a non-negative analytic function ρ on M whose zero set is contained in S and which satisfies

$$\rho g_2' = \sum_{i,j=1}^{N} \lambda_{ij} X_i g_2' X_j g_2'$$

for suitable functions λ_{ij}. By Lemma (6.4) there exists a small function β such that $\nu = \log \alpha - \beta\rho$ is analytic. Then there is a small function β' which satisfies

$$1 + \beta'\rho = e^{\beta\rho} = \alpha e^{-\gamma}.$$

Hence $(1+\beta'\rho)g_2' = \sigma' e^{-\gamma}$ is analytic. We can apply Lemma (6.1) to g_2' and $(1+\beta_\rho')g_2'$, and thus see the equivalence of the analytic functions g' and $(1+\beta'\rho)g_2'$. It is trivial that the zero sets of those functions are A_1 and A_2 respectively and that the functions are of the type mentioned in the Theorem (8.4). The corollary ensues, in view of the theorem.

In the proof above we essentially used the coherence assumption on the analytic sets when we proved the existence of an analytic function whose zero set is the given analytic set. But we cannot replace the assumption by the existence of such a function in the corollary. See the following example. On the other hand, the assumption on the type of the function germs does not seem necessary, though the author could not generalize the corollary except in special cases.

Example. — We put $M = \mathbf{R}^3$, $A_1 = \{(x, y, z) \mid x^2 + y^2(1/2 + \sin z) = 0\}$ and

$$A_2 = \{(x, y, z) \mid (x^2 + y^2(1/2 + \sin z))(x^2 + y^2(1/2 + \sin(\pi + z))) = 0\}.$$

One easily sees that A_1 and A_2 are equivalent and that locally the zero sets of the function germs are of the type specified in the theorem. But A_1 is the zero set of an irreducible analytic function. If A_1 and A_2 were analytically equivalent, then A_2 would have the same property. This is impossible.

We now consider a generalization of the conjectures I and II.

Conjecture I'. — *Let $f: M_1 \to M_2$ be a mapping of manifolds. Consider locally finite analytic closed submanifolds of* M_1. *Suppose that the germ of f at each point is transformed into a germ of an analytic mapping by a local diffeomorphism under which the submanifolds are invariant. Then f is transformed into an analytic mapping by such a diffeomorphism.*

Conjecture II'. — *With the same manifolds and submanifolds as above, if an analytic mapping is transformed into another analytic mapping by a diffeomorphism which has the same property as above, then we can choose the diffeomorphism to be analytic, the property being preserved.*

We are far from the solution. We can only prove:

Theorem **(8.7)**. — *In Conjecture* I' *we assume that for each submanifold, the Jacobian matrix of the restriction of f has constant rank except on a discrete set. Suppose further that at any point x of* M_1 *there exists a local coordinate system such that each submanifold is defined locally at x by the vanishing of some of the coordinates. Then the conjecture holds.*

Theorem **(8.8)**. — *In Conjecture* II', *if the mappings and the manifolds have the same property as above, then the conjecture is correct.*

The idea of the proof is the same for both theorems, and the proof of the first one is easier. Hence we prove only Theorem (8.8). We first establish a lemma.

Lemma **(8.9)**. — *With the same manifold* M_1 *and submanifolds* $L_1, L_2, \ldots$ *as above, there exist analytic vector fields* $Y_1, \ldots, Y_{N'}$ *on* M_1 *such that for each i the restrictions of* $Y_1, \ldots, Y_{N'}$ *to* $L_i - \bigcup_{j \neq i} L_j$ *are vector fields on* $L_i - \bigcup_{j \neq i} L_j$ *which span the tangent space at each point, and that the same property holds for* $M_1 - \bigcup_j L_j$.

Proof. — For each i we can construct vector fields $X_{i1}, \ldots, X_{iN}$ on L_i which span the tangent space at each point. We also have a non-negative analytic function φ_i on L_i whose zero set is the intersection of L_i with the union of all L_j which do not contain L_i. We assume the connectedness of all L_i. Let $1 \leq k \leq N$ be an integer and let $i_1, i_2, \ldots$ be all the integers such that the dimensions of $L_{i_1}, \ldots$ are equal to k. Let $Y_{k1}, \ldots, Y_{kN}$ be the vector fields on $\bigcup_j L_{i_j}$ defined by $Y_{k1} = \varphi_{i_j} X_{i_j 1}, \ldots$ on L_{i_j}. Then, by the statement below, we can extend Y_{ki} to the union of all L_j whose dimensions are equal to or smaller than $k+1$. We repeat the extensions up to M_1. The extensions for all k and suitable fields $\varphi X_1, \ldots, \varphi X_N$, where φ is a non-negative analytic function on M_1 such that $\varphi^{-1}(0) = \bigcup_i L_i$, provide the proof of the lemma.

" *Let* g_i *be analytic functions on* L_i *for* $i = 1, \ldots$ *such that the restrictions of* g_i *and* g_j *to* $L_i \cap L_j$ *are identical for each i and j. Then there exists an analytic function g on* M_1 *whose restriction to* L_i *is* g_i *for each i.* "

By virtue of Cartan's theorem, the problem is local. Hence we easily reduce the proof to the case where $M = \mathbf{R}^n$ with affine coordinate system $(x_1, \ldots, x_n)$ and $L_1 = \{x_1 = 0\}, \ldots, L_n = \{x_n = 0\}$. We put

$$g_{12}(x_1, \ldots, x_n) = g_1(x_2, \ldots, x_n) + g_2(x_1, x_3, \ldots) - g_1(0, x_3, \ldots)$$

$$g_{123} = g_{12} + g_3(x_1, x_2, x_4, \ldots) - g_{12}(x_1, x_2, 0, x_4, \ldots).$$

$$\vdots$$

Then $g = g_{1\ldots n}$ satisfies the conditions. Hence the lemma.

Proof of Theorem (8.8). — 1) *The case of functions.* We use the same notation as in Theorem (8.4) and its proof, and we proceed in the same way. Let $L_1, \ldots$ be the submanifolds. We assume the connectedness of all L_i. We apply the lemma above to L_i where the restriction of f_1 is not constant. We use the resulting vector fields $Y_1, \ldots, Y_{N'}$ in place of $X_1, \ldots, X_N$. Then, in the step of the reduction to the case where π is analytic near the exceptional points set, we see that, by Lemma (6.1), L_i is invariant under π if the restriction of f_1 is not constant. For another L_i

$$f_2 \circ (\pi'(x)\,\varphi(kx) + \pi(x)\,(1 - \varphi(kx))) - f_1(x) = 0$$

on L_i. Hence, by Remark (6.2.2), L_i is also invariant.

We need to modify $\pi^{(3)}$ a little, so that it leave L_i invariant, its other properties being preserved. The rest of the proof proceeds in the same way as above. We consider the intersection of $L_{i_1}, L_{i_2}, \ldots$ for any integers $i_1, \ldots$ Let L'_i for $i=0, \ldots, n$ be the union of the connected components of dimension i of all such intersections. We assume that L'_0 is contained in the set of exceptional points. We assume by induction that a modification $\pi_1^{(3)}$ is defined on the union of L'_i for $i \leq k-1$. We want to extend $\pi_1^{(3)}$ on L'_k. For this, we only have to examine the case $k=n-1$. Let $\mathfrak{p}$ be the sheaf of ideals of $\mathcal{O}$ of germs vanishing on all L_i and let F be the sum of the squares of $Y_i f_1$ for $i=1, \ldots, N'$. Then there is an exact coherent sequence

$$0 \to \mathfrak{p} \cap F\mathcal{O} \to F\mathcal{O} \to (F\mathcal{O} + \mathfrak{p})/\mathfrak{p} \to 0.$$

Regarding mappings to M_1 as to $\mathbf{R}^N$, we can analytically extend $\pi_1^{(3)}$ to M_1 by the statement in the proof of the above lemma. Then each component function of $\pi_1^{(3)} - \pi^{(3)}\Big|_{\bigcup_i L_i}$ is a small cross-section of $(F\mathcal{O} + \mathfrak{p})/\mathfrak{p}$. Hence, by Cartan's theorem, there exists a mapping $\pi_2^{(3)}$ of M_1 to $\mathbf{R}^N$ whose component functions are small cross-sections of $F\mathcal{O}$ such that $\pi_2^{(3)} = \pi_1^{(3)} - \pi^{(3)}$ on $\bigcup_i L_i$. We put $\pi_3^{(3)} = \pi^{(3)} + \pi_2^{(3)}$. Let $\pi_4^{(3)}$ be the composition of $\pi_3^{(3)}$ with an analytic projection of $\mathbf{R}^N$ on M. Then $\pi_4^{(3)}$ is an extension of $\pi_1^{(3)}$ and the difference with $\pi^{(3)}$—and hence with π—is small and divisible by F. Thus we have obtained the desired modification $\pi_4^{(3)}$ of $\pi^{(3)}$.

2) *The case of mappings.* If the ranks of the Jacobian matrices of the restrictions of f_1 to the L_i are identical for all i except on a discrete set, the proof is just the same as the above one and of 2) in Theorem (8.4). In general, we can reduce the problem to the case where $M_2 = \mathbf{R}^m$, $f_j = (f_{j1}, \ldots, f_{jm})$ for $j=1$ and 2, and the rank ℓ_i of the Jacobian matrix of the restriction of f_1 to L'_i equals that of $(f_{11}, \ldots, f_{1\ell_i})$ except on a discrete set. An outline of the proof for this case follows.

The problem is that we have to define a suitable F as above for the modification of $\pi^{(3)}$ in 1). It is trivial that $\ell_i \leq i$. If $\ell_1 = 0$, we put $F_1 = 1$. If $\ell_1 = 1$, we define F_1 by means of f_{11} in the same way as F above. We take analytic vector fields whose restrictions to $L_i - \bigcup_{j \neq i} L_j$ and L_k are vector fields on $L_i - \bigcup_{j \neq i} L_j$ and L_k, and span the tangent spaces if $\dim L_i > 2$ and $\dim L_k = 2$. We denote these fields by $Y_{N'+1}, \ldots, Y_{N''}$. Let F_2 be the sum of the squares of the $\ell_2 \times \ell_2$ submatrix determinants of $(Y_i f_{1j})_{\substack{i=1,\ldots,N'' \\ j=1,\ldots,\ell_2}}$. We define $F_3, \ldots, F_m$ in the same way, and put $F = \prod_{i=1}^{m} F_i$. We use this F for the modification of $\pi^{(3)}$. Thus we obtain a modified analytic diffeomorphism $\pi^{(3)}$ such that each component function of $f_2 \circ \pi^{(3)} - f_1$ is divisible by F and that the quotient is small. By Remark (6.7), there exists an analytic diffeomorphism $\pi^{(4)}$ such that $f_{2\ell_1} \circ \pi^{(3)} \circ \pi^{(4)} = f_{1\ell_1}$ if $\ell_1 = 1$ and that $f_2 \circ \pi^{(3)} \circ \pi^{(4)} - f_1$ is divisible by $\prod_{i=2}^{m} F_i$. By 2) *(c)* in the proof of Theorem (8.4), this implies that $f_2 \circ \pi^{(3)} \circ \pi^{(4)} = f_1$ on L'_1.

Hence the quotient of $f_2\circ\pi^{(3)}\circ\pi^{(4)}-f_1$ by $\prod_{i=2}^{m} F_i$ vanishes on L_1'. Comparing $(f_{11}, \ldots, f_{1l_2})$ and $(f_{21}, \ldots, f_{2l_2})\circ\pi^{(3)}\circ\pi^{(4)}$, we see that they are equivalent by an analytic diffeomorphism $\pi^{(5)}$. Here $\pi^{(5)}$ is chosen to be the identity on L_1, and $f_2\circ\pi^{(3)}\circ\pi^{(4)}\circ\pi^{(5)}-f_1$ is divisible by $\prod_{i=3}^{m} F_i$. Repeating this argument, we see that $f_2\circ\pi^{(3)}\circ\ldots\circ\pi^{(m+3)}=f_1$. Of course the $\pi^{(j)}$ are analytic, and the L_i are invariant under them. Hence the theorem is proved.

Remark **(8.10)**. — We need the C^∞ differentiability of the diffeomorphism in Theorem (8.4). It cannot be replaced by the C^r condition for $0\leqslant r<\infty$. For example, let

$$f_1=(x^2+y^2)^2, \quad f_2=(x^2+y^2)^2+x^{5+r}g(x,y),$$

where $g(x,y)$ is an analytic function on $\mathbf{R}^2$ sufficiently near to the zero function. Then f_1 is transformed into f_2 by a C^r diffeomorphism. But, if $g(0,0)\neq 0$, they are not equivalent.

9. Local canonical forms of mappings.

We have seen that the conjectures on functions with canonical forms are true. Because we want to generalize that result to mappings, we need to find canonical forms of mappings, particularly of analytic mappings, which correspond to those of functions.

As *canonical form* (i)′ corresponding to (i) in the introduction we take mappings whose Jacobian matrix has constant rank. If a mapping germ f takes the form (i)′, we can trivially choose local coordinate systems $x=(x_1, \ldots, x_n)$ and $y=(y_1, \ldots, y_m)$ such that

$$y=f(x)=(f_1(x), \ldots, f_m(x))=(x_1, \ldots, x_l, 0, \ldots, 0)$$

where l is the constant rank. Moreover, if f is a mapping germ to $\mathbf{R}^m$ and if $(y_1, \ldots, y_m)$ is the affine coordinate system of $\mathbf{R}^m$, there exist a local coordinate system $x=(x_1, \ldots, x_n)$ and a permutation σ of $(1, \ldots, m)$ such that

$$(f_{\sigma(1)}(x), \ldots, f_{\sigma(m)}(x))=(x_1, \ldots, x_l, g_{l+1}(x'), \ldots, g_m(x'))$$

where $g_{l+1}, \ldots, g_m$ are germs of functions of $x'=(x_1, \ldots, x_l)$. It is clear that any analytic mapping takes locally the form (i)′ except on an analytic subset of codimension at least 1.

The form (ii)′ will be defined later so that any analytic mapping takes locally (i)′ or (ii)′ except on an analytic subset of codimension at least 2.

Let f be the germ of an analytic mapping at a point a where the Jacobian matrix has maximal rank l, let S be the germ of the subset of all points where the rank is smaller than l, and let $\mathfrak{p}$ be the ideal of $\mathcal{O}_a$ generated by all the determinants of Jacobian $l\times l$ submatrices. We say that f takes *the form* (iii)′ if $\mathfrak{p}$ is generated by $x_1, \ldots, x_k$ for some local coordinate system $(x_1, \ldots, x_k, \ldots, x_n)$ at a, and if the Jacobian matrix of the restriction of f to S has constant rank. Here we remark that if the first condition is satisfied, then S is smooth, and the second condition is satisfied near a except at most on the germ of a proper analytic subset of S.

Proposition **(9.1)**. — *Let* $f: (\mathbf{R}^n, 0) \to (\mathbf{R}^m, 0)$ *be a germ of the form* (iii)′. *Then we can choose local coordinate systems* $x = (x_1, \ldots, x_n)$ *and* $y = (y_1, \ldots, y_m)$ *such that if we write*

$$f(x) = (f_1(x), \ldots, f_m(x)),$$

then

$$f_1(x) = x_1, \ldots, f_{p+q}(x) = x_{p+q},$$

$$f_{p+q+1}(x) = \sum_{i=1}^{r} \pm x_{p+q+i}^2 + \sum_{i=1}^{p} x_{q+i} x_{p+q+r+i},$$

$$f_{p+q+2}(x) = \sum_{i=1}^{r} a_{1i} x_{p+q+i}^2 + \sum_{i=1}^{p} b_{1i} x_{q+i} x_{p+q+r+i},$$

$$\vdots$$

where a_{ij} *and* b_{ij} *are germs of analytic functions in* $x_1, \ldots, x_{2p+q+r}$ *vanishing at* 0. *Here* $p+q+1 = \ell$ *and* $2p+q+r \leq n$.

Proof. — As the restriction of f to S is a submersion to the image, we have local coordinate systems $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ such that

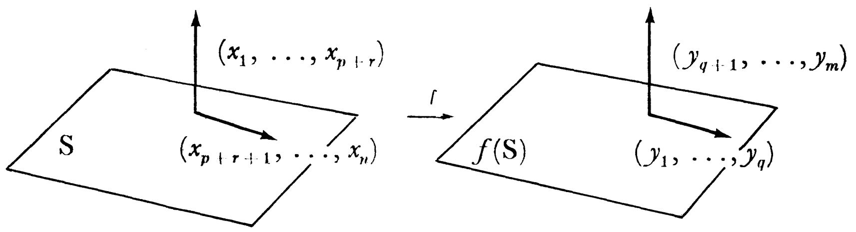
(1) $$\mathrm{S} = \{x_1 = \ldots = x_{p+r} = 0\} \quad \text{and} \quad f(\mathrm{S}) = \{y_{q+1} = \ldots = y_m = 0\}$$

where p and r will be defined later. The mapping

$$(f_1, \ldots, f_q, x_1, \ldots, x_{p+r}) : \mathbf{R}^n \to \mathbf{R}^{p+q+r}$$

is a submersion. Hence, by the implicit function theorem, we may assume

(2) $$f_1 = x_{p+r+1}, \ldots, f_q = x_{p+q+r}.$$



We remark that the rank of the Jacobian matrix of f is $\ell - 1$ on S, because, if it were not so at a point x, all the determinants of the Jacobian $\ell \times \ell$ submatrices would be 1-flat, and this contradicts the condition (iii)′. Hence the Jacobian matrix of the mapping

$(f_{q+1}, \ldots, f_m) : \mathbf{R}^n \to \mathbf{R}^{n-q}$ has rank $\ell - q - 1$ on X. We put $p =$ the rank, and $r =$ the codimension of S minus p. Then, transforming linearly $(y_{q+1}, \ldots, y_m)$, we can assume that $(f_{q+1}, \ldots, f_{q+p}) : \mathbf{R}^n \to \mathbf{R}^p$ is a submersion. It follows trivially that $(f_{q+1}, \ldots, f_{q+p}, x_{p+r+1}, \ldots, x_n)$ is a submersion too. Therefore we can take

(3) $$f_{q+1} = x_1, \ldots, f_{q+p} = x_p.$$

We have chosen the coordinate systems so that $f_{q+p+1}, \ldots, f_m$ are linear combinations of $x_1, \ldots, x_{p+r}$ with functions as coefficients (that is (1)). Moreover we can assume that they are 1-flat at 0. Let $x_1, \ldots, x_p, x_{p+r+1}, \ldots, x_{p+r+q}$ be fixed. Then the image by f is a semi-analytic set of dimension one, and so is the image by $(f_{p+q+1}, \ldots, f_m)$. Hence the restrictions of $f_{p+q+2}, \ldots, f_m$ on

$$\{x_1 = \ldots = x_p = x_{p+r+1} = \ldots = x_{p+r+q} = 0\}$$

can be assumed 2-flat at 0 after a linear transformation of $(y_{p+q+1}, \ldots, y_m)$. We write $f_{q+p+1} = g$. We have $g = \sum_{i=1}^{p+r} g_i x_i$ for some 0-flat analytic functions $g_1, \ldots, g_{p+r}$. Here $g_{p+1}, \ldots, g_{p+r}$ are chosen to be functions in the variables $x_{p+1}, \ldots, x_n$. Consider the Jacobian matrices of f and $(f_1, \ldots, f_{q+p+1})$. By condition (iii)′, both $\mathfrak{p}$ in the definition of (iii)′ are generated by $x_1, \ldots, x_{p+r}$. Hence from (2) and (3) we see that

(4) $$\frac{\partial g}{\partial x_j} \text{ for } j = p+1, \ldots, p+r, p+q+r+1, \ldots, n \text{ generate } \mathfrak{p}.$$

Putting $x_1 = \ldots = x_p = 0$, we have

$$\frac{\partial g}{\partial x_j} = \begin{cases} \sum_{i=p+1}^{p+r} \frac{\partial g_i}{\partial x_j} x_i + g_j & \text{if} \quad j = p+1, \ldots, p+q, \\ \sum_{i=p+1}^{p+r} \frac{\partial g_i}{\partial x_j} x_i & \text{if} \quad j = p+q+r+1, \ldots, n. \end{cases}$$

Hence $g_{p+1}, \ldots, g_{p+r}$ are linear combinations of $x_{p+1}, \ldots, x_{p+r}$. Therefore we have analytic functions $g_{k\ell}$ for $k, \ell = p+1, \ldots, p+r$ in the variables $x_{p+1}, \ldots, x_n$ such that

$$g = \sum_{i=1}^{p} g_i x_i + \sum_{k,\ell=p+1}^{p+r} g_{k\ell} x_k x_\ell.$$

In view of Tougeron's lemma on equivalence (1.3), (4) implies that g is equivalent to

$$\sum_{i=1}^{p} g_i x_i + \sum_{k,\ell=p+1}^{p+r} g_{k\ell}(0) x_k x_\ell$$

by a diffeomorphism which changes only $x_{p+1}, \ldots, x_{p+q}, x_{p+q+r+1}, \ldots, x_n$. Here we remark that the diffeomorphism can be chosen to be the identity on S and hence (1) remains valid. Hence, after a linear transformation of $x_{p+1}, \ldots, x_{p+r}$ we can assume that

$$g = \sum_{j=1}^{p} g_i x_i + \sum_{i=1}^{r} \pm x_{p+i}^2.$$

Apply once more Tougeron's lemma, then the expansions of g_i with respect to $x_{p+1}, \ldots, x_{p+r}$ become polynomials of degree 1 with function in other variables as coefficients. Moreover, taking $x_{p+i}+\sum_{j=1}^{p} g_{ij}x_j$ for some g_{ij} in place of x_{p+i} for $i=1, \ldots, r$, we can assume that the g_i, for $i=1, \ldots, p$, are independent of $x_{p+1}, \ldots, x_{p+r}$. Then (4) means that the rank of the matrix $\left(\frac{\partial g_i}{\partial x_j}\right)_{\substack{i=1,\ldots,p\\ j=p+q+r+1,\ldots,n}}$ is p. Hence, by the implicit function theorem we can take $g_i=x_{p+q+r+i}$ for $i=1, \ldots, p$. Thus we have

$$g=\sum_{i=1}^{p} x_{p+q+r+i}x_i+\sum_{i=1}^{r}\pm x_{p+i}^2.$$

As the rank of the Jacobian matrix of f is at most $\ell=p+q+1$, we see that, for $j=2, \ldots$, $\dfrac{\partial f_{p+q+j}}{\partial x_{p+i}}$ is divisible by x_{p+i} if $i=1, \ldots, r$ and by x_{i-q-r} if

$$i=q+r+1, \ldots, p+q+r,$$

and that it is identically 0 if $i=p+q+r+1, \ldots$ This shows that for $j=2, \ldots$

$$f_{p+q+i}=\sum_{i=1}^{p} a_{ij}x_ix_{p+q+r+i}+\sum_{i=1}^{r} b_{ij}x_{p+i}^2+g'_{p+q+j}$$

where a_{ij} and b_{ij} are functions in $x_1, \ldots, x_{2p+q+r}$, and the g'_{p+q+j} are functions in $x_1, \ldots, x_p$. Considering $y_{p+q+j}-g'_{p+q+j}(y_{q+1}, \ldots, y_{q+p})$ instead of y_{p+q+j} for $j=2, \ldots$ we have $g'_{p+q+j}=0$. We see that a_{ij} and b_{ij} vanish at 0. Therefore the proposition is proved.

Remark (**9.2**). — In the proposition above, if we admit only permutations of the index set $\{1, \ldots, m\}$ of $y=(y_1, \ldots, y_m)$ as changes of the coordinate system in $\mathbf{R}^m$, we have

$$f_1(x)=x_1, \ldots, f_q(x)=x_q,$$

$$f_{q+1}=x_{q+1}+g_1, \ldots, f_{p+q}=x_{p+q}+g_p,$$

$$f_{p+q+1}=\sum_{i=1}^{r}\pm x_{p+q+i}^2+\sum_{i=1}^{p} x_{q+i}x_{p+q+r+i}+g_{p+1}+\sum_{i=1}^{p} c_ix_{q+}$$

$$f_{p+q+2}=\sum_{i=1}^{r} a_{1i}x_{p+q+i}^2+\sum_{i=1}^{p} b_{1i}x_{q+i}x_{p+q+r+i}+g_{p+2}$$

$$\ldots$$

where $g_1, \ldots, g_{p+1}$ (resp. $g_{p+2}, \ldots, g_{m-p}$) are germs of analytic functions in $x_1, \ldots, x_q$ (resp. in $x_1, \ldots, x_{p+q}$), the c_i are constants, and a_{ij} and b_{ij} do not necessarily vanish at 0.

This follows easily from the proof above.

Let $g:\mathbf{R}^n\to\mathbf{R}^m$ be an analytic mapping and let A be the subset of $\mathbf{R}^n$ of points where the rank of the Jacobian matrix of g takes its maximum value ℓ. We want to see what form is most general on $A_0 = \mathbf{R}^n - A$. We may assume that the rank is constant on A_0, that the sum F of the squares of the determinants of all Jacobian $\ell \times \ell$ submatrices has the form (ii), and that the restriction of g to A_0 has the form (i)'. Let us consider the simplest case where $n = m = \ell = 2$. There are only two possibilities: *(a)* the restriction is an immersion or *(b)* it is a constant map. One easily proves that the germ of g at any point of A_0 is R—L equivalent to the mapping $(x, y) \to (x, y^s)$ for some integer $s > 1$ if *(a)* holds and to $(x, y) \to (x^{s_1}, g_1(x)x^{s_1} + yx^{s_2})$ for some function g_1 and integers $0 < s_1 \le s_2$ in case *(b)*. An example is $g(x, y) = (x, xy + y^3)$. In this case

$$A_0 = \{x + 3y^2 = 0\}$$

and the restriction of g to this curve is critical at (0, 0). Hence (0, 0) is an exceptional point.

We propose the following as a generalization of the above forms. We write

$$g(x) = (g_1(x), \ldots, g_m(x)) \quad \text{and} \quad x = (x_1, \ldots, x_n)$$

with

(ii)''

$$g_1(x) = x_1, \ldots, g_p(x) = x_p,$$

$$g_{p+1} = \pm x_{p+1}^{s_1},$$

$$g_{p+2} = x_{p+2}x_{p+1}^{s_2} + g_{11}x_{p+1}^{s_1},$$

$$g_{p+3} = x_{p+3}x_{p+1}^{s_3} + g_{22}x_{p+2}x_{p+1}^{s_2} + g_{21}x_{p+1}^{s_1},$$

$$\vdots$$

$$g_\ell = x_\ell x_{p+1}^{s_{\ell-p}} + g_{\ell-p-1\,\ell-p-1}x_{\ell-1}x_{p+1}^{s_{\ell-p-1}} + \ldots + g_{\ell-p-1\,1}x_{p+1}^{s_1},$$

$$g_{\ell+1} = g_{\ell-p\,\ell-p}x_\ell x_{p+1}^{s_{\ell-p}} + \ldots + g_{\ell-p\,1}x_{p+1}^{s_1},$$

$$\vdots$$

$$g_m = g_{m-p-1\,\ell-p}x_\ell x_{p+1}^{s_{\ell-p}} + \ldots + g_{m-p-1\,1}x_{p+1}^{s_1},$$

where $1 \le s_1 \le \ldots \le s_{\ell-p}$ are integers, g_{ij} is an analytic function in $x_1, \ldots, x_{p+j}$ vanishing at 0, and $g_{ij} = 0$ if $s_i = s_{i+1}$.

Definition. — Let $f\colon M_1 \to M_2$ be a mapping between manifolds, and let x be a point of M_1. We say that f takes *the form* (ii)' at x if there exist local coordinate systems $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ at x and $f(x)$ respectively, such that $x = f(x) = 0$ and that f takes the form (ii)'' in these coordinate systems.

Proposition **(9.3)**. — *Let* $f\colon M_1 \to M_2$ *be an analytic mapping between manifolds. Then the subset of* M_1 *of points at which* f *does not take the form* (ii)' *is an analytic set of codimension at least two.*

Proof. — We suppose $M_1 = \mathbf{R}^n$ and $M_2 = \mathbf{R}^m$ for the sake of brevity. We use the above notations A, A_0, ℓ and F. Let B be the subset of A_0 of points where F does not take the form (ii). Then B is an analytic set of codimension two, $A_0 - B$ is smooth, and f does not take locally the form (ii)′ on B. Let A_i for $1 \leq i \leq \ell$ be the subset of $\mathbf{R}^n$ of points where the rank of the Jacobian matrix of f is equal to or smaller than $\ell - i - 1$, and let C_i be the subset of $A_0 - B$ of points where the Jacobian matrix of the restriction of f to $A_0 - B$ has a rank equal to or smaller than $\ell - i - 1$. It is trivial that the A_i are analytic sets. We want to prove the analyticity of $B \cup C_i$. We consider the radical of the sheaf of ideals of $\mathcal{O}$ generated by F. As each stalk of the sheaf is a principal ideal, we have a finite number of cross-sections $\lambda_1, \ldots$ which are also generators. Then B is the set of common critical points of $\lambda_1, \ldots$ on A_0. We consider the vector fields

$$||\text{grad}\, \lambda_i||^2 \frac{\partial}{\partial x_j} - \left\langle \text{grad}\, \lambda_i, \frac{\partial}{\partial x_j} \right\rangle \text{grad}\, \lambda_i$$

for all i and j, where $\langle\, , \rangle$ means the inner product. Let them be $Y_1, \ldots$ They vanish on B and their restrictions to $A_0 - B$ are vector fields on $A_0 - B$ and span the tangent space at each point. Hence, the sets $B \cup C_i$ are the analytic subset of A_0 of points where the rank of the matrix $(Y_i f_j)$ is equal to or smaller than $\ell - i - 1$. Here $f = (f_1, \ldots, f_m)$. Let F_i or F'_i for $i = 1, \ldots, n$ be the sums of the squares of the determinants of all Jacobian $(\ell - i) \times (\ell - i)$ submatrices of f or of all $(\ell - i) \times (\ell - i)$ submatrices of $(Y_i f_j)$ respectively. Then A_i or $C_i \cup B$ for each i is the zero set of F_i or $F'_i + \sum_i \lambda_j^2$ respectively. Let V_i or V'_i be the subset of A_i or $B \cup C_i$ of points where F_i or F'_i does not take the form (ii) respectively. We denote by V the union of B, all V_i and all V'_i. We easily see that f does not take the form (ii)′ on V, that V is of codimension ≥ 2 and that $A_0 - V$, $A_i - V$ and $C_i - V$ are smooth and of codimension one. It is trivial that $A_i - V \supset C_{i+1} - V \supset A_{i+1} - V$ for all i. We shall study the form of f on $A_i - C_{i+1} - V$ and $C_i - A_i - V$ separately. We remark that $(A_i - C_{i+1}) \cup V$ and $(C_i - A_i) \cup V$ are analytic sets.

We first consider $A_q - C_{q+1} - V$ for each q. By definition, the rank of the Jacobian matrix of f is equal to or smaller than $\ell - q - 1$ on $A_q - C_{q+1} - V$ and that of the restriction of f to $A_q - C_{q+1} - V$ is equal to or larger than $\ell - q - 1$. Since it is trivial that the former is not smaller than the latter, they are identical. We put $\ell - q - 1 = p$. Let W_1 be the subset of $\mathbf{R}^n$ of points where the Jacobian matrix of $(f_1, \ldots, f_p)$ has rank $\leq p - 1$. Let x_0 be a point of $A_q - C_{q+1} - V - W_1$. In a suitable local coordinate system $(z_1, \ldots, z_n)$ at x_0 and by a translation of the coordinate axes $(y_1, \ldots, y_m)$ of $\mathbf{R}^m$, we assume $x_0 = f(x_0) = 0$, $f_1 = z_1, \ldots, f_p = z_p$ and $\{z_{p+1} = 0\} = A_q$. We want to find analytic functions $\varphi_i(y_1, \ldots, y_{i-1})$ for $i = p+1, \ldots, m$ such that the mapping $(f_1, \ldots, f_p, f_{p+1} - \varphi_{p+1}(f_1, \ldots, f_p), \ldots)$ takes the form (ii)″. By the definition of V, the ideal of the ring of analytic functions on a neighborhood of x_0 generated by all the determinants of the Jacobian $(p+1) \times (p+1)$ submatrices of f is the principal ideal generated by $z_{p+1}^{s_1 - 1}$ for some $s_1 > 1$. We pick up Jacobian submatrices of $(f_1, \ldots, f_p, f_{p+i})$ for $i = 1, \ldots, m - p$. Then we have

$$f_{p+i} = h'_i(z) z_{p+1}^{s_1} + h_i(z_1, \ldots, z_p)$$

for $i=1, \ldots, m-p$ and suitable functions h_i and h'_i. Calculating the determinants of other Jacobian submatrices of f, we see that $h'_i(0) \neq 0$ for at least one i. Let W_2 be the set of points where the sum of the squares of the determinants of all Jacobian $(p+1)\times(p+1)$ submatrices of $(f_1, \ldots, f_{p+1})$ is not the product of F_q (that is, the similar sum for f) by a positive function. Assume that x_0 is not contained in W_2. Then W_2 is an analytic set, and $h'_1(0) \neq 0$. Changing z_{p+1}, we get

$$f_{p+1} = \pm z_{p+1}^{s_1} + \varphi_{p+1}(f_1, \ldots, f_p).$$

Next we shall choose z_{p+2}, removing analytic sets W_3 and W_4, in such a way that

$$(*) \qquad f_{p+2} = z_{p+2} z_{p+1}^{s_2} + g_{11}(z_1, \ldots, z_{p+1}) z_{p+1}^{s_1} + \varphi_{p+2}(f_1, \ldots, f_{p+1})$$

for suitable functions φ_{p+2} and g_{11}. Let $\mathfrak{n}$ be the sheaf of submodules of $\mathcal{O}^n$ on $\mathbf{R}^n$ defined by

$$\mathfrak{n}_x = \left\{ (a_1, \ldots, a_n) \in \mathcal{O}_x^n \,\middle|\, \sum_{j=1}^n a_j \frac{\partial f_i}{\partial x_j} = 0 \quad \text{for} \quad i=1, \ldots, p+1 \right\}$$

where $(x_1, \ldots, x_n)$ is the affine coordinate system of $\mathbf{R}^n$. Then it follows that $\mathfrak{n}_x$ is generated by $n-p-1$ elements if x is a point of $A_q - C_{q+1} - V - W_1 - W_2$ and that $\mathfrak{n}$ is coherent by Oka's theorem. Hence there exist a finite number of cross-sections $Z_1, \ldots, Z_{n'}$ of $\mathfrak{n}$ which are generators of $\mathfrak{n}_x$ at any point x of $A_q - C_{q+1} - V - W_1 - W_2$. We regard an element $(a_1, \ldots, a_n)$ of $\mathfrak{n}_x$ as the tangent vector $\sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i}$. Then $Z_1, \ldots, Z_{n'}$ are vector fields on $\mathbf{R}^n$ which satisfy $Z_i f_j = 0$ for $i=1, \ldots, n'$ and $j=1, \ldots, p+1$, and which span the tangent space of $\{z_1 = \ldots = z_{p+1} = 0\}$ at x_0. Let W_3 be the set of points where the sum of the squares of $Z_i f_j$ for all i and j does not take the form (ii). We proceed in the same manner as for W_2. The set W_3 is analytic. We assume that x_0 is not contained in W_3. These facts imply that the ideal of the ring of analytic functions on a neighborhood of x_0 generated by $\frac{\partial f_i}{\partial z_j}$ for $i, j \geq p+2$ is the principal ideal generated by $z_{p+1}^{s_2}$ for some integer $s_2 > 0$. Removing an analytic set W_4 in the same way as W_2 and W_3, and changing $(z_{p+2}, \ldots, z_n)$, we can, in a neighborhood of x_0, assume that $\frac{\partial f_{p+2}}{\partial z_{p+2}}$ is the product of $z_{p+1}^{s_2}$ and a positive function. Then it follows that

$$f_{p+2} = h''_{p+2}(z) z_{p+2} z_{p+1}^{s_2} + \ldots + h''_n(z) z_n z_{p+1}^{s_2} + \varphi'_{p+2}(z_1, \ldots, z_{p+1})$$

for suitable functions h''_i and φ'_{p+2} such that $h''_{p+2}(0) \neq 0$. We substitute z_{p+2} for $h''_{p+2} z_{p+2} + \ldots + h''_n z_n$, and put $\varphi''_{p+2} = \varphi'_{p+2}(z_1, \ldots, z_p, 0)$. Then we get

$$f_{p+2} = z_{p+2} z_{p+1}^{s_2} + \varphi'_{p+2}(z_1, \ldots, z_{p+1}) - \varphi'_{p+2}(z_1, \ldots, z_{p,0}) + \varphi''_{p+2}(f_1, \ldots, f_p).$$

By the definition of s_1, $\varphi'_{p+2}(z_1, \ldots, z_{p+1}) - \varphi''_{p+2}$ is divisible by $z_{p+1}^{s_1}$ and we have $s_2 \geq s_1$.

We write the quotient as g'_{11} and put

$$g_{11} = g'_{11}(z) - g'_{11}(0)$$

and $$\varphi_{p+2}(z_1, \ldots, z_{p+1}) = \varphi''_{p+2} + g'_{11}(0) z_{p+1}.$$

Then the wanted equation (*) is achieved. It is trivial that if $s_2 = s_1$, g_{11} can be chosen identical to 0.

Applying the same method to $f_{p+3}, \ldots, f_m$, we see that there exist a finite number of analytic sets $W_5, \ldots$ such that if x_0 is not contained in the union W_{id} of all W_i, $(f_1, \ldots, f_p, f_{p+1} - \varphi_{p+1}(f_1, \ldots, f_p), \ldots)$ takes the form (ii)″ for some functions $\varphi_{p+1}, \ldots, \varphi_m$. Let σ be a permutation of $(1, \ldots, m)$. Considering the mapping

$$f_\sigma = (f_{\sigma(1)}, \ldots, f_{\sigma(m)})$$

in place of f, we get an analytic set W_σ such that f_σ takes the form of the sum of (ii)″ and a constant map in a suitable local coordinate system around each point of $A_q - C_{q+1} - V - W_\sigma$. Hence f takes locally the form (ii)′ on $A_q - C_{q+1} - V - W$ where W is the intersection of all W_σ. It is easy to see that $W \cap ((A_q - C_{q+1}) \cup V)$ is an analytic set of codimension ≥ 2 and that if f takes the form (ii)′ at a point x_0, x_0 is not contained in at least one W_σ, and hence in W.

On $C_q - A_q - V$, we obtain the same result in a similar way. Hence the set of points at which f does not take the form (ii)′ is the union of V, $W \cap ((A_q - C_{q+1}) \cup V)$ and so on. The proposition is proved.

Remark (**9.4**). — If $M = \mathbf{R}^m$, we can deal with somewhat simpler forms. Namely, let $f = (f_1, \ldots, f_m) : M_1 \to \mathbf{R}^m$ be an analytic mapping. Then the subset of M_1 of points at which the germ of f is not equivalent to any germ $g = (g_1, \ldots, g_m)$ at 0 of the form

$$g_{\sigma(1)} = x_1 + \text{const}, \ \ldots, \ g_{\sigma(p)} = x_p + \text{const},$$
$$g_{\sigma(p+1)} = \pm x_{p+1}^{s_1} + \varphi_1(x_1, \ldots, x_p),$$
$$g_{\sigma(p+2)} = x_{p+2} x_{p+1}^{s_2} + \varphi_2(x_1, \ldots, x_{p+1}), \ \ldots,$$
$$g_{\sigma(\ell)} = x_\ell x_{p+1}^{s_{\ell-p}} + \varphi_{\ell-p}(x_1, \ldots, x_{\ell-1}),$$
$$g_{\sigma(\ell+1)} = \varphi_{\ell-p+1}(x_1, \ldots, x_\ell), \ \ldots, \ g_{\sigma(m)} = \varphi_{m-p}(x_1, \ldots, x_\ell)$$

where σ is a permutation of $(1, \ldots, m)$, the φ_i are germs of analytic functions and the s are positive integers, is an analytic set of codimension at least 2.

This is shown in the same way.

10. Equivalence to analytic mappings

The following is a generalization of Theorem (7.1).

Theorem (**10.1**). — *Let f be a mapping from a manifold M_1 to another M_2. Suppose that the germ of f at each point of M_1 is equivalent to a germ of an analytic mapping. In addition,* *suppose that f takes locally the form* (i)′, (ii)′ *or* (iii)′ *in Section* 9, *except on a discrete set. Then f is equivalent to an analytic mapping.*

Proof. — We can assume that f is analytic in a neighborhood U of the discrete set. Imbedding M_2 in a Euclidean space, we assume $M_2 = \mathbf{R}^m$. We put $\dim M_1 = n$, $M_1 = M$ and denote by ℓ the maximum of the rank of the Jacobian matrix of f. Let F be the sum of the squares of the determinants of all $\ell \times \ell$ submatrices of $(X_i f_j)_{\substack{i=1,\ldots,N,\\ j=1,\ldots,m}}$ where $f = (f_1, \ldots, f_m)$. From the assumption, F is analytic on U, and for any point $x \notin U$, F_x takes the form (i), (ii) or (iii). Hence, by Theorem (7.1), there exists a diffeomorphism τ of M such that τ is analytic on $\tau^{-1}(U)$ and that $F \circ \tau$ is analytic. Considering $f \circ \tau$ instead of f, we may assume from the beginning that $S = F^{-1}(0)$ is analytic. We observe here that F is not necessarily analytic. Let U_1 be the open subset of points where the germ of f is equivalent to $(f_1, \ldots, f_m)$ in Remark (9.2) or to $(g_{\sigma(1)}, \ldots, g_{\sigma(m)})$ in Remark (9.4). We define $U_2, U_3, \ldots$ for $f_{\rho_2} = (f_{\rho_2(1)}, \ldots, f_{\rho_2(m)})$, $f_{\rho_3}, \ldots$ respectively in the same way, where ρ_i are all permutations of $(1, \ldots, m)$. Then the union of $U, U_1, \ldots$ is M. We want a diffeomorphism τ_1 such that:

(1) τ_1 is analytic on $\tau_1^{-1}(U)$,

(2) $f \circ \tau_1$ is analytic on $U \cup U_1$,

(3) τ_1 is the identity outside $U \cup U_1$, and

(4) $\tau_1(S) = S$.

If such a τ_1 exists, we can assume f to be analytic on $U \cup U_1$. Hence, repeating the argument for $(U \cup U_1, U_2)$ and so on, we obtain the theorem. To find τ_1, it is sufficient that we consider the case where $U \cup U_1 = M$ and that we replace (3) by the condition that τ_1 is sufficiently close to the identity. Moreover, for the same reason, we can assume that p, q, r and s_i of the canonical forms in the remarks (9.2) and (9.4) are fixed on $S \cap U_1$. We deal with the last canonical form only, for the sake of brevity, the proof for the first being similar. By the lemma below, we reduce the problem to the case $\ell = m$. We will shrink U by steps, but at any step the shrinked U is assumed to satisfy $U \cup U_1 = M$.

We saw that F is analytic on U and takes locally the form (i) or (ii) out of U. This implies, by Cartan's theorem, that we have an analytic function F′ whose zero set is S and whose germ at each point of $S - U$ is the square of a regular function germ. Let $Y_1, \ldots, Y_{N'}$ be analytic vector fields defined for S in the same way as in the proof of Proposition (9.4). Namely their restrictions on $S - U$ are vector fields of $S - U$ and span the tangent space at each point. Let H be the sum of the squares of the determinants of all $p \times p$ submatrices of $(Y_i f_j)_{\substack{i=1,\ldots,N'\\ j=1,\ldots,P}}$. Then Remark (6.7) shows that for a small mapping $\varphi = (\varphi_1, \ldots, \varphi_p)$, the maps $(f_1, \ldots, f_p) + H\varphi$ and $(f_1, \ldots, f_p)$ are equivalent by a diffeomorphism which satisfies (1), (3) and (4) for a shrinked U if φ is analytic on U. By Lemma (6.4), we choose the small φ so that $(f_1, \ldots, f_p)+H\varphi$ is analytic. Hence we can assume from the beginning that $(f_1, \ldots, f_p)$ is analytic.

In the same way as in the proof of Proposition (9.3), we find analytic vector fields $Z_1, \ldots, Z_{N''}$ on M such that $Z_i f_j = 0$ for $i=1, \ldots, N''$ and $j=1, \ldots, p$, and that for each point $x_0 \in U$, the tangent vectors at x_0 span the tangent space of $\{x \in M \mid (f_1, \ldots, f_p)(x) = (f_1, \ldots, f_p)(x_0)\}$. Let $\mathfrak{p}$ be the sheaf of ideals of $\mathscr{F}$ generated by $Z_i f_{p+1}$ for $i=1, \ldots, N''$. Then the zero set of $\mathfrak{p}$ is contained in $U \cup S$, and for any point x of $S-U$, $\mathfrak{p}_x$ is the (s_1-1)-th power of the ideal of germs vanishing on S_x. Let $\mathfrak{p}'$ be the intersection of $\mathfrak{p}$ and $\mathcal{O}$, let $\mathfrak{p}'' \subset \mathfrak{p}'$ be the sheaf of ideals of germs vanishing on S such that $Z_i g$ for $i=1, \ldots, N''$ belong to $\mathfrak{p}'$, and let $\mathfrak{p}^{(3)}$ be a sheaf of ideals generated by a finite number of cross-sections $g_1, \ldots, g_k$ of $\mathfrak{p}''$ such that $\mathfrak{p}'' = \mathfrak{p}^{(3)}$ on $M-U$. Then $\mathfrak{p}'$ has the same properties as $\mathfrak{p}$, and so do $\mathfrak{p}''$ and $\mathfrak{p}^{(3)}$ if we replace s_1-1 by s_1. By the assumption, for each point x_0 of $M-U$, there exist an analytic function germ φ_1 at 0 and a regular function germ φ'_1 at x_0 such that

$$f_{p+1,x_0}(x) = \varphi_1'^{s_1}(x) + \varphi_1(f_1(x)-f_1(x_0), \ldots, f_p(x)-f_p(x_0))$$

and $$\varphi_1'^{-1}(0) = S_{x_0}.$$

Hence the correspondence $x_0 \mapsto \varphi_1$ on $M-U$ and $x_0 \mapsto f_{p+1,x_0}$ on U determines an element f'_{p+1} of $\Gamma(M, \mathcal{O}/\mathfrak{p}^{(3)})$. Let f''_{p+1} be an analytic function whose image in $\Gamma(M, \mathcal{O}/\mathfrak{p}^{(3)})$ coincides with f'_{p+1}. Then $f_{p+1}-f''_{p+1}$ is a cross-section of $\mathfrak{p}^{(3)}\mathscr{F}$. By the theorem on page 82 of [12], there exist functions $g'_1, \ldots, g'_k$ analytic in a shrinked U such that

$$f_{p+1}-f''_{p+1} = \sum_{i=1}^{k} g'_i g_i.$$

Let g''_i for $i=1, \ldots, k$ be analytic approximations of g'_i. Then f_{p+1} and $f''_{p+1}+\sum_{i=1}^{k} g''_i g_i$ satisfy the conditions of Remark (6.3). Hence they are equivalent by a diffeomorphism which is close to the identity and under which $f_1, \ldots, f_p$ are invariant. Moreover, we easily see in the proof that the diffeomorphism is the identity on S and that (1) is satisfied for a shrinked U, because the g_i are all (s_1-1)-flat on S and are linear combinations of the $Z_j f$ for $j=1, \ldots, N''$ with function coefficients vanishing on S and analytic in a shrinked U. Therefore we can assume that $(f_1, \ldots, f_{p+1})$ is analytic. Repeating this argument for $f_{p+2}, \ldots, f_m$, f is transformed into an analytic mapping. The theorem is proved.

We remark that if, in the theorem above, M_2 is a Euclidean space, we can simplify the canonical form (ii)′ as in Remark (9.4).

Lemma (**10.2**). — *Let f_1 or f_2 be a mapping from a manifold* M *to* $\mathbf{R}^l$ *or* $\mathbf{R}^m$ *respectively. We put $f=(f_1, f_2)$. Assume that the maximum rank of the Jacobian matrices of f and f_1 is l, that f_1 is analytic and that the germ of f at each point of* M *is equivalent to a germ of an analytic mapping. Then f is analytic.*

Proof. — Since the problem is local, f_1 and f_2 are assumed to be germs at 0 in $\mathbf{R}^n$. By the assumption, there is a local diffeomorphism τ in the neighborhood of 0 in $\mathbf{R}^n$ such that $f \circ \tau$ is analytic. By Artin's theorem applied to the analytic germs f_1 and $f_1 \circ \tau$, there exists a local analytic diffeomorphism τ' around 0 such that $f_1 \circ \tau = f_1 \circ \tau'$ and that the jets of τ and τ' at 0 of high degree are identical. It is sufficient for the proof that $f_2 \circ \tau' = f_2 \circ \tau$. Let p be the canonical projection of $\mathbf{R}^l \times \mathbf{R}^m$ onto $\mathbf{R}^l$. Let U be an open sub-analytic set of $\mathbf{R}^n$ such that 0 is an adherence point of U, that the restriction of $f \circ \tau$ to U is a submersion onto its image, that the image is smooth, and that the restriction of p to the image is a local homeomorphism. The existence of such a U easily follows from the assumption. Let $\rho : [0, 1] \to \mathbf{R}^n$ be an analytic path such that $\rho(0) = 0$ and $\rho((0, 1]) \subset U$. If we choose τ' so that its jet at 0 of sufficiently high degree is identical with that of τ, $\tau' \circ \rho((0, 1])$ is contained in U. Hence we only have to prove that the images of $f \circ \tau \circ \rho$ and $f \circ \tau' \circ \rho$ coincide. We denote the images by A_1 and A_2 respectively, and we assume that they are different. We put

$$A = \{(y_1, y_2) \in f \circ \tau(U) \mid y_1 \in f_1 \circ \tau \circ \rho([0, 1])\}.$$

Then A is a semi-analytic set of dimension 1, and A_1 and A_2 are contained in A. By Łojasiewicz' inequality, we have $|y - y'| \geq |y|^L$ where y and y' are contained in different connected components of $A - \{0\}$ respectively and L is a positive constant. Hence it follows that

$$|f \circ \tau \circ \rho(t) - f \circ \tau' \circ \rho(t)| \geq |f \circ \tau \circ \rho(t)|^L \geq |t|^{L'}$$

for $0 \leq t \leq 1$ and a constant L'. This is impossible, because we can choose τ' so that the member of the left side is L'-flat at 0 by Artin's theorem. Hence $A_1 = A_2$. The lemma is proved.

We consider the simplest case of Conjecture II. Assume that the manifolds are one-dimensional and that the functions are not constant. Then the diffeomorphism of equivalence is always analytic. We generalize this to mappings.

Proposition **(10.3)**. — *Let f_1 and f_2 be equivalent analytic mappings from a manifold* M_1 *to another* M_2. *Assume that f_1 is an immersion on a non-empty open subset of* M_1. *Then the diffeomorphism of equivalence is always analytic.*

Proof. — The method is the same as for the proof of the lemma above. Since the problem is local, we can assume that f_1 and f_2 are mapping germs from $(\mathbf{R}^n, 0)$ to $(\mathbf{R}^m, 0)$. Let τ be a local diffeomorphism at 0 such that $f_1 = f_2 \circ \tau$. We want to prove the analyticity of τ. There exists a germ of a closed subanalytic set $S \subset f(\mathbf{R}^n)$ of dimension $< n$ such that $f_1(\mathbf{R}^n) - S$ is smooth and that the restriction of f_1 to $\mathbf{R}^n - f_1^{-1}(S)$ is a local homeomorphism. Let $U_1, \ldots, U_k$ be the connected components of the germ of $\mathbf{R}^n - f_1^{-1}(S)$, and let $\rho_1, \ldots, \rho_k$ be analytic paths such that $\rho_i(0) = 0$ and $\rho_i((0, 1]) \subset U_i$ for each i. Then for each i, $f_2^{-1}(f_1 \circ \rho_i((0, 1]))$ is a semi-analytic set of dimension one and contains the path $\tau \circ \rho_i([0, 1])$. Moreover the semi-analytic set contains $\tau' \circ \rho_i([0, 1])$

where τ' is any local diffeomorphism such that $f_1 = f_2 \circ \tau'$. Hence, applying Artin's theorem and Łojasiewicz' inequality in the same way, we have an analytic local diffeomorphism τ' such that $\tau \circ \rho_i = \tau' \circ \rho_i$ for $i = 1, \ldots, k$ and $f_1 = f_2 \circ \tau'$. If f_1 is an immersion, the proposition is trivial. This means that the equation $\tau = \tau'$ holds on $\mathbf{R}^n - f_1^{-1}(S)$. As the dimension of $f_1^{-1}(S)$ is $< n$, we see that $\tau = \tau'$. The proposition is proved.

The following is an immediate corollary of Theorem (8.4), Theorem (10.1) and the above proposition.

Corollary **(10.4).** — *If the source manifold is of dimension* ≤ 2, *the conjectures* I *and* II *hold.*

11. Other equivalence relations

If, in the conjectures I and II, we replace the equivalence by the R — L or L equivalence, the resulting conjectures are false.

Example **(11.1).** — Let us consider an immersion of $\mathbf{R}$ into $\mathbf{R}^2$ as in the figure. Then it is locally L equivalent to germs of analytic mappings but not globally R — L equivalent to any analytic mapping.

We remark that any function germ which is R — L equivalent to the germ of an analytic function is equivalent to an analytic germ.

Example **(11.2).** — Let f_1 and f_2 be strictly monotone increasing analytic functions on $\mathbf{R}$ whose germs at all critical points are R — L equivalent to each other, whose sets of critical values are $\{1/n \mid n = 1, \ldots\}$ and $\{1/n \mid n = 2, \ldots\} \cup \{2\}$ respectively and whose limits are ∞ as x tends to ∞. Then f_1 and f_2 are R — L equivalent but not analytically R — L equivalent.

We remark that if f_1 and f_2 are R — L equivalent analytic functions whose sets of critical values are discrete, then there exists an analytic diffeomorphism τ of $\mathbf{R}$ such that $\tau \circ f_1$ and f_2 are equivalent. Namely we can reduce the problem to the original conjecture II.

Remark **(11.3).** — Let f_1 and f_2 be L equivalent analytic functions on a manifold whose images are $\mathbf{R}$. Then the diffeomorphism of L equivalence is analytic.

Example **(11.4).** — In the remark above we cannot omit the condition that the images are $\mathbf{R}$. Let f_1 and f_2 be functions on $\mathbf{R}$ defined by $f_1(x) = x^2 + 1$ and $f_2(x) = f_1 \exp(1 - 1/f_1^2)$. Then the diffeomorphism τ of $[1, \infty)$ such that $\tau \circ f_1 = f_2$ is uniquely given by $\tau(y) = y \exp(1 - 1/y^2)$. Hence f_1 and f_2 are not analytically L equivalent.

Remark **(11.5).** — Any two L equivalent analytic functions on a manifold are analytically R — L equivalent.

12. Equivalence to Nash functions and the uniqueness of factorization of analytic functions

In this section we give two applications of the ideas used in the preceding sections of this chapter. First we consider the problem of when a C^∞ function on a compact Nash manifold is equivalent to a Nash function. Here a Nash manifold means an analytic manifold which is a semi-algebraic subset of a Euclidean space (see Palais [21]), and a Nash function on a Nash manifold means an analytic function whose graph is a Nash manifold. We want to prove the corresponding conjectures in some special cases. Here, we cannot use Cartan's theorem. But the theorem is not essential in some of the preceding results in this chapter, if the manifolds are compact.

Proposition (**12.1**). — *Let* M *be a compact Nash manifold, let* f *be a function on* M *and let* S *be the critical set of* f. *Let* $S_1, S_2, \dots$ *be the connected components of* S. *Suppose that for each* S_i, *there exist a Nash function* g_i *on* M, *a diffeomorphism* τ_i *of* M *and a neighborhood* U_i *of* S_i *such that*

$$g_i \circ \tau_i = f \text{ on } U_i$$

and

$$\tau_i = \text{identity on } \bigcup_{j \neq i} U_j.$$

Then f *is equivalent to a Nash function on* M.

Proof. — Let M be embedded in $\mathbf{R}^n$. Considering $f \circ \tau_1^{-1} \circ \tau_2^{-1} \dots$ instead of f, we can assume that $g_i = f$ on U_i. We let G_i denote the sum of squares of all derivatives of g_i for each i. Then S_i is a connected component of $G_i^{-1}(0)$. Let h_i be a polynomial on $\mathbf{R}^n$ which is positive on S_i, and negative on another connected component. We put

$$H_i = (h_i^2 + G_i^2)^{1/2} - h_i \quad \text{for} \quad i = 1, \dots$$

Then H_i is a Nash function whose zero set is S_i and the germ of H_i at S_i is the product of G_i and a germ of a positive function. We put

$$H_i' = \prod_{j \neq i} H_j, \qquad H_i'' = H_i'(H_i + H_i')^{-1}.$$

Then H_i'' is a Nash function whose zero set is $S - S_i$ and the germ of H'' at S_i or S_j for $j \neq i$ is of the form $1 + \Sigma a_{k\ell} \frac{\partial f}{\partial x_k} \frac{\partial f}{\partial x_\ell}$ or $\Sigma a_{k\ell} \frac{\partial f}{\partial x_k} \frac{\partial f}{\partial x_\ell}$ respectively. Multiplying by some Nash functions on H_i and H_i', we can assume that the $a_{k\ell}$ are sufficiently near to the zero function on a given small closed neighborhood of S_i or S_j. Then, by Lemma (6.1), $G = \Sigma g_i H_i''$ and f are equivalent as germs at S, and the diffeomorphism can be chosen close to the identity. Hence we can assume that $f = G$ in a neighborhood of S. Using the sum of squares of all derivatives of G in the same way as in the proof of Corollary (6.5), we prove the equivalence of f to a Nash function.

An immediate corollary is

Corollary (**12.2**). — *Let* M, *f be as above. Suppose that f has only isolated critical points and that for each point x, the germ of f is equivalent to a Nash function germ. Then f is equivalent to a Nash function on* M.

Theorem (**12.3**). — *Let* M *be a compact Nash manifold of dimension* ≤ 2 *and let f be a function on* M. *Suppose that for each point a of* M, *the germ* f_a *is equivalent to a* C^ω *function germ. Then f is equivalent to a Nash function on* M.

Proof of the theorem. — We assume $M \subset \mathbf{R}^n$ and that M is not orientable. The other case follows easily. Let $(\tilde{M}, p, M)$ be a 2-fold covering such that $\tilde{M}$ is an orientable manifold and that p is a local diffeomorphism. We can imbed $\tilde{M}$ in $\mathbf{R}^3$ so that

$$-x \in \tilde{M} \quad \text{if} \quad x \in \tilde{M},$$

and

$$p(x) = p(-x) \quad \text{for} \quad x \in \tilde{M}.$$

It follows that there exists a function φ on $\mathbf{R}^3$ such that

$$\varphi(x) = \varphi(-x), \qquad \varphi^{-1}(0) = \tilde{M},$$

and that f is regular at all points of $\tilde{M}$. Let φ' be a polynomial approximation of φ. Then

$$\varphi''(x) = [\varphi'(x) + \varphi'(-x)]/2$$

is an approximation of φ, and $\varphi''^{-1}(0)$ is diffeomorphic to $\tilde{M}$. Here the diffeomorphism commutes with the mapping $x \to -x$. Hence we may assume that $\tilde{M} \subset \mathbf{R}^3$ is a Nash manifold. Let q be the projection from a Nash tubular neighborhood of M in $\mathbf{R}^n$ onto M. Then q is a Nash mapping. Let p' be a map from $\mathbf{R}^3$ to $\mathbf{R}^n$ whose restriction on $\tilde{M}$ is p. Let p'' be a polynomial and a sufficiently strong approximation of p'. We put

$$p^* = q([p''(x) + p''(-x)]/2)\Big|_{\tilde{M}}.$$

Then p^* is a Nash approximation of p such that $p^*(x) = p^*(-x)$. Hence, we may assume that p is a Nash mapping and that $p(x) = p(-x)$. Let S be the zero set of f. We put

$$\tilde{S} = p^{-1}(S), \quad \tilde{f} = f \circ p.$$

By Proposition (12.1), we only need to show the existence of a Nash function g on M, of $\tau \in \mathrm{Diff}^\infty(M)$ and of small neighborhoods $U \subset U'$ of S in M such that

$$g \circ \tau = f \text{ on } U$$

and

$$\tau = \text{identity on } M - U'.$$

We may assume that f is analytic, by Corollary (10.5). Let $\mathcal{O}$, $\tilde{\mathcal{O}}$ be the sheaves of germs of analytic functions on M, $\tilde{M}$ respectively. There exist distinct coherent sheaves $\mathcal{H}_1, \ldots, \mathcal{H}_m$ of ideals of $\mathcal{O}$ such that

$$f\mathcal{O} = \prod_{i=1}^{m} \mathcal{H}_i^{\alpha_i}, \qquad \alpha_i > 0,$$

$$\text{the radical } \sqrt{\mathcal{H}_{ix}} = \mathcal{H}_{ix} \quad \text{for each } i \text{ and } x \in M,$$

and such that $S_i = \{x \in M \mid \mathcal{H}_{ix} \neq \mathcal{O}_x\}$ for $i = 1, \ldots, m$ are finitely many points plus the irreducible components of S. Here it is possible that $S_i = S_j$ for $i \neq j$ if S_i and S_j are points. We put

$$\tilde{\mathcal{H}}_i = p^*(\mathcal{H}_i), \qquad \tilde{S}_i = p^{-1}(S_i) \quad \text{for} \quad i = 1, \ldots, m.$$

Then we can construct functions $h_1, \ldots, h_m$ on $\tilde{M}$ such that, for each i, $\tilde{S}_i \subset h_i^{-1}(0)$, the germ h_{ix} is a generator of $\tilde{\mathcal{H}}_{ix}\tilde{\mathcal{F}}_x$ for any $x \in \tilde{S}_i$, and $h_i(x) = h_i(-x)$ or $= -h_i(-x)$ on $\tilde{M}$. Here $\tilde{\mathcal{F}}$ denotes the sheaf of germs of functions on $\tilde{M}$. We remark that if $\tilde{M}$ is not orientable, the h_i do not exist in general. We want to take Nash functions as h_i for $i = 1, \ldots, m$. Let h'_i for each i be the restriction of a polynomial on $\tilde{M}$ such that h'_i is sufficiently near to h_i, that their critical sets are the same, that their jets of large order are identical at each critical point of $\prod_{i=1}^{m} h_i$, and that $h'_i(x) = h'_i(-x)$ if $h_i(x) = h_i(-x)$ and $h'_i(x) = -h'_i(-x)$ if $h_i(x) = -h_i(-x)$. Then, by the properties of $\mathcal{H}$ and h_i, $\prod_{i=1}^{m} h_i$ and $\prod_{i=1}^{m} h'_i$ satisfy the condition of Remark (6.3) in a neighborhood of S. Hence they are equivalent as germs at S, and the diffeomorphism can be chosen near to the identity and commuting with the multiplication by -1. As the diffeomorphism can be extended to the whole of $\tilde{M}$ without losing the commutativity, we can assume from the start that the h_i are the restrictions of polynomials.

Let ψ_i for $i = 1, \ldots, m$ be Nash functions on $\tilde{M}$ such that for each i

$$\psi_i\Big|_{\tilde{S}_i} > 0, \qquad \psi_i\Big|_{h_i^{-1}(0) - \tilde{S}_i} < 0,$$

and
$$\psi_i(x) = \psi_i(-x).$$

We put

$$H_i = (h_i^2 + \psi_i^2)^{1/2} - \psi_i \quad \text{for} \quad i = 1, \ldots, m.$$

Then we have for each i

$$H_i \geqslant 0, \qquad H_i^{-1}(0) = \tilde{S}_i, \qquad H_i(x) = H_i(-x),$$

and the germ H_{ix} is a generator of $\tilde{\mathcal{H}}_{ix}^2$ for any $x \in \tilde{M}$. We put

$$\tilde{H} = \prod_{i=1}^{m} H_i^{\alpha_i}, \qquad H(x) = \tilde{H}(p^{-1}(x)) \quad \text{for} \quad x \in M.$$

From the property of $\mathscr{H}_i$, we deduce that

$$f^2\mathcal{O} = \mathrm{H}\mathcal{O}.$$

This means that

$$f^2/\mathrm{H} \in \mathbf{C}^\infty(\mathrm{M}) \quad \text{and} \quad f^2/\mathrm{H} > 0.$$

Hence there exists a Nash function H′ on M such that

$$\mathrm{H}'^2 = \mathrm{H} \quad \text{and} \quad f/\mathrm{H}' > 0.$$

It is easy to find a Nash function H″ on M such that $f/\mathrm{H}'\mathrm{H}''$ is sufficiently near to 1 and that the jet of $f/\mathrm{H}'\mathrm{H}''$ of sufficiently large order is equal to 1 at each point where f does not take the form (i) or (ii) of the introduction. Then Lemma (2.2), the proof of Lemma (2.3) and the statement in the proof of Remark (6.2) imply that f and H′H″ satisfy the conditions of Remark (6.3) in a closed neighborhood of S. Hence f and H′H″ are equivalent as germs at S, and the diffeomorphism is near to the identity. Thus the theorem is proved.

Remark **(12.4)**. — If M is not compact in the results above, f is not necessarily equivalent to a Nash function. Take for example, $\mathrm{M} = \mathbf{R}$ and $f(x) = \sin x$. But we see that for any compact subset K of M, there exist a Nash function g on M and a diffeomorphism τ of M such that f and $g \circ \tau$ coincide on K.

We can generalize the corollary (12.2) as follows in the same way.

Theorem **(12.5)**. — *Let f be a mapping from a compact Nash manifold M_1 to a Nash manifold M_2. Suppose that the rank of the differential of f is constant on M_1 except on a discrete set and that for each point x of M_1, the germ f_x is equivalent to the germ of a Nash mapping. Then f is equivalent to a Nash mapping on M_1.*

Next, as another application, we generalize Proposition (4.2) to the global case. Namely, we show that a factorization into C^∞ functions of a non-zero analytic function on a connected manifold is a factorization into analytic functions. We also consider the complex analytic case. Assume that a continuous function f on a complex analytic manifold is of class C^∞ when we regard the manifold as a real manifold. Then we call f briefly a C^∞ function.

Lemma **(12.6)**. — *Let f be a function on a manifold. Suppose that for any zero a of f, the germ f_a is the product of a function germ which takes a non-zero value at a and an analytic function germ. Then there exists a function ψ such that $\psi(x) \neq 0$ for any x and that ψf is analytic.*

*Complex case. — Let f be a **C**-valued C^∞ function on a Stein complex manifold. Suppose that for any zero a of f, f_a is the product of a **C**-valued C^∞ function germ which takes a non-zero value at a and a germ g_a of a complex analytic function. Then there exists a **C**-valued C^∞ function ψ vanishing nowhere such that ψf is complex analytic.*

Proof. — The complex case is involved in the solution of the second problem of Cousin. It is well-known that the existence of such an f proves the truth of the problem. We will repeat the proof, because the real case follows in the same way.

Complex case. — Let M be the Stein manifold on which f is defined. Let $\mathcal{O}$, $\mathcal{K}$, $\mathcal{E}$, $\mathcal{K}'$, and $\mathbf{Z}$ be the sheaves of germs of holomorphic functions, meromorphic functions, $\mathbf{C}$-valued C^∞ functions, fractions of $\mathbf{C}$-valued C^∞ nowhere flat functions and the constant $\mathbf{Z}$ sheaf on M respectively. Let $\mathcal{O}^*$, $\mathcal{E}^*$ be the subsheaves of $\mathcal{O}$, $\mathcal{E}$ respectively of invertible germs. We put $\mathcal{D}=\mathcal{K}/\mathcal{O}^*$, $\mathcal{D}'=\mathcal{K}'/\mathcal{E}^*$ viewed as multiplicative group sheaves. Then we have commutative diagrams of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{p} & \mathcal{O} & \xrightarrow{e} & \mathcal{O}^* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \\ 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{p} & \mathcal{E} & \xrightarrow{e} & \mathcal{E}^* & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}^* & \xrightarrow{q} & \mathcal{K} & \xrightarrow{r} & \mathcal{D} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \\ 0 & \longrightarrow & \mathcal{E}^* & \xrightarrow{q} & \mathcal{K}' & \xrightarrow{r} & \mathcal{D}' & \longrightarrow & 0 \end{array} \qquad (1), (2)$$

where i is the identity, j the natural injection and e the exponential mapping. Hence

$$\begin{array}{ccccccc} H^1(M, \mathcal{O}) & \xrightarrow{e^*} & H^1(M, \mathcal{O}^*) & \xrightarrow{\delta_1} & H^2(M, \mathbf{Z}) & \xrightarrow{p^*} & H^2(M, \mathcal{O}) \\ \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle j^*} \\ H^1(M, \mathcal{E}) & \xrightarrow{e^*} & H^1(M, \mathcal{E}^*) & \xrightarrow{\delta_1} & H^2(M, \mathbf{Z}) & \xrightarrow{p^*} & H^2(M, \mathcal{E}) \end{array} \qquad (3)$$

$$\begin{array}{ccccc} H^0(M, \mathcal{K}) & \xrightarrow{r^*} & H^0(M, \mathcal{D}) & \xrightarrow{\delta_2} & H^1(M, \mathcal{O}^*) \\ \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} \\ H^0(M, \mathcal{K}') & \xrightarrow{r^*} & H^0(M, \mathcal{D}') & \xrightarrow{\delta_2} & H^1(M, \mathcal{E}^*) \end{array}$$

are commutative diagrams of exact sequences. The fundamental theorem B on Stein manifolds tells us that

$$H^1(M, \mathcal{O}) = H^2(M, \mathcal{O}) = 0.$$

Now f is an element of $H^0(M, \mathcal{K}')$, and the mapping $g : a \to g_a$ is not necessarily a continuous cross section of $\mathcal{K}$. We see that r^*g is a continuous cross section of $\mathcal{D}$. We write the element of $H^0(M, \mathcal{D})$ as $[f]$. Then we have

$$j^*[f] = r^*f \text{ in } H^0(M, \mathcal{D}').$$

If we prove that $\delta_2[f]=0$, it will follow that there exists g_1 in $H^0(M, \mathcal{K})$ such that $r^*g_1=[f]$. Then g_a and g_{1a} coincide mod $\mathcal{O}^*_a$ for any a of M. This means that g_1 has no singularity and that g_1/f is a $\mathbf{C}$-valued C^∞ function vanishing nowhere on M. Hence $g_1=(g_1/f)f$ is the decomposition we want. Therefore we only need to prove the equality $\delta_2[f]=0$. We have

$$j^*\delta_2[f] = \delta_2 j^*[f] = \delta_2 r^* f = 0.$$

In the diagram (3)

$$H^1(M, \mathcal{O}^*) \xrightarrow{\delta_1} H^2(M, \mathbf{Z}), \qquad H^2(M, \mathbf{Z}) \xrightarrow{i^*} H^2(M, \mathbf{Z})$$

are isomorphisms. Hence

$$i^*\delta_1\delta_2[f] = \delta_1 j^*\delta_2[f] = 0$$

implies $\delta_2[f] = 0$. Thus we have proved the analytic case.

For the real case, we only have to consider commutative exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O} & \xrightarrow{e} & \mathcal{O}^* & \longrightarrow & \mathbf{Z}_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E} & \xrightarrow{e} & \mathcal{E}^* & \longrightarrow & \mathbf{Z}_2 & \longrightarrow & 0 \end{array}$$

instead of (1). We omit the details.

Theorem **(12.7)**. — *Let f be a non-zero analytic function on a connected manifold. Suppose that f is the product of two functions φ_1, φ_2. Then there exist two functions ψ_1, ψ_2 such that $\psi_1\psi_2 = 1$ and that $\varphi_1\psi_1$, $\varphi_2\psi_2$ are analytic.*

Complex case. — Let f be a complex analytic function on a Stein manifold. Suppose that f is the product of two **C**-*valued* C^∞ *functions φ_1, φ_2. Then there exist two* **C**-*valued* C^∞ *functions ψ_1, ψ_2 such that $\psi_1\psi_2 = 1$ and that $\varphi_1\psi_1$, $\varphi_2\psi_2$ are complex analytic.*

Proof. — Real case. Let a be a point of the manifold. By Proposition (4.2), we have two function germs g_1, g_2 at a such that $g_1 g_2 = 1$ and that $\varphi_{1a} g_1$, $\varphi_{2a} g_2$ are analytic. Hence φ_1 satisfies the condition on f in Lemma (12.6). Therefore there exists a function ψ_1 vanishing nowhere such that $\psi_1\varphi_1$ is analytic. Put $\psi_2 = 1/\psi_1$. Then ψ_1, ψ_2 are the functions we want.

Complex case. — Because of the argument above, we only need to show this theorem locally, namely, the complex case of Proposition (4.2). We can prove that case in the same way as Proposition (4.2), since the results of Tougeron and Malgrange which were used there hold true in the **C**-valued case. We omit the details.

III. — OTHER PROBLEMS OF EQUIVALENCE

13. Topological equivalence

Let f be a continuous function on a C^1 manifold M. A point x of M is called a *topologically regular* point of f if the germ of f at x is C^0 equivalent to a C^1 regular function germ. If x is not a topologically regular point, we call it *topologically singular.* The image of a topologically singular point by f is called a *topologically singular value.*

Our tools in this section are the following lemmas due to Siebenmann [29], Kirby-Siebenmann [8], Moise [16] [17], Munkres [18] [19], and Hirsch-Mazur [5].

Lemma (**13.1**). — *Given a topological manifold* X *and a codimension one foliation* $\mathscr{F}$, *one can always find a foliation* $\mathscr{F}'$ *by* 1*-manifolds transverse to* $\mathscr{F}$. *If* $\mathscr{F}''$ *is a given foliation by* 1*-manifolds transverse to* $\mathscr{F}$ *and defined near a closed set* $C \subset X$, *then one can choose* $\mathscr{F}'$ *equal to* $\mathscr{F}''$ *near* C.

Lemma (**13.2**). — *Let* Q *be a q-dimensional topological manifold without boundary, with* $q \neq 3, 4$, *and let* C *be a closed subset of* Q. *Let* Σ_0 *be a* C^∞ *structure near* C. *Let* Θ *be a* C^∞ *structure on* $Q \times \mathbf{R}$ *which agrees with* $\Sigma_0 \times \mathbf{R}$ *near* $C \times \mathbf{R}$. *Then* Q *has a* C^∞ *structure* Σ, *extending* Σ_0 *near* C, *so that there is an* ε*-isotopy* $h_t : Q_\Sigma \times \mathbf{R} \to (Q \times \mathbf{R})_\Theta$ *with* $h_0 =$ *the identity,* $h_1 =$ *a* C^∞ *diffeomorphism, and* $h_t =$ *the identity near* $C \times \mathbf{R}$. *Here* $\varepsilon : Q \times \mathbf{R} \to \mathbf{R}_+$ *is a given continuous function.*

Lemma (**13.3**). — *Any homeomorphism from a Euclidean space of dimension not* 4 *to a* C^∞ *manifold can be approximated by* C^∞ *diffeomorphisms in the* C^0 *topology.*

The topological case of the Corollaries (7.1) and (12.2) is

Theorem (**13.4**). — *Let* M *be a* C^∞ *manifold of dimension* $\neq 4, 5$. *Let* f *be a continuous unction on* M. *Suppose that* f *has only isolated topological singularities. Then* f *is* C^0 *equivalent to a* C^∞ *function on* M.

Proof. — Let S be the set of topologically singular points of f. Obviously f defines a codimension 1 topological foliation $\mathscr{F}$ on $M - S$. The Lemma (13.1) implies the existence of a foliation $\mathscr{F}'$ by 1-manifolds transverse to $\mathscr{F}$. We assume $\dim M = n+1$. Let $a \in M - S$. Then there is a neighborhood W of a and a homeomorphism $\pi : \mathbf{R}^n \times [c, d] \to W$ with $c < f(a) < d$ such that

$$\{\pi(\mathbf{R}^n \times \{t\})\}_{t \in [c,d]} = \mathscr{F}\Big|_W, \quad \{\pi(\{x\} \times [c, d])\}_{x \in \mathbf{R}^n} = \mathscr{F}'\Big|_W,$$

$$f \circ \pi(x, t) = t \quad \text{for} \quad (x, t) \in \mathbf{R}^n \times [c, d].$$

Let $\{W_i\}_{i=1,2,\dots}$ be a locally finite covering of $M-S$ by sets of type W; the mappings $\pi_i : \mathbf{R}^n \times [c_i, d_i] \to W_i$ have the same properties as π. We may assume that if

$$[c_i, d_i] \cap [c_j, d_j] \neq \varnothing \quad \text{and} \quad i<j,$$

then $\quad [c_i, d_i] \supset [c_j, d_j].$

By Lemma (13.3), there exists a C^∞ diffeomorphism $\tau_1 : W_1^\circ \to \mathbf{R}^n \times (c_1, d_1)$ such that

$$|\tau_1(x) - \pi_1^{-1}(x)| \to 0 \quad \text{when} \quad x \to \partial W_1^\circ$$

where ∂W_1° means the boundary of W_1°. Then $\pi_1 \circ \tau_1$ is a homeomorphism of W_1° such that $f \circ \pi_1 \circ \tau_1$ is C^∞ regular on W_1°. Taking a refinement of $\{W_i\}$ if necessary, we may assume that

$$\operatorname{dist}(\pi_1 \circ \tau_1(x), x) \to 0 \quad \text{when} \quad x \to \partial W_1,$$

where dist(,) is a metric on M. Then $\pi_1 \circ \tau_1$ can be extended to M so that the extension is the identity outside W_1°. We denote it by σ_1. We consider $f \circ \sigma_1$, $\sigma_1^* \mathscr{F}$ and $\sigma_1^* \mathscr{F}'$ instead of f, $\mathscr{F}$ and $\mathscr{F}'$, and we use the same notations $f, \dots$ for them.

Next we shall transform f on W_2. If $W_1 \cap W_2 = \varnothing$, we define a homeomorphism σ_2 of M in the same way as above so that $f \circ \sigma_2$ is C^∞ regular on $W_1^\circ \cup W_2^\circ$ and that $\sigma_2 =$ the identity outside W_2. If $W_1 \cap W_2 \neq \varnothing$, we have $(c_1, d_1) \supset (c_2, d_2)$. Let C be a closed subset of $\mathbf{R}^n$ such that

$$\pi_2(C \times (c_2, d_2)) \subset W_1 \cap W_2.$$

We set $W_{12} = \pi_2(C \times (c_2, d_2))$. The function f is C^∞ regular near W_{12}. We want a homeomorphism σ_2 of M such that $f \circ \sigma_2$ is C^∞ regular on W_2° and that σ_2 is the identity near W_{12} and outside W_2°. We choose C so that $W_2^\circ \cup W_{12}$ is a neighborhood of $\partial W_2 \cap \pi_1(\mathbf{R}^n \times (c_2, d_2))$. Then $f \circ \sigma_2$ is C^∞ regular on $W_2^\circ \cup W_1^\circ - \pi_2[(\mathbf{R}^n - C) \times \{c_2, d_2\}]$. Hence if we can construct σ_2 and repeat this argument for $W_3, \dots$ then we may assume that f is C^∞ regular on $M - S - \bigcup_{i=1}^{\infty} \pi_i(\mathbf{R}^n \times \{c_i, d_i\})$. We shall now construct σ_2. As f is C^∞ regular on W_1°, we have a C^∞ foliation $\mathscr{F}''$ by 1-manifolds on W_1° transverse to $\mathscr{F}$. By Lemma (13.1), there exists a foliation $\mathscr{F}^*$ by 1-manifolds on $f^{-1}[(c_1, d_1)] - S$ such that

$$\mathscr{F}^* = \begin{cases} \mathscr{F}' & \text{near } f^{-1}[(c_1, d_1)] - S - W_1 \\ \mathscr{F}'' & \text{in a sufficiently large open subset of } W_1^\circ. \end{cases}$$

We put $W_2^* = \{x \in f^{-1}[(c_2, d_2)] \mid$ the leaf of $\mathscr{F}^*$ passing through x intersects with $W_2^\circ\}$. Then $W_2^* \supset W_2^\circ$, and because of the existence of $\mathscr{F}^*$, there exists a homeomorphism $\pi_2^* : X \times (c_2, d_2) \to W_2^*$ where X is a topological manifold such that

$$f \circ \pi_2^*(x, t) = t \quad \text{for} \quad (x, t) \in X \times (c_2, d_2)$$

and that X and π_2^* are of class C^∞ near closed subsets C^* and $C^* \times (c_2, d_2)$ respectively. Here C^* has the same property as C. It is sufficient to construct σ_2 for W_2^*, π_2^* and C^*. Because W_2^* is a C^∞ manifold, we have an induced C^∞ structure on $X \times (c_2, d_2)$ by π_2^*.

The two C^∞ structures are equal near $C^* \times (c_2, d_2)$. Hence, by Lemma (13.2), there exist a C^∞ manifold Y, a closed subset D of Y, a homeomorphism $\rho : Y \to X$, and a C^∞ diffeomorphism $h : X \times (c_2, d_2) \to Y \times (c_2, d_2)$ such that

$$h^{-1} = (\rho \times \text{identity}) \text{ near } D \times (c_2, d_2),$$
$$\rho(D) = C^*$$

and that h and $\rho^{-1} \times$ identity are sufficiently close. We put

$$\sigma_2 = \begin{cases} \pi_2^* \circ (\rho \times \text{identity}) \circ h \circ \pi_2^{*-1} & \text{on } W_2^*, \\ \text{identity outside } W_2^*. \end{cases}$$

It follows that $h \circ \pi_2^{*-1}$ is a C^∞ diffeomorphism and that

$$f \circ \pi_2^* \circ (\rho \times \text{identity})(y, t) = f \circ \pi_2^*(\rho(y), t) = t$$

for $(y, t) \in Y \times (c_2, d_2)$. Hence $f \circ \sigma_2$ is C^∞ regular on W_2^*. It is trivial that σ_2 is the identity near $W_{12}^* = \pi_2^*(C^* \times (c_2, d_2))$. Thus we have constructed σ_2 as wanted.

We proved that we can assume the following: $(*)$ f is C^∞ regular on $M - S - \bigcup_{i=1}^{\infty} X_i$ where X_i, $i = 1, 2, \ldots$ are compact subsets of $M - S$ such that f is constant on each X_i and $\{X_i\}$ is locally finite in $M - S$. The above construction of homeomorphisms justifies the next remark.

Remark. — Let X be a closed subset of $M - S$ such that f is constant on X. Then there is a homeomorphism h of M such that $f \circ h$ is C^∞ regular near $h^{-1}(X)$ and satisfies the condition $(*)$ and that h is the identity outside an arbitrarily small neighborhood of X.

By this remark, we choose $\{X_i\}$ in $(*)$ so that each X_i is a connected component of $\bigcup_{i=1}^{\infty} X_i$ and that for each $a \in S$, $\bigcup_{i=1}^{\infty} X_i$ does not intersect with $f^{-1}(f(a))$ near a. We put $f(X_1) = b$. The lemma below shows the existence of $\alpha \in C^\infty(\mathbf{R})$ such that α is a homeomorphism on $\mathbf{R}$, $\alpha(b) = b$, $|\alpha(b+t) - b| \leq |t|$ for $t \in \mathbf{R}$, α is the identity outside a sufficiently small neighborhood of b, and $\alpha \circ f$ is of class C^∞ near X_1. By Lemma (13.1), we have a foliation by 1-manifolds transverse to $\mathscr{F}$ near X_1 and of class C^∞ outside a small neighborhood of X_1. This implies the existence of a homeomorphism

$$\pi' : X_1' \times (k_1, k_2) \to W'$$

where W' is a neighborhood of X_1 and X_1' is an open subset of $f^{-1}(b)$ containing X_1 such that

$$f \circ \pi'(x, t) = t \quad \text{for} \quad (x, t) \in X_1' \times (k_1, k_2),$$

that X_1' is of class C^∞ outside a small neighborhood X_1'', and that π' is of class C^∞ outside $X_1'' \times (k_1, k_2)$. We may assume that α is the identity outside (k_1', k_2') where $k_1 < k_1' < b < k_2' < k_2$. Let $\varphi \in C^0(X_1')$ be C^∞ outside X_1'', $1 \geq \varphi \geq 0$ and

$$\varphi = \begin{cases} 1 & \text{in } X_1'' \\ 0 & \text{outside } X_1^*, \end{cases}$$

where X_1^* is a small neighborhood of the closure of X_1''. We put

$$f'(x, t) = \alpha(t)\varphi(x) + (1 - \varphi(x))t \quad \text{for} \quad (x, t) \in X_1' \times (k_1, k_2).$$

Then f' is of class C^∞ and is C^0 equivalent to $f \circ \pi'$ through a homeomorphism which is the identity outside $X_1^* \times (k_1', k_2')$. We put

$$f'' = \begin{cases} f' \circ \pi'^{-1} & \text{on } W' \\ f & \text{outside } W'. \end{cases}$$

Then f and f'' are C^0 equivalent by a homeomorphism that is the identity outside a small neighborhood of X_1. We easily see that f'' is of class C^∞ outside $S \cup \bigcup_{i=2}^{\infty} X_i$. We repeat this process for other X_i. Then we have a function which is C^0 equivalent to f and of class C^∞ outside S and C^∞ regular outside $S \cup \bigcup_{i=1}^{\infty} X_i$. We use the same notation f for this function.

Next we want to transform f into a function of class C^∞ at S. For each $a \in S$ there is a neighborhood U of a such that $f\big|_{U-\{a\}}$ is C^∞ regular near $f^{-1}(f(a))$. We consider the case $U = \mathbf{R}^{n+1}$, $a = 0$, $f(a) = 0$ and $f \in C^0(\mathbf{R}^{n+1})$. By Lemma (13.1), there exists a foliation $\mathscr{F}_0$ on $\mathbf{R}^{n+1} - \{0\}$ by 1-manifolds transverse to $\mathscr{F}\big|_{U-\{0\}}$ and defined near $f^{-1}(0) - \{0\}$ by $\operatorname{grad} f$. For any $x \in \mathbf{R}^{n+1}$, Z_x denotes the intersection of $f^{-1}([-|f(x)|, |f(x)|])$ and the leaf of $\mathscr{F}_0$ passing through x. We write $D_\varepsilon = \bigcup_{|x|<\varepsilon} Z_x$ for $\varepsilon > 0$. Then $\{D_\varepsilon\}_{\varepsilon>0}$ is a fundamental system of neighborhoods of 0. The reason is the following. We only need to prove $D_\varepsilon \subset \{|x| \leq 2\}$ for sufficiently small $\varepsilon > 0$. We may assume that f is C^∞ regular on $f^{-1}([-\varepsilon, \varepsilon]) \cap \{1 \leq |x| \leq 2\}$, that $\mathscr{F}_0$ is defined by the gradient of f in the same set and that $|\operatorname{grad} f| > \delta$ there for $\delta > 2\varepsilon$. If $D_\varepsilon \not\subset \{|x| \leq 2\}$, there is a leaf in D_ε which connects two points x, y of D_ε such that $|x| = 1$, $|y| = 2$. Then we have $|f(x) - f(y)| > \delta$. But, by the definition of D_ε we always have $|f(x') - f(x'')| \leq 2\varepsilon$ for $x', x'' \in D_\varepsilon$. This is a contradiction. Thus we have proved that $\{D_\varepsilon\}$ is a fundamental system of neighborhoods of 0. This implies the following. Let α_0 be similar to the above α for X_1, that is, α_0 is a homeomorphism of $\mathbf{R}$ of class C^∞, $\alpha_0 =$ the identity on $(-\infty, -\varepsilon) \cup (\varepsilon, \infty)$, $|\alpha_0(t)| \leq |t|$ for $t \in \mathbf{R}$, and $\alpha_0 \circ f$ is of class C^∞. Let x be a point near 0. Then there exists a unique $y \in \mathbf{R}^{n+1}$ such that x and y lie on the same leaf of $\mathscr{F}_0$ and that $\alpha_0 \circ f(x) = f(y)$. Moreover the correspondence $\tau : x \to y$ is a local homeomorphism near 0. And then we easily show in the same way as in the case of X_1 that τ can be extended to a global homeomorphism so that $f \circ \tau$ is of class C^∞ on M. The proof is complete.

We used the next lemma in the proof above. The proof of the lemma is easy, and will be omitted.

Lemma **(13.5)**. — *Let* $I_n \subset \mathbf{R}$, $n = 1, 2, \ldots$ *be open intervals such that*

$$0 \in I_{n+1} \subset I_n \quad \textit{for each } n, \quad \bigcap_{n=1}^{\infty} I_n = \{0\}.$$

Let a_n, $n=1, 2, \ldots$ *be positive numbers. Then there exists a strictly increasing function* $f \in C^\infty(\mathbf{R})$ *such that* $f(0)=0$ *and*

$$\sup_{x \in I_n} |f^{(m)}(x)| \leq a_n \quad \textit{for any pair} \quad n \geq m.$$

Theorem **(13.6)**. — *Let* M *be a* C^∞ *manifold of dimension* $\neq 4, 5$, *and let* f *be a continuous function on* M. *Suppose that the set of topologically singular values of* f *has no inner point in* $\mathbf{R}$. *Then* f *is* $R-L$ C^0 *equivalent to a* C^∞ *function.*

Proof. — We may assume that f is bounded, and from the previous theorem, that f is of class C^∞ on the set R of topologically regular points of f. Let V be the C^∞ vector field grad f on R for some Riemann metric on M. Let K_i, $i=1, 2, \ldots$ be a sequence of open subsets of M such that for each i, the closure $\overline{K}_i$ of K_i is compact and contained in K_{i+1} and that M is the union of all K_i. Let S_i be the set of topologically or differentiably singular points of $\overline{K}_i$, $i=1, 2, \ldots$, and let T_i be its image by f. Since S_i is compact, so is T_i. Then the complement of T_i in $\mathbf{R}$ is the disjoint union of open intervals U_{i1}, U_{i2}, ... We put

$$W_{ij} = f^{-1}(U_{ij}) \cap K_i \quad \text{for} \quad i, j = 1, 2, \ldots$$

Adding countably many points to T_i if necessary, we may assume that, for all i

$$\inf_{x \in K_i} f(x) \quad \text{and} \quad \sup_{x \in K_i} f(x) \in T_i,$$

and that for each i, j, the function f is C^∞ regular near $\overline{W}_{ij}$ and any integral curve of V contained in W_i and passing through points of $\overline{K}_{i-1}$ does not intersect $\partial K_i = \overline{K}_i - K_i$. We still assume $T_i \subset T_{i+1}$, $i=1, 2, \ldots$ Let φ be a C^∞ function on $\mathbf{R}$ strictly increasing on $(0, 1)$, equal to 0 on $(-\infty, 0)$ and to 1 on $(1, \infty)$. We put

$$U_{ij} = (a_{ij}, b_{ij}), \qquad \varphi_{ij}(x) = \varphi[(x - a_{ij})/(b_{ij} - a_{ij})].$$

Then for each i, j, the function $\varphi_{ij} \circ f$ is of class C^∞ on K_i. Let $k_{ij} \leq 1$ be positive numbers such that for each i

$$\sum_{i=1}^{\infty} k_{ij}(b_{ij} - a_{ij}) \varphi_{ij} \circ f \Big|_{K_i}$$

converges to a C^∞ function on K_i, and that $k_{ij} \geq k_{i'j'}$ if $U_{ij} \supset U_{i'j'}$. For all i, j, i', let $c_{iji'}$ be the sum of all $k_{i'j'}(b_{i'j'} - a_{i'j'})$ with $U_{ij} \supset U_{i'j'}$ and let c_{ij} be the limit of $c_{iji'}$ when $i' \to \infty$. It is trivial that $0 \leq c_{ij} \leq k_{ij}(b_{ij} - a_{ij})$. We can choose $\{k_{ij}\}$ so that all c_{ij} are positive. This is shown as follows. First we choose k_{1j}, $j=1, 2, \ldots$ sufficiently small. Then we put

$$k_{2j} = \begin{cases} k_{1j^*} & j = 1, \ldots, \ell_2 \\ \text{sufficiently small for other } j, \end{cases}$$

where the mapping $(i,j)\to(i-1,j^*)$ is defined by $U_{ij}\subset U_{i-1j^*}$, and ℓ_2 is chosen large enough, so that c_{112} and c_{122} are sufficiently near to $k_{11}(b_{11}-a_{11})$ and $k_{12}(b_{12}-a_{12})$ respectively. Next, we put, for large ℓ_3,

$$k_{3j}=\begin{cases} k_{2j^*} & j=1,\ldots,\ell_3 \\ \text{sufficiently small for other } j, \end{cases}$$

securing that c_{113}, c_{123}, c_{133}, c_{213}, c_{223}, and c_{233} are sufficiently near to c_{112}, c_{122}, $\sum_{j^*=3} k_{2j}(b_{2j}-a_{2j})$, $k_{21}(b_{21}-a_{21})$, $k_{22}(b_{22}-a_{22})$ and $k_{23}(b_{23}-a_{23})$ respectively. We repeat the process for k_{4j}, ... Then, for each i, j, $c_{iji'}-c_{ij(i'+1)}$ is sufficiently near to 0 for $i'>i,j$, hence we have $c_{ij}\neq 0$. Observe that

$$c_{ij}=\sum_{j'^*=j} c_{(i+1)j'} \text{ for each } i,\ j \tag{$*$}$$

We shall define a homeomorphism τ of $\mathbf{R}$ such that $\tau\circ f$ is $\mathbf{C}^0$ equivalent to a $\mathbf{C}^\infty$ function on each K_i. We put

$$\tau(a_{11})=0,\qquad \tau(b_{11})=c_{11}.$$

We want to define $\tau(a_{ij})$ and $\tau(b_{ij})$ for $(i,j)\neq(1,1)$. If $a_{ij}>a_{11}$, let $\wedge_{ij}$ be the set of all j' such that $a_{11}\leq a_{ij'}<a_{ij}$. If $a_{ij}<a_{11}$, let $\wedge_{ij}=\{j'\mid a_{ij}\leq a_{ij'}<a_{11}\}$. Then we put

$$\tau(a_{ij})=\begin{cases} \sum_{\ell\in\wedge_{ij}} c_{i\ell} & \text{if} \quad a_{ij}>a_{11} \\ -\sum_{\ell\in\wedge_{ij}} c_{i\ell} & \text{if} \quad a_{ij}<a_{11}, \end{cases}$$

$$\tau(b_{ij})=\tau(a_{ij})+c_{ij}.$$

The equality $(*)$ and the assumption of the theorem ensure that τ is well defined and can be extended to a homeomorphism τ of $\mathbf{R}$. For all i, j, we put

$$g_{ij}=c_{ij}\varphi_{ij}\circ f.$$

We compare the functions $\tau\circ f\big|_{\overline{W}_{ij}}$ and $g_{ij}\big|_{\overline{W}_{ij}}+\tau(a_{ij})$. Then, by the vector field V, there exists a homeomorphism π_{ij} from $\overline{W}_{ij}\cap K_{i-1}$ into $\overline{W}_{ij}$ such that

$$\tau\circ f\circ\pi_{ij}=g_{ij}+\tau(a_{ij}) \text{ on } \overline{W}_{ij}\cap K_{i-1},$$

$$\pi_{ij}\big|_{\partial W_{ij}\cap K_{i-1}}=\text{identity}.$$

For each i, adding countably many points to T_i if necessary, we can extend π_{ij}, $j=1, 2, \ldots$ to a homeomorphism π_i from K_{i-1} into K_i so that

$$\tau\circ f\circ\pi_i=\sum_{j=1}^{\infty} g_{ij}+c_i \text{ on } K_{i-1}$$

where c_i is a constant. Because of the choice of k_{ij} and of the inequalities $k_{ij}(b_{ij}-a_{ij})\geq c_{ij}$, the function on the right hand side is of class C^∞ on K_i.

Now we want to connect π_i and π_{i+1} to each other. For each $i\geq 3$, $j\geq 1$, we have a C^∞ embedding α_{ij} of $A_{ij}\times[0, c_{ij}]$ into $\overline{W}_{ij}$ where A_{ij} is a C^∞ manifold such that the image of α_{ij} contains $\overline{W}_{ij}\cap\overline{K}_{i-1}$, that the curves $\{\alpha_{ij}(x\times[0, c_{ij}])\}_{x\in A_{ij}}$ are integral curves of V, and that

$$g_{ij}\circ\alpha_{ij}(x, t)=t \quad \text{for} \quad (x, t)\in A_{ij}\times[0, c_{ij}].$$

Let ψ_{ij} be a C^∞ function on A_{ij} such that $0\leq\psi_{ij}\leq 1$ and that

$$\psi_{ij}(x)=\begin{cases}1 & \text{if} \quad \alpha_{ij}(x\times[0, c_{ij}])\cap K_{i-2}\neq\emptyset\\ 0 & \text{if} \quad \alpha_{ij}(x\times[0, c_{ij}])\cap(K_i-K_{i-1})\neq\emptyset.\end{cases}$$

Let p be the projection $A_{ij}\times[0, c_{ij}]\rightarrow A_{ij}$. Then

$$g_{ij}\psi_{ij}\circ p\circ\alpha_{ij}^{-1}+\sum_{j'*=j}g_{(i+1)j'}(1-\psi_{ij}\circ p\circ\alpha_{ij}^{-1})$$

can be naturally extended to a function continuous on M, of class C^∞ on K_{i+1} and equal to $\sum_{j'*=j}g_{(i+1)j'}$ outside W_{ij}. We denote that function by g'_{ij}. Here we put $g'_{ij}=\sum_{j'*=j}g_{(i+1)j'}$ if $W_{ij}=\emptyset$. Then $g'_i=\sum_{j=1}^{\infty}g'_{ij}$ is a continuous function which is equal to $\sum_{j=1}^{\infty}g_{ij}$ on K_{i-2} and to $\sum_{j=1}^{\infty}g_{(i+1)j}$ outside K_{i-1}, and there exists a homeomorphism π'_i from K_i into K_{i+1} such that

$$\tau\circ f\circ\pi'_i=g'_i+c_{i+1} \text{ on } K_i,$$

$$\pi'_i=\begin{cases}\pi_i & \text{on } K_{i-2}\\ \pi_{i+1} & \text{outside } K_{i-1}.\end{cases}$$

Repeating this process, we have a continuous function on M which is C^0 equivalent to $\tau\circ f$. But, unfortunately, this function is not of class C^∞ in general. To make it differentiable we need numbers $0<k'_{ij}\leq 1$, $i, j=1, 2, \ldots$ which are similar to k_{ij}. That is, let $\{k'_{ij}\}$ satisfy

$$k'_{ij}c_{ij}=\sum_{j'*=j}k'_{(i+1)j'}c_{(i+1)j'} \quad \text{for all } i, j.$$

Then

$$k'_{ij}g_{ij}\psi_{ij}\circ p\circ\alpha_{ij}^{-1}+\sum_{j'*=j}k'_{(i+1)j'}g_{(i+1)j'}(1-\psi_{ij}\circ p\circ\alpha_{ij}^{-1})$$

is well defined C^∞ function on $\overline{W}_{ij}$ and can be naturally extended globally. We call the extension g''_{ij}. We choose $\{k'_{ij}\}$ sufficiently small so that for each i, the sum of g''_{ij},

$j=1, 2, \ldots$ converges to a C^∞ function g_i'' on K. To find the k_{ij}' is exactly the same as for k_{ij} and c_{ij} and we do not repeat the argument. Now we have, for all i

$$g_i'' = \begin{cases} \sum_{j=1}^{\infty} k_{ij}' g_{ij} + c_i' & \text{on } K_{i-2} \\ \sum_{j=1}^{\infty} k_{i+1j}' g_{i+1j} & \text{outside } K_{i-1}. \end{cases}$$

where the c_i' are constants. Hence we can connect the g_i'' as follows. We put

$$g(x) = g''(x) + c_i'' \quad \text{for} \quad x \in K_{i-1} - K_{1-2}, \quad i = 2, 3, \ldots$$

where the c_i'' are constants defined so that $g(x)$ is continuous on M. Then g is a C^∞ function on M and R — L C^0 equivalent to $\tau \circ f$. The proof is complete.

From the theorem above and Sard's theorem, we deduce:

Corollary **(13.7)**. — *Let* M *be a* C^∞ *manifold of dimension* $n \neq 4, 5$, *and let* f *be a* C^0 *function on* M. *Suppose that for each point* x *of* M, *the germ of* f *at* x *is* R — L C^0 *equivalent to a germ of a* C^n *function. Then* f *is* R — L C^0 *equivalent to a* C^∞ *function on* M.

Example. — If we modify the example in Whitney [34], we have a C^1 function on $\mathbf{R}^2$ whose set of topologically singular values is not a border set in $\mathbf{R}$. This function is not R — L C^0 equivalent to any C^∞ function on $\mathbf{R}^2$.

About the C^0 equivalence we have a simple result in the one dimensional case.

Proposition **(13.8)**. — *Let* f *be a* C^0 *function on* $I = [0, 1]$. *Let* $\bigcup_{i=1}^{\infty} (x_i, y_i)$ *be the set of topologically regular points of* f. *Then* f *is* C^0 *equivalent to a* C^r *function with* $1 \leq r \leq \infty$ *if and only if*

(1) $$\sum_{i=1}^{\infty} |f(x_i) - f(y_i)|^{1/k} < \infty$$

where $k = r$ *if* $r \neq \infty$, $k = 1, 2, \ldots$ *if* $r = \infty$,

(2) *the measure of the set of topologically singular values is* 0.

Proof of "only if". — For each i, there exists $z_i \in (x_i, y_i)$ such that

$$|f(x_i) - f(y_i)| = (y_i - x_i) |f'(z_i)|.$$

Hence

$$\sum_{i=1}^{\infty} |f(x_i) - f(y_i)| \leq \sum_{i=1}^{\infty} (y_i - x_i) \sup_{x \in I} |f'(x)| \leq \sup_{x \in I} |f'(x)| < \infty.$$

If $r \geq 2$, we have

$$|f'(z_i)| = |f'(z_i) - f'(x_i)| = (z_i - x_i) |f''(w_i)|,$$

$i = 1, 2, \ldots$ for some $w_i \in (x_i, z_i)$. Hence

$$|f(x_i) - f(y_i)|^{1/2} < (y_i - x_i) |f''(w_i)|^{1/2}.$$

This implies

$$\sum_{i=1}^{\infty} |f(x_i)-f(y_i)|^{1/2}<\infty.$$

Repeating this argument we have (1). The assertion (2) is Sard's theorem.

Proof of "if". — We assume $f(0)=0$. Let φ be a C^∞ function on $\mathbf{R}$ strictly increasing on $[0, 1]$, equal to 0 on $(-\infty, 0)$ and to 1 on $(1, \infty)$. We want to find positive numbers a_i, b_i, $i=1, 2, \ldots$ and a homeomorphism τ of I such that

(a) $\tau[(b_i, a_i+b_i)]=(x_i, y_i), \quad i=1, 2, \ldots,$

(b) $f\circ\tau=\sum_{i=1}^{\infty}(f(y_i)-f(x_i))\varphi[(x-b_i)/a_i]$ and

(c) this last function is of class C^r.

We assume $r\neq\infty$. We put

$$c_i=|f(y_i)-f(x_i)|^{1/r}.$$

Then, by (1), we have $\sum_{i=1}^{\infty} c_i<\infty$. Let d_j, $j=1, 2, \ldots$ be positive integers such that

(d) $\sum_{i=d_j}^{\infty} c_i<1/2^j.$

Then

$$\sum_{i=1}^{\infty} c_i+\sum_{i=d_1}^{\infty} c_i+\sum_{i=d_2}^{\infty} c_i+\ldots=c<\infty.$$

Let the coefficient of c_i in the above sum be α_i for each i. Then $\alpha_i\to\infty$ when $i\to\infty$. We put $a_i=\alpha_i c_i m/c$, where m is the measure of the set R of topologically regular points. Then $\sum_{i=1}^{\infty} a_i=m$. Let m_i be the measure of $[0, x_i]-R$ and let Λ_i be the set of integers j such that $x_j<x_i$. Then (2) means that

$$f(x_i)=\sum_{j\in\Lambda_i}[f(y_j)-f(x_j)].$$

We put

$$b_i=m_i+\sum_{j\in\Lambda_i} a_j.$$

Then there exists a homeomorphism τ of I which satisfies the conditions *(a)*, *(b)*. For $x\in[b_i, a_i+b_i]$, $1\leq k\leq r$, we have

$$\left|\frac{d^k}{dx^k}\sum_{j=1}^{\infty}(f(y_j)-f(x_j))\varphi[(x-b_j)/a_j]\right|$$
$$\leq[(f(y_i)-f(x_i))/a_i^k]\sup_{x\in I}|\varphi^{(k)}(x)|=(c/\alpha_i m)^k\sup_{x\in I}|\varphi^{(k)}(x)|.$$

As $\alpha_i \to \infty$ when $i \to \infty$, this simplies *(c)*. If $r=\infty$, we put $c'=|f(y_i)-f(x_i)|$, and we consider the sums

$$\sum_{i=1}^{\infty} c'_i, \quad \sum_{i=d'_1}^{\infty} c_i'^{1/2}, \quad \sum_{i=d'_2}^{\infty} c_i'^{1/3}, \quad \dots$$

instead of *(d)*. Then we obtain in the same way a_i, b_i, $i=1, 2, \dots$ and τ as required. The proposition is proved.

Example. — $f(x)=x^r \cos^2 1/x$, $x \in \mathbf{R}$, $r \geq 2$ satisfies the conditions (1) for $k<r$ and (2) in the proposition but not (1) for $k=r$. Hence f is R—L C^0 equivalent to a C^∞ function and C^0 equivalent to a C^{r-1} function but not to any C^r function.

14. C-valued C^∞ functions equivalent to C-polynomials

In this section we consider complex valued C^∞ functions on $\mathbf{R}^2$. Equivalence, R—L equivalence, etc. are defined in the same way as in § 6.

The main result of this section is the following.

Theorem **(14.1)**. — *Let f be a* **C**-*valued* C^∞ *function on* $\mathbf{R}^2$. *Suppose that f is proper and that for each* $x \in \mathbf{R}^2$, *the Taylor expansion* $T_x f$ *is equivalent to a non-constant* **C**-*polynomial. Then f is equivalent to a* **C**-*polynomial.*

Proof. — We easily see that for each $x \in \mathbf{R}^2$, the germ f_x is equivalent to a non-constant **C**-polynomial germ. If $f(x)=0$, the order of x is well defined by a holomorphic function germ equivalent to f_x. Let X be the set of singular points of f. Then X consists of a finite number of points, $a_1, \dots, a_n$ say. The reason is the following. Trivially the assumption implies this fact locally. We put

$$\mathbf{C}-f(X)=B, \qquad \mathbf{R}^2-f^{-1}(f(X))=A.$$

Then $f|_A : A \to B$ is a covering. As f is proper, the cardinality of $f^{-1}(y)$ is finite for all $y \in \mathbf{C}$. Hence the covering is p-fold for some integer p. Modifying f near each singular point, we can assume that the order of $f-f(a)$ at each $a \in X$ is always two and that

$$f(a) \neq f(a') \quad \text{for} \quad a \neq a' \in X.$$

We shall prove that $p \geq \# X+1$. Assume that there are p distinct points $a_1, \dots, a_p$ in X. Let y_0 be a point in B. Let $\nu_1, \dots, \nu_p$ be simple arcs contained in B except for their ends, from y_0 to $f(a_1), \dots, f(a_p)$ respectively, such that

$$\nu_i \cap \nu_j = \{y_0\} \quad \text{for} \quad i \neq j,$$
$$\nu_i \cap f(X) = \{f(a_i)\} \text{ for each } i.$$

Let μ_i be the connected component of $f^{-1}(\nu_i)$ containing a_i. Then

$$f\Big|_{\mu_i-\{a_i\}} : \mu_i-\{a_i\} \to \nu_i-\{f(a_i)\}$$

is twofold, hence $\mu_i \cap f^{-1}(y_0)$ consists of two points, z_i, z_i' say. Let μ be a Jordan curve contained in $\bigcup_{i=1}^{p} \mu_i$, assuming such a curve exists. Then $f(\mu)$ is contained in $\bigcup_{i=1}^{p} \nu_i$. Because f is an open mapping, the image of the interior of μ is an open set whose boundary is contained in $f(\mu)$ and in $\bigcup_{i=1}^{p} \nu_i$. This is impossible. Hence there is no Jordan curve in $\bigcup_{i=1}^{p} \mu_i$. This implies that each connected component of $\bigcup_{i=1}^{p} \mu_i$ is contractible. Therefore

$$\sharp f^{-1}(y_0) \geq \sharp\{z_i, z_i'\}_{i=1,\dots,p} > p.$$

This contradicts to the p-foldness of $f\Big|_{A}$. Hence $p \geq \sharp X + 1$.

We now regard $\mathbf{R}^2$ as being $\mathbf{C}$. If $X=\{a_1\}$, the theorem follows trivially. Hence we assume $n>1$. For each i, set $f(a_i)=b_i$, and let the order of $f-b_i$ at a_i be m_i+1. We assume $b_i \neq 0$ for all i.

First we suppose $b_i \neq b_j$ for $i \neq j$. We define a polynomial mapping $F: \mathbf{C}^n \to \mathbf{C}^n$ and a polynomial $G_x(y)$ for $x=(x_1, \dots, x_n) \in \mathbf{C}^n$ by

$$G_x(y)=\int_0^y (t-x_1)^{m_1} \dots (t-x_n)^{m_n} dt,$$

$$F(x_1, \dots, x_n)=(G_x(x_1), \dots, G_x(x_n)).$$

Then the components of F are homogeneous polynomials of the same degree. Hence, the image by F of any line passing through the origin is also such a line or $\{0\}$. If there exists $x=(x_1, \dots, x_n) \neq 0$ in $\mathbf{C}^n$ such that $F(x)=0$, then $G_x(y)$ has the critical points $x_1, \dots, x_n$ and the critical values are all 0. Hence, for each i, x_i is a zero point of order m_i+1. This means that $G_x(y)$ is of degree $\geq \sum_{i=1}^{n} (m_i+1)$. But, by definition of G_x, the degree is $\sum_{i=1}^{n} m_i + 1$. This is impossible because $n>1$. Therefore we have $F^{-1}(0)=\{0\}$, and F is proper. If F is not surjective, there is a line passing through the origin whose intersection with $F(\mathbf{C}^n)$ is $\{0\}$. Let F' be the orthogonal projection from $\mathbf{C}^n$ to the orthocomplement of the line. Then we have $(F' \circ F)^{-1}(0)=\{0\}$. This means that there exist $n-1$ polynomials on $\mathbf{C}^n$ whose common zero set is $\{0\}$. This contradicts the Hilbert Nullstellensatz. Hence F is surjective. The Jacobian matrix of F at $x=(x_1, \dots, x_n)$ is

$$\begin{pmatrix} m_1\int_0^{x_1} (t-x_1)^{m_1-1}(t-x_2)^{m_2}\dots dt, & \dots, & m_1\int_0^{x_n} (t-x_1)^{m_1-1}(t-x_2)^{m_2}\dots dt \\ \vdots & & \\ m_n\int_0^{x_1} (t-x_1)^{m_1}\dots(t-x_n)^{m_n-1} dt, & \dots, & m_n\int_0^{x_n} (t-x_1)^{m_1}\dots(t-x_n)^{m_n-1} dt \end{pmatrix}.$$

We see that the Jacobian of F is divisible by $x_i^{m_i}$ and by $(x_i-x_j)^{m_i+m_j}$ for $i \neq j$. On the other hand, the degree of the Jacobian is $n \sum_{i=1}^{n} m_i$. This means that

$$\text{Jacobian } F = K \prod_{i=1}^{n} x_i^{m_i} \prod_{i<j} (x_i - x_j)^{m_i+m_j}$$

where K is a constant. Hence the singular set of F is

$$\bigcup_{i=1}^{n} \{x_i = 0\} \cup \bigcup_{i \neq j} \{x_i = x_j\}.$$

Let R denote its complement. Then $F^{-1}(R)$ is a set of regular points. Let

$$c = (c_1, \ldots, c_n) \in \mathbf{C}^n$$

be a regular point of F. We put

$$g(x) = G_c(x), \quad g(c_i) = d_i, \quad i = 1, \ldots, n.$$

We assume $d_i \neq d_j$ for $i \neq j$. Then there exists a C^∞ simple arc $\varphi : [1, n] \to \mathbf{C}$ such that $\varphi(i) = c_i$, that $g \circ \varphi : [1, n] \to \mathbf{C}$ is also a simple arc and that the image of $g \circ \varphi$ is a C^∞ smooth manifold. The reason is the following. First we join c_1 and c_2 by a simple arc. Let S be a large circle in $\mathbf{C}$ whose interior contains all d_i. Let $y_0 \in S$ and $f^{-1}(S) = S'$. Let $\nu, \nu' : [0, 1] \to \mathbf{C}$ be simple arcs such that

$$\nu(0) = d_1, \quad \nu(1) = \nu'(0) = y_0, \quad \nu'(1) = d_2,$$

$$\text{image } \nu \cap \{d_j\} = d_1, \quad \text{image } \nu' \cap \{d_j\} = d_2,$$

and

$$\text{image } \nu \cap \text{image } \nu' = y_0.$$

Then there exist simple arcs $\mu, \mu' : [0, 1] \to \mathbf{C}$ such that

$$\mu(0) = c_1, \quad \mu'(1) = c_2,$$

and

$$g \circ \mu = \nu, \quad g \circ \mu' = \nu'.$$

If $\mu(1) = \mu'(0)$, the products $\nu^* = \nu . \nu'$ and $\mu^* = \mu . \mu'$ are the arcs we want: we have $\nu^* = g \circ \mu^*$, $\mu^*(0) = c_1$ and $\mu^*(1) = c_2$. If $\mu(1) \neq \mu'(0)$, let $\mu'' : [0, 1] \to \mathbf{C}$ be the simple arc contained in S' and joining $\mu(1)$ to $\mu'(0)$ (counter) clockwise and let $\mu^* = \mu . \mu'' . \mu$ and $\nu^* = g \circ \mu^*$. Then μ^* is simple, but ν^* is not. We want to modify ν^* to make it simple. If we move ν^* continuously in $\mathbf{C} - \{d_i\}$ preserving its ends, μ^* moves accordingly, that is, satisfying $g \circ \mu^* = \nu^*$. Assume that ν^* goes around S k-times. Then we may suppose that ν^* has $k-1$ multiple points (Figure 1). Hence there is no problem in the case $k = 1$. Assume $k = 2$ and that $\nu^*(1/3) = \nu^*(2/3)$ is the multiple point. We put

$$\nu^{**}(t) = \begin{cases} \nu^*(2t/3) & \text{for} \quad 0 \leq t \leq 1/2 \\ \nu^*(2t/3 + 1/3) & \text{for} \quad 1/2 \leq t \leq 1. \end{cases}$$

Then we have a simple arc μ^{**} in $\mathbf{C}$ such that

$$g \circ \mu^{**} = \nu^{**}, \quad \mu^{**}(0) = c_1.$$

If $\mu^{**}(1)=c_2$, μ^{**} and ν^{**} are the arcs we want. If $\mu^{**}(1)\neq c_2$, then $\mu^{**}(1)$ is a regular point of g. We put

$$\nu^{***}(t)=\begin{cases}\nu^{**}(6t) & 0\leq t\leq 1/6\\ \nu(-2t+4/3) & 1/6\leq t\leq 1/3\\ \nu(t) & 1/3\leq t\leq 1.\end{cases}$$

The set of multiple points of ν^{***} is $\nu([2/3, 1])=\nu^{***}([1/12, 1/3])$, and we have an arc μ^{***} in $\mathbf{C}$ such that

$$g\circ\mu^{***}=\nu^{***},\qquad \mu^{***}(0)=c_1,$$

$$\mu^{***}(1)=c_2\quad\text{and}\quad \mu^{***}(I^{\circ})\cap\{c_i\}=\varnothing.$$

It is easy to modify ν^{***} and μ^{***} to make them simple and satisfying

$$\mu^{***}(I)\cap\{c_i\}=\{c_1, c_2\},$$

$$\nu^{***}(I)\cap\{d_i\}=\{d_1, d_2\}\quad\text{(Figure 2).}$$

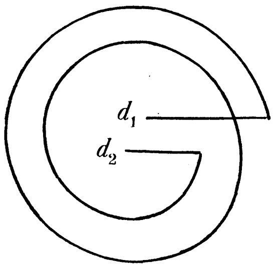


FIG. 1

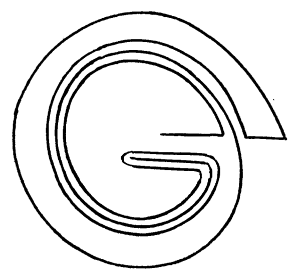


FIG. 2

If $k>2$, we repeat the argument. Thus we get a simple arc from c_1 to c_2 whose image by g is also simple and does not contain $d_3, \ldots, d_n$. After a transformation by a homeomorphism of $\mathbf{C}$, we can consider the simple arc from d_1 to d_2 as a small segment. Then we can use the method above to join d_2 and d_3 by an arc which has no common point with the segment. Repeating this process, we obtain a simple arc $\varphi: [1, n]\to\mathbf{C}$ which satisfies our conditions except that $g\circ\varphi$ is not necessarily a C^∞ manifold. But it is easy to modify φ to achieve this condition.

In the same way as above we find a C^∞ simple arc $\psi: [1, n]\to\mathbf{C}$ such that $\psi(i)=a_i$, that $f\circ\psi$ is also simple and that the image of $f\circ\psi$ is a C^∞ manifold. Here we need the fact that for a sufficiently large circle S in $\mathbf{C}$ centered at 0, $f^{-1}(S)$ is a circle too. We put

$$A_i=g\circ\varphi([i, i+1]),\qquad B_i=f\circ\psi([i, i+1])\qquad i=1, \ldots, n-1.$$

The sets $g^{-1}(\bigcup_{i=1}^{n-1} A_i)$ and $f^{-1}(\bigcup_{i=1}^{n-1} B_i)$ are contractible. We may assume that $a_i=c_i$, $f_{a_i}=g_{c_i}$, $i=1, \ldots, n$. For each i, $g^{-1}(A_{i-1}\cup A_i)$ consists of $2m_i+2$ simple arcs near c_i, and these arcs become half lines after a transformation by a suitable local diffeomorphism in a neighborhood of c_i. Then we can choose φ so that

$$\angle c'_{i-1} c_i c'_{i+1} = \pi/(m_i+1)$$

and that the direction $\overrightarrow{c'_{i-1}c'_{i+1}}$ is counterclockwise with center c_i, where c'_{i-1} and c'_{i+1} are points near c_i on $\varphi[(i-1, i)]$ and $\varphi[(i, i+1)]$ respectively. In this case we say that $A_1, \ldots, A_{n-1}$ are well-chosen arcs for g. We also choose ψ in the same way. There exists a diffeomorphism τ_1 of $\mathbf{R}^2$ such that

$$\tau_1(f^{-1}(\bigcup_{i=1}^{n-1} B_i)) = g^{-1}(\bigcup_{i=1}^{n-1} A_i).$$

Let us assume $A_i = B_i$, $i = 1, \ldots, n-1$. Then we can choose τ_1 so that

$$g \circ \tau_1 = f \quad \text{near} \quad f^{-1}(\bigcup_{i=1}^{n-1} B_i).$$

Since the restrictions of $g \circ \tau_1$ and f to $\mathbf{R}^2 - f^{-1}(\bigcup_{i=1}^{n-1} B_i)$ are $\sum_{i=1}^{n} m_i + 1$ fold coverings, there exists a unique diffeomorphism τ_2 of $\mathbf{R}^2$ such that

$$g \circ \tau_1 \circ \tau_2 = f$$

and $$\tau_2 = \text{the identity} \quad \text{near} \quad f^{-1}(\bigcup_{i=1}^{n-1} B_i).$$

Hence f is equivalent to the $\mathbf{C}$-polynomial g. Therefore we only need to find $c = (c_1, \ldots, c_n) \in \mathbf{C}^n$ such that $B_1, \ldots, B_{n-1}$ are well-chosen arcs for $g(x) = G_c(x)$. For that, we need:

Assertion. — *Let* $A'_1, \ldots, A'_{n-1}$ *be well-chosen arcs for* $g' = G_{c'}$ *where*

$$c' = (c'_1, \ldots, c'_n) \in \mathbf{C}^n, \qquad F(c') \in R.$$

Let $A''_1, \ldots, A''_{n-1}$ *be* C^∞ *simple arcs such that* $\bigcup_{i=1}^{n-1} A''_i$ *is a simple arc, and that* A''_i *is the image of a* C^∞ *mapping* p_i *from* A_i *to* $\mathbf{C}$ *sufficiently near to the identity for each* i. *Then there exists* $c'' = (c''_1, \ldots, c''_n) \in \mathbf{C}^n$ *near* C' *such that* $A''_1, \ldots, A''_{n-1}$ *are well-chosen arcs for* $g'' = G_{c''}$.

Proof of the assertion. — Let $\{d''_i, d''_{i+1}\}$ be the ends of A''_i, $i = 1, \ldots, n-1$. Then d''_i is near to d'_i for each i, where $(d'_1, \ldots, d'_n) = F(c')$. Because the restriction of F on $F^{-1}(R)$ is a covering, there exists $c'' = (c''_1, \ldots, c''_n) \in \mathbf{C}^n$ near c' such that

$$F(c'') = d'' = (d''_1, \ldots, d''_n).$$

Then, from Lemma (14.2) below remarked by Malgrange [12], $g''^{-1}(\bigcup_{i=1}^{n-1} A''_i)$ is contained in the ε-neighborhood of $g'^{-1}(\bigcup_{i=1}^{n-1} A'_i)$ for sufficiently small $\varepsilon > 0$, and there exists a C^∞ diffeomorphism π of $\mathbf{R}^2$ sufficiently close to the identity such that

$$\pi(g''^{-1}(\bigcup_{i=1}^{n-1} A''_i)) = g'^{-1}(\bigcup_{i=1}^{n-1} A'_i)$$

outside a given neighborhood of $\{c_i'\}$. Hence there exists a homeomorphism π' of $\mathbf{R}^2$ such that

$$\pi'(g''^{-1}(\bigcup_{i=1}^{n-1} A_i'')) = g'^{-1}(\bigcup_{i=1}^{n-1} A_i').$$

This means that $A_1'', \ldots, A_{n-1}''$ are well-chosen arcs $G_{c''}$. Thus the assertion is proved.

Let $\alpha : [1, 2n] \to \mathbf{C}$ be a simple arc such that

$$\alpha\Big|_{[1,n]} = f \circ \psi, \qquad 0 \notin \alpha([1, 2n])$$

and $$\alpha(t+n) = g \circ \varphi(t) \quad \text{for} \quad 1 \leq t \leq n.$$

We put

$$d_i(s) = \alpha(i+s), \qquad A_i(s) = \alpha([i+s, i+1+s]), \qquad i = 1, \ldots, n, \qquad 0 \leq s \leq n.$$

Then $d(s) = (d_1(s), \ldots, d_n(s)) \in R$ for any $0 \leq s \leq n$. Hence there exists a continuous mapping $s \in [0, n] \to c(s) = (c_1(s), \ldots, c_n(s)) \in \mathbf{C}^n$ such that

$$F(c(s)) = d(s), \qquad 0 \leq s \leq n, \qquad \text{and} \qquad c(n) = c.$$

By assumption, the set of points $s \in [0, n]$ such that $A_1(s), \ldots, A_{n-1}(s)$ are well-chosen arcs for $G_{c(s)}$ is open. Since $A_i(n) = A_i$, $A_1(n), \ldots, A_{n-1}(n)$ are well-chosen arcs for g. Hence this set is not empty. On the other hand, the proof of the assertion shows that the set is also closed. Hence $A_1(0) = B_1, \ldots, A_{n-1}(0) = B_{n-1}$ are well-chosen arcs for $G_{c(0)}$. Thus we have proved the theorem in case $b_i \neq b_j$ for $i \neq j$.

The general case. — We may assume

$$b_i \neq b_j \quad \text{for} \quad i \neq j, \quad 1 \leq i, \ j \leq m,$$

and $$\{b_1, \ldots, b_m\} = \{b_1, \ldots, b_n\}.$$

Then in the same way as above we construct a simple arc $\psi : [1, m] \to \mathbf{R}^2$ such that $\psi(i) = a_i$, $i = 1, \ldots, m$, that the image of $f \circ \psi$ is a C^∞ manifold and that the image of ψ satisfies at each $b_2, \ldots, b_{m-1}$ the same condition as the well-chosen arcs above. And there exist $c = (c_1, \ldots, c_n) \in \mathbf{C}^n$ and a simple arc $\varphi : [1, m] \to \mathbf{R}^2$ such that

$$\varphi(i) = c_i \quad \text{for} \quad 1 \leq i \leq m, \qquad g \circ \varphi = f \circ \psi,$$

$$d_j \notin g \circ \varphi([1, m]) \quad \text{for} \quad m < j \leq n,$$

and $$d_i \neq d_j \quad \text{for} \quad i \neq j \quad \text{where} \quad g = G_c, \quad d_i = g(c_i)$$

and that the image of the set $\varphi([1, m])$ by some diffeomorphism τ_1 of $\mathbf{R}^2$ is $\psi([1, m])$. We set

$$A_i = g \circ \varphi([i, i+1]) \qquad i = 1, \ldots, m-1.$$

Let

$$X_0 = \psi([1, m]),$$

X_k = the union of the connected components of $f^{-1}(A_i)$, $i=1, \ldots, m-1$, which intersect X_{k-1}, $k=1, \ldots$ We define Y_k for g in the same way. Then, as $f^{-1}(\bigcup_{i=1}^{m-1} A_i)$ is connected, we have

$$\bigcup_k X_k = f^{-1}(\bigcup_{i=1}^{m-1} A_i).$$

Let k_i, $i=m+1, \ldots, n$, be the least k such that X_k contains a_i. We may assume $k_{m+1} \leq k_{m+2} \leq \ldots$ Then there exists a homeomorphism τ_2 of $\mathbf{R}^2$ such that $\tau_2 = \tau_1$ on Y_0 and that $\tau_2\big|_{Y_k}$ is a homeomorphism onto X_k for $i \leq k \leq k_{m+1}$. Let $\varphi_{m+1} : [0, 1] \to \mathbf{R}^2$ be a simple arc such that

$$\varphi_{m+1}(0) = \pi_2^{-1}(a_{m+1}), \qquad \varphi_{m+1}(1) = c_{m+1},$$

and that $g \circ \varphi_{m+1}$ is a simple arc and does not intersect $\bigcup_{i=1}^{m-1} a_i \cup \{d_j\}$ except for the extremities. We denote the image of $g \circ \varphi_{m+1}$ by A_{m+1}. Next we define a simple arc $\varphi_{m+2} : [0, 1] \to \mathbf{R}^2$ and a set A_{m+2} such that φ_{m+2} is a path from a point of $g^{-1}(b_{m+2})$ to c_{m+2} and that $g \circ \varphi_{m+2}$ is a simple arc and does not intersect $A_{m+1} \cup \bigcup_{i=1}^{m-1} A_i \cup \{d_j\}$ except for the extremities. The point of $g^{-1}(b_{m+2})$ is chosen in the same way as for φ_{m+1} by shrinking each connected component of $g^{-1}(A_{m+1})$ to a point. We repeat this process to define $\varphi_{m+3}, \ldots, \varphi_n$, $A_{m+3}, \ldots, A_n$. Then $\bigcup_{i \neq m} A_i$ is contractible and $(\mathbf{R}^2, g^{-1}(\bigcup_{i \neq m} A_i))/\sim$ is homeomorphic to $(\mathbf{R}^2, f^{-1}(\bigcup_{i=1}^{m-1} A_i))$, where $\sim$ means the shrinking of each connected component of $g^{-1}(A_{m+i})$ to a point, $i=1, \ldots, n-m$. Let

$$d(s) = (d_1(s), \ldots, d_n(s)) \in \mathbf{C}^n, \qquad 0 \leq s \leq 1$$

be defined by

$$d_i(s) = \begin{cases} d_i & 1 \leq i \leq m, \\ g \circ \varphi_i(s) & m+1 \leq i \leq n. \end{cases}$$

Then, since we can assume that any $d_i(s)$ never takes the value 0, we have $d(s) \in R$ for $0 < s \leq 1$. Hence there exists a continuous mapping

$$[0, 1] \ni s \mapsto c(s) = (c_1(s), \ldots, c_n(s)) \in \mathbf{C}^n$$

such that $F[c(s)] = d(s)$ and $c(1) = c = (c_1, \ldots, c_n)$. If we prove $c(0) \in R$, that is, $c_i(0) \neq c_j(0)$ for $i \neq j$, then we see in the same way as for the case $b_i \neq b_j$ that f is equivalent to $G_{c(0)}$. Hence we only need to show that $c_i(0) \neq c_j(0)$ for $i \neq j$. Assume the existence of $i_0 \neq j_0$ such that $c_{i_0} = c_{j_0}$. Let

$$d_{i_0}(0) = y, \qquad G_{c(s)}^{-1}(y) = Z_s.$$

Then, by the lemma below, there are $m_{i_0}+1$ elements of Z_s near $c_{i_0}(s)$ and $m_{j_0}+1$ elements near $c_{j_0}(s)$ for small $s>0$. Because

$$W_s = G_{c(s)}^{-1}(\bigcup_{i=1}^{m-1} A_i \cup \bigcup_{i=m+1}^{n} \bigcup_{0 \leq t \leq s} d_i(t))$$

is contractible, we can join $c_{i_0}(s)$ and $c_{j_0}(s)$ by a simple arc J_s in W_s for any small $s>0$; here the mapping $(0, 1]\ni s\mapsto J_s\in C^0([0, 1], \mathbf{R}^2)$ is continuous. For any $s\geq 0$, $0\leq \ell\leq m-1$, $G_{c(s)}^{-1}(A_\ell)$ consists of $p=\sum_{i=1}^{n} m_i+1$ simple arcs $\{L_{1s}^\ell, \ldots, L_{ps}^\ell\}$ such that $G_{c(s)}\big|_{L_{ks}^\ell}$ is a homeomorphism onto A_ℓ and that $L_{ks}^\ell\cap L_{k's}^{\ell'}\subset\partial L_{ks}^\ell$ for $(\ell, k)\neq(\ell', k')$. Then, by the lemma below once more, we can order the set of arcs so that $[0, 1]\ni s\mapsto(L_{1s}^\ell, \ldots, L_{ps}^\ell)$ is continuous for each ℓ. We easily see that there exist $\ell_1, \ldots, k_1, \ldots$ such that J_s consists of $L_{k_1 s}^{\ell_1}, L_{k_2 s}^{\ell_2}, \ldots$ and a subset of $G_{c(s)}^{-1}(\bigcup_{i=m+1}^{n}\bigcup_{0\leq t\leq s} d_i(t))$. It is not possible that $J_s\subset G_{c(s)}^{-1}(\bigcup_{i=m+1}^{n}\bigcup_{0\leq t\leq s} d_i(t))$, because this would imply $a_{i_0}=a_i$ for some $i\neq i_0$. From the assumption, the extremities of J_s converge to a point when $s\to 0$. Hence $L_{k_1 0}^{\ell_1}, L_{k_2 0}^{\ell_2}, \ldots$ contains a Jordan curve. This means that $G_{c(0)}^{-1}(\bigcup_{i=1}^{m-1} A_i)$ contains that Jordan curve, which is impossible. Hence there is no $i\neq j$ such that $c_i(0)=c_j(0)$. The theorem is proved.

Lemma **(14.2)** (p. 56 *of Malgrange* [12]). — *Let z_j (resp. z'_k), $j, k=1, \ldots, p$, be the roots of the equation*

$$z^p+\sum_{i=1}^{p} c_i z^{p-i}=0 \quad (\textit{resp.}\ \ z^p+\sum_{i=1}^{p} c'_i z^{p-i}=0)$$

where the c_i, c'_i are complex numbers. Suppose that

$$|c_i|\leq K^i, \quad |c_i-c'_i|\leq K^i\delta \quad \textit{where} \quad K, \delta>0.$$

Then for any j, there exists k such that $|z_j-z'_k|\leq 2K\delta^{1/p}$.

As corollaries of the proof of the theorem we have the following.

Corollary **(14.3)**. — *Let $f(x)$, $g(x)$ be* **C**-*polynomials in a variable. Let $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_m\}$ be the singular sets of f, g respectively. Suppose that $n=m$, that the order of $f(x)-f(a_i)$ at a_i is equal to that of $g(x)-g(b_i)$ for each i and that $f(a_i)\neq f(a_j)$, $g(b_i)\neq g(b_j)$ for $i\neq j$. Then f and g are* R — L *real-analytically equivalent.*

Corollary **(14.4)**. — *Given a positive integer k. Let A_k be the set of* **C**-*polynomials in a variable of degree $\leq k$. Then the cardinality of the quotient space $A_k/\sim$ is finite, where $\sim$ means the* R — L *real analytic equivalence relation.*

Corollary **(14.5)**. — *Let $f: \mathbf{R}^2\to\mathbf{C}$ be a proper continuous function with only isolated topological singularities. Then f is C^0 equivalent to a* **C**-*polynomial.*

Example. — Let f be a **C**-polynomial in a variable of degree 6 with singular points a_1, a_2, a_3 such that $f(a_1)=f(a_2)$ and that the order of $f(x)-f(a_i)$ at a_i is 2, 2, 4 for $i=1, 2, 3$ respectively. Let B be the set of polynomials which have the same properties as f. Let X be a simple arc from $f(a_1)$ to $f(a_3)$. We remark that any two such arcs are homotopic relative to $f(a_1)$ and $f(a_3)$. Hence there are two types for the inverse image of X by f (Figure 3). Since the R—L real-analytic equivalence class of f is determined by the type, we have $\sharp B/\sim = 2$. This shows the necessity of the condition that $f(a_i) \neq f(a_j)$, $g(b_i) \neq g(b_j)$ for $b_i \neq b_j$ in the Corollary (14.3).

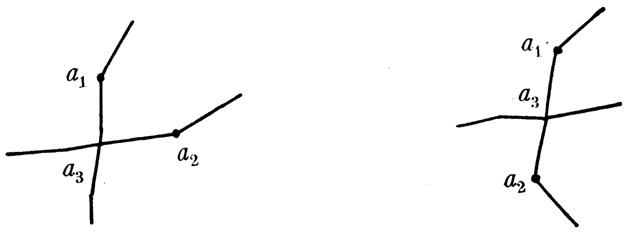


FIG. 3

Stoïlow proved that any **C**-valued light open continuous function on $\mathbf{R}^2$ is locally C^0 equivalent to a **C**-polynomial (see [36]). Hence the last corollary implies:

Corollary **(14.6)**. — *Any* **C**-*valued light open proper continuous function on* $\mathbf{R}^2$ *is* C^0 *equivalent to a* **C**-*polynomial.*

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