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ABSOLUTELY CONTINUOUS MEASURES FOR CERTAIN MAPS OF AN INTERVAL

by MICHAŁ MISIUREWICZ*

0. Introduction

Dynamical properties of mappings of an interval into itself are intensely studied from both " theoretical " and " experimental " (numerical experiments) points of view. One of the most important theoretical problems (but also closely related to the problem of reliability of numerical results) is to establish for which mappings there exist invariant probabilistic measures, absolutely continuous with respect to the Lebesgue measure, how many of them, and what are their ergodic properties. The aim of this paper is to answer these questions for a certain class of mappings. They are essentially the piecewise monotone mappings with non-positive Schwarzian derivative, no sinks and trajectories of critical points staying far from critical points (the exact conditions can be found in section 3: conditions (i)-(vi)). For a slightly similar class of mappings M. Jakobson [3] proved the existence of an absolutely continuous invariant measure. Our technique is quite different and it enables us to obtain much information about our measures.

After the preliminary results of Sections 1-5, we prove the main theorems (theorems (6.2) and (6.3)) in Section 6. For a mapping from our class, there exist a finite number (but at least one) of ergodic invariant probabilistic measures, absolutely continuous with respect to the Lebesgue measure. Their densities are continuous on an open dense set. Images of every finite measure, absolutely continuous with respect to the Lebesgue measure, under nk-th iterations of the mapping (for a certain k), converge strongly (as $n \to \infty$) to a linear combination of those measures. The mapping with every one of those measures is a skew product of a permutation of a finite set (in the base) and an exact transformation.

In Section 7 we show that for most widely considered one-parameter families of mappings (like $x \mapsto 4\alpha x(1-x)$), our conditions ((i)-(vi)) are satisfied for a set of parameters of power the continuum. The question, whether the measure of this set of parameters is zero or positive, remains open. However, there is some evidence that

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it is zero for generic families. Namely, it can be decomposed into a countable union of subsets, each of them arising from a point " running through " a Cantor set, which also depends on a parameter (one of these subsets is considered in Section 7). This Cantor set has always measure zero and there are reasons to believe that our point " runs too fast " to stay in this Cantor set for a time of positive measure (in a generic case) ([1]).

In sections 8 and 9 we study the problem; when is the measure-theoretical entropy (computable in numerical experiments as a characteristic exponent) equal to the topological entropy, which measures complexity of dynamics from the topological point of view? The answer for the maps of our class is: almost never (we obtain infinitely many independent necessary conditions).

The proof of the main results bases on the properties of the Schwarzian derivative. It is defined as $Sf=\frac{f'''}{f'}-\frac{3}{2}\left(\frac{f''}{f'}\right)^2$ and the idea of applying it to mappings of an interval belongs to D. Singer [10]. Two main properties are:

(1) $S(f\circ g)=(g')^2.(Sf\circ g)+Sg$, and consequently if $Sf\leq 0$ and $Sg\leq 0$ then $S(f\circ g)\leq 0$. In particular, if $Sf\leq 0$ then $S(f^n)\leq 0$ for all $n\geq 0$.

(2) If $Sf\leq 0$ then $\frac{1}{\sqrt{|f'|}}$ is convex on the components of the complement of the set of critical points (see (3.1)) and consequently, $|f'|$ has no positive strict local minima.

One can find examples of mappings with negative Schwarzian derivative for example in [10] and [5], and more properties of these mappings in [2].

With some additional work, it is possible to generalize the result of the paper, replacing the hypotheses C^3 and $Sf\leq 0$ by C^1 with f' Lipschitz and $\frac{1}{\sqrt{|f'|}}$ convex on the components of the complement of the set of critical points of f (i.e. on the components of $I\setminus A$).

Throughout the whole paper we denote by λ the Lebesgue measure, by f^n the n-th iterate of f, and by $\overline{E}$ the closure of E.

I would like to acknowledge very helpful discussions with J. Guckenheimer and Z. Nitecki.

1. Stretching far from critical points

In this section, I will be a closed interval, U and V two open subsets of I consisting of a finite number of intervals each, and such that U contains the endpoints of I and $U\cup V=I$, and $f: V\to I$ will be a continuous mapping.

([1]) Recently Jakobson proved that the absolutely continuous invariant measures exist for a set of parameters of positive measure (" Absolutely continuous invariant measures for one-parameter families of one-dimensional maps ", to appear in *Commun. Math. Phys.*). His set of parameters is much larger and his arguments seem to confirm our conjecture.

We shall call an open interval $J \subset V$ a *homterval* if for all positive integers n, f^n maps homeomorphically J onto its image. We shall say that f *has no sinks* if there does not exist an interval $J \subset V$ and a positive integer n such that f^n maps J homeomorphically into J.

Lemma (**1.1**). — *If f has no sinks and* J *is a homterval, then the intervals* $f^n J$, $n=0, 1, 2, \ldots$, *are pairwise disjoint.*

Proof. — Suppose that for some $n \geq 0$ and $k > 0$, $f^n(J)$ and $f^{n+k}(J)$ are not disjoint. Then for every $p \geq 0$, $f^{n+pk}(J)$ and $f^{n+(p+1)k}(J)$ are also not disjoint. Therefore, the set $K = \bigcup_{p=0}^{\infty} f^{n+pk}(J)$ is an interval. For every p the map

$$f^k\Big|_{f^{n+pk}(J)} = \left(f^{n+(p+1)k}\Big|_J\right) \circ \left(f^{n+pk}\Big|_J\right)^{-1}$$

is a homeomorphism, and thus $f^k\big|_K$ is a homeomorphism. But $f^k(K) \subset K$ and hence f has sinks. ■

Notice that any image of a homterval is also a homterval.

The following theorem may be considered as a generalization of the Denjoy theorem for a circle. It was proved by Z. Nitecki.

Theorem (**1.2**). — *Let f have no sinks, be of class* C^1, $f'(x) \neq 0$ *for all* $x \in V$ *and* $\log|f'|$ *be a Lipschitz function on components of* V. *Then for every homterval* J *there exists* $m \geq 0$ *such that* $\overline{f^m(J)} \subset U$.

Proof. — Denote the Lipschitz constant of $\log|f'|$ by γ. We shall show first:

(1.1) If K is an interval such that f^n is defined on K then:

$$\log \frac{\sup_K |(f^n)'|}{\inf_K |(f^n)'|} \leq \gamma \sum_{k=0}^{n-1} \lambda(f^k(K)).$$

Indeed, if $a, b \in K$ then we have:

$$\log \frac{|(f^n)'(a)|}{|(f^n)'(b)|} = \sum_{k=0}^{n-1} [\log|f'(f^k(a))| - \log|f'(f^k(b))|].$$

Since the set $f^k(K)$ is an interval and is contained in V, it is contained in some component of V. Hence:

$$\log|f'(f^k(a))| - \log|f'(f^k(b))| \leq \gamma|a-b| \leq \gamma\lambda(f^k(K)),$$

and we obtain (1.1).

Set $\beta = \mathrm{dist}(I \setminus U, I \setminus V)$. Since the sets $I \setminus U$ and $I \setminus V$ are compact and disjoint, we have $\beta > 0$.

Suppose that there exists a homterval J such that $\overline{f^n(\mathrm{J})}\setminus\mathrm{U} \neq \emptyset$ for every $n \geq 0$. We claim that:

(1.2) there exists n_0 such that for each $n \geq n_0$ every homterval containing $f^n(\mathrm{J})$ is disjoint from U.

Suppose that (1.2) is false. Then there exist $n \geq 0$, $k > 0$ and homtervals K and L such that $f^n(\mathrm{J}) \subset \mathrm{K}$, $f^{n+k}(\mathrm{J}) \subset \mathrm{L}$ and both $\overline{\mathrm{K}}$ and $\overline{\mathrm{L}}$ contain the same endpoint of some component of U and a piece of U adjacent to this endpoint. Then the interval $\mathrm{K} \cup \mathrm{L}$ is a homterval and $f^k(\mathrm{K} \cup \mathrm{L})$ intersects $\mathrm{K} \cup \mathrm{L}$. This contradicts Lemma (1.1). Hence (1.2) is true.

Let M be a maximal homterval containing $f^{n_0}(\mathrm{J})$. By (1.2) we have:

(1.3) For each $n \geq 0$ every homterval containing $f^n(\mathrm{M})$ is disjoint from U.

Now take an open interval L containing M, not equal to M and such that:

(1.4) $$\frac{\lambda(\mathrm{L})}{\lambda(\mathrm{M})} < \frac{\beta}{\lambda(\mathrm{I})} e^{-\gamma(\beta + \lambda(\mathrm{I}))} + 1.$$

We shall prove by induction that for every k:

(1.5) $$\frac{\lambda(f^k(\mathrm{L}))}{\lambda(f^k(\mathrm{M}))} < \frac{\beta}{\lambda(\mathrm{I})} + 1 \quad \text{and } f^k\big|_{\mathrm{L}} \text{ is a homeomorphism.}$$

For $k = 0$ it is obvious. Suppose that (1.5) holds for $k = 0, 1, \ldots, n-1$. We then have $\lambda(f^k(\mathrm{L})\setminus f^k(\mathrm{M})) < \frac{\beta}{\lambda(\mathrm{I})}\lambda(f^k(\mathrm{M})) \leq \beta$. From this, (1.3) and the definition of β it follows that $f^k(\mathrm{L}) \subset \mathrm{V}$ for $k = 0, \ldots, n-1$. Hence (since L is an interval), $f^n\big|_{\mathrm{L}}$ is a homeomorphism. By (1.1) and Lemma (1.1) we obtain:

$$\frac{\lambda(f^n(\mathrm{L}))}{\lambda(f^n(\mathrm{M}))} - 1 = \frac{\lambda(f^n(\mathrm{L}\setminus\mathrm{M}))}{\lambda(f^n(\mathrm{M}))} < \frac{\lambda(\mathrm{L}\setminus\mathrm{M})}{\lambda(\mathrm{M})} e^{\gamma \sum_{k=0}^{n-1} \lambda(f^n(\mathrm{L}))}$$
$$\leq \frac{\beta}{\lambda(\mathrm{I})} e^{-\gamma(\beta + \lambda(\mathrm{I}))} e^{\gamma \sum_{k=0}^{n-1} \left(\frac{\beta}{\lambda(\mathrm{I})} + 1\right) \lambda(f^n(\mathrm{M}))} \leq \frac{\beta}{\lambda(\mathrm{I})} e^{\gamma\left[\left(\frac{\beta}{\lambda(\mathrm{I})} + 1\right)\lambda(\mathrm{I}) - \beta - \lambda(\mathrm{I})\right]} = \frac{\beta}{\lambda(\mathrm{I})}.$$

This ends the proof of (1.5).

Since $f^n\big|_{\mathrm{L}}$ is a homeomorphism for each $n \geq 0$, L is a homterval. This contradicts the maximality of M. ■

Theorem **(1.3)**. — *Let f have no sinks, f be of class C^3, $f'(x) \neq 0$ for all $x \in \mathrm{V}$ and $\mathrm{S}f \leq 0$. Then there exists $m \geq 1$ such that, if $f^j(x) \notin \mathrm{U}$ for $j = 0, \ldots, m-1$, one has $|(f^m)'(x)| > 1$.*

Proof. — Suppose that for every $n \geq 1$ there exists x_n such that:

$$(1.6) \qquad |(f^n)'(x_n)| \leq 1$$

and

$$(1.7) \qquad f^j(x_n) \notin U \quad \text{for} \quad j = 0, \ldots, n-1.$$

Take the maximal open interval J_n containing x_n and such that f^n is defined on J_n. The point x_n divides J_n into two subintervals. Since $S(f^n) \leq 0$, on one of these subintervals we have $|(f^n)'| \leq 1$. Denote this subinterval by L_n. By the maximality of J_n, there exists $k(n) < n$ such that

$$(1.8) \qquad f^{k(n)}(L_n) \text{ has a common endpoint with some component of V.}$$

We claim that:

$$(1.9) \qquad \lambda(L_n) \to 0 \quad \text{as} \quad n \to \infty.$$

If not, there exists a sequence $(n_i)_{i=1}^{\infty}$, $x_0 \in I$ and $\varepsilon > 0$ such that $n_i \to \infty$ and $x_{n_i} \to x_0$ as $i \to \infty$, $\lambda(L_{n_i}) > \varepsilon$ for every i, and all intervals L_{n_i} are on the same side of x_{n_i}. For every j we have $f^j(x_{n_i}) \notin U$ if $n_i > j$ and hence $f^j(x_0) \notin U$. But one of the intervals $(x_0 - \varepsilon, x_0)$ and $(x_0, x_0 + \varepsilon)$ is a homterval, which contradicts Theorem (1.2). This proves (1.9).

By (1.6) we have $\lambda(f^n(L_n)) \leq \lambda(L_n)$, and therefore:

$$(1.10) \qquad \lambda(f^{n-k(n)}(f^{k(n)}(L_n))) \to 0 \quad \text{as} \quad n \to \infty.$$

In view of (1.8) and the fact that $f^{k(n)}(x_n) \notin U$, $f^{k(n)}(L_n)$ contains some component of the set $U \cap V$. Hence, there exists a component K of $U \cap V$ and a sequence $(n_i)_{i=1}^{\infty}$ such that $n_i \to \infty$ as $i \to \infty$ and $f^{k(n_i)}(L_{n_i}) \supset K$ for every i. From (1.10) we get $\lambda(f^{n_i - k(n_i)}(K)) \to 0$ as $i \to \infty$ and therefore:

$$(1.11) \qquad n_i - k(n_i) \to \infty \quad \text{as} \quad i \to \infty.$$

Let a be some point of condensation of the sequence $(f^{k(n_i)}(x_{n_i}))_{i=1}^{\infty}$. Since $f^j(f^{k(n_i)}(x_{n_i})) \notin U$ for $n_i - k(n_i) > j$, we obtain in view of (1.11):

$$(1.12) \qquad f^j(a) \notin U \quad \text{for every } j.$$

Let M be the minimal open interval such that $K \cup \{a\} \subset \overline{M}$. Then, by (1.11), M is a homterval. This contradicts (1.12) and Theorem (1.2). ∎

2. Estimates I

We want to obtain estimates, which enable us to prove the convergence of the images of the Lebesgue measure to an absolutely continuous measure. In particular, we want to estimate measures of inverse images of neighbourhoods of the set of critical values of f^n. First we shall deal with the part of estimates which can be done in the context similar to that of Section 1.

In this section U is an open subset of I consisting of a finite number of intervals and such that the endpoints of I belong to U. We denote by $f: I \setminus U \to I$ a map of class C^1 such that:

$$|f'| \geq \alpha > 1 \tag{2.1}$$

and the function $\log|f'|$ is Lipschitz on the components of $I \setminus U$, and by B a subset of $I \setminus U$ such that $f(B) \subset B$ and:

$$\beta = \operatorname{dist}(B, U) > 0. \tag{2.2}$$

Define $E_n = \{x \in I : f^k(x) \notin U \text{ for } k = 0, \ldots, n-1\}$ (notice that E_n is a domain of f^n). Now fix an element a of B and a number $\zeta \in [0, 1)$ and define:

$$\varphi(x) = \begin{cases} |x-a|^{-\zeta} & \text{for } x > a, \\ 0 & \text{for } x \leq a, \end{cases}$$

or

$$\varphi(x) = \begin{cases} |x-a|^{-\zeta} & \text{for } x < a, \\ 0 & \text{for } x \geq a. \end{cases}$$

Proposition **(2.1)**. — *There exist constants (independent on a and on whether* supp φ *is to the right or to the left of a)* $\eta \in (0, 1)$, $\xi \in (0, 1)$, $\theta > 0$ *such that for every* $n \geq 0$:

$$\lambda(E_n) \leq \eta^n \lambda(I), \tag{2.3}$$

$$\int_{E_n} \varphi \, d\lambda \leq (n+1)\theta\xi^n. \tag{2.4}$$

Proof. — We can, instead of U, take a slightly smaller set and extend f onto the complement of this set in such a way that all hypotheses still hold (perhaps α will be slightly smaller and the Lipschitz constant of $\log|f'|$ slightly larger) and also the image of every component of the domain of f is the whole interval. The new sets E_n will be perhaps slightly larger, but at least not smaller. Therefore we may simply assume that the image of every component of $I \setminus U$ under f is the whole interval I.

We shall first prove (2.3). Let K be a component of E_n, let γ be a Lipschitz constant for $\log|f'|$. By the same arguments as in the proof of Theorem (1.2), (1.1) holds. By (2.1) we have $\lambda(I) \geq \lambda(f^n(K)) \geq \alpha^{n-k}\lambda(f^k(K))$, and thus:

$$\sum_{k=0}^{n-1} \lambda(f^k(K)) \leq \lambda(I) \sum_{k=0}^{n-1} \alpha^{k-n} \leq \lambda(I) \sum_{j=1}^{\infty} \alpha^{-j} = \frac{\lambda(I)}{\alpha - 1}.$$

Set $\delta = e^{\frac{\gamma\lambda(I)}{\alpha-1}}$. In such a way we obtain:

$$\frac{\sup_K |(f^n)'|}{\inf_K |(f^n)'|} \leq \delta. \tag{2.5}$$

We have $f^n(K)=I$ and $f^n(K\setminus E_{n+1})=U$. Therefore $\lambda(I)\geq\lambda(K)\inf_K|(f^n)'|$ and $\lambda(U)\leq\lambda(K\setminus E_{n+1})\sup_K|(f^n)'|$. By (2.5) we get:

$$\frac{\lambda(K)}{\lambda(K\cap E_{n+1})}\geq 1+\frac{\lambda(K\setminus E_{n+1})}{\lambda(K)}\geq 1+\frac{\lambda(U)\inf_K|(f^n)'|}{\lambda(I)\sup_K|(f^n)'|}\geq 1+\frac{\lambda(U)}{\delta\lambda(I)}.$$

Set $\eta=\dfrac{1}{1+\dfrac{\lambda(U)}{\delta\lambda(I)}}$. Clearly $0<\eta<1$. We have $\lambda(K\cap E_{n+1})\leq\eta\lambda(K)$ for every component K of E_n. Summing over components of E_n, we get $\lambda(E_{n+1})\leq\eta.\lambda(E_n)$. Hence, by induction we obtain (2.3).

Now we shall prove (2.4). Since $f(B)\subset B$, we have $B\subset E_n$ for all n. For $k=0, 1, 2, \ldots$, denote by J_k the component of $E_k\cap\operatorname{supp}\varphi$ containing a. One of the endpoints of J_k is a; denote the other one by a_k. We have:

$$E_n\cap\operatorname{supp}\varphi=\bigcup_{k=0}^{n-1}H_k\cup J_n \tag{2.6}$$

where $H_k=E_n\cap(J_k\setminus J_{k+1})$.

The endpoints of $f^k(J_{k+1})$ are $f^k(a)\in B$ and $f^k(a_{k+1})\in\overline{U}$. From this, using (2.2) (which is clearly true also for subintervals of K instead of K) and (2.5), we obtain:

$$|a-a_{k+1}|\geq\frac{\beta}{\sup_{J_{k+1}}|(f^k)'|}\geq\frac{\beta}{\sup_{J_k}|(f^k)'|}\geq\frac{\beta}{\delta\inf_{J_k}|(f^k)'|}.$$

Consequently, since $f^k(H_k)\subset f^k(E_n)\subset E_{n-k}$, we have:

$$\int_{H_k}\varphi\,d\lambda\leq\lambda(H_k)\sup_{H_k}\varphi\leq\frac{\lambda(f^k(H_k))}{\inf_{H_k}|(f^k)'|}|a-a_{k+1}|^{-\zeta} \tag{2.7}$$
$$\leq\lambda(E_{n-k})\beta^{-\zeta}\delta^{\zeta}(\inf_{J_k}|(f^k)'|)^{\zeta-1}\leq\eta^{n-k}\lambda(I)\beta^{-\zeta}\delta^{\zeta}(\alpha^{\zeta-1})^k.$$

Since $|a-a_n|\leq\frac{1}{\alpha^n}|f^n(a)-f^n(a_n)|\leq\frac{\lambda(I)}{\alpha^n}$, we have:

$$\int_{J_n}\varphi\,d\lambda=\int_0^{|a-a_n|}x^{-\zeta}\,dx=\frac{1}{1-\zeta}|a-a_n|^{1-\zeta}\leq\frac{(\lambda(I))^{1-\zeta}}{1-\zeta}(\alpha^{\zeta-1})^n. \tag{2.8}$$

Set $\xi=\max(\eta, \alpha^{\zeta-1})$ and $\theta=\max\left(\lambda(I)\beta^{-\zeta}\delta^{\zeta}, \dfrac{(\lambda(I))^{1-\zeta}}{1-\zeta}\right)$.

Clearly, $0<\xi<1$ and $\theta>0$. Now (2.4) follows from (2.6), (2.7) and (2.8). ∎

3. Estimates II

In this section (and the next ones) I will be a closed interval, A a finite subset of I, containing its endpoints, and $f\,|\,I\setminus A\to I$ a continuous map, strictly monotone on components of $I\setminus A$.

Sometimes (especially in later sections) we shall pretend that f is defined on the whole of I. When speaking about $f^n(x)$ for an x such that f^n is in fact not defined at x, we shall mean a one-sided limit (it is usually clear which one).

However, in order to be more rigorous, we introduce (and sometimes use) the following notations:

$$\hat{\mathrm{I}} = \mathrm{I} \times \{+, -\} \setminus \{(\text{left endpoint of I}, -), (\text{right endpoint of I}, +)\},$$

$$\hat{\mathrm{I}} \ni (x, +) = x_+, \quad \hat{\mathrm{I}} \ni (x, -) = x_-,$$

$$\hat{f}: \hat{\mathrm{I}} \to \hat{\mathrm{I}} \text{ given by } \hat{f}(x, \varepsilon) = (x', \varepsilon')$$

where x' is a limit of $f(y)$ as y tends to x from the $\begin{Bmatrix} \text{right if } \varepsilon = + \\ \text{left if } \varepsilon = - \end{Bmatrix}$ and $\varepsilon' = \varepsilon$ if and only if f preserves the orientation in a $\begin{Bmatrix} \text{right if } \varepsilon = + \\ \text{left if } \varepsilon = - \end{Bmatrix}$ neighbourhood of x. If $x \in \hat{\mathrm{I}}$ then $\check{x}$ is its first coordinate (i.e. $(y_+)\check{} = (y_-)\check{} = y$). We set $\hat{\mathrm{A}} = \hat{\mathrm{I}} \cap (\mathrm{A} \times \{+, -\})$.

If the reader becomes confused about the use of the above notations, he can always think about the case of f continuous and omit all $\hat{}$'s and $\check{}$'s.

Now we make further assumptions on f.

(i) f is of class C^3,

(ii) $f' \neq 0$,

(iii) $Sf \leq 0$.

Consider f on a component of I$\setminus$A. Set $g = \dfrac{1}{\sqrt{|f'|}}$. We have:

$$g' = -\frac{1}{2} g \frac{f''}{f'},$$

$$g'' = -\frac{1}{2}\left(-\frac{1}{2} g \left(\frac{f''}{f'}\right)^2 + g \frac{f'''f' - (f'')^2}{(f')^2}\right) = -\frac{1}{2} g Sf.$$

Hence:

$$\left(\frac{1}{\sqrt{|f'|}}\right)'' = -\frac{Sf}{2\sqrt{|f'|}}. \tag{3.1}$$

This means that (iii) is equivalent to the following condition:

(iii') $\dfrac{1}{\sqrt{|f'|}}$ is convex on components of I$\setminus$A.

From this it follows that one-sided limits of f' at the elements of A exist. We shall denote them as $\hat{f}'$ at the corresponding points. Clearly $\hat{f}'$ exists also outside of A.

(iv) If $\hat{f}^p(x) = x$ then $(\hat{f}^p)'(x) > 1$.

From (iv) it easily follows that f has no sinks. Indeed, if J is a subinterval of I and f^n maps homeomorphically J into itself, then there exists a point $x \in \mathrm{J} \times \{+, -\}$ with $\hat{f}^{2n}(x) = x$ and $(\hat{f}^{2n})'(x) \leq 1$.

(v) There exists a neighbourhood U of A such that for every $a \in \hat{\mathrm{A}}$ and $n \geq 0$, $(\hat{f}^n(a))^{\vee} \in \mathrm{A} \cup (\mathrm{I} \setminus \mathrm{U})$.

(vi) For every $a \in \hat{\mathrm{A}}$ there exist constants $\delta, \alpha, \omega > 0$ and $u \geq 0$ such that:

$$\alpha |x - \check{a}|^u \leq |f'(x)| \leq \omega |x - \check{a}|^u$$

for every $x \in \begin{cases} (\check{a}, \check{a} + \delta) & \text{if} \quad a = \check{a}_+, \\ (\check{a} - \delta, \check{a}) & \text{if} \quad a = \check{a}_-. \end{cases}$

Taking smaller α and larger ω we can obtain the above inequalities on the whole corresponding component of I\A. Clearly (vi) is satisfied if f has non-zero one-sided derivatives (first, second or higher) at all elements of A.

We make also two additional assumptions:

(vii) $|f'| > 1$ on I\U,

(viii) if $a \in \hat{\mathrm{A}}$ is a periodic point for $\hat{f}$, then it is a fixed point for $\hat{f}$.

Lemma **(3.1)**. — *If f satisfies conditions* (i)-(vi) *then some iterate satisfies conditions* (i)-(viii) *(perhaps with a different set* A*)*.

Proof. — Let $m \geq 1$, $\tilde{f} = f^m$, $\tilde{\mathrm{A}} = \bigcup_{k=0}^{m-1} f^{-k}(\mathrm{A})$. It is easy to see that (i)-(vi) are also satisfied by $\tilde{f}$, $\tilde{\mathrm{A}}$ instead of f, A. In (v) we take $\tilde{\mathrm{U}} = \bigcup_{k=0}^{m-1} f^{-k}(\mathrm{U})$ instead of U. In (vi) we use a simple computation showing that if $\alpha_i x^{u_i} \leq g_i'(x) \leq \omega_i x^{u_i}$ for $i = 1, 2$, this is also true for $i = 3$ with some α_3, ω_3, u_3 where $g_3 = g_1 \circ g_2$ (we can take:

$$u_3 = (u_1 + 1)(u_2 + 1) - 1).$$

It is also clear that if f satisfies also (vii) (resp. (viii)) then so does $\tilde{f}$.

Now it remains to show that if f satisfies (i)-(vi) then some iterate satisfies (vii) and some (perhaps an other) one satisfies (viii). But the first fact follows from Theorem (1.3) and the second one is trivial (notice that an image of a periodic point is periodic). ∎

Set: $\mathrm{A}_1 = \{a \in \hat{\mathrm{A}} : \hat{f}(a) = a\}$, $\mathrm{A}_2 = \hat{\mathrm{A}} \setminus \mathrm{A}_1$, $\mathrm{C}_n = \bigcup_{i=1}^{n} \hat{f}^i(\hat{\mathrm{A}})$, $\mathrm{C} = \bigcup_{i=1}^{\infty} \hat{f}^i(\hat{\mathrm{A}})$, $\mathrm{B} = \bar{\check{\mathrm{C}}}$.

For a measurable function φ on I we denote by $\varphi\lambda$ the measure which is absolutely continuous with respect to λ and with the density (i.e. Radon-Nikodym derivative) φ.

For a measure μ and a map g, $g^*(\mu)$ denotes the image of μ under g, i.e. a measure such that for every measurable set E:

$$(3.2) \qquad (g^*(\mu))(\mathrm{E}) = \mu(g^{-1}(\mathrm{E})).$$

For an absolutely continuous map g, g_* denotes its Perron-Frobenius operator, i.e. for a measurable function φ:

$$(g_*(\varphi)).\lambda = g^*(\varphi.\lambda).$$

Notice that by (3.2), $\int_{g^{-1}(E)} \varphi\, d\lambda = \int_E g_*(\varphi)\, d\lambda$.

It is easy to check that we have (if g is differentiable):

$$g_*(\varphi)(x) = \sum_{y \in g^{-1}(x)} \frac{\varphi(y)}{|g'(y)|}.$$

Proposition **(3.2)**. — *If f satisfies* (i)-(iii) *then for every* $x \in I \setminus \check{C}_n$:

$$f_*^n(1)(x) \leq \frac{\lambda(I)}{\operatorname{dist}(x, \check{C}_n)}. \tag{3.3}$$

Proof. — Let J be a component of the set on which f^n is defined (i.e. $I \setminus \bigcup_{k=0}^{n-1} f^{-k}(A)$). The set $f^n(J)$ is an interval; denote it by (a, b). Set $g = f^n\big|_J$. The map g is a homeomorphism and $Sg \geq 0$. By the formula for the Schwarzian derivative of a composition we have $0 = S(\mathrm{id}) = S(g \circ g^{-1}) = S(g^{-1}) + ((g^{-1})')^2 (Sg \circ g^{-1})$. Hence $S(g^{-1}) \geq 0$. We have $g_*(1) = \dfrac{1}{|g'| \circ g^{-1}} = |(g^{-1})'|$. By (3.1) applied for g^{-1}, the function $\dfrac{1}{\sqrt{|(g^{-1})'|}}$ is concave. Therefore the function $\sqrt{|(g^{-1})'|}$ is convex, and consequently $g_*(1)$ is also convex (a reader not convinced by this argument may look at Lemma (4.3)). Thus:

$$g_*(1)(x) \leq \left\{ \begin{array}{c} \dfrac{1}{x-a} \displaystyle\int_a^x g_*(1)\, d\lambda = \dfrac{\lambda(g^{-1}(a, x))}{x-a} \\ \text{or} \\ \dfrac{1}{b-x} \displaystyle\int_x^b g_*(1)\, d\lambda = \dfrac{\lambda(g^{-1}(x, b))}{b-x} \end{array} \right\} \leq \frac{\lambda(J)}{\operatorname{dist}(x, \check{C}_n)}$$

(because $a, b \in \check{C}_n$).

Summing over all components of $I \setminus \bigcup_{k=0}^{n-1} f^{-k}(A)$, we obtain (3.3). ■

Lemma **(3.3)**. — *Let* $g : (b, c) \to \mathbf{R}$ *be a map of class* C^1, $g' > 0$. *Let* $\alpha, \omega > 0$ *and* $u \geq 0$ *be such that* $\alpha(x-b)^u \leq g'(x) \leq \omega(x-b)^u$ *for every* $x \in (b, c)$. *Let* $0 \leq \zeta < 1$ *and let a function* $\varphi : (b, c) \to \mathbf{R}$ *be given by a formula* $\varphi(x) = (x-b)^{-\zeta}$. *Then for some constants* $\delta > 0$, $0 \leq \xi < 1$ *we have* $g_*(\varphi)(y) \leq \delta(y - g(b))^{-\xi}$ *for every* $y \in (g(b), g(c))$.

Proof. — We have $g_*(\varphi)(g(x)) = \dfrac{\varphi(x)}{g'(x)} \leq \dfrac{(x-b)^{-\zeta}}{\alpha(x-b)^u} = \dfrac{1}{\alpha}(x-b)^{-\zeta-u}$. But:

$$g(x) - g(b) = \int_b^x g'(t)\,dt \leq \int_b^x \omega(t-b)^u\, dt = \omega \int_0^{x-b} t^u\, dt = \frac{\omega}{u+1}(x-b)^{u+1},$$

and hence $x-b\geq\left(\frac{u+1}{\omega}(g(x)-g(b))\right)^{\frac{1}{u+1}}$. Putting $y=g(x)$ we obtain:

$$g_*(\varphi)(y)\leq\frac{1}{\alpha}\left(\frac{u+1}{\omega}(g(x)-g(y))\right)^{\frac{-\zeta+u}{1+u}}.$$

Thus we put $\xi=\frac{\zeta+u}{1+u}$, $\delta=\frac{1}{\alpha}\left(\frac{\omega}{u+1}\right)^{\xi}$. Clearly $0\leq\xi<1$, $\delta>0$. ■

Now we assume that f satisfies (i)-(viii). By (v), $B\setminus A$ is disjoint from U. Since $Sf\leq 0$ and by the continuity of f', there exist open intervals U_a, $a\in A$, such that $\bigcup_{a\in\hat{A}}\overline{U}_a$ is disjoint from $B\setminus A$, $|f'|\geq\alpha>1$ on $I\setminus\bigcup_{a\in\hat{A}}\overline{U}_a$ and $\breve{a}$ is the $\left\{\begin{matrix}\text{left} & \text{if } a=\breve{a}_+\\ \text{right} & \text{if } a=\breve{a}_-\end{matrix}\right\}$ endpoint of U_a. It is easy to see that we can also have $|f'|\geq\beta>1$ on U_a if $a\in A_1$.

Lemma **(3.4)**. — *For every* $a\in A_2$ *there exist constants* γ, $\delta>0$ *and* ζ, $\xi\in[0, 1)$ *such that*:

$$\textbf{(3.4)}\qquad f_*^n(1)(x)\leq\gamma|x-\breve{a}|^{-\zeta}\quad \textit{for every } n\geq 0 \textit{ and } x\in U_a,$$

$$\textbf{(3.5)}\qquad \left(f\Big|_{U_a}\right)_*(f_*^n(1))(y)\leq\delta|y-(f(\hat{a}))^{\breve{}}|^{-\xi}\quad \textit{for every } n\geq 0,\ y\in f(U_a).$$

Proof. — By (viii) and the definition of A_2 we can assume that $A_2=\{a_1, \dots, a_r\}$ and no iterate of $\hat{f}$ takes a_i to a_j for $j\leq i$. Then we use induction, proving first (3.4) for a_1, then (3.5) for a_1, then (3.4) for a_2, then (3.5) for a_2, etc.

To prove (3.4) for a_i we write $f_*^n(1)$ in the form:

$$f_*^n(1)=\left(f\Big|_G\right)_*(f_*^{n-1}(1))+\left(f\Big|_{I\setminus G}\right)_*(f_*^{n-1}(1))$$

where $G=\bigcup_{j\in T}U_{a_j}$, $T=\{j:\hat{f}(a_j)=a_i\}$. Then we obtain an estimation of the first summand from (3.5) for $j\in T$, and the second one from Proposition (3.2) and the fact that by the definition of U_a, $\operatorname{dist}(B\setminus A, \bigcup_{a\in\hat{A}}U_a)>0$. We get a finite sum of expressions of the form $\gamma|x-\breve{a}_i|^{-\zeta}$ (notice that a constant is also of this form for $\zeta=0$), and the sum is not greater than some function of the same form. Thus we obtain (3.4), but in general only for x from some semi-neighbourhood of $\breve{a}_i$ smaller than U_{a_i}. Using once more Proposition (3.2) we obtain an estimation by a constant on the rest of U_a. Again, the sum of a function of the form $\gamma|x-\breve{a}_i|^{-\zeta}$ and a constant is not greater than some function of the form $\gamma|x-\breve{a}_i|^{-\zeta}$.

(3.5) for a_i follows from (3.4) for a_i and Lemma (3.3). ■

Lemma **(3.5)**. — *Let* $H\subset I$ *and* $H_k=\{x:f^i(x)\in H$ *for* $i=0, \dots, k-1\}$. *Then for every* s, m *we have*:

$$\int_{H_s}f_*^m(1)d\lambda\leq\sum_{k=s}^{\infty}\int_{H_k}\sup_{n\geq 0}\left(f\Big|_{I\setminus H}\right)_*(f_*^n(1))d\lambda+\lambda(H_{s+m}).$$

Proof. — *Let* $G_k = f^{-1}(H_k) \setminus H$. We have $f^{-1}(H_k) = G_k \cup H_{k+1}$, and by induction we get:

$$f^{-m}(H_s) = \bigcup_{k=s}^{s+m-1} f^{-s-m+1+k}(G_k) \cup H_{s+m}.$$

Hence:

$$\begin{aligned}\int_{H_s} f_*^m(1)\,d\lambda &= \lambda(f^{-m}(H_s)) \leq \sum_{k=s}^{s+m-1} \lambda(f^{-s-m+1+k}(G_k)) + \lambda(H_{s+m}) \\ &= \sum_{k=s}^{s+m-1} \int_{f(G_k)} \left(f\Big|_{G_k}\right)_* (f_*^{s+m-1-k}(1))\,d\lambda + \lambda(H_{s+m}) \\ &\leq \sum_{k=s}^{\infty} \int_{H_k} \sup_{n\geq 0} \left(f\Big|_{I\setminus H}\right)_* (f_*^n(1))\,d\lambda + \lambda(H_{s+m}). \quad \blacksquare\end{aligned}$$

Lemma (**3.6**). — *For every* $a \in A_1$ *and* $\varepsilon > 0$ *there exists a neighbourhood* W *of* $\breve{a}$ *such that*:

$$\int_{W \cap U_a} f_*^n(1)\,d\lambda < \varepsilon \quad \textit{for every} \quad n \geq 0.$$

Proof. — Let $a \in A_1$. There exists a constant $\beta > 1$ such that:

(3.6) $$|f(x) - \breve{a}| \geq \beta\,|x - \breve{a}| \quad \text{for all} \quad x \in U_a.$$

The set $V_k = \{x : f^i(x) \in U_a \text{ for } i = 0, \ldots, k-1\}$ is a neighbourhood of $\breve{a}$ in $\overline{U}_a$, and by (3.6) we have:

(3.7) $$\lambda(V_k) \leq \frac{\lambda(I)}{\beta^k}.$$

Therefore it is enough to prove that:

(3.8) $$\sup_{m \geq 0} \int_{V_s} f_*^m(1)\,d\lambda \to 0 \quad \text{as} \quad s \to \infty.$$

First we write, as in the proof of Lemma (3.4):

$$\left(f\Big|_{I\setminus U_a}\right)_* (f_*^n(1)) = \left(f\Big|_{G}\right)_* (f_*^n(1)) + \left(f\Big|_{I\setminus(G\cup U_a)}\right)_* (f_*^n(1)),$$

where $G = \bigcup_{b \in R} U_b$, $R = \{b \in \hat{A} \setminus \{a\} : \hat{f}(b) = a\}$. The same arguments as in the proof of Lemma (3.4) show that:

$$\sup_{n\geq 0} \left(f\Big|_{I\setminus U_a}\right)_* (f_*^n(1))(y) \leq \delta\,|y - a|^{-\xi}$$

for some constants $\delta > 0$, $0 \leq \xi < 1$ (notice that $R \subset A_2$, so we can use (3.5)). In view of Lemma (3.5) (for $H = U_a$; then $H_k = V_k$) and (3.7) we get:

$$\begin{aligned}\sup_{m\geq 0} \int_{V_s} f_*^m(1)\,d\lambda &\leq \sum_{k=s}^{\infty} \delta \int_0^{\frac{\lambda(I)}{\beta^k}} t^{-\xi}\,dt + \frac{\lambda(I)}{\beta^{s+m}} \\ &\leq \delta \sum_{k=s}^{\infty} \frac{1}{1-\xi} \left(\frac{\lambda(I)}{\beta^k}\right)^{1-\xi} + \frac{\lambda(I)}{\beta^s} \leq \frac{\delta}{1-\xi} \lambda(I)^{1-\xi} \sum_{k=s}^{\infty} (\beta^{\xi-1})^k + \frac{\lambda(I)}{\beta^s} \to 0\end{aligned}$$

as $s \to \infty$. This proves (3.8). ■

Lemma **(3.7).** — *For every* $\varepsilon>0$ *there exists a neighbourhood* W *of* B\A *such that*:

$$\int_W f_*^n(1)\,d\lambda<\varepsilon \quad \text{for every } n\geq 0.$$

Proof. — If we put $A\cup\bigcup_{a\in\hat{A}} U_a$ instead of U and B\A instead of B, then the hypotheses of Section 2 are satisfied. Let E_n be defined as in Section 2. Clearly, E_n is a neighbourhood of B\A. Hence, it is enough to prove that:

$$\textbf{(3.9)}\qquad \sup_{m\geq 0}\int_{E_s} f_*^m(1)\,d\lambda\to 0 \quad \text{as} \quad s\to 0.$$

By Proposition (2.1) and Lemma (3.4) there exist constants $\eta,\xi\in(0,1)$ and $\theta>0$ such that $\lambda(E_s)\leq\eta^s\lambda(I)$ for all $s\geq 0$ and:

$$\int_{E_s}\sup_{m\geq 0}\Big(f\Big|\bigcup_{a\in\hat{A}}U_a\Big)_*(f_*^n(1))\,d\lambda\leq(s+1)\theta\xi^s \quad \text{for all } s\geq 0.$$

Hence, by Lemma (3.5) (for $H=I\setminus\bigcup_{a\in\hat{A}}U_a$, $H_k=E_k$) we obtain:

$$\sup_{m\geq 0}\int_{E_s} f_*^m(1)\,d\lambda\leq\sum_{k=s}^{\infty}(s+1)\theta\xi^s+\lambda(I)\eta^s\to 0$$

as $s\to\infty$. This proves (3.9). ∎

Proposition **(3.8).** — *If f satisfies* (i)-(vi) *then for every* $\varepsilon>0$ *there exists* $\delta>0$ *such that if* $G\subset I$ *and* $\lambda(G)<\delta$ *then* $\int_G f_*^n(1)\,d\lambda<\varepsilon$ *for all n.*

Proof. — Suppose first that f satisfies (i)-(viii). Since a function of the form $\varphi(x)=\gamma|x-\breve{a}|^{-\zeta}$ $(0\leq\zeta<1)$ is integrable at $\breve{a}$, from Lemma (3.4) it follows that the statement of Lemma (3.6) holds also for all $a\in A_2$. This and Lemmata (3.6) and (3.7) imply that for every $\varepsilon>0$ there exists a neighbourhood W of B such that:

$$\int_W f_*^n(1)\,d\lambda<\varepsilon \quad \text{for all } n.$$

Now suppose that f satisfies (i)-(vi). By Lemma (3.1), there exists $m\geq 1$ such that f^m satisfies (i)-(viii). In order to distinguish the sets A and B defined for different iterates of f we shall use the symbols $A(f^k)$ and $B(f^k)$ for those sets defined for f^k. There exists $q\geq 1$ such that the set on which f^k is defined (i.e. $I\setminus A(f^k)$) consists of at most q components for $k=0,1,\ldots,m-1$.

Fix $\varepsilon>0$. Since f^m satisfies (i)-(viii), there exists an open neighbourhood W of $B(f^m)$ such that:

$$\textbf{(3.10)}\qquad \int_W f_*^{mn}(1)\,d\lambda<\frac{\varepsilon}{3} \quad \text{for all } n\geq 0.$$

Take:

$$\textbf{(3.11)}\qquad \eta=\frac{\varepsilon\,\mathrm{dist}(B(f^m),\,I\setminus W)}{3\lambda(I)}.$$

Clearly $\eta > 0$. There exist neighbourhoods U_k of $A(f^k)$, $k = 0, 1, \ldots, m-1$ such that:

$$(3.12) \qquad \lambda(U_k) \leq \eta.$$

Take:

$$(3.13) \qquad \delta = \frac{\eta}{q} \min_{0 \leq k \leq m-1} \inf_{I \setminus U_k} |(f^k)'|.$$

By (ii), also $\delta > 0$.

Suppose now that $G \subset I$ and $\lambda(G) < \delta$. Fix $k \in [0, m-1]$. We want to estimate $\int_G f_*^{mn+k}(1)d\lambda$. Let $J_1, \ldots, J_p$ be the components of $I \setminus A(f^k)$. By the definition of q, we have $p \leq q$. We have:

$$(3.14) \qquad \int_G f_*^{mn+k}(k)d\lambda = \int_{f^{-k}(G)} f_*^{mn}(1)d\lambda$$
$$\leq \int_W f_*^{mn}(1)d\lambda + \int_{U_k \setminus W} f_*^{mn}(1)d\lambda + \sum_{i=1}^{p} \int_{f^{-k}(G) \cap J_i \setminus (U_k \cup W)} f_*^{mn}(1)d\lambda.$$

By (3.11), (3.12) and Proposition (3.2):

$$(3.15) \qquad \int_{U_k \setminus W} f_*^{mn}(1)d\lambda \leq \lambda(U_k) \cdot \frac{\lambda(I)}{\operatorname{dist}(I \setminus W, B(f^m))} \leq \frac{\varepsilon}{3}.$$

By (3.11), (3.12), (3.13) and Proposition (3.2):

$$(3.16) \qquad \sum_{i=1}^{p} \int_{f^{-k}(G) \cap J_i \setminus (U_k \cup W)} f_*^{mn}(1)d\lambda \leq p . \lambda(f^{-k}(G) \cap J_i \setminus U_k) \cdot \frac{\lambda(I)}{\operatorname{dist}(I \setminus W, B(f^m))}$$
$$\leq q \cdot \frac{\lambda(G)}{\inf_{J_i \setminus U_k} |(f^k)'|} \cdot \frac{\lambda(I)}{\operatorname{dist}(I \setminus W, B(f^m))} < \frac{q . \delta}{\inf_{I \setminus U_k} |(f^k)'|} \cdot \frac{\lambda(I)}{\operatorname{dist}(I \setminus W, B(f^m))} \leq \frac{\varepsilon}{3}.$$

Now from (3.14), (3.10), (3.15) and (3.16) we obtain $\int_G f_*^{mn+k}(1)d\lambda < \varepsilon$. ■

Lemma **(3.9)**. — *If f satisfies* (i)-(vi) *then* $\lambda(B) = 0$.

Proof. — By Lemma (3.1), there exists $m \geq 1$ such that f^m satisfies (i)-(viii). We may assume that $A(f^m) = \bigcup_{i=0}^{m-1} f^{-i}(A(f))$. Then:

$$C(f) \subset \bigcup_{i=1}^{m} \hat{f}^i(\hat{A}(f)) \cup \bigcup_{i=0}^{m-1} \hat{f}^i(C(f^m)).$$

The set $\bigcup_{i=1}^{m} \hat{f}^i(\hat{A}(f))$ is finite, and hence it is enough to prove that $\lambda(\overline{\bigcup_{i=0}^{m-1} (\hat{f}^i(C(f^m)))^\vee}) = 0$. But:

$$\overline{\bigcup_{i=0}^{m-1} (\hat{f}^i(C(f^m)))^\vee} \subset \bigcup_{i=0}^{m-1} \hat{f}^i(B(f^m)) \cup \{(\hat{f}^i(a))^\vee : a \in \hat{A}(f),\ 0 \leq i < m\}.$$

We have $\lambda(\bigcup_{i=0}^{m-1} \hat{f}^i(B(f^m))) = 0$ because by Lemma (3.7) (for $n = 0$), $\lambda(B(f^m)) = 0$. The set $\{(\hat{f}^i(a))^\vee : a \in \hat{A}(f), 0 \leq i < m\}$ is finite. ■

4. Functional spaces

If f satisfies (i)-(vi) then the existence of an invariant probabilistic measure, absolutely continuous with respect to λ, follows already from Proposition (3.8). It is enough to take any weak-* limit of a subsequence of $\left(\frac{1}{n}\sum_{k=0}^{n-1} f^*(\lambda)\right)_{n=1}^{\infty}$. However, we want to know something about ergodic properties of such measures, their densities and how many of them there exist. Thus we are going to establish several auxilliary facts.

Let J be an open interval. We denote by $\mathscr{D}_r(\mathrm{J})$ the set consisting of all C^r positive functions τ on J such that $\frac{1}{\sqrt{\tau}}$ is concave, and the function 0.

Lemma **(4.1)**. — *An integrable non-zero function* τ *belongs to* $\mathscr{D}_2(\mathrm{J})$ *if and only if* $\tau = g_*(1)$ *for some* C^3 *diffeomorphism* $g:\mathrm{J}^*\to\mathrm{J}$ *with non-positive Schwarzian derivative.*

Proof. — Let J^* be an open interval, $g:\mathrm{J}^*\to\mathrm{J}$ a C^3 diffeomorphism, $\tau = g_*(1)$. As in the proof of Proposition (3.2) we deduce that $\frac{1}{\sqrt{\tau}}$ is concave if and only if $\mathrm{S}g\leq 0$. This proves that if $\mathrm{S}g\leq 0$ then $\tau\in\mathscr{D}_2(\mathrm{J})$.

Now assume that $\tau\in\mathscr{D}_2(\mathrm{J})$, $\tau\neq 0$ and τ is integrable. Let $\tilde{g}(x)=\int_a^x \tau(t)\,dt$, where a is a left-hand endpoint of J. Set $\mathrm{J}^*=\tilde{g}(\mathrm{J})$, $g=\tilde{g}^{-1}$. Clearly, we have $\tau = g_*(1)$, g is a C^3 diffeomorphism and $\mathrm{S}g\leq 0$. ■

Lemma **(4.2)**. — *If* $\tau, \psi\in\mathscr{D}_r(\mathrm{J})$ *then* $\tau+\psi\in\mathscr{D}_r(\mathrm{J})$.

Proof. — Take $[a, b]\subset \mathrm{J}$, $t\in(0,1)$, $c=ta+(1-t)b$. We want to show that (if $\tau, \psi\neq 0$):

$$\textbf{(4.1)}\qquad \frac{1}{\sqrt{\tau(c)+\psi(c)}}\geq\frac{t}{\sqrt{\tau(a)+\psi(a)}}+\frac{1-t}{\sqrt{\tau(b)+\psi(b)}}.$$

Take affine functions ρ, σ such that:

$$\textbf{(4.2)}\qquad \rho(a)=\frac{1}{\sqrt{\tau(a)}},\quad \rho(b)=\frac{1}{\sqrt{\tau(b)}},\quad \sigma(a)=\frac{1}{\sqrt{\psi(a)}},\quad \sigma(b)=\frac{1}{\sqrt{\psi(b)}}.$$

Since $\tau, \psi\in\mathscr{D}_r(\mathrm{J})$, we have:

$$\rho(c)\leq\frac{1}{\sqrt{\tau(c)}},\qquad \sigma(c)\leq\frac{1}{\sqrt{\psi(c)}}.$$

Hence $\tau(c)+\psi(c)\leq(\rho(c))^{-2}+(\sigma(c))^{-2}$, i.e.:

$$\textbf{(4.3)}\qquad \varphi(c)\leq\frac{1}{\sqrt{\tau(c)+\psi(c)}},$$

where $\varphi(x)=\frac{1}{\sqrt{(\rho(x))^{-2}+(\sigma(x))^{-2}}}$.

Since ρ and σ are affine functions, their derivatives are constants. Denote these derivatives by α and β respectively. We have:

$$\varphi'(x) = -\frac{1}{2}(\varphi(x))^3.(-2)(\alpha(\rho(x))^{-3}+\beta(\sigma(x))^{-3})$$
$$= (\varphi(x))^3(\alpha(\rho(x))^{-3}+\beta(\sigma(x))^{-3}),$$
$$\varphi''(x) = 3(\varphi(x))^2.(\varphi(x))^3.(\alpha(\rho(x))^{-3}+\beta(\sigma(x))^{-3})^2$$
$$+(\varphi(x))^3(-3)(\alpha^2(\rho(x))^{-4}+\beta^2(\sigma(x))^{-4})$$
$$= 3(\varphi(x))^5\left[(\alpha(\rho(x))^{-3}+\beta(\sigma(x))^{-3})^2 - \frac{\alpha^2(\rho(x))^{-4}+\beta^2(\sigma(x))^{-4}}{(\varphi(x))^2}\right]$$
$$= -3(\varphi(x))^5(\rho(x))^{-2}(\sigma(x))^{-2}\left(\frac{\alpha}{\rho(x)}-\frac{\beta}{\sigma(x)}\right)^2 < 0,$$

i.e. the function φ is concave on $[a, b]$. Hence, $\varphi(x) \geq t\varphi(a)+(1-t)\varphi(b)$. By (4.2), $\varphi(a)=\dfrac{1}{\sqrt{\tau(a)+\psi(a)}}$, $\varphi(b)=\dfrac{1}{\sqrt{\tau(b)+\psi(b)}}$. Together with (4.3), this gives (4.1). ∎

Lemma **(4.3)**. — *Every element of* $\mathscr{D}_0(\mathrm{J})$ *is a convex function.*

Proof. — Take $[a, b] \subset \mathrm{J}$, $t \in (0, 1)$, $c = ta+(1-t)b$, $\tau \in \mathscr{D}_0(\mathrm{J}) \setminus \{0\}$. We have:

$$\frac{1}{\sqrt{\tau(c)}} \geq \frac{t}{\sqrt{\tau(a)}} + \frac{1-t}{\sqrt{\tau(b)}} = \frac{1}{\sqrt{\varphi(t)}},$$

where $\varphi(x)=\left[\left(\dfrac{1}{\sqrt{\tau(a)}}-\dfrac{1}{\sqrt{\tau(b)}}\right)x+\dfrac{1}{\sqrt{\tau(b)}}\right]^{-2}.$

We have:

$$\varphi'(x) = -2(\varphi(x))^{3/2}.\left(\frac{1}{\sqrt{\tau(a)}}-\frac{1}{\sqrt{\tau(b)}}\right),$$
$$\varphi''(x) = 6(\varphi(x))^2\left(\frac{1}{\sqrt{\tau(a)}}-\frac{1}{\sqrt{\tau(b)}}\right)^2 > 0$$

and therefore the function φ is convex. Thus:

$$\tau(c) \leq \varphi(t) \leq t\varphi(1)+(1-t)\varphi(0) = t\tau(a)+(1-t)\tau(b). \quad ∎$$

Now we consider again a mapping f which satisfies (i)-(vi). Denote by $\mathscr{D}_r$ the set of all functions τ on $\mathrm{I} \setminus \mathrm{B}$ such that $\tau\big|_{\mathrm{J}} \in \mathscr{D}_r(\mathrm{J})$ for all components J of $\mathrm{I} \setminus \mathrm{B}$. We shall consider on $\mathscr{D}_0$ the topology of uniform convergence on compact subsets of $\mathrm{I} \setminus \mathrm{B}$ (shortly u.c.s. topology). Clearly, $\mathscr{D}_r \subset \mathscr{D}_0$ for all r.

Lemma (4.4). — *The set* $\mathscr{D}_0$ *is closed in the space of all continuous functions on* I\B *with the u.c.s. topology.*

Proof. — Let J be a component of I\B, $\tau_n \in \mathscr{D}_0(J)$ for $n = 1, 2, \ldots$ and $\tau_n \to \tau$ uniformly on compact subsets of J. Suppose that $x, y \in J$, $\tau(x) = 0$, $\tau(y) > 0$. Then $\frac{1}{\sqrt{\tau_n(x)}} \to +\infty$, $\frac{1}{\sqrt{\tau_n(y)}} \to \frac{1}{\sqrt{\tau(y)}} < +\infty$, and from the concavity of $\frac{1}{\sqrt{\tau_n}}$ it follows that $\frac{1}{\sqrt{\tau_n(z)}} \to -\infty$ for all z belonging to the component of $J \backslash \{y\}$ which does not contain x. Since $\frac{1}{\sqrt{\tau_n(z)}} \geq 0$, this is impossible. Therefore either $\tau = 0$ on the whole of J or τ is positive and $\frac{1}{\sqrt{\tau_n}} \to \frac{1}{\sqrt{\tau}}$ uniformly on compact subsets of J. But then $\frac{1}{\sqrt{\tau}}$ is concave, i.e. $\tau \in \mathscr{D}_0(J)$.

If we have a sequence of elements of $\mathscr{D}_0$ convergent in the u.c.s. topology, then we apply the above argument to every component J of I\B and conclude that the limit function is also an element of $\mathscr{D}_0$. ■

Proposition (4.5). — *Let* ρ *be a* C^2 *real function on* I. *Let* H *be the closure of the convex hull of the set* $\{f_*^n(\rho)\}_{n=0}^{\infty}$ *in the u.c.s. topology. Then:*

(a) *if* $\rho \in \mathscr{D}_0$ *then* $H \subset \mathscr{D}_0$;

(b) *every element of* H *is continuous on* I\B;

(c) H *is compact*;

(d) $H \subset L^1(\lambda)$;

(e) *the* L^1 *topology and the u.c.s. topology coincide on* H.

Proof. — Suppose first that $\rho \in \mathscr{D}_2$. By Lemmata (4.1) and (4.2), all functions $f_*^n(\rho)$ belong to $\mathscr{D}_2$, and therefore to $\mathscr{D}_0$. Hence, their convex combinations belong to $\mathscr{D}_0$. Now (a) follows from Lemma (4.4).

By Lemma (4.3), all functions $f_*^n(\rho)$, and therefore also their convex combinations, are convex on the components of I\B. By Proposition (3.2), they are all bounded by the function $\sigma(x) = \sup_I \rho \,.\, \frac{\lambda(I)}{\operatorname{dist}(x, B)}$. Hence they are equicontinuous on compact subsets of I\B (cf. [1]). By Arzelà-Ascoli's theorem, from every sequence of these functions we can choose a u.c.s. convergent subsequence. This proves (c) in the case $\rho \in \mathscr{D}_2$.

Let now ρ be an arbitrary C^2 real function on I. If for a C^2 positive function τ on I we have $\tau'' - \frac{3(\tau')^2}{2\tau} > 0$ then $\left(\frac{1}{\sqrt{\tau}}\right)'' = -\frac{1}{2}\tau^{-3/2}\left(\tau'' - \frac{3(\tau')^2}{2\tau}\right) < 0$, and consequently $\tau \in \mathscr{D}_2$. Therefore $\mathscr{D}_2$ contains an open cone in the space of C^2 functions on I. This cone is non-empty, for instance because $\tau(x) = \frac{1}{x - a}$, where a is a point to the left of I, belongs to it. Hence, there exist two C^2 functions on I, φ, ψ, which also belong to $\mathscr{D}_2$ and such that $\varphi-\psi=\rho$.

Thus, by (a), every convex combination of functions of the form $f_*^n(\rho)$ is a difference of two elements of $\mathscr{D}_0$, and therefore is continuous on $I\setminus B$. Now (b) follows from the fact that the u.c.s. limit of continuous functions is continuous.

If we have a sequence of convex combinations of functions of the form $f_*^n(\rho)$, then we can write every function as a difference of convex combinations of functions of the form $f_*^n(\varphi)$ and $f_*^n(\psi)$, respectively. Since we know already that (c) holds for elements of $\mathscr{D}_2$, we can find a convergent subsequence of our sequence. This proves (c).

The function ρ is bounded on I and hence $\rho\in L^1(\lambda)$. We have:

$$\textbf{(4.4)}\qquad \int |f_*^n(\rho)|\,d\lambda=\int\left|\sum_{y\in f^{-n}(x)}\frac{\rho(y)}{|(f^n)'(y)|}\right|d\lambda(x)$$
$$\leq\int\sum_{y\in f^{-n}(x)}\frac{|\rho(y)|}{|(f^n)'(y)|}=\int f_*^n|\rho|\,d\lambda=\int|\rho|\,d\lambda$$

and hence all functions $f_*^n(\rho)$ (and consequently also their convex combinations) belong to $L^1(\lambda)$.

Let now $(\sigma_n)_{n=1}^{\infty}$ be a sequence of convex combinations of functions of the form $f_*^k(\rho)$, and let $\sigma_n\to\sigma_0$ in the u.c.s. topology. We shall show that $\sigma_n\to\sigma_0$ also in the L^1 topology. By Proposition (3.8) and Lemma (3.9), for every $\varepsilon>0$ there exists an open neighbourhood W of B such that:

$$\textbf{(4.5)}\qquad \int_W f_*^n(1)\,d\lambda<\frac{\varepsilon}{3\sup_I|\rho|}\qquad\text{for all } n\geq 0.$$

Since $I\setminus W$ is a compact subset of $I\setminus B$, there exists n_0 such that:

$$\textbf{(4.6)}\qquad \int_{I\setminus W}|\sigma_n-\sigma_0|\,d\lambda<\frac{\varepsilon}{3}\qquad\text{for all } n\geq n_0.$$

From (4.5), in view of Fatou's lemma, it follows that:

$$\textbf{(4.7)}\qquad \int_W|\sigma_0|\,d\lambda\leq\liminf_{n\to\infty}\int_W|\sigma_n|\,d\lambda<\frac{\varepsilon}{3}.$$

If $n\geq n_0$, then from (4.5), (4.6) and (4.7) we obtain $\int|\sigma_n-\sigma_0|\,d\lambda<\varepsilon$. Hence $\sigma_n\to\sigma_0$ in L^1.

Now (d) follows from (4.4) whereas (e) follows from the fact that the identity mapping from H with the u.c.s. topology onto H with the L^1 topology is continuous, and from (c). ■

Theorem **(4.6)**. — *If f satisfies* (i)-(vi), *then there are no homtervals for f.*

Proof. — Suppose that $J\subset I$ is a homterval. Take a C^2 non-negative function ρ on I such that $\operatorname{supp}\rho\subset J$, $\int\rho\,d\lambda>0$. By Proposition (4.5), there exists a subsequence $(f_*^{n_i}(\rho))_{i=1}^{\infty}$ convergent to some function $\bar{\rho}$ in $L^1(\lambda)$. Then $\int \bar{\rho}\, d\lambda = \int \rho\, d\lambda > 0$. But f has no sinks and hence by Lemma (1.1) all intervals $f^{n_i}(J)$ are pairwise disjoint. Hence $f_*^{n_i}(\rho)$ converges to 0 pointwise—a contradiction. ∎

Lemma (**4.7**). — *Let* $U \subset I$ *be an open non-empty set. Then there exists* $n \geq 0$ *such that* $f^n(U)$ *contains some component of* $I \setminus B$.

Proof. — By Theorem (4.6), there exists an interval $(x, y) \subset U$ and $k, m \geq 0$ such that $f^k(x), f^m(y) \in A$. Let $n = \max(k, m) + 1$. Take the maximal interval (x, z) disjoint from $A(f^n) = \bigcup_{i=0}^{n-1} f^{-1}(A)$. Clearly $z \leq y$ and hence $(x, z) \subset U$. But $f^n(x, z)$ is an interval and its endpoints belong to B. ∎

5. Spectral Decomposition

In this section we still assume that f satisfies (i)-(vi). We shall decompose the possible support of absolutely continuous invariant measures in a way in some sense similar to the spectral decomposition for Axiom A diffeomorphisms. However, the reader should remember that we are interested in absolutely continuous measures. Therefore a part of the non-wandering set (a Cantor set of measure 0) can remain outside our set (i.e. the set $\bigcup \mathscr{H}$).

Denote the set of all components of $I \setminus B$ by $\mathscr{J}$. Since $f^n(B) \subset B$, if for some $J, L \in \mathscr{J}$ the set $f^n(J)$ intersects L, then it contains L. Hence the reader can think about our system as a topological Markov chain with a countable number of states or as a walking on an oriented graph with a countable number of vertices.

We define two relations on $\mathscr{J}$:

$J \sim L$ if and only if there exists $n \geq 0$ such that $f^n(J) \supset L$,

$J \approx L$ if and only if there exists $n \geq 0$ such that $f^n(J) \supset J \cup L$.

Let $\mathscr{H}$ be the set of all elements $J \in \mathscr{J}$ such that, for every $L \in \mathscr{J}$, if $J \sim L$ then $L \sim J$.

Lemma (**5.1**). — *The relations* $\sim$ *and* $\approx$ *are equivalence relations on* $\mathscr{H}$.

Proof. — 1°. $f^0(J) \supset J = J \cup J$.

2°. If $J \sim L$ then $L \sim J$ by the definition of $\mathscr{H}$. Let $J \approx L$. Then there exist $k, n \geq 0$ such that $f^k(J) \supset J \cup L$, $f^n(L) \supset J$. Hence $f^{n+k}(L) \supset L \cup J$, i.e. $L \approx J$.

3°. If $J \sim L$ and $L \sim M$ then clearly $J \sim M$. Let $J \approx L$ and $J \approx M$. Then there exist $k, m, n \geq 0$ such that $f^k(J) \supset J \cup L$, $f^m(J) \supset J \cup M$, $f^n(L) \supset J$. Hence $f^{k+m+n}(L) \supset L \cup M$, i.e. $L \approx M$. ∎

Lemma (**5.2**). — *The number of equivalence classes of the relation* $\sim$ *is not greater than* Card $A - 2$.

Proof. — Let $\mathscr{G} \subset \mathscr{H}$ be an equivalence class of the relation $\sim$. We shall show that the set $K = \overline{\bigcup \mathscr{G}}$ contains in its interior an element of A which is not an endpoint of I.

Suppose that K does not contain such a point. Let J be an open interval contained in $K \setminus A$. Since f is a homeomorphism on intervals disjoint from A and maps them onto intervals, and since $f(K \setminus A) \subset A$, we obtain by induction that $f^n(J)$ is contained in $K \setminus A$ and $f^n\big|_J$ is a homeomorphism for every $n \geq 0$. This contradicts Theorem (4.6). ■

Set $\mathscr{K} = \{\overline{\bigcup \mathscr{G}} : \mathscr{G}$ is an equivalence class of $\approx$ in $\mathscr{H}\}$. We shall use the notation $f^k(K)$ instead of the more precise $(\hat{f}^k(\hat{K}))^{\vee}$.

Lemma (**5.3**). — *The set* $\mathscr{K}$ *is finite, and f maps elements of* $\mathscr{K}$ *onto elements of* $\mathscr{K}$. *Moreover, for every* $K \in \mathscr{K}$ *there exists* $n \geq 1$ *such that* $f^n(K) = K$.

Proof. — Let $J, L, M, R \in \mathscr{H}$, $J \approx L$, $f(J) \supset M$, $f(L) \supset R$. Then there exist $k, m \geq 0$ such that $f^k(J) \supset J \cup L$ and $f^m(M) \supset J$. Hence, $f^{k+m+1}(M) \supset M \cup R$, i.e. $M \approx R$. This shows that f maps elements of $\mathscr{K}$ into elements of $\mathscr{K}$.

Let $K \in \mathscr{K}$. In view of the definition of $\mathscr{H}$ there exists $n \geq 1$ such that $f^n(K) \subset K$. Then $\bigcup_{k=1}^{n} f^k(K)$ must be a closure of a union of all elements of some equivalence class of $\sim$. Hence, $f^n(K) = K$ and also $f^k(K)$, $k = 1, \ldots, n-1$, are all elements of $\mathscr{K}$. By Lemma (5.2), $\mathscr{K}$ is finite. ■

Lemma (**5.4**). — *Let* $K \in \mathscr{K}$ *and let* $J \in \mathscr{J}$, $J \subset K$. *Then there exists* $n \geq 0$ *such that* $f^n(J) \supset K$.

Proof. — By Theorem (1.3), there exists $m \geq 1$ such that if $L \in \mathscr{J}$ and the sets $\bar{L}$, $\overline{f(L)}, \ldots, \overline{f^{m-1}(L)}$ are disjoint from A, then $\lambda(f^m(L)) > \lambda(L)$. Using this argument repeatedly, we see that for $n = 0, 1, 2, \ldots$, the set $f^{nm}(J)$ is a union of a finite number of intervals, each not shorter than $\alpha = \min\Big(\lambda(J), \min\limits_{\substack{M \in \mathscr{J} \\ \bar{M} \cap A \neq \emptyset \\ 0 \leq k \leq m}} \lambda(f^k(M))\Big) > 0$.

Since $J \in \mathscr{J}$ and $J \subset K$, also $J \in \mathscr{H}$. Hence there exists $k \geq 1$ such that $f^k(J) \supset J$. If $r \geq \ell$ then $f^{km(r-\ell)}(J) \supset J$ and thus $f^{kmr}(J) \supset f^{km\ell}(J)$. Hence $(f^{kmn}(J))_{n=0}^{\infty}$ is an ascending sequence of sets. If $L \in \mathscr{J}$, $L \subset K$, then there exists $n \geq 0$ such that $f^n(J) \supset J \cup L$ and therefore also $f^{kmn}(J) \supset L$. Consequently, we obtain the following situation: K is the closure of the union of an ascending sequence of sets $(f^{kmn}(J))_{n=0}^{\infty}$, and every term of this sequence is the union of a finite number of intervals, each not shorter than α.

Since inside I there is enough room only for a finite number of such intervals, after some n_0 their number must stabilize (although a priori it can be smaller for $\bigcup_{n=0}^{\infty} f^{kmn}(J)$). As $n (> n_0)$ tends to infinity, these intervals can only become longer and longer. Denote the limit intervals by $K_1, \ldots, K_s$. If $x_n \in K \setminus f^{kmn}(J)$ and $x_n \to \bar{x}$, then $\bar{x}$ must be an endpoint of a certain K_i.

Let $a \in \hat{I}$ be an endpoint of some K_i ($a = \check{a}_+$ if it is a left endpoint and $a = \check{a}_-$ if it is a right one). We shall show that there exists a point $b \in \hat{I}$ such that $\check{b}$ is an interior point of a certain K_j, and $\hat{f}^{kmn}(b) = a$ for a certain $n \geq 0$. Suppose that this is not true. Then a is an element of a periodic $\hat{f}^{km}$-orbit consisting of endpoints of K_j's and every element of this orbit has only one preimage (under $\hat{f}^{km}$). Together with the fact that a is repelling, this implies that there exists a (one-sided) neighbourhood U of $\check{a}$ and an open non-empty set $V \subset K$ such that $f^{kmn}(V)$ is disjoint from U for every $n \geq 0$. But by Lemma (4.7), some $f^{km\ell}(V)$ contains an element of $\mathscr{J}$. Since $f^{km}(K) \subset K$, this element is contained in K. Hence there exists $r \geq 0$ such that $f^{kmr}(V) \supset J$, and consequently $f^{kmn}(V)$ intersects U for a certain n—a contradiction.

Consequently, there exists $\ell(a) \geq 0$ such that $\check{a} \in \overline{K_i \cap f^{km\ell(a)}(J)}$. Now if we take $\ell = \max\{\ell(a) : a$ is an endpoint of K_i; $i = 1, \ldots, s\}$, we have $K = f^{km\ell}(J)$. ■

Corollary **(5.5)**. — *Each* $K \in \mathscr{K}$ *is a union of a finite number of intervals.*

Remark **(5.6)**. — If f is also continuous then each $K \in \mathscr{K}$ is an interval.

Notice that in the case of f continuous, the proof of Lemma (5.4) becomes much easier (each $f^{kmn}(J)$ is an interval).

From Lemmata (4.7) and (5.4) it follows that:

Proposition **(5.7)**. — If $K \in \mathscr{K}$ and $f^k(K) = K$, then for every open non-empty set $U \subset \bigcup_{i=0}^{\infty} f^{-ik}(K)$ there exists $n \geq 0$ such that $f^{nk}(U) \supset K$. In particular, $f^k\big|_K$ is topologically exact.

6. Absolutely Continuous Invariant Measures

Now we shall apply the results of Sections 4 and 5 to obtain the main results of the paper. As usual, f is a mapping satisfying (i)-(vi). By $\|\cdot\|$ we shall denote the norm in the space $L^1(\lambda)$, and by χ_M the characteristic function of a set M. Notice that by (4.4), the L^1 norm of the operator f_* is not greater than 1.

Lemma **(6.1)**. — *Let* $K \in \mathscr{K}$, $f^k(K) = K$, $M = \bigcup_{n=0}^{\infty} f^{-nk}(K)$. *Let* φ *be a non-negative continuous function on* $I \setminus B$ *such that* $0 < \int \varphi \, d\lambda < +\infty$, $\operatorname{supp} \varphi \subset K$, $f_*^k(\varphi) = \varphi$. *Then for every* C^2 *function* ρ *on* I, $\lim_{n \to \infty} \chi_M f_*^{nk}(\rho) = \dfrac{\int_M \rho \, d\lambda}{\int \varphi \, d\lambda} \varphi$ *in* $L^1(\lambda)$ *and u.c.s.*

Proof. — Let ρ be a C^2 function on I.

Suppose that $\chi_M f_*^{n_i k}(\rho) \to \bar{\rho}$ as $i \to \infty$, in the L^1 topology. Set:

$$\psi_i = \chi_M f_*^{n_i k}(\rho) - \frac{\int_M \rho\, d\lambda}{\int \varphi\, d\lambda}\varphi \quad \text{and} \quad \psi = \bar{\rho} - \frac{\int_M \rho\, d\lambda}{\int \varphi\, d\lambda}\varphi.$$

Since $f^{-n_i k}(M) = M$, we have:

$$\int \psi_i\, d\lambda = \int_M f_*^{n_i k}(\rho)\, d\lambda - \frac{\int_M \rho\, d\lambda}{\int \varphi\, d\lambda}\int \varphi\, d\lambda = 0.$$

If $n_j - n_i = \ell > 0$, then $f_*^{\ell k}\psi_i = \psi_j$. Clearly $\psi_i \to \psi$ in the L^1 topology.

Suppose that ψ is not equal a.e. to 0. By Proposition (4.5) we can assume that ψ is continuous on $I \setminus B$. Then there are open non-empty sets $U, V \subset M$ such that $\psi\big|_U > 0$, $\psi\big|_V < 0$. By Proposition (5.7), there exists $\ell_0 \geq 0$ such that $f^{\ell_0 k}(U) \supset K$ and $f^{\ell_0 k}(V) \supset K$. Hence, for every $x \in K \setminus B$ and $\ell \geq \ell_0$ we have $|f_*^{\ell k}(\psi)(x)| < f_*^{\ell k}(|\psi|)(x)$. Denote $\varepsilon = ||\psi|| - ||f_*^{\ell_0 k}(\psi)||$. Since $\lambda(K) > 0$, we have:

$$\varepsilon = \int |\psi|\, d\lambda - \int |f_*^{\ell_0 k}(\psi)|\, d\lambda = \int (f_*^{\ell_0 k}(|\psi|) - |f_*^{\ell_0 k}(\psi)|)\, d\lambda > 0.$$

If $\ell \geq \ell_0$ then $||f_*^{\ell k}(\psi)|| = ||f_*^{(\ell - \ell_0)k}(f_*^{\ell_0 k}(\psi))|| \leq ||f_*^{\ell_0 k}(\psi)||$, and thus:

(6.1) $$||\psi|| - ||f_*^{\ell k}(\psi)|| \geq \varepsilon \quad \text{for all } \ell \geq \ell_0.$$

But there exist i, j such that $\ell = n_j - n_i \geq \ell_0$, $||\psi_i - \psi|| < \frac{\varepsilon}{2}$ and $||\psi_j - \psi|| < \frac{\varepsilon}{2}$. Hence:

$$\begin{aligned} ||\psi|| - ||f_*^{\ell k}(\psi)|| &\leq ||\psi - f_*^{\ell k}(\psi)|| \leq ||\psi - \psi_j|| + ||\psi_j - f_*^{\ell k}(\psi)|| \\ &\leq ||\psi - \psi_j|| + ||f_*^{\ell k}(\psi_i - \psi)|| < \varepsilon. \end{aligned}$$

This contradicts (6.1). Hence, $\psi = 0$, and consequently, $\chi_M f_*^{nk}(\rho) \to \frac{\int_M \rho\, d\ell}{\int \varphi\, d\lambda}\varphi$ in the L^1 topology. Now the u.c.s. convergence follows from Proposition (4.5). ■

Theorem **(6.2)**. — *Let f satisfy* (i)-(vi), *and let* $K \in \mathscr{K}$. *Then there exists* $k \geq 1$ *and a probability measure* μ_K, *absolutely continuous with respect to the Lebesgue measure and* f^k*-invariant such that:*

(a) $f^k(K) = K$ *and the interiors of* K *and* $f^i(K)$ *are disjoint for* $i = 1, \ldots, k-1$,

(b) $\operatorname{supp} \mu_K = K$,

(c) $\frac{d\mu_K}{d\lambda} \in \mathscr{D}_0$,

(d) $\inf_{K \setminus B} \frac{d\mu_K}{d\lambda} > 0$,

(e) *if* $M = \bigcup_{n=0}^{\infty} f^{-nk}(K)$ *and* $\chi_M \cdot \rho \in L^1(\lambda)$ *then* $\lim_{n\to\infty} \chi_M \cdot f_*^{nk}(\rho) = \left(\int_M \rho \, d\lambda\right) \cdot \frac{d\mu_K}{d\lambda}$ *in* $L^1(\lambda)$; *if additionally* ρ *is continuous on* M *then the convergence is also u.c.s.*,

(f) *the system* $\left(K, f^k\Big|_K, \mu_K\right)$ *is exact,*

(g) μ_K *is the unique probability measure on* K, *absolutely continuous with respect to* λ *and* f^k*-invariant.*

Proof. — Let k be the smallest positive integer such that $f^k(K) = K$. By Lemma (5.3), such an integer exists. Now (a) follows from Lemma (5.3).

By Proposition (4.5), the closed convex hull of $\{f_*^{kn}(1)\}_{n=0}^{\infty}$ is compact in $L^1(\lambda)$ and hence, by Markov-Kakutani theorem, it contains a fixed point φ of f_*^k (if the reader does not want to use strong theorems, he can take instead a point of condensation of the sequence $\left(\left\{\frac{1}{n}\sum_{i=0}^{n-1} f_*^{k_i}(1)\right\}_{n=1}^{\infty}\right)$. We set $\mu_K = \frac{\varphi \cdot \chi_K}{\int_K \varphi \, d\lambda} . \lambda$. Clearly $\mu_k(I) = 1$. Since $f^k(K) = K$, we have

$$(f^k)^*(\mu_K) = f_*^k\left(\frac{\varphi \cdot \chi_K}{\int_K \varphi \, d\lambda}\right) . \lambda = \frac{\chi_K}{\int_K \varphi \, d\lambda} f_*^k(\varphi) . \lambda = \mu_K, \quad \text{i.e. } \mu_K \text{ is } f^k\text{-invariant.}$$

From the definition of μ_K it follows that $\operatorname{supp} \mu_K \subset K$. Proposition (4.5) implies (c). Since $\mu_K(K) = 1$, $\inf_J \frac{d\mu_K}{d\lambda} > 0$ for a certain component J of K\B. By Proposition (5.7), there exists $n \geq 0$ such that $f^{nk}(J) \supset K$. But $|f'|$ is bounded and consequently $\inf_{f^{nk}(J)} \frac{d\mu_K}{d\lambda} > 0$.

This proves (d), and also ends the proof of (b).

We shall prove (e). Let $M = \bigcup_{n=0}^{\infty} f^{-nk}(K)$, and let ρ be a function on I such that $\chi_M \cdot \rho \in L^1(\lambda)$. Let $\varepsilon > 0$. There exists a C^2 function ψ such that $\int_M |\rho - \psi| \, d\lambda < \frac{\varepsilon}{3}$. By Lemma (6.1) there exists n_0 such that for all $n \geq n_0$ we have:

$$\left\| \chi_M \cdot f_*^{nk}(\psi) - \left(\int_M \psi \, d\lambda\right) \frac{d\mu_K}{d\lambda} \right\| < \frac{\varepsilon}{3}.$$

If $n \geq n_0$ then:

$$\begin{aligned} &\left\| \chi_M \cdot f_*^{nk}(\rho) - \left(\int_M \rho \, d\lambda\right) \frac{d\mu_K}{d\lambda} \right\| \\ &\quad \leq \| \chi_M \cdot f_*^{nk}(\rho - \psi) \| + \left\| \chi_M \cdot f_*^{nk}(\psi) - \left(\int_M \psi \, d\lambda\right) \frac{d\mu_K}{d\lambda} \right\| \\ &\qquad + \left\| \left(\int_K (\psi - \rho) d\lambda\right) \frac{d\mu_K}{d\lambda} \right\| < \varepsilon. \end{aligned}$$

This proves the convergence in $L^1(\lambda)$.

Let now ρ be also continuous. Let $\varepsilon>0$ and let E be a compact subset of $M\setminus B$. Set $\xi=\sup_E \frac{d\mu_K}{d\lambda}$. Clearly $\xi<+\infty$. Hence there exists a C^2 function ψ such that $\sup_M|\rho-\psi|<\delta$ where $\delta=\frac{\varepsilon}{2(\lambda(M)+1)\xi}$. By Lemma (6.1) there exists n_0 such that if $n\geq n_0$ then:

$$\left|f_*^{kn}(1)-\lambda(M)\frac{d\mu_K}{d\lambda}\right|\leq 1 \text{ on E} \quad \text{and} \quad \left|f_*^{kn}(\psi)-\left(\int_M\psi\, d\lambda\right)\frac{d\mu_K}{d\lambda}\right|\leq\delta \text{ on E.}$$

If $n\geq n_0$ then:

$$|f_*^{kn}(\rho-\psi)|\leq f_*^{km}(|\rho-\psi|)\leq\delta f_*^{kn}(1)\leq\left(\lambda(M)\frac{d\mu_K}{d\lambda}+1\right)\delta\leq(\lambda(M)\xi+1)\delta \text{ on E,}$$

and consequently:

$$\left|f_*^{kn}(\rho)-\left(\int_M\rho\, d\lambda\right)\frac{d\mu_K}{d\lambda}\right|\leq|f_*^{kn}(\rho)-f_*^{kn}(\psi)|$$
$$+\left|f_*^{kn}(\psi)-\left(\int_M\psi\, d\lambda\right)\frac{d\mu_K}{d\lambda}\right|+\left|\int_M\psi\, d\lambda-\int_M\rho\, d\lambda\right|\frac{d\mu_K}{d\lambda}$$
$$<(\lambda(M)\xi+1)\delta+\delta+\delta\lambda(M)\xi=\varepsilon.$$

This proves the u.c.s. convergence.

We shall prove (f). Suppose that the system $\left(K, f^k\big|_K, \mu_K\right)$ is not exact. Then there exists a set $E\subset K$ such that $E=K\cap f^{-nk}(f^{nk}(E))$ for all $n\geq 0$ and $0<\mu_K(E)<1$. Then for every $n\geq 0$ we have $f_*^{nk}(\chi_E)=0$ outside $f^{nk}(E)$ and hence:

$$\left\|f_*^{nk}(\chi_E)-\lambda(E)\frac{d\mu_K}{d\lambda}\right\|\geq\int_{K\setminus f^{nk}(E)}\lambda(E)\frac{d\mu_K}{d\lambda}d\lambda$$
$$=\lambda(E)\mu_K(K\setminus f^{nk}(E))=\lambda(E)\mu_K(K\setminus E)>0.$$

This contradicts (e). Hence, the system $\left(K, f^k\big|_K, \mu_K\right)$ is exact.

At last, (g) follows from (e). ■

From Theorem (6.2), Lemma (5.3) and Lemma (5.2) we obtain easily

Theorem **(6.3)**. — *Let f satisfy* (i)-(vi). *Then there exist probability f-invariant measures $\mu_1, \ldots, \mu_s$, absolutely continuous with respect to the Lebesgue measure, and a positive integer k such that*:

(a) $\operatorname{supp}\mu_i=\overline{\bigcup\mathcal{G}_i}$ *for certain equivalence classes $\mathcal{G}_i$ of the relation $\sim$,* $i=1, \ldots, s$,

(b) $\operatorname{supp}\mu_i\cap\operatorname{supp}\mu_j$ *is a finite set if* $i\neq j$,

(c) $1\leq s\leq \operatorname{Card} A-2$,

(d) μ_i *is ergodic,* $i=1, \ldots, s$,

(e) $\frac{d\mu_i}{d\lambda}\in\mathcal{D}_0$, $i=1, \ldots, s$,

(f) $\inf \frac{d\mu_i}{d\lambda}$ *over the set* $\operatorname{supp} \mu_i \setminus B$ *is positive,* $i = 1, \ldots, s$,

(g) *if* $\rho \in L^1(\lambda)$ *then* $\lim_{n \to \infty} \sum_{=1}^{k} f_*^{n+j}(\rho) = \sum_{i=1}^{s} \alpha_i \frac{d\mu_i}{d\lambda}$ *in* $L^1(\lambda)$, *where* $\alpha_i = \int_{\bigcup_{n=0}^{\infty} f^{-n}(\operatorname{supp} \mu_i)} \rho \, d\lambda$; *if moreover* ρ *is continuous, then the convergence is also u.c.s.,*

(h) *for every finite Borel measure* ν, *absolutely continuous with respect to* λ *and f-invariant, one has* $\nu = \sum_{i=1}^{s} \alpha_i \mu_i$, *where* $\alpha_i = \nu(\bigcup_{n=0}^{\infty} f^{-n}(\operatorname{supp} \mu_i))$.

7. Examples

We are going to show that for a large class of one-parameter families of mappings (looking like $4\alpha x(1-x)$) conditions (i)-(vi) are satisfied for a set of parameters of power the continuum.

Let $F : [0, 1] \times I \to I$ be a continuous mapping. For $\alpha \in [0, 1]$ we denote by $f_\alpha : I \to I$ the mapping given by $f_\alpha(x) = F(\alpha, x)$. Let $I = [a_0, a_1]$. We assume that:

(a) $f_\alpha(a_0) = f_\alpha(a_1) = a_0$ for all $\alpha \in [0, 1]$,

(b) $f_0(x) = a_0$ for all $x \in I$,

(c) for every $\alpha \in (0, 1]$ there exists $c_\alpha \in I$ such that f_α is strictly increasing on $[a_0, c_\alpha]$ and strictly decreasing on $[c_\alpha, a_1]$,

(d) $f_1(c_1) = a_1$.

Lemma **(7.1)**. — *The mapping* $\alpha \mapsto c_\alpha$ *is continuous.*

Proof. — Fix $\alpha \in (0, 1]$ and $\varepsilon > 0$. By (c) there exists $\eta > 0$ such that

(7.1) if $f_\alpha(x) > f_\alpha(c_\alpha) - \eta$ then $|x - c_\alpha| < \varepsilon$.

The mapping F is continuous and $[0, 1] \times I$ is compact. Therefore F is uniformly continuous. Hence, there exists $\delta > 0$ such that:

(7.2) if $|\alpha - \beta| < \delta$ then $|f_\alpha(x) - f_\beta(x)| < \frac{\eta}{2}$ for all $x \in I$.

Let $|\alpha - \beta| < \delta$. By (7.2) we have:

$$f_\alpha(c_\beta) > f_\beta(c_\beta) - \frac{\eta}{2} \geq f_\beta(c_\alpha) - \frac{\eta}{2} > f_\alpha(c_\alpha) - \eta,$$

and by (7.1), $|c_\beta - c_\alpha| < \varepsilon$. ∎

Now define $\alpha_0 = \sup\{\alpha : f_\alpha(c_\alpha) \leq c_\alpha\}$.

Lemma **(7.2)**. — $\alpha_0 < 1$.

Proof. — By (a) and (d), we have $c_1 < a_1$. The mapping $\alpha \mapsto f_\alpha(c_\alpha) - c_\alpha$ is continuous in view of Lemma (7.1). Hence the set $\{\alpha : f_\alpha(c_\alpha) > c_\alpha\}$ is open and contains 1. ∎

Lemma **(7.3)**. — *For every* $\alpha\in(\alpha_0, 1]$ *there exists exactly one* $b_\alpha\in(c_\alpha, a_1)$ *and exactly one* $b'_\alpha\in(a_0, c_\alpha)$ *such that* $f_\alpha(b'_\alpha)=f_\alpha(b_\alpha)=b_\alpha$.

Proof. — Let $\alpha\in(\alpha_0, 1]$. Then $f_\alpha(c_\alpha)-c_\alpha>0$, $f_\alpha(a_1)-a_1<0$ and therefore there exists $b_\alpha\in(c_\alpha, a_1)$ such that $f_\alpha(b_\alpha)=b_\alpha$. It is unique because f_α is decreasing on (c_α, a_1). Now $f_\alpha(a_0)<b_\alpha$, $f_\alpha(c_\alpha)>f_\alpha(b_\alpha)=b_\alpha$ and therefore there exists $b'_\alpha(a_0, c_\alpha)$ such that $f_\alpha(b'_\alpha)=b_\alpha$. It is unique because f_α is strictly monotone on (a_0, c_α). ■

Now we must make a new assumption on f:

(e) there exists a neighbourhood U of the set $\{(\alpha, c_\alpha) : \alpha\in(0, 1]\}$ in $(0, 1]\times I$ such that for every $\alpha\in(0, 1]$, $U_\alpha=\{x : (\alpha, x)\in U\}$ is an interval and f_α satisfies the Lipschitz condition with the constant $\sqrt{2}$ on U_α.

Lemma **(7.4)**. — *There exists* $\alpha>\alpha_0$ *such that* $f^2_\alpha(c_\alpha)\geq b'_\alpha$.

Proof. — By (e), there exists $\varepsilon>0$ such that if $|\alpha-\alpha_0|<\varepsilon$ then f_α satisfies the Lipschitz condition with the constant $\sqrt{2}$ on the interval $[c_\alpha-\varepsilon, c_\alpha+\varepsilon]$. By Lemmata (7.1) and (7.2), there exists $\alpha>\alpha_0$ such that $|\alpha-\alpha_0|<\varepsilon$ and $|c_\alpha-f_\alpha c_\alpha|<\varepsilon$. Then f_α satisfies the Lipschitz condition with the constant $\sqrt{2}$ on the interval $[c_\alpha, f_\alpha(c_\alpha)]$ and b_α belongs to this interval. Hence:

$$\frac{1}{2}|f^2_\alpha(c_\alpha)-b_\alpha|\leq\frac{\sqrt{2}}{2}|f_\alpha(c_\alpha)-b_\alpha|\leq b_\alpha-c_\alpha. \tag{7.3}$$

If $b_\alpha-c_\alpha\geq c_\alpha-b'_\alpha$ then $0\leq c_\alpha-b'_\alpha\leq f_\alpha(c_\alpha)-c_\alpha<\varepsilon$ and f_α satisfies the Lipschitz condition with the constant $\sqrt{2}$ also on the interval $[b'_\alpha, c_\alpha]$, and hence also:

$$\frac{1}{2}|f^2_\alpha(c_\alpha)-b_\alpha|\leq c_\alpha-b'_\alpha. \tag{7.4}$$

But if $b_\alpha-c_\alpha<c_\alpha-b'_\alpha$ then (7.4) follows from (7.3).

Summing (7.3) and (7.4) we get $|f^2_\alpha(c_\alpha)-b_\alpha|\leq b_\alpha-b'_\alpha$, and therefore $f^2_\alpha(c_\alpha)\geq b'_\alpha$. ■

Now we define $\alpha_1=\sup\{\alpha : f^2_\alpha(c_\alpha)\geq b'_\alpha\}$. By Lemma (7.4), we have $\alpha_1>\alpha_0$.

Let $\xi=(\xi(n))^\infty_{n=0}$ be a 0-1 sequence. Let $D^0_\alpha=[b'_\alpha, c_\alpha]$, $D^1_\alpha=[c_\alpha, b_\alpha]$ (for $\alpha>\alpha_0$).

Lemma **(7.5)**. — *There exists a descending sequence* $(P_k)^\infty_{k=0}$ *of closed intervals contained in* $[\alpha_1, 1]$, *such that:*

$$f^{3+2n}_\alpha(c_\alpha)\in D^{\xi(n)}_\alpha \quad \textit{for } n=0, 1, \ldots, k \textit{ and for every } \alpha\in P_k, \tag{7.5}$$

$$\textit{there exist } \beta(k), \gamma(k)\in P_k \textit{ such that } f^{3+2k}_{\beta(k)}(c_{\beta(k)})=c_{\beta(k)} \tag{7.6}$$

$$f^{3+2k}_{\gamma(k)}(c_{\gamma(k)})=\begin{cases} b'_{\gamma(k)} & \textit{if } \xi(k)=0\\ b_{\gamma(k)} & \textit{if } \xi(k)=1.\end{cases}$$

Proof. — We shall use induction.

The functions $\alpha\mapsto b'_\alpha$, $\alpha\mapsto c_\alpha$, $\alpha\mapsto b_\alpha$ and $\alpha\mapsto f^3_\alpha(c_\alpha)$ are continuous on $[\alpha_1, 1]$ and we have $b'_\alpha < c_\alpha < b_\alpha$, $f^3_{\alpha_1}(c_{\alpha_1}) = b_{\alpha_1}$, $f^3_1(c_1) < b'_1$. Hence it is obvious that there exists an interval P_0 satisfying (7.5) and such that its endpoints (as $\beta(0)$ and $\gamma(0)$) satisfy (7.6).

Suppose now that we have found already P_{m-1} such that (7.5) and (7.6) hold for $k = m-1$. The function $\alpha\mapsto f^{3+2m}_\alpha(c_\alpha)$ is continuous. We have by (7.6):

$$f^{3+2m}_{\gamma(m-1)}(c_{\gamma(m-1)}) = b_{(m-1)} \quad \text{and} \quad f^{3+2m}_{\beta(m-1)}(c_{\beta(m-1)}) = f^2_{\beta(m-1)}(c_{\beta(m-1)}) < b'_{\beta(m-1)}.$$

Hence, as before, there exists an interval $P_m \subset P_{m-1}$ satisfying (7.5) for $k = m$ and such that its endpoints (as $\beta(m)$ and $\gamma(m)$) satisfy (7.6) for $k = m$. ■

From Lemma (7.5) it follows immediately that:

Proposition **(7.6)**. — *There exists* $\alpha(\xi)\in[\alpha_1, 1]$ *such that* $f^{3+2n}_{\alpha(\xi)}(c_{\alpha(\xi)}) \in D^{\xi(n)}_{\alpha(\xi)}$ *for* $n = 0, 1, 2, \ldots$

Proposition **(7.7)**. — *There is no homterval joining* $c_{\alpha(\xi)}$ *with a periodic point of* $f_{\alpha(\xi)}$.

Proof. — Denote $f = f_{\alpha(\xi)}$, $c = c_{\alpha(\xi)}$, $b = b_{\alpha(\xi)}$, $b' = b'_{\alpha(\xi)}$. Suppose that J is a homterval joining c and p and $f^n(p) = p$. Then $f^{2n}(p) = p$ and f^{2n} preserves an orientation at p (and hence on J). We may assume that $2n \geq 4$ (otherwise take 4 instead of $2n$). Then, by the definition of $\alpha(\xi)$, $f^{2n}(c)\in f[b', b]$, and hence $f^{2n}(c) \geq b$. Clearly $p \neq c$. If $p > c$ then $p = f^{2n}(p) > f^{2n}(c) \geq b$, and consequently $b\in J$. Then $b = f^{2n}(b) > f^{2n}(c) \geq b$- a contradiction.

If $p < c$ then $c < b \leq f^{2n}(c)$, i.e. $c\in f^{2n}(J)$. This contradicts the assumption that J is a homterval. ■

Proposition **(7.8)**. — *The point* $c_{\alpha(\xi)}$ *does not belong to the closure of the set* $\{f^n_{\alpha(\xi)}(c_{\alpha(\xi)})\}^\infty_{n=1}$.

Proof. — We use the same notation as in the preceeding proof. If $\alpha(\xi) = \alpha_1$ then $f^3(c) = b$ and $\overline{\{f^n(c)\}^\infty_{n=1}} = \{f(c), b', b\} \not\ni c$.

Consider the case $\alpha(\xi) > \alpha_1$. We have $f^2(c) < b'$, and hence there exists an open interval $V\ni c$ such that if $x\in V$ then $f^2(x) < b'$. Thus $f^n(c)\notin V$ for n odd. For n even and greater than 2 we have $f^n(c)\in f([b', b]) \subset [b, a_1]$. For $n = 2$, $f^n(c) < b'$. Consequently, since $V \subset (b', b)$, $f^n(c) \notin V$ for all $n \geq 1$. ■

Theorem **(7.9)**. — *Let the family* $\{f_\alpha\}_{\alpha\in[0,1]}$ *of mappings of* $I = [a_0, a_1]$ *into itself satisfy the following conditions:*

(1) $(\alpha, x)\mapsto f_\alpha(x)$ *is continuous,*

(2) f_α *is of class* C^3 *for every* $\alpha\in[0, 1]$,

(3) $(\alpha, x)\mapsto f'_\alpha(x)$ *is continuous,*

(4) $f''_\alpha(x) < 0$ *for every* $\alpha\in(0, 1]$, $x\in I$,

(5) $Sf_\alpha \leq 0$ *for every* $\alpha\in[0, 1]$,

(6) $f_\alpha(a_0) = f_\alpha(a_1) = a_0$ *for every* $\alpha \in [0, 1]$,
(7) $\sup f_0 = a_0$,
(8) $\sup f_1 = a_1$.

Then there exists a set $\Theta \subset [0, 1]$ *of power the continuum such that* f_α *satisfies conditions* (i)-(vi) *for every* $\alpha \in \Theta$.

Proof. — Clearly, our family satisfies conditions (a)-(d). Fix $\beta \in (0, 1]$. By (3), there exists $\varepsilon > 0$ such that if $|\alpha - \beta| < \varepsilon$ and $|x - c_\beta| < \varepsilon$ then $|f'_\alpha(x)| < \sqrt{2}$. But there exists $\delta > 0$ such that if $|\alpha - \beta| < \delta$ then $|c_\alpha - c_\beta| < \frac{\varepsilon}{2}$. Hence, if $|\alpha - \beta| < \min(\delta, \varepsilon)$ and $|x - c_\alpha| < \frac{\varepsilon}{2}$, then $|f'_\alpha(x)| < \sqrt{2}$. This proves (e).

Now we take a sequence ξ and $\alpha(\xi)$ obtained in Proposition (7.6). We shall prove that $f_{\alpha(\xi)}$ satisfies (i)-(vi). As before, we denote $f = f_{\alpha(\xi)}$, $c = c_{\alpha(\xi)}$.

Conditions (i)-(iii) are satisfied by (2), (4) and (5) (notice that (ii) means that $f'(x) \neq 0$ at $x \in I \setminus \{a_0, a_1, c\}$). Suppose that (iv) is not satisfied. Then there exists a periodic point p of period n such that $|(f^n)'(p)| \leq 1$. If $p = a_0$ then by (4), $f(x) < x$ for all $x > a_0$. By (5), $|(f^{2n})'|$ has no positive strict local minima and hence there exists an open interval J such that p is one of the endpoints of J and $0 < |(f^{2n})'|\big|_J \leq 1$. We can take as J a maximal such interval. Then the other endpoint of J is either an endpoint of I or a point at which $(f^{2n})'$ is 0. In the first case we obtain a contradiction, because $f^{2n}(J) \subset J$ and consequently $|(f^{2kn})'(x)| \leq 1$ for every k at the endpoint x of J. But since $f'(a_0) > 1$, no image of x is a_0. In the second case, some image of J is a homterval joining c with a periodic point. This contradicts Proposition (7.7). Hence (iv) holds.

Condition (v) follows from Proposition (7.8). Condition (vi) follows from the fact that $f'(a_0) \neq 0$, $f'(a_1) \neq 0$, and $f''(c) \neq 0$ (we take $u = 0$ for a_0 and a_1 and $u = 1$ for c). ■

8. Entropy

We still assume that f satisfies (i)-(vi). If f is also continuous then the topological entropy of f is equal to the topological entropy of the corresponding symbolic system (see e.g. [6]). Here we use for coding the partition into components of $I \setminus A$. Therefore, in the case of f not necessarily continuous we can define $h(f)$ simply as the topological entropy of the corresponding symbolic system (cf. [8]). It is easy to see that both systems are conjugate to each other after removing a countable number of points from both spaces. Hence there is a one-to-one correspondence between probabilistic invariant non-atomic measures, and thus the topological entropy so defined is equal to the supremum of metric entropies.

The goal of this section is to give necessary conditions for an absolutely continuous measure to be a measure with maximal entropy.

The first reason why an absolutely continuous measure may be not a measure with maximal entropy, is that it often happens that $h\left(f\big|_{\operatorname{supp}\mu}\right)<h(f)$. This reason was pointed out by J. Guckenheimer.

Notice that if $K\in\mathscr{K}$, $f^k(K)=K$ and $\frac{1}{k}\sum_{i=0}^{k-1} f^*(\mu_K)$ is a measure with maximal entropy for f, then μ_K is a measure with maximal entropy for f^k.

Theorem **(8.1)**. — *Let f satisfy* (i)-(vi), $K\in\mathscr{K}$, $f^k(K)=K$. *If the absolutely continuous measure μ_K is a measure with maximal entropy for $f^k\big|_K$, then for every periodic point $x\in K\setminus B$ of period n for f^k, $|(f^{nk})'(x)|=\beta^n$ where $h\left(f^k\big|_K\right)=\log\beta$.*

Proof. — In view of Corollary (5.5), K is a union of a finite number of intervals. Since K is f^k-invariant, we can " cut out " the gaps between them and we obtain a piecewise continuous mapping of an interval onto itself. By Proposition (5.7), $f\big|_K$ is strongly transitive (see [8]). By Theorem (6.3) (f), $h\left(f^k\big|_K\right)\geq h_{\mu_K}\left(f^k\big|_K\right)>0$. Hence, by the theorem of Parry [8], $f^k\big|_K$ is conjugate to a piecewise linear mapping g such that $|g'|=\beta$ (in the case of f continuous, this follows also from [4]). Denote this conjugacy by σ. Clearly, g satisfies (i)-(vi), and it has only one set in its " spectral decomposition "—the whole interval. Therefore there exists a unique g-invariant probabilistic measure ν, absolutely continuous with respect to λ. Clearly $\frac{d\nu}{d\lambda}\circ\sigma$ is continuous on $K\setminus B$.

Denote by $\varkappa$ the measure $(\sigma^{-1})^*(\nu)$, and by ψ and φ the measure theoretical jacobians of μ_K and $\varkappa$ respectively (see [7]). We have:

(8.1)
$$\psi=\frac{\left(\frac{d\mu_K}{d\lambda}\circ f^k\right)\cdot|(f^k)'|}{\frac{d\mu_K}{d\lambda}},$$

(8.2)
$$\varphi=\frac{\left(\frac{d\nu}{d\lambda}\circ g\right)\cdot\beta}{\frac{d\nu}{d\lambda}}\circ\sigma=\beta\frac{\frac{d\nu}{d\lambda}\circ\sigma\circ f^k}{\frac{d\nu}{d\lambda}\circ\sigma}.$$

For every probabilistic f^k-invariant measure ξ on K we have, in view of (8.2):

(8.3)
$$\int\log\varphi\,d\xi=\log\beta+\int\log\left(\frac{d\nu}{d\lambda}\circ\sigma\right)\circ f^k\,d\xi-\int\log\left(\frac{d\nu}{d\lambda}\circ\sigma\right)d\xi=\log\beta.$$

In view of Theorem (4.6), our systems have one-sided generators, and hence (see [7]) the measure-theoretic entropy is equal to the integral of the logarithm of the measure theoretic jacobian. Thus, by (8.3) we get:

(8.4)
$$h_{\mu_K}(f^k)-h_\varkappa(f^k)=\int\log\psi\,d\mu_K-\int\log\varphi\,d\varkappa$$
$$=\left(\int\log\psi\,d\mu_K-\log\beta\right)+\left(\log\beta-\int\log\varphi\,d\mu_K\right)=\int\log\frac{\psi}{\varphi}\,d\mu_K.$$

If a function ρ belongs to $L^1(\mu_K)$ then for the conditional expectation of ρ with respect to the inverse image of the whole σ-field under f^k, we have the formula:

$$E_{\mu_K}(\rho\,|\,f^{-k}(\mathscr{B}))(x)=\sum_{y\in f^{-k}(f^k(x))}\frac{\rho(y)}{\psi(y)}.$$

Hence:

(8.5)
$$\int\left(\frac{\psi}{\varphi}-1\right)d\mu_K=\int E_{\mu_K}\left(\frac{\psi}{\varphi}-1\,|\,f^{-k}(\mathscr{B})\right)d\mu_K$$
$$=\int\sum_{y\in f^{-k}(f^k(x))}\left(\frac{1}{\varphi(y)}-\frac{1}{\psi(y)}\right)d\mu_K(x)=\int(1-1)\,d\mu_K=0.$$

Assume now that μ_K is a measure with maximal entropy for $f^k\big|_K$, and that $x\in K\setminus B$, $f^{kn}(x)=x$. By (8.4) and (8.5), we have $\varphi=\psi$ μ_K-almost everywhere. The whole trajectory of x (under f^k) is disjoint from B and hence, in view of (8.1), (8.2) and Theorem (6.2) (c), φ and ψ are continuous in some neighbourhood of the trajectory of x. Thus, they are equal on the whole trajectory of x. Therefore by (8.1) and (8.2) we obtain:

$$|(f^{nk})'(x)|=\prod_{i=0}^{n-1}|(f^k)'(f^{ik}(x))|=\prod_{i=0}^{n-1}\psi(f^{ik}(x))=\prod_{i=0}^{n-1}\varphi(f^{ik}(x))=\beta^n.\quad\blacksquare$$

9. Entropy for quadratic maps

The best known (and easiest for computations) family of maps satisfying the hypotheses of Theorem (7.9) is the family of maps $f_\alpha:[0,1]\to[0,1]$ given by the quadratic polynomials $f_\alpha(x)=4\alpha x(1-x)$. Comparison of the graphs of topological entropy [4] and the characteristic exponents [9] of these maps suggests that the only case when the metric and topological entropies are equal is $\alpha=1$. In this case they are clearly equal since f_1 is smoothly conjugate to the piecewise linear map with slope ± 2.

It often happens that the topological entropy on the support of the absolutely continuous measure is smaller that the topological entropy on the whole interval. As J. Guckenheimer pointed out, this effect takes place e.g. when there is an interval J containing $c_\alpha\left(=\frac{1}{2}\right)$ invariant under f_α^3 and with 0-th, 1-st and 2-nd images disjoint.

Then the absolutely continuous measure is supported by $J \cup f_\alpha(J) \cup f_\alpha^2(J)$ (or even a smaller set). But we have then $h\left(f_\alpha\Big|_J\right) \leq \frac{1}{3}\log 2 < \log\frac{1+\sqrt{5}}{2} = h(f_\alpha)$. For our family this happens for $\alpha \in \left[\frac{1+2\sqrt{2}}{4}, 0.964200...\right]$ (of course, we consider only those α, for which an absolutely continuous invariant measure exists).

This effect is unavoidable for such families. However, the question remains, what happens in the case when the support of an absolutely continuous measure is an interval.

We are interested in the maps f_α for which $\frac{1}{2} \notin \overline{\left\{f_\alpha^n\left(\frac{1}{2}\right)\right\}_{n=1}^{\infty}}$. Then the probabilistic invariant absolutely continuous measure is unique (since $\operatorname{Card} A = 3$). We shall denote this measure by μ_α. We shall use the notations of Section 7 (remember that $c_\alpha = \frac{1}{2}$ for all α).

Proposition **(9.1)**. — *The support of $\mu_{\alpha(\xi)}$ is an interval.*

Proof. — Let $f = f_{\alpha(\xi)}$, $b = b_{\alpha(\xi)}$, $b' = b'_{\alpha(\xi)}$, $\mu = \mu_{\alpha(\xi)}$. In the proof of Proposition (7.8) we can take as V the maximal interval with the given properties. We have $V \subset \operatorname{supp}\mu$ and hence $\overline{f^3(V)} \subset \operatorname{supp}\mu$. But $\overline{f^2(V)} \ni b'$ and hence $\overline{f^3(V)} \ni b$. Since there are no homtervals, there exists n such that $f^{n+3}(V) \ni \frac{1}{2}$. Therefore:

$$\operatorname{supp}\mu \supset \left[f^2\left(\frac{1}{2}\right), f\left(\frac{1}{2}\right)\right].$$

But this interval is invariant, and hence it must be equal to $\operatorname{supp}\mu$. ■

We are going to show that $\mu_{\alpha(\xi)}$ is not a measure with maximal entropy for an uncountable set of sequences ξ. To simplify the computations we will work first with the equivalent family of maps g_β, where $g_\beta(x) = x^2 - \beta$, $\beta \in [0, 2]$. It is easy to check that g_β maps the interval $\left[-\frac{1+\sqrt{1+4\beta}}{2}, \frac{1+\sqrt{1+4\beta}}{2}\right]$ into itself.

Lemma **(9.2)**. — *Let $0 \leq \gamma < \delta \leq 2$ be such that g_γ has no periodic point of prime period 3 but g_δ does. Then $\gamma < 7/4 \leq \delta$.*

Proof. — Denote $W_\beta(x) = g_\beta(x) - x = x^2 - x - \beta$. It is easy to check that:

(9.1) $$g_\beta^3(x) - x = W_\beta(x) \cdot P_\beta(x)$$

where

(9.2) $$\begin{aligned} P_\beta(x) &= W_\beta(x) \cdot (W_\beta(x) + 2x + 1)^2 + 2x(W_\beta(x) + 2x + 1) + 1 \\ &= x^6 + x^5 + (1 - 3\beta)x^4 + (1 - 2\beta)x^3 + (1 - 3\beta + 3\beta^2)x^2 \\ &\quad + (1 - 2\beta + \beta^2)x + (1 - \beta + 2\beta^2 - \beta^3). \end{aligned}$$

From (9.2) it follows that if $W_\beta(x)=0$ then:

$$P_\beta(x)=2x(2x+1)+1=4x^2+2x+1=3x^2+(x+1)^2>0.$$

Therefore x is a periodic point of prime period 3 if and only if $P_\beta(x)=0$.

If $x=\frac{1+\sqrt{1+4\beta}}{2}$ then $g_\beta(x)=x$ and hence $P_\beta(x)>0$. If $x=-\frac{1+\sqrt{1+4\beta}}{2}$ then $P_\beta(x)=\frac{g_\beta^3(x)-x}{g_\beta(x)-x}=1>0$. Hence, there exists $\alpha\in(\gamma,\delta]$ and:

$$y\in\left(-\frac{1+\sqrt{1+4\alpha}}{2}, \frac{1+\sqrt{1+4\alpha}}{2}\right)$$

such that $P_\alpha(y)=0$ and $P'_\alpha(y)=0$.

We also have $P_\alpha(g_\alpha(y))=P_\alpha(g_\alpha^2(y))=0$ and the points y, $g_\alpha(y)$, $g_\alpha^2(y)$ are distinct. We have:

$$(g_\alpha^3)'(y)-1=(g_\alpha^3(x)-x)'\Big|_{x=y}=(W_\alpha.P_\alpha)'(y)=0$$

and hence also $(g_\alpha^3)'(g_\alpha(y))=(g_\alpha^3)'(g_\alpha^2(y))=1$. Thus:

$$(W_\alpha.P'_\alpha)(g_\alpha(y))=(g_\alpha^3)'(g_\alpha(y))-1-(W'_\alpha.P_\alpha)(g_\alpha(y))=0,$$

and analogically $(W_\alpha.P'_\alpha)(g_\alpha^2(y))=0$. Consequently, $P'_\alpha(g_\alpha(y))=P'_\alpha(g_\alpha^2(y))=0$.

Hence, the polynomial P_α has three double zeros. Since its degree is 6, it is a square of a certain polynomial Q of degree 3. The coefficient of x^6 in P_α is 1 and therefore we may assume that also the coefficient of x^3 in Q is 1. Let:

$$Q(x)=x^3+\zeta_2x^2+\zeta_1x+\zeta_0.$$

Comparing coefficients of x^5, x^4, x^3 and x in P_α and Q^2 we obtain:

$$\left.\begin{aligned} &2\zeta_2=1\\ &2\zeta_1+\zeta_2^2=1-3\alpha\\ &2\zeta_0+2\zeta_1\zeta_2=1-2\alpha\\ &2\zeta_0\zeta_1=1-2\alpha+\alpha^2\end{aligned}\right\}$$

From the first three equations we obtain successively: $\zeta_2=\frac{1}{2}$, $\zeta_1=\frac{3}{8}-\frac{3}{2}\alpha$, $\zeta_2=\frac{5}{16}-\frac{1}{4}\alpha$.

Then from the fourth equation we get $\frac{1}{4}\alpha^2-\frac{7}{8}\alpha+\frac{49}{64}=0$, i.e. $\alpha=\frac{7}{4}$. ■

Proposition **(9.3)**. — *g_β has a periodic point of prime period 3 if and only if $\beta\geq\frac{7}{4}$.*

Proof. — We get from (9.2): $P_0(x)=x^6+x^5+x^4+x^3+x^2+x+1=\frac{x^7-1}{x-1}$. Since P_0 has no real zeros, g_0 has no periodic points of prime period 3. If $z\in\mathbf{C}$, $|z|=1$, then $2\,\mathrm{Re}(z^2)=(2\,\mathrm{Re}\,z)^2-2=g_2(2\,\mathrm{Re}\,z)$. Hence, if we take as z a complex root of 1 of degree 7, then $2\,\mathrm{Re}\,z$ will be a periodic point of g_2 of prime period 3.

Now, if $\beta \geq \frac{7}{4}$ then g_β has a periodic point of prime period 3 by Lemma (9.2) applied to $\gamma=\beta$, $\delta=2$; if $\beta<\frac{7}{4}$ then g_β has no periodic point of prime period 3 by Lemma (9.2) applied to $\gamma=0$, $\delta=\beta$. ■

Since g_β is (linearily) conjugate to f_α, where $\alpha=\frac{1}{4}(1+\sqrt{1+4\beta})$, we obtain

Corollary **(9.4).** — *f_α has a periodic point of prime period 3 if and only if* $\alpha \geq \frac{1}{4}(1+2\sqrt{2})$.

Lemma **(9.5).** — *If* $\alpha(\xi)<\frac{1}{4}(1+2\sqrt{2})$ *then* $h_{\mu_{\alpha(\xi)}}(f_{\alpha(\xi)})<\log\frac{1+\sqrt{5}}{2}$.

Proof. — Let $f=f_{\alpha(\xi)}$, $\mu=\mu_{\alpha(\xi)}$. By Proposition (9.1), supp μ is an interval. Hence (see [8]), $f\big|_{\operatorname{supp}\mu}$ is conjugate to a piecewise linear map $\widetilde{f}$ with $|\widetilde{f}'|=\beta$, where $h\Big(f\big|_{\operatorname{supp}\mu}\Big)=\log\beta$. It is easy to see that $\widetilde{f}$ is linearily conjugate to the map g given by

$$g(x)=\begin{cases}\beta x & \text{if } x\leq 1\\ \beta(2-x) & \text{if } x\geq 1\end{cases}$$

on some subinterval of $[0,2]$, containing 1. Suppose that $h_\mu(f)\geq\log\frac{1+\sqrt{5}}{2}$. Then $\beta\geq\log\frac{1+\sqrt{5}}{2}$. The trajectory $\frac{2\beta}{1+\beta^3}, \frac{2\beta^2}{1+\beta^3}, \frac{2\beta^3}{1+\beta^3}$ is a periodic trajectory of prime period 3. In order to see this, it is enough to notice that $\frac{2\beta}{1+\beta^3}<\frac{2\beta^2}{1+\beta^3}\leq 1$ and $1<\frac{2\beta^3}{1+\beta^3}\leq\beta=g(1)$. Hence f has also a periodic trajectory of prime period 3. This contradicts Corollary (9.4). ■

Theorem **(9.6).** — *For the family* $\{f_\alpha\}_{\alpha\in[0,1]}$, $f_\alpha(x)=4\alpha x(1-x)$, $f_\alpha:[0,1]\to[0,1]$, *there exists a set* $\Lambda\subset\Theta$ *of power the continuum such that if* $\alpha\in\Lambda$ *then the absolutely continuous measure is not a measure with maximal entropy.*

Proof. — Let Λ be the set of all $\alpha(\xi)$ such that there exists k with the property that all blocks of 1's appearing in ξ are shorter than k.

Let $\alpha\in\Lambda$, $f=f_\alpha$, $\mu=\mu_\alpha$, $b=b_\alpha$, $b'=b'_\alpha$ $\left(\text{remember that } \frac{1}{2}=c_\alpha\right)$. Since

$$f^{3+2n}(b)\in D^1_\alpha \quad \text{for all } n,$$

then b does not belong to $\overline{\left\{f^n\left(\frac{1}{2}\right)\right\}_{n=1}^{\infty}}$. Suppose that μ is a measure with maximal entropy. Then, by Theorem (8.1), we have $|f'(b)|=\beta$, where $\log\beta=h_\mu(f)$. We have $4\alpha b(1-b)=b$, $b\neq 0$ and thus $4\alpha(1-b)=1$, i.e. $b=1-\frac{1}{4\alpha}$. Then

$$\beta=|f'(b)|=|4\alpha-8b\alpha|=|4\alpha-8\alpha+2|=4\alpha-2. \tag{9.3}$$

Suppose first that $\alpha < \frac{1}{4}(1+2\sqrt{2})$. By Lemma (9.5) we have $\beta < \frac{1+\sqrt{5}}{2}$, and hence by (9.3):

(9.4) $$\alpha < \frac{5+\sqrt{5}}{8}.$$

We have $\alpha = \alpha(\xi) \geq \alpha_1$ and $f^2_{\alpha_1}\left(\frac{1}{2}\right) = b'_{\alpha_1}$. Since:

$$b'_{\alpha_1} = 1 - b_{\alpha_1} = \frac{1}{4\alpha_1} \quad \text{and} \quad f^2_{\alpha_1}\left(\frac{1}{2}\right) = 4\alpha_1^2(1-\alpha_1),$$

α_1 is a zero of a polynomial $Q(x) = 16x^4 - 16x^3 + 1$. By (9.4), $\alpha_1 < \frac{5+\sqrt{5}}{8}$. Since $Q\left(\frac{5+\sqrt{5}}{8}\right) = \frac{7-5\sqrt{5}}{32} < 0$, $Q\left(\frac{1}{2}\right) = 0$ and $Q'(x) = 16x^2(4x-3)$, we must have $\alpha_1 = \frac{1}{2}$. But $f_{\frac{1}{2}}\left(\frac{1}{2}\right) = \frac{1}{2}$ and hence $\frac{1}{2} \leq \alpha_0$. Consequently, $\alpha_1 \leq \alpha_0$. This contradicts Lemma (7.4).

Suppose now that $\alpha \geq \frac{1}{4}(1+2\sqrt{2})$. Then, by (9.3):

(9.5) $$\beta \geq 2\sqrt{2} - 1.$$

We have $\log \beta = h_\mu(f) \leq h(f)$. Take the intervals $J_1 = \left[f^2\left(\frac{1}{2}\right), b'\right]$, $J_2 = \left[b', \frac{1}{2}\right]$, $J_3 = \left[\frac{1}{2}, b\right]$, $J_4 = \left[b, f\left(\frac{1}{2}\right)\right]$. It is easy to see that the set of non-wandering points of f is contained in $\{0\} \cup \bigcup_{i=1}^{4} J_i$. The map $f\big|_{J_i}$ is monotone for $i = 1, 2, 3, 4$, and therefore $h(f)$ is equal to the topological entropy of the symbolic system obtained by coding with respect to $\{J_i\}_{i=1}^4$ (see e.g. [6]). We have $f(J_1) \subset J_2 \cup J_3$, $f(J_2) \subset J_4$, $f(J_3) \subset J_4$, $f(J_4) \subset J_1 \cup J_2 \cup J_3$. Hence $h(f)$ is not larger than the topological entropy of the topological Markov chain with the transition matrix $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$. The characteristic polynomial of this matrix is $x.P(x)$, where $P(x) = x^3 - 2x - 2$. Hence β is not larger than the largest zero of P. We have:

$$P'(x) = 3x^2 - 2 = 3\left(x - \sqrt{\frac{2}{3}}\right)\left(x + \sqrt{\frac{2}{3}}\right),$$

and $$P\left(\sqrt{\frac{2}{3}}\right) = -\frac{4}{3}\sqrt{\frac{2}{3}} - 2 < 0.$$

By (9.5), $\beta \geq 2\sqrt{2}-1 > \sqrt{\frac{2}{3}}$. Hence $P(2\sqrt{2}-1)<0$. But:

$$P(2\sqrt{2}-1)=18\sqrt{2}-25>0,$$

a contradiction. ■

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