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COHOMOLOGY OF FINITE GROUPS OF LIE TYPE, I

by EDWARD CLINE, BRIAN PARSHALL, *and* LEONARD SCOTT

In this paper we determine $H^1(G, V)$ for G a finite Chevalley group over $k = GF(q)$, $q > 3$, and V belonging to the class of " minimal " irreducible kG-modules. The modules under consideration are described precisely in § 1; they include all the standard and spin modules, as well as some adjoint modules and exterior products. Many occur naturally as sections in Chevalley groups of larger rank (1).

Our approach is thematically Lie-theoretic, and relatively free of explicit calculation. All lower bounds on cohomology are determined by examining indecomposable modules for G constructed from appropriate irreducible modules for the corresponding complex Lie algebra (cf. (1.2) and (4.2*c*)); in particular, we never have to explicitly exhibit any cocycles. Upper bounds are obtained by studying interactions between the " roots " and " weights " for G which arise from their analogues in the algebraically closed case (cf. § 2, (4.2*c*), and § 5).

Many of the adjoint modules were treated by Hertzig [14], and certain of the classical cases have been studied by D. Higman [15], H. Pollatsek [22], and O. Taussky and H. Zassenhaus [27]. It should be noted that these papers contain results for more general fields than we consider, especially the fields of 2 and 3 elements. Nevertheless for $k = GF(q)$, $q > 3$, our results include all the above with the exception of the adjoint module of type F_4 (2).

We include (cf. § 5) a proof of a result on $\mathrm{Ext}^1(U, V) = H^1(G, \hat{U} \otimes V)$ for $G = SL(2, 2^n)$ stated by G. Higman in his notes on odd characterizations [16] and used there in the analysis of certain 2-local subgroups.

One can also obtain lower bounds on cohomology by means of the Cartan-Eilenberg stability theorem [5; p. 259]. An interesting by-product of our investigation is a new interpretation of this theorem in terms of a previously unnoticed *action* of Hecke algebras on cohomology (cf. § 6).

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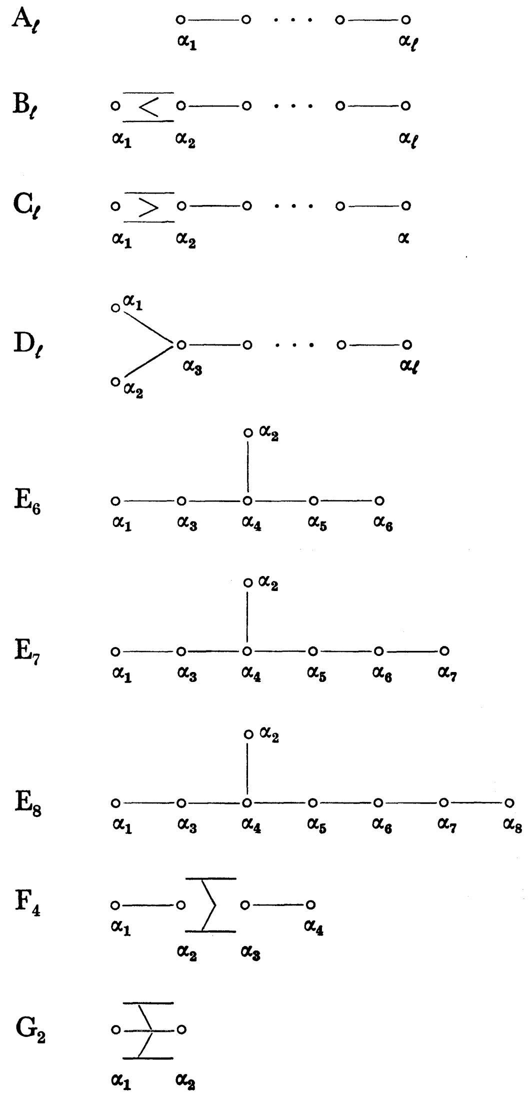
(1) Also, they include (essentially) the modules V for which (G, V) is a quadratic pair in the sense of Thompson [28].

(2) Since the writing of this paper, the fields of 2 and 3 elements have been treated by Wayne Jones in his thesis. By means of an elegant theorem describing the behavior of restriction to a Levi complement in a suitable maximal parabolic subgroup, he is able to reduce the question of upper bound to low rank cases.

1. Representations (3).

Let Σ be a finite root system in a $\mathbf{Q}$-vector space E endowed with a positive definite symmetric bilinear form (,) invariant under the Weyl group W of Σ. For α, $\beta \in E$, $\widehat{(\alpha, \beta)}$ denotes the angle formed by α and β. Also, Σ_{short} denotes the set of roots in Σ of minimal length, while Σ_{long} denotes the set of roots in Σ which are not short. It will be convenient for later reference to fix the following notation for a fundamental system Δ of the indecomposable root systems:



(3) More details concerning the representation theory of algebraic and Chevalley groups can be found in [9], [25], [26], and [29].

For $1 \leq i \leq \ell$, $\lambda_{\alpha_i} = \lambda_i$ denotes the fundamental dominant weight corresponding to the root $\alpha_i \in \Delta$. Recall $\lambda_i \in E$ is defined by the condition that $(\lambda_i, \alpha_j^v) = \delta_{ij}$, where $\alpha_j^v = 2\alpha_j/(\alpha_j, \alpha_j)$ is the coroot corresponding to α_j. The dominant weights are the non-zero elements of E of the form $\lambda = \sum_i n_i \lambda_i$, n_i a non-negative integer.

Minimal weights

We partially order the vector space E by the relation $w \geqq v$ iff $w - v$ is a non-negative integral combination of the α_i. A dominant weight λ is called *minimal* if it is minimal relative to this partial order. We will write $\Lambda = \Lambda(\Sigma)$ for the set of dominant weights in E, and denote the set of minimal elements of Λ by Λ_m.

Let $\mathfrak{L}$ be the complex semisimple Lie algebra with root system Σ relative to a fixed Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{L}$. Let $\lambda \in \Lambda$, and let $\mathfrak{M} = \mathfrak{M}(\lambda)$ denote the irreducible $\mathfrak{L}$-module of dominant weight λ. *Suppose* $\omega \in \Lambda \cup \{0\}$ *satisfies* $\omega \leqq \lambda$. Choose a weight ω' of $\mathfrak{H}$ in $\mathfrak{M}$ minimal with respect to $\omega' \geqq \omega$, and write $\tilde{\omega} = \omega' - \omega = \sum_{i=1}^{n} \beta_i$, where the β_i are fundamental roots. If ω is not a weight of $\mathfrak{H}$ in $\mathfrak{M}$, then $\tilde{\omega} \neq 0$, so

$$0 < (\tilde{\omega}, \tilde{\omega}) = (\tilde{\omega}, \sum_{i=1}^{n} \beta_i),$$

whence $(\tilde{\omega}, \beta_j) > 0$ for some j. Since $(\omega, \beta_k) \geqq 0$ for all k, $(\omega', \beta_j) = (\tilde{\omega} + \omega, \beta_j) > 0$. This means [17; Theorem 1, p. 112] that $\omega' - \beta_j$ is a weight of $\mathfrak{H}$ in $\mathfrak{M}$, contradicting the minimality of ω'. *Hence,* ω *is in fact a weight of* $\mathfrak{H}$ *in* $\mathfrak{M}$. This slightly generalizes —by essentially the same argument—a result of Freudenthal [12].

We can now determine the elements in Λ_m. Let $\lambda \in \Lambda_m$. First, consider the case when λ belongs to the root lattice $\mathbf{Z}\Sigma$ of E. From the previous paragraph, 0 is a weight of $\mathfrak{H}$ in $\mathfrak{M}$, whence, by irreducibility, ν is a weight of $\mathfrak{H}$ in $\mathfrak{M}$ for some root ν. Replacing ν by a suitable W-conjugate, we can assume that $\nu \in \Lambda$. The minimality of λ implies then that $\lambda = \nu$. If Σ has one root length, $\lambda = \nu$ is the maximal (relative to $\geqq$) root, while if Σ has two root lengths, $\lambda = \nu$ is the maximal short root. *Conversely, this shows the maximal short root belongs to* Λ_m. We enumerate these elements of Λ_m:

$$A_\ell,\ \lambda_1 + \lambda_\ell;\ B_\ell,\ \lambda_\ell;\ C_\ell,\ \lambda_{\ell-1};\ D_\ell,\ \lambda_{\ell-1};\ E_6,\ \lambda_2;\ E_7,\ \lambda_1;\ E_8,\ \lambda_8;\ F_4,\ \lambda_4;\ G_2,\ \lambda_2.$$

Next, suppose $\lambda \in \Lambda_m$, $\lambda \notin \mathbf{Z}\Sigma$. Let Σ^v be the dual root system to Σ [2; p. 144], and let γ^v be the maximal root in Σ^v (relative to the positive system defined by Δ^v). Then [2; Ex. 24, p. 226] λ is minimal iff $(\gamma^v, \lambda) = 1$. *We can therefore enumerate these additional elements of* Λ_m *as follows*:

$$A_\ell, \lambda_i,\ 1 \leqq i \leqq \ell;\ B_\ell, \lambda_1;\ C_\ell, \lambda_2;\ D_\ell, \lambda_i,\ i = 1, 2, \ell;\ E_6, \lambda_1, \lambda_6;\ E_7, \lambda_7\ (4).$$

(4) The weights in this list are the "minimal" dominant weights of Chevalley [7; Exp. 21].

For future reference we list here the maximal roots which are not minimal dominant weights:

$$B_\ell,\ \lambda_{\ell-1};\ C_\ell,\ 2\lambda_\ell;\ F_4,\ \lambda_1;\ G_2,\ \lambda_1.$$

These roots together with the maximal short roots listed above are precisely the elements of $\Sigma \cap \Lambda$.

Modules

Fix $\lambda \in \Lambda_m$. Let $\mathfrak{U}$ be the universal enveloping algebra of $\mathfrak{L}$, and let $\{X_\alpha, H_\beta \mid \alpha \in \Sigma, \beta \in \Delta\}$ be a Chevalley basis for $\mathfrak{L}$ [26; p. 6]. $\mathfrak{U}_{\mathbf{Z}}$ and $\mathfrak{U}_{\mathbf{Z}}^-$ denote the $\mathbf{Z}$-subalgebras of $\mathfrak{U}$ generated by the $X_\alpha^m/m!$ $(m \in \mathbf{Z}^+)$ for $\alpha \in \Sigma$ and $\alpha \in \Sigma^-$, respectively. Fix $v \neq 0$ in the λ-weight space $\mathfrak{M}_\lambda$ of $\mathfrak{H}$ in $\mathfrak{M}$, and set $M = v\mathfrak{U}_{\mathbf{Z}}^-$, a $\mathfrak{U}_{\mathbf{Z}}$-stable lattice in $\mathfrak{M}$.

Let G^* be the universal (or simply connected) Chevalley group over the algebraic closure K of k defined by $\mathfrak{L}$ (or Σ). Let T^* be the maximal k-split torus of G^* corresponding to $\mathfrak{H}$, and let $X^*(T^*)$ be the character module for T^* [1; p. 199] ([5]). We identify Σ with the root system of T^* in G^*, so $\Sigma \subseteq X^*(T^*) \subseteq X^*(T^*) \otimes \mathbf{Q} = E$. For $\alpha \in \Sigma$, U_α^* denotes the corresponding one-dimensional root subgroup (normalized by T^*), and $x_\alpha : K \to U_\alpha^*$ is the isomorphism of [26; p. 21].

Since $\mathfrak{U}_{\mathbf{Z}}$ stabilizes M, G^* acts in a natural fashion on the K-space

$$S^* = K \otimes M = K v G^* = K v U^{*-},$$

where U^{*-} denotes the unipotent radical of the Borel subgroup B^{*-} defined by T^* and $-\Delta$. S^* is an indecomposable G^*-module of dominant weight λ, and if X^* is a maximal submodule, S^*/X^* is the irreducible KG^*-module of dominant weight λ. If $X^* \neq 0$, let ω be a maximal weight of T^* in X^*. Then $\omega \leqq \lambda$. Since the λ-weight space S_λ^* of T^* in S^* is one-dimensional, $X^* = 0$ and S^* is irreducible when $\lambda \notin \mathbf{Z}\Sigma$. When $\lambda \in \mathbf{Z}\Sigma$, $\omega = 0$ and X^* is contained in the zero weight space S_0^* of T^* in S^*.

Assume $\lambda \in \Lambda_m \cap \mathbf{Z}\Sigma$, i.e. λ is the maximal short root ν in Σ. We claim X^* consists of the set Y^* of vectors $w \in S_0^*$ fixed by the root subgroups U_α^*, $-\alpha \in \Delta$. Indeed, T^* *and the* U_α^* *generate* B^{*-} (this follows by a standard argument from the formulas [26; pp. 148, 151]), hence $Y^* = \{w \in S_0^* \mid wB^{*-} = w\}$. Clearly, $X^* \subseteq Y^*$, while if w is fixed by B^{*-}, it is fixed by G^* (the morphism $G^* \to S^*$ defined by $g \mapsto wg$ factors through the complete variety G^*/B^{*-}, and hence is constant [20; p. 104]).

Write $V^* = S^*/X^*$. Let $\Sigma' = \{\alpha \in \Sigma \mid \alpha$ is a weight of T^* in $V^*\}$, and set $\Delta' = \Sigma' \cap \Delta$. When Σ has only one root length, $\Delta' = \Delta$, while if Σ has two root lengths, Δ' consists of the set of short roots in Δ. The non-zero weights of T^* in S^* are precisely the elements of Σ', and for $\alpha \in \Sigma'$, the corresponding weight space S_α^* is *one-dimensional*. Also,

([5]) We recall briefly the interpretation of T^* in the notation of [26; p. 43]. When G^* is universal, T^* is the direct product of the subgroups $\{h_\alpha(t) \mid t \in K^\times\}$ for $\alpha \in \Delta$. $X^*(T^*)$ identifies naturally with the lattice L_1: if $\mu \in L_1$, we view μ as the character $\mu : T^* \to K^\times$ defined by $\mu(\prod_{i=1}^{\ell} h_{\alpha_i}(t_i)) = \prod_{i=1}^{\ell} t_i^{(\mu, \alpha_i^\vee)}$.

$\dim S_0^* = \dim \mathfrak{M}_0 = |\Delta'|$ [18; Eq. 50, p. 261]. Let $C' = C(\Delta')$ denote the Cartan matrix of Δ'. Let $r(p)$ denote the rank of C' modulo p, where p is the characteristic of k. We can now state [6]:

Theorem (**1.1**). — *Let* $\lambda = \nu$, S^*, X^*, V^* *be as above. Then*

$$\dim_K X^* = \dim_K S_0^* - \dim_K V_0^* = |\Delta'| - r(p).$$

Proof. — For $\alpha \in \Delta$, fix $0 \neq v_\alpha \in M$ so that $\mathbf{Z}v_\alpha = M \cap \mathfrak{M}_\alpha$. Because $S^* = KvU^{*-}$ and because the U_α^*, for $\alpha \in -\Delta$, generate U^{*-}, we have by [26; Lemma 72, p. 209] that $S_0^* = \langle v_\alpha X_{-\alpha} \mid \alpha \in \Delta' \rangle$. But $\dim S_0^* = \dim \mathfrak{M}_0 = |\Delta'|$, so $\{v_\alpha X_{-\alpha} \mid \alpha \in \Delta'\}$ form a basis for S_0^*. For $\alpha, \beta \in \Delta'$, write $v_\alpha X_{-\alpha} X_\beta = \langle \alpha, \beta \rangle v_\beta$. Then it is easy to see [21; proof of Prop. 2] that the v_γ, for $\gamma \in \Delta'$, can be adjusted so that the matrix $-(\langle \alpha, \beta \rangle)$ is the matrix C'. Thus, $\sum_{\alpha \in \Delta'} c_\alpha v_\alpha X_{-\alpha} \in X^*$ (for $c_\alpha \in K$) iff for all $\beta \in \Delta'$

$$0 = \Big(\sum_{\alpha \in \Delta'} c_\alpha v_\alpha X_{-\alpha}\Big) X_\beta = \Big(\sum_{\alpha \in \Delta'} c_\alpha \langle \alpha, \beta \rangle\Big) v_\beta,$$

hence iff

$$0 = \sum_{\alpha \in \Delta'} c_\alpha \langle \alpha, \beta \rangle$$

for all $\beta \in \Delta'$. It follows that $\dim X^* = |\Delta'| - r(p)$, as desired. Q.E.D.

For convenience we tabulate the number $\dim_k X^*$ for those Σ and p where it is nonzero:

Σ	A_ℓ	B_ℓ	C_ℓ	$D_{2\ell}$	$D_{2\ell+1}$	E_6	E_7	F_4	G_2
p	$p \mid \ell+1$	2	$p \mid \ell$	2	2	3	2	3	2
$\dim_k X^*$	1	1	1	2	1	1	1	1	1

For a k-subgroup H^* of G^* (e.g. G^*, T^*, U^{*-}, B^{*-}, etc.) we shall denote by H the subgroup H_k^* of k-rational points of H^*. It should be noted here that since G^* is universal, G is just the corresponding universal Chevalley group over k [26; Cor. 3, p. 65].

If W^* is a K-vector space defined over k, we let W denote the k-subspace of k-rational points W_k^*. The G^*-module S^* above is endowed with a natural k-structure $S = k \otimes M$, and this induces k-structures on X^* and V^*. Note V^* is an irreducible k-rational G^*-module and remains irreducible upon restriction to G. This follows from Steinberg's theorem [26; Th. 43] since here the dominant weight $\lambda = \sum_i n_i \lambda_i$ satisfies $0 \leq n_i \leq q-1$, except when $G = SL(2, 2)$, $\lambda = \alpha_1$. When we wish to emphasize the dependence of S, X, and $V = S/X$ on λ, we denote them by $S(\lambda)$, $X(\lambda)$, and $V(\lambda)$ respectively.

(6) A similar result is stated in [3; p. 15].

We claim S *is indecomposable when* $q>2$. Let N be a direct summand of S which covers V. Recall $M=v\mathfrak{U}_{\mathbf{Z}}^{-}$; hence for some $v_0\in X$, $v'=v+v_0\in N$. Since v has weight λ and $q>2$, there is a $t\in T$ with $\lambda(t)\neq 1$, except when Σ is of type A_1 and $q=3$. With this exception, N contains $(1-\lambda(t))v$, whence contains v, thus $N=V$; otherwise $X=0$ by (1.1) and $S=V$ is irreducible, so indecomposable (7).

A lower bound for H^1

Let $q>2$ and let $\hat{S}$ be the dual module to S. Let $r'=|\Delta'|-r(p)$. Since S has a unique r'-dimensional submodule, $\hat{S}$ has a unique submodule $\hat{W}=X^{\perp}$ of codimension r'. Then $\hat{S}/\hat{W}=\hat{Z}$ is isomorphic to the dual module of X, and hence is a trivial kG-module. We note also that 0 is the only fixed point of G in $\hat{S}$, else S would contain a submodule of codimension 1, which is absurd. Hence (see (2.3*e*)) from the following exact sequence of G-modules

$$0\to\hat{W}\to\hat{S}\to\hat{Z}\to 0$$

we get the exact sequence of cohomology groups:

$$0=\hat{S}^G\to\hat{Z}^G=\hat{Z}\to H^1(G,\hat{W}).$$

When $\lambda=\nu$ is the maximal short root, it is stable under the opposition involution ι of Δ ($\iota=-w_0$, where w_0 is the unique element in W such that $w_0(\Delta)=-\Delta$), so V^* is self-dual as a G^*-module. Thus, V is self-dual as a G-module. Since $\hat{W}$ is the dual module to $S/X=V$, we obtain the following, using the fact that $X=0$ if $\lambda\notin\mathbf{Z}\Sigma$:

Theorem **(1.2)**. — *Let* $q>2$ *and let* λ *be a minimal dominant weight. Then*

$$\dim_k X(\lambda)\leq\dim_k H^1(G, V(\lambda)).$$

Recall that $\dim_k X(\lambda)=\dim_k X^*(\lambda)$ is 0 unless $\lambda=\nu$, in which case it is given by Theorem (1.1) and the table which accompanies it.

2. Cohomology.

In this section we outline some basic homological results and apply these to the cohomology of Chevalley groups.

Let A be a finite group, and V a kA-module. The first cohomology group $H^1(A, V)$ is defined to be $Z^1(A, V)/B^1(A, V)$ where

$$Z^1(A, V)=\{\gamma : A\to V\,|\,\gamma(xy)=\gamma(x)y+\gamma(y)\},$$
$$B^1(A, V)=\{\gamma : A\to V\,|\,\gamma(x)=v-vx \text{ for some fixed } v\in V\}.$$

(7) When $q=2$, S may not be indecomposable, e.g. $G=SL(2,2)$, $\lambda=\alpha_1$. This is the only case in which S is not indecomposable [30].

The elements of $Z^1 = Z^1(A, V)$ are called cocycles and those of B^1 are called coboundaries. Two cocycles γ, γ' are said to be cohomologous ($\gamma \sim \gamma'$) if $\gamma - \gamma' \in B^1$, or equivalently, if γ and γ' determine the same cohomology class $[\gamma] = [\gamma']$ in H^1.

There are other ways to think of H^1. For instance we can associate with each $\gamma \in Z^1$ a complement A_γ to V in the split extension A.V by the formula

(2.1) $$A_\gamma = \{x\gamma(x) \mid x \in A\},$$

and conversely each complement determines a cocycle. Two complements $A_{\gamma'}$, A_γ are conjugate in A.V iff $\gamma \sim \gamma'$; more precisely, $A_\gamma^v = A_{\gamma'}$ for an element $v \in V$ iff $\gamma'(x) = \gamma(x) + v - vx$. Hence $H^1(A, V)$ may be regarded as the collection of conjugacy classes of complements to V in the split extension A.V.

For some calculations it is more useful to think in terms of the homomorphism $f_\gamma : A \to A_\gamma \leq A.V$ defined by

(2.2) $$f_\gamma(x) = x\gamma(x).$$

In general if $f : A \to A.V$ is any function such that $f(x) \equiv x \pmod{V}$ and γ is defined by (2.2), then γ is a cocycle iff f is a homomorphism.

(2.3) We list some standard properties of H^1 (8).

a) $H^1(A, V)$ is a k-vector space in a natural way. In fact, for $\gamma \in Z^1$ and $c \in k$ we have $c\gamma \in Z^1$ defined by $(c\gamma)(x) = c\gamma(x)$. Clearly B^1 is a *k-subspace* of Z^1.

b) More generally if $f : V \to V'$ is any homomorphism of kA-modules (9) we can define $\gamma f \in Z^1(A, V')$ by the formula $(\gamma f)(x) = \gamma(x) f$. The resulting map

$$Z^1(A, V) \to Z^1(A, V')$$

induces a map on H^1.

c) If $\varphi : A' \to A$ is a homomorphism of groups, then φ induces a natural kA'-module structure on V, and for $\gamma \in Z^1(A, V)$ we can define $\varphi\gamma \in Z^1(A', V)$ by $(\varphi\gamma)(x') = \gamma(x'^\varphi)$, $x' \in A'$. The resulting map $Z^1(A, V) \to Z^1(A', V)$ induces a map on H^1.

This map is called inflation when φ is the natural projection of A' onto a quotient group A, and restriction ($\gamma|_{A'}$) when φ is the natural inclusion from a subgroup A' to A.

In case $a \in A$, $B \leq A$, and $\varphi = a^{-1} : B^a \to B$, then φ does not induce a map $Z^1(B, V) \to Z^1(B^a, V)$ by the formula above, since it is required there that V be given a "new" kB^a-module structure (induced by φ). However the map $v \mapsto va$ is an isomorphism from this "new" kB^a-module back to the "old". Applying paragraph *b)*, we obtain a legitimate map $a^{-1}\gamma a$ from $Z^1(B, V)$ to $Z^1(B^a, V)$, which induces a map on H^1.

(8) A general reference for the cohomology of groups is [5; Ch. 12], though the reader can doubtless supply proofs here without difficulty.

(9) Similar statements apply if f is just a homomorphism of ZA-modules; in particular cohomology groups of Galois conjugate modules are isomorphic abelian groups.

We shall denote $a^{-1}\gamma a$ by γ^a, and record here the formula

$$\gamma^a(x)=\gamma(x^{a^{-1}})a \qquad (x\in B^a).$$

Also we note that $\gamma^a\sim\gamma$ when $a\in B$.

d) If $\delta\in H^1(A, V)$ and $B\leq A$ then the class $\varepsilon=\delta|_B\in H^1(B, V)$ has the property of " stability ":

$$\varepsilon|_{B\cap B^a}=\varepsilon^a|_{B\cap B^a} \quad \text{for each} \quad a\in A.$$

e) If W is a kA-submodule of V we have a natural exact sequence [10]

$$0\to W^A\to V^A\to (V/W)^A\to H^1(A, W)\to H^1(A, V)\to H^1(A, V/W).$$

Also, we have an exact sequence of cocycles

$$0\to Z^1(A, W)\to Z^1(A, V)\to Z^1(A, V/W).$$

f) If $B\lhd A$ we have an exact " inflation-restriction " sequence

$$0\to H^1(A/B, V^B)\to H^1(A, V)\to H^1(B, V)^{A/B}.$$

g) If $B\leq A$ and the characteristic p of k does not divide the index $[A : B]$, then the restriction map $H^1(A, V)\to H^1(B, V)$ is injective. If in addition $B\lhd A$, then $H^1(A, V)\to H^1(B, V)^{A/B}$ is an isomorphism (see also § 6).

h) If V is a projective kA-module, then $H^1(A, V)=0$.

The following two propositions extend the results of § 4 to central factors and direct products; they play no further role in this paper.

Proposition (**2.4**). — *Suppose* $V^A=0$ *and* B *is a subgroup of the center of* A. *Then* [11]

a) *If* $V^B=V$, *then* $H^1(A, V)\cong H^1(A/B, V)$,

b) *If* $V^B=0$, *then* $H^1(A, V)=0$ [12].

Proof. — By (2.3*f*) we have an exact sequence

$$(*) \qquad 0\to H^1(A/B, V^B)\to H^1(A, V)\to H^1(B, V)^{A/B}.$$

If $V^B=V$, then $B^1(B, V)=0$ and $Z^1(B, V)$ is just the collection of group homomorphisms from B to V; $H^1(B, V)^{A/B}$ may be identified with the A-homomorphisms from B to V, whence is 0 since $V^A=0$ and B is central. Thus, $H^1(A, V)=H^1(A/B, V)$.

If $V^B=0$, then certainly $V^{B_1}=0$ where $B=B_1\times B_0$, B_1 a p'-group and B_0 a p-group. Applying the sequence (*) with B_1 in place of B gives $H^1(A, V)=0$ since $H^1(B_1, V)=0$ [13].

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(10) Here $(\ )^A$ denotes the fixed points of A, e.g. $V^A=\{v\in V\,|\,va=v \text{ for all } a\in A\}$.

(11) If V is irreducible then of course $V^B=V$ or $V^B=0$.

(12) A number of special cases of the results in § 4 can be obtained effortlessly from this fact.

(13) An alternate argument, well-known to finite group theorists, can be made from (2.1) and the fact that $C_{A\cdot V}(B_1)=C_A(B_1)$.

Proposition (**2.5**). — *Suppose* $A = A_1 \times A_2$ *and* $V = V_1 \otimes V_2$ *where* V_1 *is a* kA_1*-module and* V_2 *is a* kA_2*-module. Assume* $V_1^{A_1} = 0$. *Then*

$$H^1(A_1 \times A_2, V_1 \otimes V_2) \simeq H^1(A_1, V_1) \otimes V_2^{A_2} \ (^{14}).$$

Proof. — Again (2.3*f*) gives an exact sequence

$$0 \to H^1(A_1, (V_1 \otimes V_2)^{A_2}) \to H^1(A_1 \times A_2, V_1 \otimes V_2) \to H^1(A_2, V_1 \otimes V_2)^{A_1}.$$

It is easily checked that $(V_1 \otimes V_2)^{A_2} = V_1 \otimes V_2^{A_2}$, $H^1(A_1, V_1 \otimes V_2^{A_2}) \simeq H^1(A_1, V_1) \otimes V_2^{A_2}$, and $H^1(A_2, V_1 \otimes V_2)^{A_1} \simeq (V_1 \otimes H^1(A_2, V_2))^{A_1} \simeq V_1^{A_1} \otimes H^1(A_2, V_2) = 0$. The result follows.

Notation. — For the rest of this section, G denotes, as in § 1 (see especially footnote 5), the group of k-rational points of a universal Chevalley group G^* over K. Similar conventions hold for B, T, U, U_α ($\alpha \in \Sigma$). Thus, B is the semi-direct product T.U, $T = \{\prod_{i=1}^{\ell} h_{\alpha_i}(\zeta_i) \mid \zeta_i \in k^\times, 1 \leq i \leq \ell\}$, U is a Sylow p-subgroup of G, and $U_\alpha = \{x_\alpha(\xi) \mid \xi \in k\}$.

Weights of T; *Galois equivalence*

Let X(T) denote the collection of homomorphisms (or " weights ") $\omega : T \to k^\times$. For $\omega \in X(T)$ we let (as in § 1) V_ω denote the associated " weight space "

$$\{v \in V \mid vt = \omega(t)v \text{ for all } t \in T\}.$$

We think of X(T) additively, so for example V_0 denotes the weight space associated with the trivial weight $\omega(t) = 1$ ($t \in T$).

A root $\alpha \in \Sigma$ determines an element, still denoted α, of X(T) by the formula $x_\alpha(\xi)_t = x_\alpha(\alpha(t)\xi)$, $\xi \in k$, $t \in T$. In § 3 the extent to which this notation is ambiguous is exactly determined, that is, when distinct elements of Σ determine the same element of X(T).

We shall say two weights $\omega_1, \omega_2 \in X(T)$ are *Galois equivalent* ($\omega_1 \sim \omega_2$) if $\omega_1^\sigma = \omega_2$ for some automorphism σ of k, or more precisely, if $\omega_1(t)^\sigma = \omega_2(t)$ for all $t \in T$. The importance of this concept lies in the following result.

Proposition (**2.6**). — *Let* L *be a* kTU_α*-module which is* 1*-dimensional over* k *and on which* T *acts with weight* ω. *Then*

$$\dim_k Z^1(U_\alpha, L)^T = \begin{cases} 1 & \text{if } \omega \sim \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — U_α acts trivially on L of course, since L is 1-dimensional. Thus $Z^1(U_\alpha, L)^T$ is the collection of k_0T-homomorphisms from U_α to L, where $k_0 = GF(p)$. In all cases U_α is an irreducible k_0T-module, so $k \otimes_{k_0} U_\alpha$ is a sum of distinct Galois conjugates. Thus $Z^1(U_\alpha, L)^T = \mathrm{Hom}_{k_0T}(U_\alpha, L) \simeq \mathrm{Hom}_{kT}(k \otimes_{k_0} U_\alpha, L)$ has dimension 1 or 0 over k, depending on whether $\omega \sim \alpha$ or not.

(14) Suitably interpreted this result is just a very special case of the Künneth formula [19; p. 166].

1-parameter cohomology

The result just established readily yields information concerning the cohomology of kTU_α-modules L of arbitrary finite dimension.

Lemma (**2.7**). — *Let* L *be a finite-dimensional* kTU_α*-module. Then*

a) $\dim_k Z^1(U_\alpha, L)^T \leq \sum_{\omega\sim\alpha} \dim_k L_\omega$,

b) $\dim_k B^1(U_\alpha, L)^T = \dim_k L_0/L_0^{U_\alpha}$,

c) $\dim_k H^1(U_\alpha, L)^T \leq \sum_{\omega\sim\alpha} \dim_k L_\omega - \dim_k L_0/L_0^{U_\alpha}$.

Proof. — *c)* is of course a trivial consequence of *a)* and *b)*, and the fact that T is a p'-group.

a) is trivial if $L=0$. So assume $L\neq 0$ and let M be a maximal kTU_α-submodule of L. By (2.3*e*) we have an exact sequence

$$0\to Z^1(U_\alpha, M)^T \to Z^1(U_\alpha, L)^T \to Z^1(U_\alpha, L/M)^T.$$

M has codimension 1, since U_α is a p-group and all irreducible kT-modules have dimension 1. Applying (2.6) we obtain easily that

$$\dim_k Z^1(U_\alpha, L/M)^T = \sum_{\omega\sim\alpha} \dim_k L_\omega - \sum_{\omega\sim\alpha} \dim_k M_\omega,$$

from which *a)* follows by induction.

To prove *b)* we observe that the map $L_0\to B^1(U_\alpha, L)^T$ which sends ℓ_0 in L_0 to the coboundary $u\mapsto \ell_0-\ell_0 u$ $(u\in U_\alpha)$ is surjective (the result then follows since the kernel of this map is obviously $L_0^{U_\alpha}$): Let $u\mapsto \ell-\ell u$ be an element of $B^1(U_\alpha, L)^T$; thus

$$\ell-\ell u=(\ell-\ell tut^{-1})t$$

for each $t\in T$, $u\in U_\alpha$, or $\ell-\ell t=(\ell-\ell t)u$. Hence ℓ is a fixed point of T modulo L^{U_α}. Since T is a p'-group there exists $\ell_0\in L^T=L_0$ with $\ell-\ell_0\in L^{U_\alpha}$, that is, the original coboundary $u\mapsto \ell-\ell u$ is equal to the coboundary $u\mapsto \ell_0-\ell_0 u$. Thus $L_0\to B^1(U_\alpha, L)^T$ is surjective.

Upper bounds for $H^1(G, V)$

We can now prove the main theorem of this section.

Theorem (**2.8**). — *Suppose* $\psi\subseteq\Sigma^+$ *is a set of roots whose corresponding root groups generate* U, *that is,* $U=\langle U_\alpha \mid \alpha\in\psi\rangle$. *Then*

$$\dim_k H^1(G, V) \leq \sum_{\alpha\in\psi} \dim_k Z^1(U_\alpha, V)^T - \dim_k V_0 + \dim_k V^B \text{ (15)}.$$

Proof. — The defining formula $\gamma(xy)=\gamma(x)y+\gamma(y)$ for elements γ of $Z^1(U, V)$ and the fact that $U=\langle U_\alpha \mid \alpha\in\psi\rangle$ insure that the natural product of restriction maps

(15) Usually $V^B=0$ in this paper in view of (3.5) and [26, Theorem 46, p. 239].

$Z^1(U, V) \to \prod_{\alpha\in\psi} Z^1(U_\alpha, V)$ is injective. Hence $Z^1(U, V)^T \to \prod_{\alpha\in\psi} Z^1(U_\alpha, V)^T$ is injective, and $\dim_k Z^1(U, V)^T \leq \sum_{\alpha\in\psi} \dim_k Z^1(U_\alpha, V)^T$ (16).

On the other hand, the map $V_0 \to B^1(U, V)^T$ which takes ℓ_0 in V_0 to the coboundary $u \mapsto \ell_0 - \ell_0 u$ has kernel $V_0^U = V^B$, hence $\dim_k V_0 - \dim_k V^B \leq \dim_k B^1(U, V)^T$. Thus by (2.3$g$)

$$\begin{aligned}\dim_k H^1(G, V) \leq \dim_k H^1(B, V) &= \dim_k H^1(U, V)^T \\ &= \dim_k Z^1(U, V)^T - \dim_k B^1(U, V)^T \\ &\leq \sum_{\alpha\in\psi} \dim_k Z^1(U_\alpha, V)^T - \dim_k V_0 + \dim_k V^B.\end{aligned}$$

Q.E.D.

For $\alpha \in \Sigma \cup \{0\}$ set $n_\alpha = \sum_{\omega\sim\alpha} \dim_k V_\omega$.

Corollary **(2.9)**. — *Under the hypothesis of* (2.8), *if* $V^B = 0$ *then*

$$\dim_k H^1(G, V) \leq \sum_{\alpha\in\psi} n_\alpha - n_0 \text{ (17)}.$$

Proof. — Apply (2.7a) and (2.8).

3. Restrictions of roots and Galois equivalences (18).

As in § 1, G^* denotes the universal Chevalley group over K defined by Σ; $X^* = X^*(T^*)$ is the free abelian group generated by the fundamental dominant weights. Since T^* is k-split, T has exponent $q-1$, hence any homomorphism $\chi : T^* \to K^\times$ maps T into $k^\times$, and so there is a natural restriction homomorphism $\rho : X^*(T^*) \to X(T)$ (19).

Proposition **(3.1)** (20). — *Assume* Σ *is indecomposable, and* $q > 3$. *Let* $\beta \neq \gamma$ *be elements of* $\Sigma \cup \{0\}$. *Then one of the following holds*:

a) $\rho(\beta) \neq \rho(\gamma)$;

b) $q = 4$, Σ *is of type* G_2, β, γ *are long, and* $\widehat{(\beta, \gamma)} = \frac{2\pi}{3}$;

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(16) Similar considerations appear in Hertzig [13].

(17) In particular $H^1(G, V) = 0$ if all n_α's are 0. We mention here that T. A. Springer has obtained a splitting criterion (for exact sequences of modules of a semisimple simply connected algebraic group over an algebraically closed field of characteristic p) which also is described in terms of weights in the modules involved [24; Proposition 4.5].

(18) Parts of Propositions (3.1) and (3.3) in this section are contained in Chevalley [8; Lemma 11].

(19) For $\alpha \in \Sigma \subseteq X^*(T^*)$, $\rho(\alpha)(t)$ is $\alpha(t)$ as defined in § 2 (see footnote 5 and [26; Lemma 19c, p. 27]).

(20) The fibres of ρ restricted to $\Sigma_0 = \Sigma \cup \{0\}$ partition Σ_0 as $\Sigma_0 = \mathfrak{o}_0 \cup \ldots \cup \mathfrak{o}_s$, with $0 \in \mathfrak{o}_0$. When $q=2$, $s=0$, $\Sigma = \mathfrak{o}_0$. When $q=3$, the situation—now more complicated—entails the following alternatives: 1) Σ is of type B_ℓ ($\ell \geq 3$), $s = \frac{\ell(\ell+1)}{2}$, $\mathfrak{o}_0 = \{0\}$, $\mathfrak{o}_i = \{\pm\alpha_2, \pm(2\alpha_1+\alpha_2)\}w$ for some $w \in W$, $1 \leq i \leq \binom{\ell}{2}$, and $\mathfrak{o}_j = \{\pm\alpha\}$ for some α short, $\binom{\ell}{2} < j \leq s$; 2) Σ is of type C_ℓ ($\ell \geq 1$), $s = \binom{\ell}{2}+1$, $\mathfrak{o}_0 = \{0\} \cup \Sigma_{\text{long}}$, $\mathfrak{o}_i = \{\pm\alpha_2, \pm(\alpha_1+\alpha_2)\}w$ for some $w \in W$, for $i > 1$; 3) Σ is of type D_ℓ ($\ell \geq 3$), $s = \binom{\ell}{2}$, $\mathfrak{o}_0 = \{0\}$, $\mathfrak{o}_i = \{\pm\alpha_1, \pm\alpha_2\}w$ for some $w \in W$, $i > 1$; 4) Σ is of type F_4, $s = 15$, $\mathfrak{o}_0 = \{0\}$, $\mathfrak{o}_i = \{\pm\alpha_1\}\langle w_{\alpha_3}, w_{\alpha_4}\rangle w$ for some $w \in W$, $i = 1, 2, 3$, $\mathfrak{o}_j = \{\pm\alpha\}$ for some short α, $j > 3$; 5) Σ is of type G_2, $s = 3$, $\mathfrak{o}_0 = \{0\}$, $\mathfrak{o}_i = \{\pm\alpha_1, \pm(\alpha_1+2\alpha_2)\}w$ for some $w \in W$ for $i > 5$; 6) in the remaining cases $s = |\Sigma|/2$ and each $\mathfrak{o}_i$ is of the form $\{\pm\alpha\}$. A proof of this statement can be based on the fact that the fibres of ρ on Σ form systems of imprimitivity for the action of W. All such systems can be easily determined, since the stabilizer W_ζ of ζ = maximal long or maximal short root is generated by the fundamental reflections it contains.

c) [21] $q=4$, Σ *is of type* A_2, $\widehat{\langle \beta, \gamma \rangle}=2\pi/3$;
d) $q=5$, Σ *is of type* C_ℓ, $\beta=-\gamma$ *is long.*

Proof. — Since $\ker(\rho)=(q-1)X^*$, we need only consider the map

$$\hat{\rho} : X^* \rightarrow X^*/(q-1)X^*.$$

Write $\zeta \in X^*$ as a **Z**-linear combination of fundamental weights

$$\zeta = \sum_i m_{\zeta,i}\lambda_i \quad (m_{\zeta,i}=(\zeta, \alpha_i^\vee))$$

then $\hat{\rho}(\zeta)=0$ iff $m_{\zeta,i}\equiv 0 \pmod{q-1}$ for each i.

Let $\alpha \in \Sigma$, and $\beta \in \Sigma \cup \{0\}$ be distinct elements such that $\hat{\rho}(\alpha)=\hat{\rho}(\beta)$. We may assume α is a dominant weight (see the lists in § 1). Clearly $\hat{\rho}(\alpha)\neq 0$, and if Σ is not of type G_2 or A_ℓ, the congruences $m_{\alpha,i}\equiv m_{\beta,i} \pmod{q-1}$ for all i, together with $q>3$, imply β is a multiple of α, hence $\beta=-\alpha$. Evidently this implies $m_{\alpha,i}>1$ for the non-zero $m_{\alpha,i}$, so by inspection Σ is of type C_ℓ, $\alpha=2\lambda_\ell=-\beta$, $q=5$ and *d)* holds. If Σ is of type G_2, *b)* follows by inspection. If Σ is of type A_ℓ, $\ell>1$, then $\beta=m_1\lambda_1+m_\ell\lambda_\ell$ where $m_i\equiv 1 \pmod{q-1}$ for $i=1,\ell$. This implies $q=4$ and $m_i-1=0$ or -3, whence either $\beta=-\alpha$ or $\beta\in\{\lambda_1-2\lambda_\ell, \lambda_\ell-2\lambda_1\}$. Clearly $\beta\neq-\alpha$, and since $\lambda_1-2\lambda_\ell$, $\lambda_\ell-2\lambda_1$ are roots only when $\ell=2$, *c)* holds. Q.E.D.

The Galois equivalence relation on $X(T)$ induces a similar relation on X^*, *viz.*, for $\lambda, \mu \in X^*(T^*)$ we say λ is *Galois equivalent* to μ $(\lambda\sim\mu)$ provided $\rho(\lambda)\sim\rho(\mu)$ in the sense of § 2. If $\lambda\sim\mu$ and $q=p^n$, then

(3.2) $$p^j m_{\lambda,i}\equiv m_{\mu,i} \pmod{p^n-1} \text{ for some } j,\ 0\leq j<n, \text{ all } i.$$

We proceed to determine Galois equivalences among roots.

Proposition **(3.3)**. — *Assume* Σ *is indecomposable and* $q>3$. *If* $\alpha, \beta\in\Sigma$ *are distinct Galois equivalent roots, then one of the following occurs:*

a) $q=4$, Σ *is not of type* G_2 *or* A_2, $\alpha=-\beta$;

b) $q=4$, Σ *is of type* G_2, $\alpha=-\beta$ *or* α, β *are both long (all long roots are equivalent here)*;

c) $q=4$, Σ *is of type* A_2, *all roots are Galois equivalent*;

d) $q=5$, Σ *is of type* C_ℓ, $\alpha=-\beta$ *is long*;

e) $q=9$, Σ *is of type* C_ℓ $(\ell\geq 1)$, $\alpha=-\beta$ *is long.*

Proof. — If q is prime, $\zeta\sim\xi$ iff $\rho(\zeta)=\rho(\xi)$, since $k=GF(q)$ has no non-trivial automorphism; hence *d)* holds if $q=5$. If $q=4$ and Σ is of type G_2 or A_2, *b)* and *c)* hold by inspection. We may assume henceforth $q\neq 5$ and Σ is not of type G_2 or A_2 when $q=4$.

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[21] The authors thank the referee for pointing out this case.

Also we may assume α is a dominant weight, and $\beta \sim \alpha$ via a non-trivial automorphism of $GF(q)$.

Suppose $m_{\alpha,i} \leq 1$ for all i. Then (3.2) reads

$$m_{\beta,i} \equiv \begin{cases} p^j & \text{if } m_{\alpha,i} = 1 \\ 0 & \text{otherwise} \end{cases} \pmod{p^n - 1}, \text{ for some } 0 < j < n.$$

Since $|m_{\beta,i}| \leq 2$ when Σ is not of type G_2, β must be a multiple of α; hence $\beta = -\alpha$, $p^j + 1 \equiv 0 \pmod{p^n - 1}$, $j = 1$, $n = 2$, $p = 2$, and *a)* holds.

Otherwise Σ is of type C_ℓ and $\alpha = 2\lambda_\ell$; then (3.2) reads

$$m_{\beta,i} \equiv \begin{cases} 2p^j & \text{if } i = \ell \\ 0 & \text{if } i \neq \ell \end{cases} \pmod{p^n - 1},$$

for some $0 < j < n$. Since $|m_{\beta,i}| \leq 2$, β must be a multiple of α, so $\beta = -\alpha$, and either *a)* or *e)* holds. Q.E.D.

Now we consider the Galois equivalences between roots and minimal dominant weights. We may assume the minimal dominant weight λ is not in the root lattice. Then $\lambda = \lambda_i$ is fundamental and Σ is not of type G_2. Thus for $\alpha \in \Sigma$, $|m_{\alpha,j}| \leq 2$. If we assume $q > 3$, the congruences $m_{\alpha,j} \equiv 0 \pmod{q-1}$ for $j \neq i$ imply $\alpha = m\lambda$ for $m = \pm 2$. By the lists in § 1, Σ must be of type C_ℓ $(\ell \geq 1)$. Since $p^j - m \equiv 0 \pmod{p^n - 1}$ for some $0 \leq j < n$, $p = 2$. If $m = -2$, necessarily $n = 2$. We summarize:

Proposition **(3.4)**. — *Assume Σ is indecomposable and $q > 3$* [22]. *If $\lambda \in \Lambda_m$, $\lambda \notin \mathbf{Z}\Sigma$, and if $\lambda \sim \alpha \in \Sigma$, then Σ is of type C_ℓ $(\ell \geq 1)$, $\lambda = \lambda_\ell$, and $p = 2$. If $q = 4$, $\alpha = \pm 2\lambda_\ell$, while if $q > 4$, $\alpha = 2\lambda_\ell$.*

(3.5) We remark here that, for $q > 3$ and $\lambda \in \Lambda_m$, $\lambda \nsim 0$. It follows that

$$\dim_k V_0^* = \dim_k V_0$$

where V_0^* is as in § 1 and V_0 is as in § 2.

4. Examples.

With the bounds developed so far, it is an easy matter to compute $H^1(G, V)$ for irreducible kG-modules $V = V(\lambda)$ where λ is a minimal dominant weight. We have the following possibilities for λ:

(4.1*a*) λ is not Galois equivalent to a root;

(4.1*b*) λ is not a root, but is Galois equivalent to a root;

(4.1*c*) λ is the maximal short root.

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[22] When $k = GF(3)$ the Galois equivalences are covered by footnote [20].

If $p=2$, G^* is universal of type C_ℓ $(\ell \geq 1)$, and $\widetilde{G}^*$ is universal of type B_ℓ; let $\iota : \widetilde{G}^* \to G^*$ be the isogeny defined by the special isomorphism $X^*(T^*) \otimes \mathbf{Q} \xrightarrow{\varphi} X^*(\widetilde{T}^*) \otimes \mathbf{Q}$ given by $\varphi(\alpha_1) = 2\widetilde{\alpha}_1$ and $\varphi(\alpha_i) = \widetilde{\alpha}_i$ for $i > 1$ [7; Exp. 18, 23]. Then $\varphi(\lambda_\ell) = \widetilde{\lambda}_\ell$ if $\ell > 1$, while $\varphi(\lambda_1) = 2\widetilde{\lambda}_1 = \widetilde{\alpha}_1$ if $\ell = 1$. Also, the restriction of ι to $\widetilde{G}$ is an isomorphism from $\widetilde{G}$ onto G. *Now by* (3.4), (4.1*b*) *occurs only when* $p=2$, Σ *is of type* C_ℓ *and* $\lambda = \lambda_\ell$; *hence in order to treat case* (4.1*b*), *it suffices to consider the cases when* G *is of type* $B_\ell(2^n)$ *and* $\lambda = \lambda_\ell$ *for* $\ell > 1$, *or of type* $A_1(2^n)$, $\lambda = 2\lambda_1 = \alpha_1$ *for* $\ell = 1$. This reduces (4.1*b*) to (4.1*c*).

Theorem **(4.2)**. — *Assume* $q > 3$. *If* λ *satisfies*

a) (4.1*a*), *then* $H^1(G, V) = 0$;

b) (4.1*b*), *then* $H^1(G, V) \simeq X(\lambda)$ *where* $\lambda = \lambda_\ell$ *is a fundamental weight of the dual system of type* B_ℓ (see the above paragraph);

c) (4.1*c*), *then* $H^1(G, V) \simeq X(\lambda)$, *or* $q = 5$ *and* $G = A_1(5)$ *in which case* $H^1(G, V)$ *is* 1-*dimensional*.

Proof. — Since λ is minimal, every nonzero weight of T in V is W-conjugate to λ. If λ satisfies (4.1*a*) and $\beta \in \Sigma$, then $n_\beta = 0$ and (2.9) implies $H^1(G, V) = 0$.

For the remaining cases, *we assume first that* $q > 4$, *and if* $q = 5$ *or* 9, G *is not of type* A_1. Since we have shown that (4.1*b*) reduces to (4.1*c*), we assume λ is the maximal short root in Σ. By (1.2)

$$\dim_k X(\lambda) \leq \dim_k H^1(G, V).$$

If Σ has two root lengths and $\gamma \in \Sigma_{\text{long}}$, (3.3) implies $n_\gamma = 0$, while if $\gamma \in \Sigma_{\text{short}}$, $n_\gamma = 1$ by (3.3). Hence by (2.9) ([23]), if $\Delta' = \Delta \cap \Sigma_{\text{short}}$

$$\dim_k H^1(G, V) \leq \sum_{\delta \in \Delta'} n_\delta - n_0.$$

By (1.1) and (3.5), $|\Delta'| = n_0 + \dim_k X(\lambda)$; hence

$$\dim_k X(\lambda) \leq \dim_k H^1(G, V) \leq \dim_k X(\lambda).$$

Now assume $q = 4$ and G is not of type A_2. We show

$$\textbf{(4.3)} \qquad \dim_k Z^1(U_\alpha, V)^T \leq \begin{cases} 1 & \text{if} \quad \alpha \in \Sigma_{\text{short}}, \\ 0 & \text{otherwise.} \end{cases}$$

If $\alpha \in \Sigma_{\text{long}}$, (4.3) follows from (3.3). Assume $\alpha \in \Sigma_{\text{short}}$. Let $G_\alpha = \langle U_\alpha, U_{-\alpha} \rangle$ and $V_{(\alpha)} = V_\alpha G_\alpha$, then $V_{(\alpha)} \subseteq V_{-\alpha} \oplus V_0 \oplus V_\alpha$, and every proper G-submodule of $V_{(\alpha)}$ is contained in $(V_{(\alpha)})_0$; consequently $V_{(\alpha)}$ has a unique maximal proper submodule $X \subseteq (V_{(\alpha)})_0$. By (2.7*a*), $Z^1(U_\alpha, X)^T = 0$, hence (2.3*e*) yields an injection

$$0 \to Z^1(U_\alpha, V_{(\alpha)})^T \to Z^1(U_\alpha, \overline{V}_{(\alpha)})^T$$

([23]) Here we are using the fact that $U = \langle U_\alpha | \alpha \in \Delta \rangle$ when $q > 3$. It suffices to consider the rank 2 case. It is almost trivial for A_2 and C_2, while it follows for G_2 by (3.3) and the fact that T normalizes $\langle U_\alpha | \alpha \in \Delta \rangle$.

where $\overline{V}_{(\alpha)}=V_{(\alpha)}/X$. By (2.7a) again, $Z^1(U_\alpha, V/V_{(\alpha)})^T=0$, hence (2.3e) yields

$$Z^1(U_\alpha, V)^T \simeq Z^1(U_\alpha, V_{(\alpha)})^T,$$

so it suffices to show

$$\dim_k Z^1(U_\alpha, \overline{V}_{(\alpha)})^T \leq 1. \tag{4.4}$$

Now $\overline{V}_{(\alpha)}$ is an irreducible kG_α-module. Since the torus $T_\alpha=T\cap G_\alpha$ acts with weight α on V_α, it follows $\overline{V}_{(\alpha)}$ is 2-dimensional. Set $\overline{V}_{\pm\alpha}=(\overline{V}_{(\alpha)})_{\pm\alpha}$.

If $\gamma \in Z^1(U_\alpha, \overline{V}_{(\alpha)})^T$, the homomorphism $u \mapsto u\gamma(u)$ of (2.2) yields $(u\gamma(u))^2=1$, whence u fixes $\gamma(u)$. Writing $\gamma(u)=v_\alpha(u)+v_{-\alpha}(u)$ where $v_{\pm\alpha}(u) \in \overline{V}_{\pm\alpha}$, we see that $v_{-\alpha}(u)$ is fixed by u since $v_\alpha(u)$ is automatically fixed by u. Since T acts irreducibly on U_α, $v_{-\alpha}(u)$ is fixed by U_α. But since $\overline{V}_{(\alpha)}$ is 2-dimensional and U_α is non-trivial, U_α fixes a unique line in $\overline{V}_{(\alpha)}$. Necessarily $v_{-\alpha}(u)=0$ for all $u \in U_\alpha$. By (2.3e)

$$0 \to Z^1(U_\alpha, \overline{V}_\alpha)^T \to Z^1(U_\alpha, \overline{V}_{(\alpha)})^T \xrightarrow{\pi} Z^1(U_\alpha, \overline{V}_{(\alpha)}/\overline{V}_\alpha)^T$$

is exact. We have just shown π is the zero map, so (4.4) follows from (2.6).

By (2.8)

$$\dim_k H^1(G, V) \leq \sum_{\alpha \in \Delta'} \dim_k Z^1(U_\alpha, V)^T - n_0.$$

Hence $\dim_k H^1(G, V) \leq |\Delta'| - n_0 = \dim_k X(\lambda)$.

For the remainder of the proof, we consider the exceptional cases $A_2(4)$, $A_1(5)$ and $A_1(9)$. *Assume first that* $G=A_2(4)$, $\lambda=\lambda_1+\lambda_2$. Let $\gamma \in Z^1(U, V)^T$. The homomorphism (2.2) and the fact that U_β has exponent 2 imply $\gamma(x_\beta(\xi)) \in V^{U_\beta}$ for each $\beta \in \Sigma$, $\xi \in k$. This yields the following form for γ:

$$\begin{aligned}
&\gamma(x_{\alpha_1}(\xi)) = v_{\alpha_1, 0}(\xi) + v_{\alpha_1, \alpha_1}(\xi) + v_{\alpha_1, \alpha_1+\alpha_2}(\xi) + v_{\alpha_1, -\alpha_2}(\xi),\\
&\gamma(x_{\alpha_2}(\xi)) = v_{\alpha_2, 0}(\xi) + v_{\alpha_2, \alpha_2}(\xi) + v_{\alpha_2, \alpha_1+\alpha_2}(\xi) + v_{\alpha_2, -\alpha_1}(\xi),\\
&\gamma(x_{\alpha_1+\alpha_2}(\xi)) = v_{\alpha_1+\alpha_2, 0}(\xi) + v_{\alpha_1+\alpha_2, \alpha_1+\alpha_2}(\xi) + v_{\alpha_1+\alpha_2, \alpha_1}(\xi) + v_{\alpha_1+\alpha_2, \alpha_2}(\xi),
\end{aligned} \tag{4.5}$$

where for each appropriate pair $\beta, \nu \in \Sigma \cup \{0\}$

$$v_{\beta,\nu} : k \to (V_\nu^*)_k$$

is an additive homomorphism which, because of the T-stability of γ, satisfies

$$v_{\beta,\nu}(\beta(t)\xi) = \nu(t) v_{\beta,\nu}(\xi) \quad \text{for} \quad t \in T. \tag{4.6}$$

Thus, $v_{\beta,0}=0$ for $\beta>0$.

Next consider the maps $\theta_{\beta,\xi} : (V_0^*)_k \to (V_\beta^*)_k$ defined for $\beta>0$, $\xi \in k$ by

$$\theta_{\beta,\xi}(w) = w - wx_\beta(\xi) \quad \text{for} \quad w \in (V_0^*)_k.$$

If $\beta>0$ and $\xi, \tau \in k^\times$, then

$$\ker(\theta_{\beta,\xi}) = \ker(\theta_{\beta,\tau}) \neq 0$$

while

$$\ker(\theta_{\alpha_1,\xi}) \oplus \ker(\theta_{\alpha_2,\tau}) = (V_0^*)_k.$$

It follows that we may adjust γ by a T-stable 1-coboundary to eliminate v_{α_1,α_1} and v_{α_2,α_2}; thus

$$\text{(4.5')}\qquad \begin{aligned} \gamma(x_{\alpha_1}(\xi)) &= v_{\alpha_1,\alpha_1+\alpha_2}(\xi)+v_{\alpha_1,-\alpha_2}(\xi),\\ \gamma(x_{\alpha_2}(\tau)) &= v_{\alpha_2,\alpha_1+\alpha_2}(\tau)+v_{\alpha_2,-\alpha_1}(\tau),\\ \gamma(x_{\alpha_1+\alpha_2}(\xi\tau)) &= v_{\alpha_1+\alpha_2,\alpha_1+\alpha_2}(\xi\tau)+v_{\alpha_1+\alpha_2,\alpha_1}(\xi\tau)+v_{\alpha_1+\alpha_2,\alpha_2}(\xi\tau),\end{aligned}$$

for $\xi, \tau\in k$.

If we assume $v_{\alpha_1,-\alpha_2}\neq 0$, and compute the V_{α_1}-component of $\gamma([x_{\alpha_1}(\xi), x_{\alpha_2}(\tau)])$ using (2.2), (4.5'), (4.6) and [26, Lemma 72, p. 209] (or the fact that we know explicitly the action of G on V here), we obtain a non-zero vector $w\in V_{\alpha_1}$ and constants A, B, C with $B\neq 0$ such that

$$v_{\alpha_1+\alpha_2,\alpha_1}(\xi\tau)=(A\xi^2\tau^2)w=(B\xi^3\tau+C\xi^2\tau^2)w$$

for all $\xi, \tau\in k$. This is a contradiction, hence $v_{\alpha_1,-\alpha_2}=0$. A symmetric argument shows $v_{\alpha_2,-\alpha_1}=0$.

Since $V_{\alpha_1+\alpha_2}\subseteq V^U$, a recomputation of $\gamma([x_{\alpha_1}(\xi), x_{\alpha_2}(\tau)])$ shows $\gamma\equiv 0$ on $U_{\alpha_1+\alpha_2}$, thus

$$\text{(4.5'')}\qquad \begin{aligned} \gamma(x_{\alpha_1}(\xi)) &= v_{\alpha_1,\alpha_1+\alpha_2}(\xi),\\ \gamma(x_{\alpha_2}(\xi)) &= v_{\alpha_2,\alpha_1+\alpha_2}(\xi),\\ \gamma(x_{\alpha_1+\alpha_2}(\xi)) &= 0, \quad \text{for} \quad \xi\in k.\end{aligned}$$

Now let $[\tilde\gamma]\in H^1(G, V)$, and choose $\tilde\gamma$ so its restriction γ to U is T-stable, hence, after adjustment by a T-stable coboundary, so that γ satisfies (4.5''). By (2.3*d*), $\tilde\gamma^w|_{U\cap U^w}\sim\tilde\gamma|_{U\cap U^w}$ for every $w\in W$. Let $w=w_{\alpha_2}w_{\alpha_1}$; then

$$U\cap U^w=U_{\alpha_2} \quad \text{and} \quad \mu=(\tilde\gamma^w-\tilde\gamma)|_{U_{\alpha_2}}$$

is a T-stable coboundary, hence there exists a vector $v\in V$ such that

$$\mu(x_{\alpha_2}(\xi))=v-vx_{\alpha_2}(\xi) \quad \text{for} \quad \xi\in k,$$

and $\qquad vt-v\in V^{U_{\alpha_2}} \quad$ for $\quad t\in T$.

Since T is a 2'-group, we may choose v so that $v\in(V_0^*)_k$, whence

$$\mu(U_{\alpha_2})\subseteq(V^*_{\alpha_2})_k.$$

However, direct computation of μ shows $\mu(U_{\alpha_2})\subseteq(V^*_{-\alpha_1})_k+(V^*_{\alpha_1+\alpha_2})_k$, hence $\mu=0$. Now from (4.5'')

$$\tilde\gamma^w(U_{\alpha_2})=\tilde\gamma(U_{\alpha_2})\subseteq(V^*_{-\alpha_1})_k\cap(V^*_{\alpha_1+\alpha_2})_k=0,$$

so $\tilde\gamma|_{U_{\alpha_2}}=0$. By a symmetric argument, $\tilde\gamma|_{U_{\alpha_1}}=0$, so it follows from (2.3*g*) that $H^1(G, V)=0$ completing the proof in this case.

Suppose next $G=A_1(5)$, *and* $\lambda=\alpha_1$. By (2.9) and (3.3), we have

$$\dim_k H^1(G, V)\leq n_\alpha-n_0=1.$$

On the other hand, a computation shows that the module $S(6\lambda)$ formed as in § 1 is indecomposable, and has a unique submodule J of dimension 4 which contains V. J is indecomposable and an application of the long exact sequence (2.3*e*) as in (1.2) yields a lower bound of 1 on $\dim_k H^1(G, V)$ completing the proof in this case ([24]).

Finally suppose $G=A_1(9)$, *and* $\lambda=\alpha_1$. Let $\gamma\in Z^1(U, V)^T$. Then by (2.3*g*) it suffices to show $\gamma\sim 0$. As in the case $G=A_2(4)$, write

$$\gamma(x_{\alpha_1}(\xi))=v_{\alpha_1}(\xi)+v_0(\xi)+v_{-\alpha_1}(\xi)$$

where $v_\beta : k\to(V^*_\beta)_k$ is a function satisfying $v_\beta(\alpha_1(t)\xi)=\beta(t)v_\beta(\xi)$, for $\xi\in k$, $t\in T$. Since $V/(V^*_{\alpha_1}+V^*_0)_k$ is a trivial U-module, $v_{-\alpha_1}$ is additive. If θ is a primitive 4th root of unity in GF(9) and if we write $\xi=a+b\theta$ $(a, b\in GF(3))$, then

$$v_{-\alpha_1}(\xi)=(a+b\theta^{-1})v_{-\alpha_1}(1)=\xi^3v_{-\alpha_1}(1).$$

By computing the component of $\gamma(x_{\alpha_1}(\xi+\tau))$ in $(V^*_0)_k$ in two ways for ξ, $\tau\in GF(9)$, we obtain

$$v_0(\xi+\tau)=v_0(\xi)+v_0(\tau)+\xi^3\tau v_{-\alpha_1}(1)X_{\alpha_1}.$$

If $v_{-\alpha_1}(1)\neq 0$, $v_{-\alpha_1}(1)X_{\alpha_1}\neq 0$, hence by the symmetry of $v_0(\xi+\tau)$ in ξ and τ, $\xi^3\tau=\tau^3\xi$ for all $\xi, \tau\in k$, a contradiction. Hence $v_{-\alpha_1}=0$. Also v_0 is a $Z_3(T)$-homomorphism, hence $v_0=0$. For some $\xi\in k^\times$, we may choose $v_0\in(V^*_0)_k$ such that

$$\gamma(x_{\alpha_1}(\xi))=v_{\alpha_1}(\xi)=v_0-v_0x_{\alpha_1}(\xi).$$

By T-stability, $\gamma(x_{\alpha_1}(\rho))=v_0-v_0x_{\alpha_1}(\rho)$ for all ρ, so $\gamma\sim 0$. Q.E.D

We summarize our results in a table:

TABLE (4.5)

Type	Char $k=p$ $q>3$	Dominant Weight	$\dim_k V$	$\dim_k H^1(G, V)$
A_1	2	$\alpha_1=2\lambda_1(\lambda_1)$	2	1(1)
	$2\nmid q$: $q=5$	$\alpha_1=2\lambda_1(\lambda_1)$	3(2)	1(0)
	$2\nmid q$: $q>5$	$\alpha_1=2\lambda_1(\lambda_1)$	3(2)	0(0)
A_ℓ $(\ell>1)$	arbitrary	λ_i	$\binom{\ell+1}{i}$	0
	$(\ell+1, p)=1$	$\mu=\lambda_1+\lambda_\ell$	$(\ell+1)^2-1$	0
	$(\ell+1, p)=p$	μ	$(\ell+1)^2-2$	1

([24]) An alternate proof in this case is as follows: If R denotes the permutation representation of $A_1(5)$ on 5 letters over $k=GF(5)$, then its restriction to a Sylow 5-subgroup is evidently the regular representation, hence R is indecomposable. It evidently contains unique submodules M and m of dimensions 1 and 4 respectively, and $M/m\simeq V$. Now the exact sequence (2.2) applied to R/m yields a lower bound of 1 for the dimension of $H^1(G, V)$ as in (1.2).

TABLE (4.5) *(suite)*

Type	Char $k=p$ $q>3$	Dominant Weight	$\dim_k V$	$\dim_k H^1(G, V)$
B_ℓ $(\ell>2)$	arbitrary	λ_1	2^ℓ	0
	2	$\nu=\lambda_\ell$	2ℓ	1
	odd	ν	$2\ell+1$	0
C_ℓ $(\ell>1)$	2	λ_ℓ	2ℓ	1
	odd	λ_ℓ	2ℓ	0
	$(\ell,p)=p$	$\nu=\lambda_{\ell-1}$	$(\ell-1)(2\ell+1)-1$	1
	$(\ell,p)=1$	ν	$(\ell-1)(2\ell+1)$	0
D_ℓ $(\ell>3)$	arbitrary	λ_i; $i=1, 2, \ell$	2ℓ; $i=\ell$	0
			$2^{\ell-1}$; $i\neq\ell$	0
	odd	$\mu=\lambda_{\ell-1}$	$(2\ell-1)\ell$	0
$D_{2\ell}$	2	μ	$2\ell(4\ell-1)-2$	2
$D_{2\ell+1}$	2	μ	$(2\ell+1)(4\ell+1)-1$	1
E_6	arbitrary	λ_1, λ_6	27	0
	3	$\mu=\lambda_2$	77	1
	$(3, p)=1$	μ	78	0
E_7	arbitrary	λ_7	56	0
	2	$\mu=\lambda_1$	132	1
	$(2, p)=1$	μ	133	0
E_8	arbitrary	$\mu=\lambda_8$	248	0
F_4	3	$\nu=\lambda_4$	25	1
	$(p, 3)=1$	ν	26	0
G_2	2	$\nu=\lambda_2$	6	1
	$(p, 2)=1$	ν	7	0

5. A result on Ext for $SL(2, 2^n)$.

Let $G = SL(2, 2^n)$, $n \geq 1$. Let σ be the automorphism of G induced by the field automorphism $t \mapsto t^2$, $t \in k = GF(2^n)$. Let ρ be the irreducible representation of G over k defined by the dominant weight $\lambda = \lambda_1$ (§ 1), and set $\rho_i = \rho \circ \sigma^i$ for $0 \leq i < n$. Let M_i be the kG-module corresponding to ρ_i.

Theorem **(5.1)** [16; p. 29]. — $Ext^1(M_i, M_j) = 0$, *for all* i, j.

Proof. — Since $Ext^0(V, V') \simeq Hom_G(V, V') \simeq (\hat{V} \otimes V)^G \simeq H^0(G, \hat{V} \otimes V')$ for all finite dimensional kG-modules V, V', we have $Ext^1(V, V') \simeq H^1(G, \hat{V} \otimes V')$ by a standard dimension shift. Since each of the modules M_j is self-dual

$$Ext^1(M_i, M_j) \simeq H^1(G, M_i \otimes M_j).$$

Assume first that $i \neq j$, say $i > j$. The weights of T in M_i are $\pm 2^i \lambda$, hence the weights of T in $V = M_i \otimes M_j$ are $\mu_1 = (2^i + 2^j)\lambda$, $\mu_2 = (2^i - 2^j)\lambda$, $\mu_3 = -\mu_2$, $\mu_4 = -\mu_1$. This module V is the irreducible kG-module defined by the dominant weight $(2^i + 2^j)\lambda$ ([26; Thm. 43] or [25]). Since $i > j$,

(5.2a) $\qquad \mu_1 \nsim \alpha_1 = \alpha;$

(5.2b) $\qquad \mu_2 \sim \alpha \sim \mu_3$ iff $n = 2$, $i = 1$, and $j = 0$ [25].

Since V is self-dual, it has [9; Cor. 1.5g] unique kU_α-submodules $m = V_{\mu_1}$ and M of dimensions 1 and 3 respectively. If $H^1(G, V) \neq 0$, the same is true of $H^1(U_\alpha, V)^T$ by (2.3g). Since T is a 2'-group, there exists a non-zero T-stable cocycle $\gamma : U_\alpha \to V$.

If $c \in k$, $(x_\alpha(c)\gamma(x_\alpha(c)))^2 = 1$ by (2.2), hence $\gamma(x_\alpha(c))$ lies in the centralizer $C_V(x_\alpha(c))$.

(5.3) $\qquad$ If $c \neq 0$, $C_V(x_\alpha(c))$ is not T-stable.

Otherwise, for $t \in T$, $C_V(x_\alpha(c))^t = C_V(x_\alpha(c)^t) = C_V(x_\alpha(\alpha(t)c))$, hence

$$C_V(x_\alpha(c)) = C_V(U_\alpha) = m$$

for every $c \neq 0$. Now $\gamma : U_\alpha \to m$ is in $Z^1(U_\alpha, m)^T$, contradicting (2.7a) and (5.2a).

We leave the exceptional case $n = 2$, $i = 1$, $j = 0$ of (5.2b) to the reader [26], and assume henceforth $\mu_\ell \nsim \alpha$ for some $\ell = 2$ or 3. By (5.3) (or by tensor product considerations) V is not a uniserial kTU_α-module, and so there is a 2-dimensional kTU_α-submodule V_ℓ of M with $(M/V_\ell)_{\mu_\ell} = M/V_\ell$. Now $\gamma(x_\alpha(c))$ lies in $C_V(x_\alpha(c)) \subset M$,

---

[25] If $\mu_1 \sim \alpha$, then $2^\ell(2^i + 2^j) \equiv 1 \pmod{2^n - 1}$ for some ℓ, $0 \leq \ell < n$. If r, s are least residues of $i + \ell$ and $j + \ell$ respectively, then $2^r + 2^s - 1 \equiv 0 \pmod{2^n - 1}$. This implies $r = s$ and $i \equiv j \pmod{n}$, whence $i = j$, a contradiction.

Suppose $\mu_2 \sim \alpha \sim \mu_3$; then for some $0 \leq \ell$, $m < n$, we have $2^\ell(2^i - 2^j) \equiv 1 \equiv 2^m(2^j - 2^i) \pmod{2^n - 1}$. Setting $e = i - j$ and $f = |\ell - m|$ we obtain $2^e - 1 \equiv 2^u \pmod{2^n - 1}$ for some $0 \leq u < n$, and $(2^f + 1)(2^e - 1) \equiv 0 \pmod{2^n - 1}$. The first congruence implies $2^f + 1 \equiv 0 \pmod{2^n - 1}$, whence since $0 \leq f < n$, $f = 1$ and so $n = 2$ as required.

[26] Actually it is obvious that $H^1(G, V) = 0$ here, since V is the Steinberg module, which is projective [10].

and by (2.7a) the projection of γ on M/V_ℓ is 0, whence γ takes values in V_ℓ. Because of (5.3), we must have $V_\ell \cap C_V(x_\alpha(c)) = m$. But now γ takes values in m, again contradicting (2.7a) and (5.2a).

Assume $i=j$, and $q>2$ (we leave the case $q=2$ to the reader ([27])), then $V = M_i \otimes M_i$ is a Galois conjugate of $M_0 \otimes M_0 = \hat{M}_0 \otimes M_0 \cong \mathrm{Hom}_k(M_0, M_0)$. By footnote 9 (or (2.3c)) it suffices to treat the case $V = \mathrm{Hom}_k(M_0, M_0)$. Evidently V has unique submodules m and M of dimensions 1 and 3, and M/m is a Galois conjugate of M_0. Applying (2.3e) to $0 \to m \to V \to V/m \to 0$ gives $0 \to H^1(G, V) \to H^1(G, V/M)$ exact. Applying (2.3e) again to the sequence $0 \to M/m \to V/m \to V/M \to 0$ yields $\dim_k H^1(G, V/m) = \dim_k H^1(G, M/m) - 1$. By (4.2c) $\dim_k H^1(G, M/m) = 1$ ([28]). Q.E.D.

6. Action of Hecke algebras on cohomology.

Let A be a finite group and V a kA-module. When B is a subgroup of A whose index $[A:B]$ is not divisible by the characteristic p of k, the restriction map $H^1(A, V) \to H^1(B, V)$ is injective and its image consists of stable classes, as mentioned in (2.3g, d). Denote here the collection of stable classes in $H^1(B, V)$ by $H^1(B, V)^{B\backslash A/B}$. Then the stability theorem of Cartan-Eilenberg [5; p. 259] asserts

$$H^1(A, V) \cong H^1(B, V)^{B\backslash A/B};$$

indeed Cartan-Eilenberg prove the analogous result for n-dimensional cohomology, $n \in \mathbf{Z}$.

Our notation suggests that the stable classes are the " fixed points " for an action of the Hecke algebra on $H^1(B, V)$. In fact the Hecke algebra $B\backslash A/B$ does act naturally on $H^n(B, V)$ for all integers n, all subgroups B of A, and all **Z**A-modules V. If, as in the present case, V is a p-group and $p \nmid [A:B]$, then the stable classes may be interpreted as " fixed points " of this action.

We sketch the details. The **Z**-linear combinations in the rational group algebra **Q**A of elements $\frac{1}{|B|}\underline{BaB}$ form a **Z**-algebra $B\backslash A/B$. Here $\underline{BaB}$ denotes the sum of all members of the B, B double coset BaB of A. For $\mu \in H^n(B, V)$ define $\mu\left(\frac{1}{|B|}\underline{BaB}\right) = \mu^a|_{B^a \cap B}|^B$, where $|^B$ is corestriction (defined in dimension 0 to be a sum of multiplications by coset representatives) ([29]). It is easily checked that this defines an *action* of $B\backslash A/B$ on $H^0(B, V)$ for all V, also $\hat{H}^0(B, V)$, hence on $H^n(B, V)$ by dimension shifting.

When V is a p-group and $p \nmid [A:B]$ the stable classes may be interpreted as " fixed points " for $B\backslash A/B$ in the following way: There is a natural homomorphism

([27]) Again this case is immediate since M_0 is the Steinberg module.

([28]) It should be noted that, for $q>4$, the upper bound in (4.2c) depends only on (2.8).

([29]) A similar definition appears in Shimura, *Introduction to the arithmetic theory of automorphic functions*, Princeton University Press, 1971.

from the Hecke algebra $B\backslash A/B$ into $\mathbf{Z}$ which may be obtained from the augmentation $\mathbf{Q}A \to \mathbf{Q}$; the value of this homomorphism on $\frac{1}{|B|}\underline{BaB}$ is $[B : B^a \cap B]$. So if M is any module for $B\backslash A/B$ it is reasonable to call $m \in M$ a " fixed point " for $B\backslash A/B$ provided $m\left(\frac{1}{|B|}\underline{BaB}\right) = [B : B^a \cap B]m$ for all $a \in A$. Now observe that in case M is the set of B-fixed points of an A-module V we have $m\left(\sum_a \frac{1}{|B|}\underline{BaB}\right) = m|^A|_B$ for any $m \in M$ (in the sum a ranges over a set of B, B double coset representatives). It follows easily that multiplication by $\sum_a \frac{1}{|B|}\underline{BaB}$ is the same as $|^A|_B$ on $H^n(B, V)$ for any n. So if $m \in H^n(B, V)$ is a " fixed point " for the action of $B\backslash A/B$ we have $m|^A|_B = \sum_a [B : B^a \cap B]m = [A : B]m$. In the present case $(p \nmid [A : B])$ this implies that m is the *restriction of an element of* $H^n(A, V)$, and so certainly a stable class. On the other hand all stable classes are obviously " fixed points " for the action of $B\backslash A/B$, since restriction followed by corestriction is multiplication by the index. *Thus we have shown that, when* V *is a p-group and* $p \nmid [A : B]$, *the stable classes are precisely the " fixed points " of the Hecke algebra* $B\backslash A/B$.

The homological " explanation " for this Hecke algebra action seems to be the fact that $\mathrm{Ext}^n_A(T|^A, V) \simeq \mathrm{Ext}^n_B(T, V|_B)$. Here $|^A$ and $|_B$ denote induction and restriction. $\mathrm{Ext}^n_A(T|^A, V)$ is obviously a $\mathrm{Hom}_A(T|^A, T|^A)$ module. In case $T = \mathbf{Z}$ with trivial B-action, $\mathrm{Hom}_A(T|^A, T|^A)$ is well-known to be isomorphic to the Hecke algebra, while $\mathrm{Ext}^n_B(T, V|_B) \cong H^n(H, V|_B)$.

Finally we mention that there seems to be no corresponding Hecke algebra action for algebraic K-theory. It is of course possible to formally reproduce the definition $\mu\left(\frac{1}{|B|}\underline{BaB}\right) = \mu^a|_{B^a \cap B}|^B$ but this simply does not define an *action* of the Hecke algebra $B\backslash A/B$—not even on $K_0(\mathbf{C}B)$ where $\mathbf{C}$ = complex numbers, B = cyclic group of order two, A = symmetric group on three letters.

We conclude this section with an application of the Hecke algebra action. A somewhat less obvious proof can be given directly.

Corollary **(6.1)** [30]. — *Let* G *be a finite group with a* BN-*pair* (see [2; Chapter IV]), *and let* V *be a k*G-*module. Let* $W = N/(B \cap N)$ *be the Weyl group of* G, *and let* $\{w_\alpha\}_{\alpha \in \Delta}$ *be the associated set of fundamental reflections. Assume that the characteristic p of k does not divide* [G : B].

Then a necessary and sufficient condition that a class

$$\mu \in H^n(B, V)$$

be stable $(\mu^w|_{B^w \cap B} = \mu|_{B^w \cap B}$ *for all* $w \in W)$ *is that* $\mu^{w_\alpha}|_{B^{w_\alpha} \cap B} = \mu|_{B^{w_\alpha} \cap B}$ *for* $\alpha \in \Delta$.

[30] This result has been obtained independently by George Glauberman.

Proof. — The hypothesis implies

$$\mu\left(\frac{1}{B}\underline{Bw_\alpha B}\right)=[B:B\cap B^{w_\alpha}]\mu$$

for each $\alpha\in\Delta$. Hence μ is a "fixed point" for the subalgebra generated by the various $\frac{1}{|B|}\underline{Bw_\alpha B}$. It is an easy exercise ([31]) to show from the axioms for a BN-pair that this subalgebra is in fact the full Hecke algebra. Q.E.D.

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([31]) It is enough to show $\frac{1}{|B|}\underline{BwB}\,\frac{1}{|B|}\underline{Bw_rB}=\frac{1}{|B|}\underline{Bww_rB}$ whenever $\ell(ww_r)>\ell(w)$ and w_r is a fundamental reflection. Since $(BwB)(Bw_rB)=Bww_rB$ here, the above equation is certainly true up to a scalar multiple of the right hand side. A formal calculation shows this multiple is $\frac{1}{|B|}|Bw_rB\cap Bw^{-1}Bww_r|$. If this number is not 1, then $B^w\cap BB^{w_r}\nsubseteq B$, so $B^w\cap Bw_rB\neq\varnothing$, which is impossible since $(BwB)(Bw_rB)\cap BwB=\varnothing$.

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